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Two-Grid Finite Element Methods for the Steady Navier-Stokes/Darcy Model

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Abstract. Two-grid finite element methods for the steady Navier-Stokes/Darcy model are considered. Stability and optimal error estimates in the H^1-norm for velocity and piezometric approximations and the L^2-norm for pressure are established under mesh sizes satisfying $h = H^2$. A modified decoupled and linearised two-grid algorithm is developed, together with some associated optimal error estimates. Our method and results extend and improve an earlier investigation, and some numerical computations illustrate the efficiency and effectiveness of the new algorithm.

AMS subject classifications: 65N15, 65N30, 76D07

Key words: Navier-Stokes equations, Darcy's law, multimodeling problems, two-grid method.

1. Introduction

A two-grid method is an efficient numerical scheme for partial differential equation (PDE) based on two spaces with different meshes, first introduced by Xu [26, 27] for both linear and nonlinear elliptic PDE. Two-grid schemes have since been studied by many researchers. For example, Dawson *et al.* [6, 7] studied nonlinear parabolic equations using both finite element and finite difference methods. For the Navier-Stokes equations, we refer to Refs. [14, 17–21] and references therein. Recently, we have studied the stability and convergence of a two-grid finite volume method for nonlinear parabolic problems in semi-discrete and fully discrete formulations — cf. Refs. [28, 29, 31], respectively.

In recent years, the coupling of incompressible fluid flow with porous media flows has also been researched extensively [8–10, 15, 22, 30]. The fluid flow and the porous media flow are respectively modelled by the Navier-Stokes equations and Darcy's law, with the interface coupled via certain conditions. Several numerical schemes have been proposed for this model [5, 11], but any implementation for the coupled nonlinear discrete problem is

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usually complicated and difficult. In order to solve the Navier-Stokes/Darcy model model efficiently, Cai *et al.* [4] developed a decoupled linearised method, and established the following convergence result for velocity and pressure:

$$
\begin{aligned}
&\|\mathbf{u}_f - \mathbf{u}_{fh}^d\|_{(H^1(\Omega_f))^d} + \|p_f - p_{fh}^d\|_{L^2(\Omega_f)} \\
\le& C(h + H^{3/2})\Big(\|\mathbf{u}_f\|_{H^2(\Omega_f)^d} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)}\Big). \qquad (1.1)
\end{aligned}
$$

Here and below, C (with or without a subscript) denotes a positive constant, independent of the two mesh sizes h and H where $h \ll H$. The estimates for $\mathbf{u}_f$ and p_f are not optimal, as suggested by numerical experiments [4], and the estimate (1.1) might be improved to $O(H^2)$. This motivated us to propose our modified decoupled two-grid finite element numerical scheme, and derive the optimal estimates of $O(h)$ for both $\mathbf{u}_f$ and p_f with $H = \sqrt{h}$.

The classical two-grid technique we adopt to treat the steady Navier-Stokes/Darcy problem involves: (I) a coupled and nonlinear problem on a coarse grid with mesh size H; and (II) a coupled and linear problem on fine mesh with mesh size $h = H^2$, using Newton iteration to linearise the nonlinear term. Consequently, in lieu of (1.1) we have

$$
\begin{aligned}
&\|\mathbf{u}_f - \mathbf{u}_{fh}^c\|_{(H^1(\Omega_f))^d} + \|\phi - \phi_h^c\|_{H^1(\Omega_p)} + \|p_f - p_{fh}^c\|_{L^2(\Omega_f)} \\
\le& C(h + H^2)\Big(\|\mathbf{u}_f\|_{H^2(\Omega_f)^d} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)}\Big). \qquad (1.2)
\end{aligned}
$$

Secondly, we develop a modified decoupled and linearised two-grid algorithm. Our numerical scheme is thus: (I) a coupled Navier-Stokes/Darcy model to be solved on the coarse mesh; (II) a Darcy problem with the coarse grid approximation $(\mathbf{u}_{fH}, p_H, \psi_H)$ to the interface coupling conditions on the fine mesh; and (III) a linearised Navier-Stokes problem treated with the numerical solution of the Darcy problem on the fine mesh and the interface coupling conditions. Consequently, we then obtain

$$
\begin{aligned}
&\|\mathbf{u}_f - \mathbf{u}_{fh}^{md}\|_{(H^1(\Omega_f))^d} + \|p_f - p_{fh}^{md}\|_{L^2(\Omega_f)} \\
\le& C(h + H^2)\Big(\|\mathbf{u}_f\|_{H^2(\Omega_f)^d} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)}\Big). \qquad (1.3)
\end{aligned}
$$

Comparing (1.1) with (1.2) and (1.3), in using the two mesh sizes satisfying $h = H^2$ we evidently have improved the error estimates in Ref. [4] for both $\mathbf{u}_f$ and p_f.

The rest of this article is organised as follows. The coupled steady Navier-Stokes/Darcy model and associated properties are described in Section 2. In Section 3, the coupled two-grid finite element method is developed and its convergence is established. In Section 4, the decoupled linearised two-grid scheme in Ref. [27] is reviewed and our modified decoupled and linearised algorithm to improve the computational efficiency is proposed. Some enhancements of our decoupled two-grid method are discussed in Section 5, and numerical results that verify the performance of our developed numerical schemes are presented in Section 6. Brief concluding remarks are made in Section 7.

2. The Steady Coupled Navier-Stokes/Darcy Model

We consider coupled fluid and porous media flows on a domain $\Omega \subset R^d$ where $d = 2$ or 3, consisting of a fluid region Ω_f and a porous media region Ω_p separated by an interface Γ — i.e. such that $\Omega = \Omega_f \cup \Omega_p$, $\Omega_f \cap \Omega_p = \emptyset$ and $\overline{\Omega}_f \cap \overline{\Omega}_p = \Gamma$. Let $\mathbf{n}_f$ and $\mathbf{n}_p$ denote the unit outward normal directions on the boundaries $\partial\Omega_f$ and $\partial\Omega_p$ of Ω_f and Ω_p, respectively. The interface Γ is assumed to be sufficiently smooth, as in Refs. [4,15,22].

The fluid motion is governed by the steady Navier-Stokes equations [25]:

$$\begin{cases} -\nabla\cdot \mathbf{T}(\mathbf{u}_f, p_f) + \rho_f(\mathbf{u}_f\cdot\nabla)\mathbf{u}_f = \mathbf{g}_f & \text{in } \Omega_f\,, \\ \nabla\cdot\mathbf{u}_f = 0 & \text{in } \Omega_f\,, \end{cases} \tag{2.1}$$

where ρ_f denotes the density of the fluid, $\mathbf{u}_f$ the velocity in Ω_f, p_f the kinetic pressure, $\mathbf{g}_f$ is the external force, and

$$\mathbf{T}(\mathbf{u}_f, p) = 2\nu\mathbf{D}(\mathbf{u}_f) - p_f\mathbf{I}$$

is the stress tensor, where $\nu > 0$ is the viscosity coefficient and

$$\mathbf{D}(\mathbf{u}_f) = \frac{1}{2}(\nabla\mathbf{u}_f + \nabla^T\mathbf{u}_f)$$

is the deformation tensor. The porous media flow is governed by Darcy's law [23], involving the piezometric head ϕ and the discharge vector $\mathbf{q}$ proportional to the velocity $\mathbf{u}_p$ — i.e. $\mathbf{q} = n\mathbf{u}_p$, where n defines the volumetric porosity. Thus we have

$$\begin{cases} \nabla\cdot\mathbf{q} = g_p & \text{in } \Omega_p\,, \\ \mathbf{q} = -\mathbf{K}\cdot\nabla\phi & \text{in } \Omega_p \quad \text{(Darcy's law)}\,, \end{cases} \tag{2.2}$$

where $\mathbf{K}$ is the hydraulic conductivity tensor of the porous medium, the source g_p satisfies the solvability condition

$$\int_{\Omega_p} g_p = 0\,,$$

and $\phi = z + p_p/(\rho_f g)$ where z is the elevation from a reference level, p_p is the pressure in Ω_p and g is the gravity acceleration. We assume that the representation of $\mathbf{K}$ is a symmetric positive definite matrix uniformly bounded above and below — i.e. there exist constants $\lambda_{\min} > 0$ and $\lambda_{\max} > 0$ such that

$$a.e.\;\; \mathbf{x}\in\Omega_p\,, \quad \lambda_{\min}\mathbf{x}\cdot\mathbf{x} \le \mathbf{K}\mathbf{x}\cdot\mathbf{x} \le \lambda_{\max}\mathbf{x}\cdot\mathbf{x}\,. \tag{2.3}$$

From Darcy's law, the other (continuity) equation in (2.2) can be rewritten in the elliptic form

$$-\nabla\cdot(\mathbf{K}\cdot\nabla\phi) = g_p \qquad \text{in } \Omega_p\,. \tag{2.4}$$

A key aspect in any mixed model is the interface coupling, and we retain the following interface conditions on Γ [2,12,13,15,24]:

$$\begin{cases} \mathbf{u}_f \cdot \mathbf{n}_f + \mathbf{u}_p \cdot \mathbf{n}_p = \mathbf{0}, \\ \mathbf{n}_f \cdot [-\mathbf{T}(\mathbf{u}_f, p) \cdot \mathbf{n}_f] = \rho_f g \phi, \\ \tau_i \cdot [-\mathbf{T}(\mathbf{u}_f, p) \cdot \mathbf{n}_f] = \dfrac{\alpha}{\sqrt{\tau_i \cdot \nu \mathbf{K} \cdot \tau_i}} \mathbf{u}_f \cdot \tau_i, \quad i = 1, \cdots, d-1, \end{cases} \tag{2.5}$$

where α is a positive parameter depending on the properties of the porous medium that is experimentally determined, and $\{\tau_i\}_{i=1}^{d-1}$ are linearly independent unit tangential vectors on Γ (where d denotes the spacial dimension as above). The first interface condition ensures mass conservation across the interface Γ, the second the balance of normal forces across the interface, and the third states that the slip velocity along Γ is proportional to the shear stress along Γ. Certain mathematical features have been introduced for the third condition in the literature to expedite the mathematical analysis (e.g. an inertial energy term $\rho_f \mathbf{u}_f \cdot \mathbf{u}_f / 2$ has been included in Ref. [5]), but their physical justification is unclear.

For simplicity, we assume that n, ρ_f and g are constants. Several types of boundary conditions for this coupled model are discussed in Ref. [2], and here we choose to adopt homogeneous Dirichlet boundary conditions — viz.

$$\mathbf{u}_f = \mathbf{0} \ \text{ on } \partial\Omega_f \backslash \Gamma \quad \text{and} \quad \phi = 0 \ \text{ on } \partial\Omega_p \backslash \Gamma.$$

On denoting $W = H_f \times H_p$ and $Q = L^2(\Omega_f)$ where

$$H_f = \left\{ \mathbf{v} \in (H^1(\Omega_f))^d \,|\, \mathbf{v} = 0 \text{ on } \partial\Omega_f \right\} \quad \text{and} \quad H_p = \left\{ \psi \in H^1(\Omega_p) \,|\, \psi = 0 \text{ on } \partial\Omega_p \right\},$$

we obtain the weak formulation for the steady coupled Navier-Stokes/Darcy problem:

For $f \in W'$, find $u = (\mathbf{u}_f, \phi) \in W$, $p_f \in Q$ such that

$$\begin{cases} a(u, v) + d(v, p_f) + b(\mathbf{u}_f, \mathbf{u}_f, \mathbf{v}) = f(v) & \forall \ v = (\mathbf{v}, \psi) \in W, \\ d(u, q) = 0 & \forall \ q \in Q, \end{cases} \tag{2.6}$$

where $a(u, v) = a_\Omega(u, v) + a_\Gamma(u, v)$ involves

$$a_\Omega(u, v) = a_{\Omega_f}(\mathbf{u}_f, \mathbf{v}) + a_{\Omega_p}(\phi, \psi), \qquad f(v) = n \int_{\Omega_f} \mathbf{g}_f \cdot \mathbf{v} + \rho_f g \int_{\Omega_p} g_p \psi,$$

$$a_{\Omega_f}(\mathbf{u}_f, \mathbf{v}) = n \int_{\Omega_f} 2\nu \mathbf{D}(\mathbf{u}_f) : \mathbf{D}(\mathbf{v}) + n \sum_{i=1}^{d-1} \int_\Gamma \frac{\alpha}{\sqrt{\tau_i \cdot \nu \mathbf{K} \cdot \tau_i}} (\mathbf{u}_f \cdot \tau_i) \cdot (\mathbf{v} \cdot \tau_i),$$

$$a_{\Omega_p}(\phi, \psi) = \rho_f g \int_{\Omega_p} \mathbf{K} \cdot \nabla\phi \cdot \nabla\psi, \qquad a_\Gamma(u, v) = n \rho_f g \int_\Gamma (\phi \mathbf{v} \cdot \mathbf{n}_f - \psi \mathbf{u}_f \cdot \mathbf{n}_f),$$

$$d(v, p_f) = -n \int_{\Omega_f} p_f \nabla \cdot \mathbf{v}, \qquad b(\mathbf{u}_f, \mathbf{u}_f, \mathbf{v}) = n \rho_f \int_{\Omega_f} (\mathbf{u}_f \cdot \nabla) \mathbf{u}_f \cdot \mathbf{v}.$$

We recall the following Poincare and Korn's inequalities, trace and Sobolev inequalities. There exist constants $\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3, C_1, C_2, C_3, C_4$ and C_5 that depend on Ω_f such that for all $\mathbf{v} \in H_f$, $\phi \in H_p$ we have

$$\begin{aligned}
&\|\mathbf{v}\|_{L^2(\Omega_f)} \leq \mathscr{P}_1 \|\nabla \mathbf{v}\|_{L^2(\Omega_f)}, && \|\mathbf{v}\|_{L^4(\Omega_f)} \leq \mathscr{P}_2 \|\nabla \mathbf{v}\|_{L^2(\Omega_f)}, \\
&\|\phi\|_{L^2(\Omega_p)} \leq \mathscr{P}_3 \|\nabla \phi\|_{L^2(\Omega_p)}, && \|\nabla \mathbf{v}\|_{L^2(\Omega_f)} \leq C_1 \|\mathbf{D}(\mathbf{v})\|_{L^2(\Omega_f)}, \\
&\|\mathbf{v}\|_{L^2(\Gamma)} \leq C_2 \|\nabla \mathbf{v}\|_{L^2(\Omega_f)}, && \|\mathbf{v}\|_{L^4(\Gamma)} \leq C_3 \|\nabla \mathbf{v}\|_{L^2(\Omega_f)}, \\
&C_4 \|\mathbf{K}^{1/2} \nabla \phi\|^2_{L^2(\Omega_p)} \leq a_{\Omega_p}(\phi, \phi), && \|\phi\|_{L^2(\Omega_p)} \leq C_5 \|\mathbf{K}^{1/2} \nabla \phi\|_{L^2(\Omega_p)}.
\end{aligned} \tag{2.7}$$

Given (2.3), for all $\phi \in H^1(\Omega_p)$ we have

$$\frac{1}{\sqrt{\lambda_{\max}}} \|\mathbf{K}^{1/2} \nabla \phi\|_{L^2(\Omega_p)} \leq \|\nabla \phi\|_{L^2(\Omega_p)} \leq \frac{1}{\sqrt{\lambda_{\min}}} \|\mathbf{K}^{1/2} \nabla \phi\|_{L^2(\Omega_p)}.$$

Furthermore, the following estimates on $b(\cdot,\cdot,\cdot)$ are useful in our analysis (cf. [5,25]):

$$\begin{aligned}
|b(\mathbf{u},\mathbf{v},\mathbf{w})| &\leq n\rho_f \|\mathbf{u}\|_{L^4(\Omega_f)} \|\nabla \mathbf{v}\|_{L^2(\Omega_f)} \|\mathbf{w}\|_{L^4(\Omega_f)} \\
&\leq n\rho_f \mathscr{P}_2^2 \|\nabla \mathbf{u}\|_{L^2(\Omega_f)} \|\nabla \mathbf{v}\|_{L^2(\Omega_f)} \|\nabla \mathbf{w}\|_{L^2(\Omega_f)},
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
|b(\mathbf{u},\mathbf{v},\mathbf{w})| &\leq n\rho_f \|\mathbf{u}\|_{L^2(\Omega_f)} \|\nabla \mathbf{v}\|_{L^4(\Omega_f)} \|\mathbf{w}\|_{L^4(\Omega_f)} \\
&\leq n\rho_f \mathscr{P}_2^2 \|\mathbf{u}\|_{L^2(\Omega_f)} \|\mathbf{v}\|_{H^2(\Omega_f)} \|\nabla \mathbf{w}\|_{L^2(\Omega_f)}.
\end{aligned} \tag{2.9}$$

By using the nonlinear Steklov-Poincare operator under the conditions that the normal velocity across the interface is sufficiently small and the viscosity is sufficiently large, Badea *et al.* [1] and Discacciati [10] have established well-posedness for the coupled Navier-Stokes/Darcy model (2.6).

Theorem 2.1 (cf. [1,11])**.** *Assume that the data satisfies:*

$$\frac{\mathscr{P}_1^2 C_1^2}{\nu} \|\mathbf{g}_f\|^2_{L^2(\Omega_f)} + \frac{2\mathscr{P}_3^2}{\lambda_{\min}} \|g_p\|^2_{L^2(\Omega_p)} < \frac{\nu^3}{C_1^6 \mathscr{P}_2^4}.$$

Then the problem (2.6) *has at most one weak solution satisfying*

$$\|\nabla \mathbf{u}_f\|^2_{L^2(\Omega)} \leq \frac{\mathscr{P}_1^2 C_1^2}{2\nu^2} \|\mathbf{g}_f\|^2_{L^2(\Omega_f)} + \frac{\mathscr{P}_3^2}{\nu \lambda_{\min}} \|g_p\|^2_{L^2(\Omega_p)}.$$

Similarly to Ref. [11], we can readily verify that: (i) $a(\cdot,\cdot)$ is continuous and coercive on W; and (ii) $b(\cdot,\cdot)$ is continuous on $W \times Q$, and satisfies the well-known Brezzi-Babuska condition — i.e. there exists a constant $\beta > 0$ such that for all $q \in Q$ there exists some $w \in W$ where $d(w,q) \geq \beta \|w\|_W \|q\|_Q$.

Remark 2.1. The weak formulation of (2.6) can be rewritten as the following coupled system: For all $(\mathbf{v},q) \in H_f \times Q$, $\psi \in H_p$, find $(\mathbf{u}_f, p_f) \in H_f \times Q$, $\phi \in H_p$ such that

$$\begin{cases}
a_{\Omega_f}(\mathbf{u}_f, \mathbf{v}) + d(\mathbf{v}, p_f) + b(\mathbf{u}_f, \mathbf{u}_f, \mathbf{v}) = (n\mathbf{g}_f, \mathbf{v}) - \int_\Gamma n\rho_f g \phi \mathbf{v} \cdot \mathbf{n}_f, \\
d(\mathbf{u}_f, q) = 0, \\
a_{\Omega_p}(\phi, \psi) = (\rho_f g g_p, \psi) + \int_\Gamma n\rho_f g \psi \mathbf{u}_f \cdot \mathbf{n}_f.
\end{cases} \tag{2.10}$$

3. Coupled Two-Grid Method for the Navier-Stokes/Darcy Problem

We now consider the classical two-grid method for the steady Navier-Stokes/Darcy problem, and determine the convergence of the numerical solutions. Let the triangulation of the global domain $\mathscr{T}_h$ be regular, compatible and quasi-uniform on Γ as described in Ref. [9], and let $W_h = H_{fh} \times H_{ph} \subset W$ and $Q_h \subset Q$ be two finite element spaces. The finite element discretisation applied to the model problem of (2.6) leads to a coupled discrete problem:

Find $u_h = (\mathbf{u}_{fh}, \phi_h) \in W_h,\ p_{fh} \in Q_h$ such that

$$\begin{cases} a(u_h, v_h) + d(v_h, p_{fh}) + b(\mathbf{u}_{fh}, \mathbf{u}_{fh}, \mathbf{v}_h) = f(v_h) & \forall\ v_h = (\mathbf{v}_h, \psi_h) \in W_h\,, \\ d(u_h, q_h) = 0 & \forall\ q_h \in Q_h\,. \end{cases} \tag{3.1}$$

The construction of the finite element spaces W_h and Q_h is as follows. The finite element spaces H_{fh} and Q_h approximating the velocity and pressure in the fluid region are assumed to satisfy the discrete inf-sup condition —- viz. there exists a constant $\beta^* > 0$ independent of h, such that $\forall\ \mathbf{v}_h \in H_{fh}, q_h \in Q_h$

$$d(\mathbf{v}_h, q_h) \geq \beta^* \|\mathbf{v}_h\|_{H_f} \|q_h\|_Q\,. \tag{3.2}$$

Several families of finite element spaces for the Navier-Stokes problem have been provided [3], all of which satisfy the discrete inf-sup condition of (3.2) and can be applied for H_{fh} and Q_h. For the porous media region H_{ph}, standard finite element approximations of $H^m(\Omega_p)$ can be used, such as piecewise linear elements for $m = 1$. By using a similar argument to the one used in Refs. [5, 10], we can establish well-posedness for the coupled discrete model (3.1) in a conforming finite element formulation (we omit the proof here).

Theorem 3.1 (cf. [5]). *Let*

$$\mathscr{R} = \left(\max\left(\frac{3}{4}, \frac{C_4}{2}\right)\right)^{1/2} \left(\frac{\mathscr{P}_1^2 C_1^2}{2\nu} \|\mathbf{g}_f\|^2_{L^2(\Omega_f)} + \frac{2C_5^2}{C_4} \|g_p\|^2_{L^2(\Omega_p)}\right)^{1/2}.$$

Under the conditions

$$\mathscr{R}^2 \leq \frac{32\nu^3}{C_1^6 C_2^2 C_3^4} \quad \textit{and} \quad \frac{C_1^3}{\sqrt{2}} \Big(\mathscr{P}_2^2 + \frac{1}{2} C_2 C_3^2\Big) \mathscr{R} < \nu^{3/2},$$

the problem (3.1) *admits a unique solution satisfying*

$$2\nu \|\nabla \mathbf{u}_{fh}\|^2_{L^2(\Omega_f)} + \|\mathbf{K}^{1/2} \nabla \phi_h\|^2_{L^2(\Omega_p)} \leq \mathscr{R}^2.$$

For convenience, henceforth we use $x \lesssim y$ to denote that there exists a constant C such that $x \leq Cy$. To obtain the error estimates, we assume the regularity of $u \in (H^2(\Omega_f))^d \times H^2(\Omega_p)$ and $p_f \in H^1(\Omega_f)$, and that the finite element spaces as described above of first order in h

are used for the fluid and porous media regions. The error analysis for the coupled model of (3.1) then yields the estimates (cf. [4])

$$\begin{aligned}&\|\mathbf{u}_f-\mathbf{u}_{fh}\|_{H^1(\Omega_f)}+\|\phi-\phi_h\|_{H^1(\Omega_p)}+\|p_f-p_{fh}\|_{L^2(\Omega_f)}\\ \le& Ch\Big(\|\mathbf{u}_f\|_{H^2(\Omega_f)^d}+\|\phi\|_{H^2(\Omega_p)}+\|p_f\|_{H^1(\Omega_f)}\Big).\end{aligned}\tag{3.3}$$

Furthermore, the extended framework of the Aubin-Nitsche duality technique [3] yields the L^2-norm estimate for the coupled model (3.1) as follows.

Theorem 3.2 (see [4]). *Letting* $(\mathbf{u}_f,\phi,p_f)\in H^2(\Omega_f)^d\times H^2(\Omega_p)\times H^1(\Omega_f)$ *be the solution of the Navier-Stokes/Darcy model* (2.6)*, and* $(\mathbf{u}_{fh},\phi_h,p_{fh})$ *be the finite element solution of* (3.1)*, for* ν *sufficiently large we have the* L^2*-error estimate*

$$\|\mathbf{u}_f-\mathbf{u}_{fh}\|_{L^2(\Omega_f)}+\|\phi-\phi_h\|_{L^2(\Omega_p)}\le Ch^2\Big(\|\mathbf{u}_f\|_{H^2(\Omega_f)^d}+\|\phi\|_{H^2(\Omega_p)}+\|p_f\|_{H^1(\Omega_f)}\Big).$$

By combining (3.3) and Theorem 3.2, for $h\ll H$ we then have

$$\begin{cases}\|\mathbf{u}_{fh}-\mathbf{u}_{fH}\|_{(H^1(\Omega_f))^d}\le CH\Big(\|\mathbf{u}_f\|_{H^2(\Omega_f)^d}+\|\phi\|_{H^2(\Omega_p)}+\|p_f\|_{H^1(\Omega_f)}\Big),\\ \|\mathbf{u}_{fh}-\mathbf{u}_{fH}\|_{(L^2(\Omega_f))^d}\le CH^2\Big(\|\mathbf{u}_f\|_{H^2(\Omega_f)^d}+\|\phi\|_{H^2(\Omega_p)}+\|p_f\|_{H^1(\Omega_f)}\Big),\\ \|\phi_h-\phi_H\|_{H^1(\Omega_p)}\le CH\Big(\|\mathbf{u}_f\|_{H^2(\Omega_f)^d}+\|\phi\|_{H^2(\Omega_p)}+\|p_f\|_{H^1(\Omega_f)}\Big),\\ \|\phi_h-\phi_H\|_{L^2(\Omega_p)}\le CH^2\Big(\|\mathbf{u}_f\|_{H^2(\Omega_f)^d}+\|\phi\|_{H^2(\Omega_p)}+\|p_f\|_{H^1(\Omega_f)}\Big).\end{cases}\tag{3.4}$$

From (3.1), we know that the variables $\mathbf{u}_{fh}$, p_{fh} and ϕ_h are coupled together by the boundary condition. In Ref. [5], the finite element discretisation in the flow region and a discontinuous Galerkin discretisation in the porous media region were adopted; and in Ref. [11], discontinuous Galerkin discretisation was used in both regions. However, these approaches result in discrete problems that are both coupled and nonlinear, where numerical difficulties increase as the mesh size decreases. In addition, the implementation is complicated and difficult. Henceforth, H and $h\ll H$ are taken to be two real positive parameters tending to 0. A coarse mesh triangulation of $\mathcal{T}_H$ is made as before, and a fine triangulation $\mathcal{T}_h$ is generated by a mesh refinement process to $\mathcal{T}_H$. The finite element spaces (W_H,Q_H) and $(W_h,Q_h)\subset(W_H,Q_H)$ are based on the triangulations $\mathcal{T}_H$ and $\mathcal{T}_h$, respectively. Adopting these finite element spaces, we consider the following two-grid finite element method.

`Coupled Two-Grid method (CTGM)`

Step 1. Solve the coupled problem (3.1) on a coarse grid with mesh size H as follows:
Find $u_H=(\mathbf{u}_{fH},\phi_H)\in W_H\subset W_h,\ p_{fH}\in Q_H\subset Q_h$ such that

$$\begin{cases}a(u_H,v_H)+d(v_H,p_{fH})+b(\mathbf{u}_{fH},\mathbf{u}_{fH},\mathbf{v}_H)=f(v_H) & \forall\ v_H=(\mathbf{v}_H,\psi_H)\in W_H,\\ d(u_H,q_H)=0 & \forall\ q_H\in Q_H.\end{cases}\tag{3.5}$$

Step 2. Solve the fine grid linearised problem using Newton iteration for the nonlinear term as follows:

Find $u_h^c=(\mathbf{u}_{fh}^c,\phi_h^c)\in W_h,\ p_h^c\in Q_h$ such that for all $v_h=(\mathbf{v}_h,\psi_h)\in W_h$ and $q_h\in Q_h$

$$\begin{cases} a(u_{fh}^c,v_h)+d(v_h,p_{fh}^c)+b(\mathbf{u}_{fH},\mathbf{u}_{fh}^c,\mathbf{v}_h)+b(\mathbf{u}_{fh}^c,\mathbf{u}_{fH},\mathbf{v}_h)=f(v_h)+b(\mathbf{u}_{fH},\mathbf{u}_{fH},\mathbf{v}_h), \\ d(u_{fh}^c,q_h)=0. \end{cases} \tag{3.6}$$

By the Lax-Milgram theorem, from the continuities and coercivity of $a_{\Omega_f}(\cdot,\cdot)$ and $a_{\Omega_p}(\cdot,\cdot)$ and (3.2) we know that problem (3.6) has unique solution. Furthermore, we obtain the boundedness of the coarse numerical solutions of problem (3.5) under the conditions stated in Theorem 3.1, and we may consequently consider the convergence of the coupled two-grid finite element solution (u_h^c,p_h^c) to (u,p) in some norm. To do this, we subtract (3.1) from (3.6) and so obtain the following error equations for all $(v_h,q_h)\in W_h\times Q_h$:

$$\begin{cases} a(u_h-u_{fh}^c,v_h)+d(v_h,p_h-p_{fh}^c)+b(\mathbf{u}_{fh},\mathbf{u}_{fh},\mathbf{v}_h)-b(\mathbf{u}_{fH},\mathbf{u}_{fh}^c,\mathbf{v}_h) \\ \qquad -b(\mathbf{u}_{fh}^c,\mathbf{u}_{fH},\mathbf{v}_h)+b(\mathbf{u}_{fH},\mathbf{u}_{fH},\mathbf{v}_h)=0, \\ d(u_h-u_{fh}^c,q_h)=0. \end{cases} \tag{3.7}$$

Theorem 3.3. *Let $(\mathbf{u}_{fh},p_{fh},\phi_h)$ and $(\mathbf{u}_{fh}^c,p_{fh}^c,\phi_h^c)$ be defined by the discrete models* (3.1) *and* (3.6) *on the fine grid, respectively. Under the conditions of Theorem* 3.1*, we have*

$$\begin{aligned} &\|\mathbf{u}_{fh}-\mathbf{u}_{fh}^c\|_{H_f}+\|p_{fh}-p_{fh}^c\|_Q+\|\phi_h-\phi_h^c\|_{H_p} \\ \le& CH^2\left(\|\mathbf{u}_f\|_{H^2(\Omega_f)^d}+\|\phi\|_{H^2(\Omega_p)}+\|p_f\|_{H^1(\Omega_f)}\right). \end{aligned}$$

Proof. For the trilinear terms in (3.7), it is easy to verify that

$$\begin{aligned} &b(\mathbf{u}_{fh},\mathbf{u}_{fh},\mathbf{v}_h)-\left[b(\mathbf{u}_{fH},\mathbf{u}_{fh}^c,\mathbf{v}_h)+b(\mathbf{u}_{fh}^c,\mathbf{u}_{fH},\mathbf{v}_h)-b(\mathbf{u}_{fH},\mathbf{u}_{fH},\mathbf{v}_h)\right] \\ =& b(\mathbf{u}_{fH},\mathbf{u}_{fh}-\mathbf{u}_{fh}^c,\mathbf{v}_h)+b(\mathbf{u}_{fh}-\mathbf{u}_{fH},\mathbf{u}_{fh}-\mathbf{u}_{fH},\mathbf{v}_h)+b(\mathbf{u}_{fh}-\mathbf{u}_{fh}^c,\mathbf{u}_{fH},\mathbf{v}_h). \end{aligned} \tag{3.8}$$

Choosing $v_h=u_h-u_{fh}^c$ and $q_h=p_h-p_{fh}^c$ in (3.7) and using (3.8), we have

$$\begin{aligned} &n\nu\|\nabla(\mathbf{u}_{fh}-\mathbf{u}_{fh}^c)\|_{L^2(\Omega_f)}^2+n\sum_{i=1}^{d-1}\frac{\alpha}{\sqrt{\tau_i\cdot\nu\mathbf{K}\tau_i}}\|(\mathbf{u}_{fh}-\mathbf{u}_{fh}^c)\cdot\tau_i\|_{L^2(\Gamma)}^2+\rho_f g\|\mathbf{K}^{1/2}\nabla(\phi_h-\phi_h^c)\|_{L^2(\Omega_p)}^2 \\ \le& n\rho_f\mathscr{P}_2^2\Big(\|\nabla(\mathbf{u}_{fh}-\mathbf{u}_{fh}^c)\|_{L^2(\Omega_f)}\|\nabla(\mathbf{u}_{fh}-\mathbf{u}_{fH})\|_{L^2(\Omega_f)}^2 \\ &+\|\nabla(\mathbf{u}_{fh}-\mathbf{u}_{fh}^c)\|_{L^2(\Omega_f)}^2\|\nabla\mathbf{u}_{fH}\|_{L^2(\Omega_f)}\Big). \end{aligned}$$

Under the conditions of Theorem 3.1, on applying triangle inequality and (3.4) we obtain

$$\begin{aligned} &\|\mathbf{u}_{fh}-\mathbf{u}_{fh}^c\|_{H_f}+n\sum_{i=1}^{d-1}\frac{\alpha}{\sqrt{\tau_i\cdot\nu\mathbf{K}\tau_i}}\|(\mathbf{u}_{fh}-\mathbf{u}_{fh}^c)\cdot\tau_i\|_{L^2(\Gamma)}+\|\phi_h-\phi_h^c\|_{H_p} \\ \le& CH^2\left(\|\mathbf{u}_f\|_{H^2(\Omega_f)^d}+\|\phi\|_{H^2(\Omega_p)}+\|p_f\|_{H^1(\Omega_f)}\right). \end{aligned} \tag{3.9}$$

Furthermore, thanks to the discrete inf-sup condition (3.2) one finds

$$\begin{aligned}\|p_h - p^c_{fh}\| \lesssim \tfrac{d(v_h, p_h - p^c_{fh})}{\|\nabla v_h\|_0} \lesssim n\nu \|\nabla(\mathbf{u}_{fh} - \mathbf{u}^c_{fh})\|_{L^2(\Omega_f)} \\ + n\textstyle\sum_{i=1}^{d-1} \tfrac{\alpha}{\sqrt{\tau_i \cdot \nu \mathbf{K}\tau_i}} \|(\mathbf{u}_{fh} - \mathbf{u}^c_{fh}) \cdot \tau_i\|_{L^2(\Gamma)} + \rho_f g \|\mathbf{K}^{1/2}\nabla(\phi_h - \phi^c_h)\|_{L^2(\Omega_p)} \\ + n\rho_f \mathscr{P}_2^2 \left(2\|\nabla \mathbf{u}_{fH}\|_0 \|\nabla(\mathbf{u}_{fh} - \mathbf{u}^c_{fh})\|_0 + \|\nabla(\mathbf{u}_{fh} - \mathbf{u}_{fH})\|_0^2 \right),\end{aligned}$$

so from Theorem 3.1, (3.3), (3.4) and (3.9) the proof is complete. □

4. Decoupled Two-Grid Methods for the Navier-Stokes/Darcy Problem

From the description of the coupled two-grid method (3.6), a large linear system needs to be treated on the fine mesh, and compared with the standard Galerkin finite element method a lot of computational cost is to be saved. In view of the large-scale of coefficient matrix obtained from the discrete algebraic equations, we decouple the equations into several subproblems, where each subproblem can be solved easily due to its small scale. Cai *et al,* [4] developed an efficient two-grid method, by decoupling the interface coupling conditions to treat the Navier-Stokes/Darcy problem, and their numerical scheme can be described as follows.

`Decoupled Two-Grid method (DTGM)`

Step 1. Solve the coupled problem (3.1) on the coarse grid with spacing H:
Find $u_H = (\mathbf{u}_{fH}, \phi_H) \in W_H \subset W_h,\ p_{fH} \in Q_H \subset Q_h$ by (3.5).

Step 2. Solve a modified fine grid linearized problem by using Newton iteration for the nonlinear term:
Find $u^d_{fh} = (\mathbf{u}^d_{fh}, \phi^d_h) \in W_h,\ p^d_{fh} \in Q_h$ such that for all $v_h = (\mathbf{v}_h, \psi_h) \in W_h$ and $q_h \in Q_h$

$$\left\{\begin{aligned} & a(u^d_{fh}, v_h) + d(v_h, p^d_{fh}) + b(\mathbf{u}_{fH}, \mathbf{u}^d_{fh}, \mathbf{v}_h) + b(\mathbf{u}^d_{fh}, \mathbf{u}_{fH}, \mathbf{v}_h) \\ & \qquad\qquad = f(v_h) - a_\Gamma(u_H, v_h) + b(\mathbf{u}_{fH}, \mathbf{u}_{fH}, \mathbf{v}_h), \\ & d(u^d_{fh}, q_h) = 0. \end{aligned}\right. \tag{4.1}$$

The discrete model of (4.1) is equivalent to two linearized problems — viz. the Navier-Stokes problem on Ω_f and the Darcy problem on Ω_p with the boundary conditions defined by u_H on Γ, respectively. Specifically, the discrete Darcy problem on the porous media region Ω_p is:

Find $\phi^d_h \in H_{ph}$ such that $\forall\ \psi_h \in H_{ph}$

$$a_{\Omega_p}(\phi^d_h, \psi_h) = (\rho_f g g_p, \psi_h) + \int_\Gamma n\rho_f g \psi_h \mathbf{u}_{fH} \cdot \mathbf{n}_f .$$

The linearised Navier-Stokes problem based on Newton iteration on the fluid region Ω_f is:

Find $\mathbf{u}_{fh}^d \in H_{fh},\ p_{fh}^d \in Q_h$ such that for any $(\mathbf{v}_h, q_h) \in H_{fh} \times Q_h$

$$\begin{cases} a_{\Omega_f}(\mathbf{u}_{fh}^d, \mathbf{v}_h) + d(\mathbf{v}_h, p_{fh}^d) + b(\mathbf{u}_{fH}, \mathbf{u}_{fh}^d, \mathbf{v}_h) + b(\mathbf{u}_{fh}^d, \mathbf{u}_{fH}, \mathbf{v}_h) \\ \qquad\qquad = (n\mathbf{g}_f, \mathbf{v}_h) - \int_\Gamma n\rho_f g \phi_H \mathbf{v}_h \cdot \mathbf{n}_f + b(\mathbf{u}_{fH}, \mathbf{u}_{fH}, \mathbf{v}_h), \\ d(\mathbf{u}_{fh}^d, q_h) = 0. \end{cases}$$

Furthermore, for the decoupled scheme (4.1), we recall the following results.

Theorem 4.1 (cf. [4]). *Let* $(\mathbf{u}_{fh}, p_{fh}, \phi_h)$ *and* $(\mathbf{u}_{fh}^d, p_{fh}^d, \phi_h^d)$ *be the solutions defined by* (3.1) *and* (4.1) *on the fine grid, respectively. Under the conditions of Theorem* 3.1 *and* $h \ll H$,

$$\|\phi_h - \phi_{fh}^d\|_{H^1(\Omega_p)} \le CH^2 \left(\|\mathbf{u}_f\|_{H^2(\Omega_f)^d} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)} \right),$$
$$\|\mathbf{u}_{fh} - \mathbf{u}_{fh}^d\|_{(H^1(\Omega_f))^d} + \|p_{fh} - p_{fh}^d\|_{L^2(\Omega_f)} \le CH^{3/2} \left(\|\mathbf{u}_f\|_{H^2(\Omega_f)^d} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)} \right).$$

The error estimates in Theorem 4.1 for $\mathbf{u}_{fh} - \mathbf{u}_{fh}^d$ and $p_{fh} - p_{fh}^d$ may not be optimal, as the numerical experiments in Ref. [4] suggest. We now proceed to our modified decoupled algorithm for the mixed model (2.6), and establish the corresponding error estimates. To provide a clear comparison, we also adopt the Newton iteration to deal with the nonlinear term, although of course other iterative schemes such as simple or Oseen iteration can be used to linearise the nonlinear term (and similar results obtained).

`Modified Decoupled Two-Grid Mehtod (MDTGM)`

Step 1. Solve the coupled problem (3.1) on the coarse grid with spacing H:
Find $u_H = (\mathbf{u}_{fH}, \phi_H) \in W_H \subset W_h,\ p_{fH} \in Q_H \subset Q_h$ by (3.5).

Step 2. Solve the discrete Darcy problem for the porous media region Ω_p on the fine grid:
Find $\phi_h^{md} \in H_{ph}$ such that

$$a_{\Omega_p}(\phi_h^{md}, \psi_h) = (\rho_f g g_p, \psi_h) + \int_\Gamma n\rho_f g \psi_h \mathbf{u}_{fH} \cdot \mathbf{n}_f \qquad \forall\ \psi_h \in H_{ph}. \tag{4.2}$$

Step 3. Solve the discrete linearised Navier-Stokes problem on the fluid region Ω_f on the fine grid:
Find $\mathbf{u}_{fh}^{md} \in H_{fh},\ p_{fh}^{md} \in Q_h$ such that for all $(\mathbf{v}_h, q_h) \in H_{fh} \times Q_h$

$$\begin{cases} a_{\Omega_f}(\mathbf{u}_{fh}^{md}, \mathbf{v}_h) + d(\mathbf{v}_h, p_{fh}^{md}) + b(\mathbf{u}_{fH}, \mathbf{u}_{fh}^{md}, \mathbf{v}_h) + b(\mathbf{u}_{fh}^{md}, \mathbf{u}_{fH}, \mathbf{v}_h) \\ \qquad\qquad = (n\mathbf{g}_f, \mathbf{v}_h) - \int_\Gamma n\rho_f g \phi_h^{md} \mathbf{v}_h \cdot \mathbf{n}_f + b(\mathbf{u}_{fH}, \mathbf{u}_{fH}, \mathbf{v}_h), \\ d(\mathbf{u}_{fh}^{md}, q_h) = 0. \end{cases} \tag{4.3}$$

From the continuities and coercivity of $a_{\Omega_f}(\cdot,\cdot)$ and $a_{\Omega_p}(\cdot,\cdot)$ and (3.2), we know problems (4.2) and (4.3) have unique solutions. Furthermore, both DTGM and MDTGM have some common advantages. Firstly, they are numerically efficient, as one can apply an efficient optimised local linear solver on the fine grid that has been well developed for the Navier-Stokes and Darcy equations. Secondly, two kinds of numerical schemes allow easy and efficient implementation and software reuse, in existing local solvers from available software resources. Finally, with a properly chosen coarse grid we can prove that these decoupled two-grid algorithms keep the same order of approximation accuracy as the coupled and nonlinear algorithm (3.1). Moreover, the DTGM can be used in parallelism based on the solutions on the coarse mesh, while the MDTGM improves the accuracy of numerical solutions at the price of losing parallelism. We use the numerical solution ϕ_h^{md} obtained on the finer mesh in Step 2 to approximate the interface boundary condition of Step 3.

Let us now analyse the convergence of our modified decoupled two-grid method.

Theorem 4.2. *Let* $(\mathbf{u}_{fh}, p_{fh}, \phi_h)$ *and* $(\mathbf{u}_{fh}^{md}, p_{fh}^{md}, \phi_h^{md})$ *be defined by the discrete models* (3.1) *and* (4.2)-(4.3) *on the fine grid, respectively. Under the conditions of Theorem* 3.1 *and*

$$\left(\max\left(\frac{3}{4},\frac{C_4}{2}\right)\right)^{\frac{1}{2}}\left(\frac{\mathscr{P}_1^2C_1^2}{2\nu}\|\mathbf{g}_f\|^2_{L^2(\Omega_f)}+\frac{2C_5^2}{C_4}\|g_p\|^2_{L^2(\Omega_p)}\right)^{\frac{1}{2}}\leq\frac{\sqrt{2\nu}}{8}, \tag{4.4}$$

we have

$$\begin{aligned}&\|\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md}\|_{H_f}+\|p_{fh}-p_{fh}^{md}\|_Q+\|\phi_h-\phi_h^{md}\|_{H_p}\\ \leq& CH^2\left(\|\mathbf{u}_f\|_{H^2(\Omega_f)^d}+\|\phi\|_{H^2(\Omega_p)}+\|p_f\|_{H^1(\Omega_f)}\right).\end{aligned}$$

Proof. By comparing the discrete models (3.1) with $(\mathbf{v}_h, q_h)=0$ and (4.2) on the fine grid, we have that for any $\phi_h\in H_{ph}$

$$a_{\Omega_p}(\phi_h-\phi_h^{md},\psi_h)=\int_\Gamma n\rho_f g\psi_h(\mathbf{u}_{fh}-\mathbf{u}_{fH})\cdot\mathbf{n}_f\,.$$

In particular, taking $\psi_h=\phi_h-\phi_h^{md}$ we get

$$a_{\Omega_p}(\phi_h-\phi_h^{md},\phi_h-\phi_h^{md})=\int_\Gamma n\rho_f g(\phi_h-\phi_h^{md})(\mathbf{u}_{fh}-\mathbf{u}_{fH})\cdot\mathbf{n}_f\,.$$

Setting $\theta\in H^1(\Omega_f)$ be a harmonic extension of $\phi_h-\phi_h^{md}$ to the fluid flow region satisfying

$$\begin{cases}-\Delta\theta=0 & \text{in } \Omega_f\,,\\ \theta=\phi_h-\phi_h^{md} & \text{on } \Gamma\,,\\ \theta=0 & \text{on } \partial\Omega_f\backslash\Gamma\end{cases} \tag{4.5}$$

and letting $H_{00}^{1/2}(\Gamma)$ denote the interpolation space (cf. [4, 15, 22]), we have

$$H_{00}^{1/2}(\Gamma)=\left[L^2(\Gamma),H_0^1(\Gamma)\right]_{1/2}.$$

Then the following inequality holds:

$$\|\theta\|_{H^1(\Omega_f)}\lesssim\|\phi_h-\phi_h^{md}\|_{H_{00}^{1/2}(\Gamma)}\lesssim\|\phi_h-\phi_h^{md}\|_{H_p}\,.$$

It follows from the second equality in (4.5) for any $q_H \in Q_H$ that

$$\begin{aligned}
&\int_\Gamma n\rho_f g(\phi_h - \phi_h^{md})(\mathbf{u}_{fh} - \mathbf{u}_{fH}) \cdot \mathbf{n}_f \\
=&\int_{\partial\Omega_f} n\rho_f g\theta(\mathbf{u}_{fh} - \mathbf{u}_{fH}) \cdot \mathbf{n}_f \\
=&\int_{\Omega_f} (n\rho_f g\theta) \cdot \nabla\cdot(\mathbf{u}_{fh} - \mathbf{u}_{fH}) + \int_{\Omega_f} \nabla(n\rho_f g\theta) \cdot (\mathbf{u}_{fh} - \mathbf{u}_{fH}) \\
=&n\rho_f g\left[\int_{\Omega_f} (\theta - q_H) \cdot \nabla\cdot(\mathbf{u}_{fh} - \mathbf{u}_{fH}) + \int_{\Omega_f} \nabla\theta \cdot (\mathbf{u}_{fh} - \mathbf{u}_{fH})\right].
\end{aligned}$$

Note that in the last equality we use the discrete divergence-free property for $\mathbf{u}_{fh}$ and $\mathbf{u}_{fH}$ — i.e. namely, for any $q_H \in Q_H$,

$$\int_{\Omega_f} q_H \cdot \nabla\cdot(\mathbf{u}_{fh} - \mathbf{u}_{fH}) = 0.$$

Thus we deduce the desired result for $\phi_h - \phi_h^{md}$ using the inequality

$$\begin{aligned}
\|\phi_h - \phi_h^{md}\|_{H_p}^2 \lesssim& a_{\Omega_p}(\phi_h - \phi_h^{md}, \phi_h - \phi_h^{md}) \\
\lesssim& \inf_{\forall q_H \in Q_H} \left|\int_{\Omega_f} (\theta - q_H) \cdot \mathrm{div}(\mathbf{u}_{fh} - \mathbf{u}_{fH})\right| + \left|\int_{\Omega_f} \nabla\theta \cdot (\mathbf{u}_{fh} - \mathbf{u}_{fH})\right| \\
\lesssim& \|\mathbf{u}_{fh} - \mathbf{u}_{fH}\|_{H_f} \|\theta - q_H\|_{L^2(\Omega_f)} + \|\mathbf{u}_{fh} - \mathbf{u}_{fH}\|_{(L^2(\Omega_f))^d} \|\theta\|_{H^1(\Omega_f)} \\
\lesssim& \left(H\|\mathbf{u}_{fh} - \mathbf{u}_{fH}\|_{H_f} + \|\mathbf{u}_{fh} - \mathbf{u}_{fH}\|_{(L^2(\Omega_f))^d}\right) \|\theta\|_{H^1(\Omega_f)} \\
\lesssim& H^2 \|\phi_h - \phi_h^{md}\|_{H_{00}^{1/2}(\Gamma)} \\
\le& CH^2 \left(\|\mathbf{u}_f\|_{H^2(\Omega_f)^d} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)}\right) \|\phi_h - \phi_h^{md}\|_{H_p}. \qquad (4.6)
\end{aligned}$$

On the other hand, from the discrete equations (3.1) with $\psi_h = 0$ and (4.3) we have

$$\begin{aligned}
&a_{\Omega_f}(\mathbf{u}_{fh} - \mathbf{u}_{fh}^{md}, \mathbf{v}_h) + d(\mathbf{v}_h, p_{fh} - p_{fh}^{md}) \\
=&\int_\Gamma n\rho_f g(\phi_h - \phi_h^{md})\mathbf{v}_h \cdot \mathbf{n}_f - b(\mathbf{u}_{fh}, \mathbf{u}_{fh}, \mathbf{v}_h) \\
&+ \left[b(\mathbf{u}_{fH}, \mathbf{u}_{fh}^{md}, \mathbf{v}_h) + b(\mathbf{u}_{fh}^{md}, \mathbf{u}_{fH}, \mathbf{v}_h) - b(\mathbf{u}_{fH}, \mathbf{u}_{fH}, \mathbf{v}_h)\right]. \qquad (4.7)
\end{aligned}$$

Choosing $\mathbf{v}_h = \mathbf{u}_{fh} - \mathbf{u}_{fh}^{md}$ in (4.7) and using the discrete divergence-free property of $\mathbf{u}_{fh}$

and $\mathbf{u}_{fh}^{md}$, we obtain

$$\begin{aligned}
&a_{\Omega_f}(\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md},\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md})\\
=&\int_\Gamma n\rho_f g(\phi_h-\phi_{fh}^{md})(\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md})\cdot\mathbf{n}_f+b(\mathbf{u}_{fH},\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md},\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md})\\
&+b(\mathbf{u}_{fh}-\mathbf{u}_{fH},\mathbf{u}_{fh}-\mathbf{u}_{fH},\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md})+b(\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md},\mathbf{u}_{fH},\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md}).
\end{aligned}$$

Hence from (2.7) and (2.8) we have

$$\begin{aligned}
&\|\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md}\|_{H_f}^2\\
\lesssim&\|\mathbf{D}(\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md})\|_{L^2(\Omega_f)}^2\lesssim a_{\Omega_f}(\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md},\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md})\\
=&\int_\Gamma n\rho_f g(\phi_h-\phi_h^{md})(\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md})\cdot\mathbf{n}_f+b(\mathbf{u}_{fH},\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md},\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md})\\
&+b(\mathbf{u}_{fh}-\mathbf{u}_{fH},\mathbf{u}_{fh}-\mathbf{u}_{fH},\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md})+b(\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md},\mathbf{u}_{fH},\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md})\\
\lesssim&\|\phi_h-\phi_h^{md}\|_{L^2(\Gamma)}\|\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md}\|_{(L^2(\Gamma))^d}+\|\nabla(\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md})\|_{L^2(\Omega_f)}\|\nabla(\mathbf{u}_{fh}-\mathbf{u}_{fH})\|_{L^2(\Omega_f)}^2\\
&+\|\nabla(\mathbf{u}_{fH})\|_{L^2(\Omega_f)}\|\nabla(\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md})\|_{L^2(\Omega_f)}^2\\
\lesssim&\left(\|\phi_h-\phi_h^{md}\|_{L^2(\Gamma)}+\|\mathbf{u}_{fh}-\mathbf{u}_{fH}\|_{H_f}^2\right)\|\nabla(\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md})\|_{L^2(\Omega_f)}\\
&+\|\nabla(\mathbf{u}_{fH})\|_{L^2(\Omega_f)}\|\nabla(\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md})\|_{L^2(\Omega_f)}^2.
\end{aligned}\tag{4.8}$$

With the help of the trace theorem and (4.6), we obtain

$$\begin{aligned}
\|\phi_h-\phi_h^{md}\|_{L^2(\Gamma)}\lesssim&\|\phi_h-\phi_h^{md}\|_{L^2(\Omega_p)}+\|\phi_h-\phi_h^{md}\|_{H_p}\\
\leq&CH^2\left(\|\mathbf{u}_f\|_{H^2(\Omega_f)^d}+\|\phi\|_{H^2(\Omega_p)}+\|p_f\|_{H^1(\Omega_f)}\right),
\end{aligned}\tag{4.9}$$

and combining (4.4), (4.8) and (4.9) we arrive at

$$\|\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md}\|_{H_f}\leq CH^2\left(\|\mathbf{u}_f\|_{H^2(\Omega_f)^d}+\|\phi\|_{H^2(\Omega_p)}+\|p_f\|_{H^1(\Omega_f)}\right).\tag{4.10}$$

From the discrete Brezzi-Babuska condition on Ω_f, for $q_h=p_{fh}-p_{fh}^{md}\in Q_h$ there exists a $\mathbf{v}_h\in H_{f,h}$ such that

$$\|p_{fh}-p_{fh}^{md}\|_{L^2(\Omega_f)}\lesssim\frac{-\int_{\Omega_f}n(p_{fh}-p_{fh}^{md})\mathrm{div}\mathbf{v}_h}{\|\mathbf{v}_h\|_{H_f}}.\tag{4.11}$$

Combining (4.7) and (4.11), we can treat the first term of the left-hand side of (4.7) as

$$|a_{\Omega_f}(\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md},\mathbf{v}_h)|\lesssim\|\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md}\|_{H_f}\|\mathbf{v}_h\|_{H_f}.$$

For the term on the the right-hand side of (4.7), we have

$$\begin{aligned}\left|\int_\Gamma n\rho_f g(\phi_h-\phi^h_{A3.2})\mathbf{v}_h\cdot\mathbf{n}_f\right| &\lesssim \|\phi_h-\phi_h^{md}\|_{L^2(\Gamma)}\|\mathbf{v}_h\|_{(L^2(\Gamma))^d}\\ &\lesssim \|\phi_h-\phi_h^{md}\|_{L^2(\Gamma)}\|\mathbf{v}_h\|_{H_f}\,.\end{aligned}$$

Now, for the trilinear term we have

$$\begin{aligned}&\left|b(\mathbf{u}_{fH},\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md},\mathbf{v}_h)+b(\mathbf{u}_{fh}-\mathbf{u}_{fH},\mathbf{u}_{fh}-\mathbf{u}_{fH},\mathbf{v}_h)+b(\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md},\mathbf{u}_{fH},\mathbf{v}_h)\right|\\ \lesssim&2\|\nabla(\mathbf{u}_{fH})\|_{L^2(\Omega_f)}\|\nabla(\mathbf{u}_{fh}-\mathbf{u}_{fh}^{md})\|_{L^2(\Omega_f)}\|\nabla\mathbf{v}_h\|_{L^2(\Omega_f)}+\|\nabla(\mathbf{u}_{fh}-\mathbf{u}_{fH})\|^2_{L^2(\Omega_f)}\|\nabla\mathbf{v}_h\|_{L^2(\Omega_f)}\,.\end{aligned}$$

The proof is completed by combining the inequalities immediately above with (4.9)-(4.11). □

Corollary 4.1. *Let $\mathbf{u}_{fh}^{md}\in H_{fh}$, $p_{fh}^{md}\in Q_h$ and $\phi_h^{md}\in H_{ph}$ be the solutions of the modified decoupled two-grid method with $H=\sqrt{h}$. Under the conditions of Theorem* 3.1 *and* (4.4)*, we have*

$$\begin{aligned}&\|\mathbf{u}_f-\mathbf{u}_{fh}^{md}\|_{H_f}+\|p_f-p_{fh}^{md}\|_Q+\|\phi-\phi_h^{md}\|_{H_p}\\ \le&C(h+H^2)\Big(\|\mathbf{u}_f\|_{H^2(\Omega_f)^d}+\|\phi\|_{H^2(\Omega_p)}+\|p_f\|_{H^1(\Omega_f)}\Big)\,.\end{aligned}$$

5. Improvements

Theorem 4.1 determines that the convergence of the decoupled two-grid method is O($H^{3/2}$) for velocity and pressure, and we investigated improving the convergence of DTGM by adding a correction based on solving a linearized Navier-Stokes problem on the coarse mesh that produced an improved algorithm as follows.

`Improved Decoupled Two-Grid Method (IDTGM)`

Step 1. Solve coupled problem (3.1) on the coarse grid with spacing H:
Find $u_H=(\mathbf{u}_{fH},\phi_H)\in W_H\subset W_h$, $p_{fH}\in Q_H\subset Q_h$ using (3.5).

Step 2. Solve a modified fine grid linearized problem using Newton iteration for the nonlinear term:
Find $u^d_{fh}=(\mathbf{u}^d_{fh},\phi^d_h)\in W_h$, $p^d_{fh}\in Q_h$ such that for all $v_h=(\mathbf{v}_h,\psi_h)\in W_h$ and $q_h\in Q_h$

$$\begin{cases}a(u^d_{fh},v_h)+d(v_h,p^d_{fh})+b(\mathbf{u}_{fH},\mathbf{u}^d_{fh},\mathbf{v})+b(\mathbf{u}^d_{fh},\mathbf{u}_{fH},\mathbf{v})\\ \qquad\qquad =f(v_h)-a_\Gamma(u_H,v_h)+b(\mathbf{u}_{fH},\mathbf{u}_{fH},\mathbf{v}),\\ d(u^d_{fh},q_h)=0\,.\end{cases}$$

Step 3. Solve the discrete linearized Navier-Stokes problem in the fluid region Ω_f on the coarse grid:

Find $\mathbf{u}_{fH}^{Id} \in H_{fH}$, $p_{fH}^{Id} \in Q_H$ such that for all $(\mathbf{v}_H, q_H) \in H_{fH} \times Q_H$

$$\begin{cases} a_{\Omega_f}(\mathbf{u}_{fH}^{Id}, \mathbf{v}_H) + d(\mathbf{v}_H, p_{fH}^{Id}) + b(\mathbf{u}_{fh}^{d}, \mathbf{u}_{fH}^{Id}, \mathbf{v}_H) + b(\mathbf{u}_{fH}^{Id}, \mathbf{u}_{fh}^{d}, \mathbf{v}_H) \\ \qquad = (n\mathbf{g}_f, \mathbf{v}_H) - \int_\Gamma n\rho_f g\phi_{fh}^d \mathbf{v}_H \cdot \mathbf{n}_f + b(\mathbf{u}_{fh}^d, \mathbf{u}_{fh}^d, \mathbf{v}_H), \\ d(\mathbf{u}_{fH}^{Id}, q_H) = 0. \end{cases} \tag{5.1}$$

Theorem 5.1. *Let* $(\mathbf{u}_{fh}, p_{fh})$ *be the solution of problem* (3.1) *on a fine grid, and* $(\mathbf{u}_{fH}^{Id}, p_{fH}^{Id})$ *be defined by the discrete model* (5.1) *on a coarse grid. Under the conditions of Theorem* 3.1*,*

$$\|\mathbf{u}_{fh} - \mathbf{u}_{fH}^{Id}\|_{H_f} + \|p_{fh} - p_{fH}^{Id}\|_Q \le CH^2 \Big(\|\mathbf{u}_f\|_{H^2(\Omega_f)^d} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)} \Big).$$

Proof. From Theorem 4.1, $\|\phi_h - \phi_{fh}^d\|_{H^1(\Omega_p)} \lesssim H^2$, and we proceed to use this result to fill the gap of order $H^{1/2}$ for the velocity in the H^1 norm and the pressure in the L^2 norm.

Combining (3.1) with $\psi_h = 0$ and (5.1), we have

$$\begin{aligned} & a_{\Omega_f}(\mathbf{u}_{fh} - \mathbf{u}_{fH}^{Id}, \mathbf{v}_H) + d(\mathbf{v}_H, p_{fh} - p_{fH}^{Id}) \\ = & \int_\Gamma n\rho_f g(\phi_h - \phi_{fh}^d)\mathbf{v}_H \cdot \mathbf{n}_f - b(\mathbf{u}_{fh}, \mathbf{u}_{fh}, \mathbf{v}_H) \\ & + \Big[b(\mathbf{u}_{fh}^d, \mathbf{u}_{fH}^{Id}, \mathbf{v}_H) + b(\mathbf{u}_{fH}^{Id}, \mathbf{u}_{fh}^d, \mathbf{v}_H) - b(\mathbf{u}_{fd}^d, \mathbf{u}_{fh}^d, \mathbf{v}_H) \Big]. \end{aligned} \tag{5.2}$$

Choosing $\mathbf{v}_H = \mathbf{u}_{fh} - \mathbf{u}_{fH}^{Id}$ in (5.2) and using the discrete divergence-free property of $\mathbf{u}_{fh}$ and $\mathbf{u}_{fH}^{Id}$,

$$\begin{aligned} & a_{\Omega_f}(\mathbf{u}_{fh} - \mathbf{u}_{fH}^{Id}, \mathbf{u}_{fh} - \mathbf{u}_{fH}^{Id}) \\ = & \int_\Gamma n\rho_f g(\phi_h - \phi_{fh}^d)(\mathbf{u}_{fh} - \mathbf{u}_{fH}^{Id}) \cdot \mathbf{n}_f + b(\mathbf{u}_{fh}^d, \mathbf{u}_{fh} - \mathbf{u}_{fH}^{Id}, \mathbf{u}_{fh} - \mathbf{u}_{fH}^{Id}) \\ & + b(\mathbf{u}_{fh} - \mathbf{u}_{fh}^d, \mathbf{u}_{fh} - \mathbf{u}_{fh}^d, \mathbf{u}_{fh} - \mathbf{u}_{fH}^{Id}) + b(\mathbf{u}_{fh} - \mathbf{u}_{fH}^{Id}, \mathbf{u}_{fh}^d, \mathbf{u}_{fh} - \mathbf{u}_{fH}^{Id}). \end{aligned}$$

From (2.7) and above identity, similarly to (4.8) we therefore get

$$\begin{aligned} \|\mathbf{u}_{fh} - \mathbf{u}_{fH}^{Id}\|_{H_f}^2 \lesssim & a_{\Omega_f}(\mathbf{u}_{fh} - \mathbf{u}_{fH}^{Id}, \mathbf{u}_{fh} - \mathbf{u}_{fH}^{Id}) \\ & \lesssim \Big(\|\phi_h - \phi_{fh}^d\|_{L^2(\Gamma)} + \|\mathbf{u}_{fh} - \mathbf{u}_{fh}^d\|_{H_f}^2 \Big) \|\nabla(\mathbf{u}_{fh} - \mathbf{u}_{fH}^{Id})\|_{L^2(\Omega_f)} \\ & \quad + \|\nabla \mathbf{u}_{fh}^d\|_{L^2(\Omega_f)} \|\nabla(\mathbf{u}_{fh} - \mathbf{u}_{fH}^{Id})\|_{L^2(\Omega_f)}^2. \end{aligned} \tag{5.3}$$

With the help of the trace theorem and Theorem 4.1,

$$\begin{aligned} \|\phi_h - \phi_{fh}^d\|_{L^2(\Gamma)} \lesssim & \|\phi_h - \phi_{fh}^d\|_{L^2(\Omega_p)} + \|\phi_h - \phi_{fh}^d\|_{H_p} \\ \le & CH^2 \Big(\|\mathbf{u}_f\|_{H^2(\Omega_f)^d} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)} \Big). \end{aligned} \tag{5.4}$$

Combining (5.3) and (5.4), and following the proof of (4.11), we complete the proof. $\square$

Remark 5.1. In (5.3) we have used the boundedness of $\|\nabla \mathbf{u}_{fh}^d\|_{L^2(\Omega_f)}$, which can be proven analogously to the proofs of Theorems 8-10 in [5]. From Theorem 5.1, we can see that optimal error estimates are established by adding a correction on the coarse mesh. Of course, there is some increased computational cost — but the increased amount is controllable, due to the small scale of the discrete problem on the coarse grid in Step 3.

Remark 5.2. Combining (3.3) with Theorem 5.1, we improve the results provided in Theorem 4.1, and obtain the improved estimate with mesh sizes satisfy $h = H^2$:

$$
\begin{aligned}
&\|\mathbf{u}_f - \mathbf{u}_{fH}^{Id}\|_{H^1(\Omega_f)} + \|\phi - \phi_{fh}^d\|_{H^1(\Omega_p)} + \|p_f - p_{fH}^{Id}\|_{L^2(\Omega_f)} \\
\le & C(h+H^2)\Big(\|\mathbf{u}_f\|_{H^2(\Omega_f)^d} + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)}\Big) \,.
\end{aligned}
$$

6. Numerical Experiments

To provide further insight on the theoretical results discussed in previous sections, we undertook some numerical tests. All of the numerical experiments reported here were performed on a PC with an i5-3210M core processor and 4GB of random access memory, mainly to check the relative performance of a usual two-grid finite element method versus our modified decoupled two-grid method. The steady mixed Navier-Stokes/Darcy model was defined on the convex domain $\Omega = [0,1] \times [0,2]$ with the interface $\Gamma = [0,1] \times \{1\}$ — cf. Fig. 1(a). Unstructured triangulation of Ω into triangles were adopted — cf. Fig. 1(b). The GMRES solver was used, and the model parameters ρ_f, g, n and α simply set to 1. The boundary conditions and the functions on right-hand side in the model were selected such that the exact solutions are given by

$$
\begin{cases}
u = x^2(y-1)^2 + y\,, \\
v = -\dfrac{2}{3}x(y-1)^3 + 2 - \pi\sin(\pi x)\,, \\
p_f = \big(2 - \pi\sin(\pi x)\big)\sin\left(\dfrac{1}{2}\pi y\right)\,, \\
\phi = \big(2-\pi\sin(\pi x)\big)\big(1 - y - \cos(\pi y)\big)\,,
\end{cases}
$$

where the components of $\mathbf{u}_f$ are denoted by (u, v).

We compared the performance of our modified decoupled two-grid schemes (4.2)-(4.3) with results obtained from the standard Galerkin method (3.1), a usual two-grid method (3.6), and a decoupled two-grid method (4.1). We used a stable finite element pair to approximate numerical solutions — specifically, the MINI elements and piecewise linear polynomials are used for the Navier-Stokes equations in Ω_f and the Darcy flow in Ω_p, respectively. In the following tables, we use $\|\cdot\|_0$ to denote the L^2-norm and $\|\nabla\cdot\|_0$ to denote the H^1-semi-norm.

In Table 1, we show the relative errors of the coupled algorithm between the exact solution of the mixed model and its finite element approximations. The errors for $\frac{\|\nabla(\mathbf{u}_f - \mathbf{u}_{fh})\|_0}{\|\nabla \mathbf{u}_f\|_0}$,

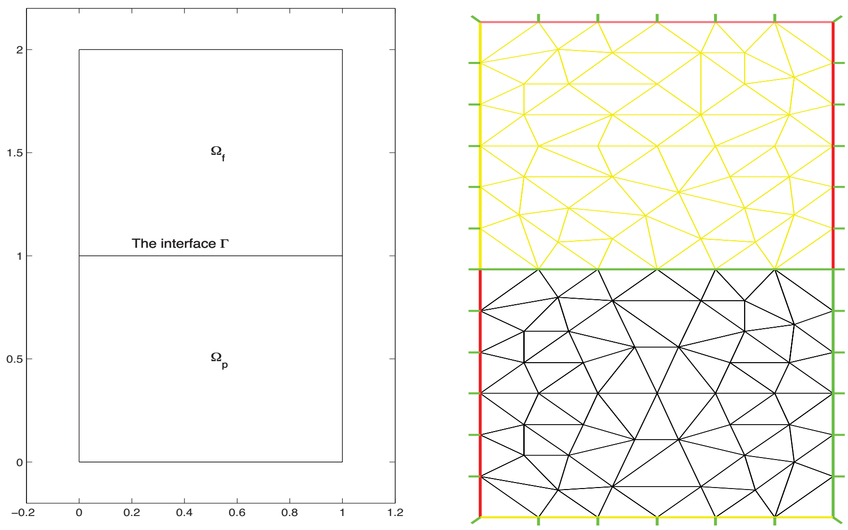

Figure 1: (a) The domain Ω and interface Γ, (b) Unstructured triangulation of Ω into triangles.

Table 1: Errors in the standard Galerkin finite element for the steady Navier-Stokes/Darcy problem.

$1/h$	$\frac{\|\nabla(\mathbf{u}_f-\mathbf{u}_{fh})\|_0}{\|\nabla\mathbf{u}_f\|_0}$	Rate	$\frac{\|p_f-p_{fh}\|_0}{\|p_f\|_0}$	Rate	$\frac{\|\nabla(\phi-\phi_{fh})\|_0}{\|\nabla\phi\|_0}$	Rate	CPU(S)
4	0.274386	-	2.39623	-	0.311874	-	0.831
9	0.0951824	1.3056	0.436885	2.0988	0.131439	1.0655	4.358
16	0.0497091	1.1061	0.208657	1.2844	0.07337	1.0133	22.375
25	0.0331143	0.9102	0.139515	0.9019	0.0478867	0.9561	83.016
36	0.0222924	1.0852	0.086767	1.3025	0.0325863	1.0557	308.109
49	0.0167132	0.9343	0.064530	0.9604	0.0239182	1.0031	380.983

Table 2: Errors in the usual two-grid method for the steady Navier-Stokes/Darcy problem.

$1/H$	$1/h$	$\frac{\|\nabla(\mathbf{u}_f-\mathbf{u}^c_{fh})\|_0}{\|\nabla\mathbf{u}_f\|_0}$	Rate	$\frac{\|p_f-p^c_{fh}\|_0}{\|p_f\|_0}$	Rate	$\frac{\|\nabla(\phi-\phi^c_h)\|_0}{\|\nabla\phi\|_0}$	Rate	CPU(S)
2	4	0.275137		2.36175		0.334666		0.064
3	9	0.0939132	1.3253	0.472663	2.0356	0.139822	1.0786	0.435
4	16	0.0523482	1.0163	0.246283	1.1294	0.0803606	0.9619	2.23
5	25	0.0328793	1.0422	0.136344	1.3709	0.0508308	1.0292	6.02
6	36	0.0226157	1.0257	0.0826363	1.4969	0.0349108	1.0324	15.666
7	49	0.016694	0.9867	0.0630421	0.8454	0.0256915	0.9874	35.102

$\frac{\|\nabla(\phi-\phi_h)\|_0}{\|\nabla\phi\|_0}$ and $\frac{\|p_f-p_{fh}\|_0}{\|p_f\|_0}$ are seen to become smaller and smaller as the mesh is refined, and also that the corresponding errors are $O(h)$. For contrast, we show the errors between the exact solution and the numerical solutions of the usual two-grid methods — i.e. for the

Table 3: Errors in the decoupled two-grid method for the steady Navier-Stokes/Darcy problem.

$1/H$	$1/h$	$\frac{\Vert\nabla(\mathbf{u}_f-\mathbf{u}_{fh}^d)\Vert_0}{\Vert\nabla\mathbf{u}_f\Vert_0}$	Rate	$\frac{\Vert p_f-p_{fh}^d\Vert_0}{\Vert p_f\Vert_0}$	Rate	$\frac{\Vert\nabla(\phi-\phi_h^d)\Vert_0}{\Vert\nabla\phi\Vert_0}$	Rate	CPU(S)
2	4	0.275044		2.43852		0.334665		0.062
3	9	0.0940614	1.3232	0.469813	2.0308	0.139822	1.0762	0.517
4	16	0.0523525	1.0184	0.265427	0.9924	0.0803617	0.9626	1.903
5	25	0.0328918	1.0414	0.131887	1.5671	0.0508308	1.0263	5.525
6	36	0.0226327	1.0252	0.0834871	1.2540	0.0349108	1.0303	14.311
7	49	0.0166941	0.9872	0.0614422	0.9945	0.0256918	0.9946	31.682

Table 4: Errors in the modified decoupled two-grid method for the steady Navier-Stokes/Darcy problem.

$1/H$	$1/h$	$\frac{\Vert\nabla(\mathbf{u}_f-\mathbf{u}_{fh}^{md})\Vert_0}{\Vert\nabla\mathbf{u}_f\Vert_0}$	Rate	$\frac{\Vert p_f-p_{fh}^{md}\Vert_0}{\Vert p_f\Vert_0}$	Rate	$\frac{\Vert\nabla(\phi-\phi_h^{md})\Vert_0}{\Vert\nabla\phi\Vert_0}$	Rate	CPU(S)
2	4	0.275118		2.4447		0.334665		0.064
3	9	0.0940548	1.3255	0.464689	1.9839	0.139822	1.0762	0.465
4	16	0.0523595	1.0158	0.265168	1.1330	0.0803604	0.9626	1.98
5	25	0.0328892	1.0421	0.131852	1.3249	0.0508308	1.0263	6.02
6	36	0.0226288	1.0262	0.082578	1.3732	0.0349108	1.0303	14.888
7	49	0.016697	0.9847	0.0609507	1.0360	0.0256915	0.9946	33.804

Table 5: Errors in the modified decoupled two-grid method for the steady Navier-Stokes/Darcy problem with fixed $h=1/100$.

$\frac{1}{H}$	$\frac{\Vert\nabla(\mathbf{u}_f-\mathbf{u}_{fh}^{md})\Vert_0}{\Vert\nabla\mathbf{u}_f\Vert_0}$	Rate	$\frac{\Vert p_f-p_{fh}^{md}\Vert_0}{\Vert p_f\Vert_0}$	Rate	$\frac{\Vert\nabla(\phi-\phi_h^{md})\Vert_0}{\Vert\nabla\phi\Vert_0}$	Rate
2	0.0302149		0.395089		0.0570869	
3	0.0143143	0.9213	0.151092	1.1853	0.0275339	0.8992
4	0.00895545	0.8151	0.0812323	1.0786	0.0167849	0.8602
5	0.00841317	-	0.0446173	-	0.0143814	-
6	0.00840814	-	0.0426251	-	0.0130662	-
7	0.00838343	-	0.0425451	-	0.0123221	-

decoupled two-grid method and modified decoupled two-grid method in Tables 2-4, respectively. We inserted n points on the horizontal boundaries and $2n$ points on the vertical boundaries, and write $h=1/n$ to characterise the mesh size. The accuracy of these two-grid methods is comparable with that of the standard Galerkin finite element method with the same mesh sizes where $1/h=4,9,16,25,36,49$. As expected, the numerical results confirm our theoretical findings very well.

The CPU time for the different methods is also compared in Tables 1-4, where we can see that the modified decoupled two-grid method needed less time than the standard Galerkin finite element method. Both the coupled two-grid method and the modified decoupled two-grid method have almost the same accuracy with meshes $h = H^2$, while the MDTGM saves a little computational time. Compared with the decoupled two-grid method, the MDTGM improves the numerical results as anticipated in our theoretical analysis.

Table 5 shows some numerical results from the MDTGM for various coarse meshes ($1/H = 2,3,4,5,6,7$) but a fixed fine mesh $h = 1/100$. The rates of convergence with respect to the mesh size H are given by the formula $\log(E_i/E_{i+1})/\log(H_i^2/H_{i+1}^2)$ where E_i and E_i+1 are the relative errors corresponding to the mesh of sizes H_i and H_{i+1}, respectively. It is seen that the errors decrease as H becomes smaller and the convergence orders are about 1, which verifies the theoretical findings. The coarse numerical solutions provide some initial data for the linearised discrete problem on fine grid. As the coarse grid H becomes smaller, the accuracy reduction is not so obvious. The reason for this may be that the differences between h and H^2 become smaller and smaller as H decreases.

7. Conclusions

We have considered two-grid methods for the steady Navier-Stokes/Darcy problem. The corresponding error estimates have been established, and our theoretical analysis and numerical results demonstrate the efficiency and effectiveness of the two-grid algorithms. Our modified decoupled two-grid method improves the existing results for optimal estimates of the order $h = H^2$. Furthermore, by adding a correction based on solving a linearised Navier-Stokes problem on the coarse mesh, we improve the results of Ref. [4].

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