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Distributed Control of the Stochastic Burgers Equation with Random Input Data

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Abstract. We discuss a control problem involving a stochastic Burgers equation with a random diffusion coefficient. Numerical schemes are developed, involving the finite element method for the spatial discretisation and the sparse grid stochastic collocation method in the random parameter space. We also use these schemes to compute closed-loop suboptimal state feedback control. Several numerical experiments are performed, to demonstrate the efficiency and plausibility of our approximation methods for the stochastic Burgers equation and the related control problem.

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1. Introduction

In the study of turbulence phenomena, the Burgers equation provides a simplified and interesting model. For a better understanding of the important problem of the control of turbulence, it has been suggested that involving this equation can be the first step towards application to fluid mechanics problems. Following this strategy, our aim is to study control problems for this equation and to develop computational tools which are powerful enough so that they can be used for the Navier-Stokes equations.

We consider the stochastic Burgers equation with a random coefficient and its distributed control problem, where we want to find an optimal control $f^*(t)$ which minimises the cost functional

$$J(f) = \mathbb{E}\left[\int_0^\infty \left(||u(t)||^2_{L^2(D)} + \beta||f(t)||^2_{L^2(D)}\right) dt\right], \tag{1.1}$$

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subject to

$$\begin{aligned}&\frac{\partial}{\partial t}u(\omega,t,x)-\frac{\partial}{\partial x}\left(a(\omega,x)\frac{\partial}{\partial x}u(\omega,t,x)\right)-u(\omega,t,x)\frac{\partial}{\partial x}u(\omega,t,x)\\ =&f(\omega,t,x) \text{ in } (0,\infty)\times D\,, \qquad (1.2)\\ &u|_{\{0\}\times D}=u_0(x)\,,\qquad u|_{(0,T]\times\partial D}=0\,,\end{aligned}$$

where D is $[0,1]$ and $\beta>0$ is a weight. The diffusion coefficient $a(\omega,x)$ and the force term $f(\omega,t,x)$ are random processes on the spatial domain and temporal-spatial domain, respectively. Here $u_0(x)$ is considered to be a deterministic data function and $\mathbb{E}$ denotes an expected value, which is defined as the Lebesque integral in a complete probability space $(\Omega,\mathscr{F},\mathbb{P})$ where Ω is any set, $\mathscr{F}$ is a σ-algebra of subsets of Ω and $\mathbb{P}$ is a probability measure on $\mathscr{F}$ [11,20].

Control problems of the deterministic Burgers equation have been studied by many authors [1,3,4,17,19,21,26,27], and stochastic control problems with additive white noise in Refs. [9,12]. Here we focus on the case of a random process acting on a diffusion coefficient [13,14]. Although the Burgers equation is often considered as the prototype for fluid flow, this equation can also be used as a reasonable mathematical model in other physical contexts such as traffic flow, supersonic flow about airfoils, acoustic transmission, and turbulence in hydrodynamic flows. In brief, the Burgers equation can be regarded as a suitable model for nonlinear wave propagation problems subject to dissipation [15]. Depending on the model problem, this dissipation may result from viscosity, heat conduction, mass diffusion, thermal radiation, chemical reaction, etc.. Thus we also take the viewpoint that the Burgers equation is a variation of the linear heat equation, and an heterogeneous body can be modelled with a *variable thermal conductivity coefficient* (diffusion coefficient) dependent on spatial position. Moreover, when there is a lack of information or uncertainty in the input data, this coefficient can be represented as a random field with estimated statistics.

Our goal here is to develop numerical schemes for a feedback control in minimising a cost function (1.1) subject to the stochastic Burgers equation (1.2). We adopt the finite element method in the spatial discretisation, and sparse grid stochastic collocation in a parameter space where random variables are involved. Later, we illustrate that the sparse grid collocation method is efficient in the optimal choice between the number of nodes and the error in a high-dimensional parameter space to obtain appropriate statistical information. For the optimal control of the Burgers equation, we introduce a feedback law for a linearised equation (i.e. a linear parabolic equation), obtained from a closed-loop system relating the linear quadratic regulator (LQR) theory and the linear quadratic estimation (LQE) problem. The feedback control law from the linearised problem produces the desired extent of stability for the closed-loop nonlinear system, although from the viewpoint of control theory this kind of strategy is actually suboptimal.

In Section 2, we introduce some function spaces, notations and assumptions needed throughout this article. In Section 3, we present a variational formulation for the stochastic burgers equation in order to apply the finite element approximation, and then employ a computable discretisation of the spatial domain in the stochastic sense. We briefly describe

how the sparse grid stochastic collocation is constructed under the Smolyak formula in Section 4. In Section 5, we consider a stochastic optimal control problem constrained by a stochastic Burgers equation, formulate a stochastic control problem using the suboptimal state feedback control for stochastic version, and explain how that strategy is used to solve the control problem approximately. Finally, we provide some computational experiments to demonstrate our arguments in Section 6, followed by our concluding remarks in Section 7.

2. Preliminaries

As usual, let the function space $H_0^1(D)$ be the subspace of $H^1(D)$ consisting of functions vanishing at the boundary of D equipped with the norm $||v||_{H_0^1(D)} = \left[\int_D |v_x|^2 dx\right]^{1/2}$. We also define a Hilbert space

$$L^2(0,\infty;H_0^1(D)) = \left\{ w(t) \in H_0^1(D) \;\middle|\; \int_0^\infty ||w(t)||^2_{H_0^1(D)} dt < \infty \right\}$$

equipped with the norm

$$||w||_{L^2(0,\infty;H_0^1(D))} = \left[\int_0^\infty ||w(t)||^2_{H_0^1(D)} dt\right]^{1/2} = \left[\int_0^\infty \int_D |w_x(t,x)|^2 dx dt\right]^{1/2}.$$

We use a variational formulation to determine a finite element method to approximate Eq. (1.2) later. Before that, we make several assumptions for the conditions on the random coefficient $a(\omega,x)$ and random force $f(\omega,t,x)$ [7, 10, 18].

Assumption 2.1. The random process $a : \Omega \times D \to \mathbb{R}$ is bounded and uniformly coercive almost surely — i.e.

$$\exists\, a_{min}, a_{max} \in (0,\infty) \;:\quad \mathbb{P}(\, a_{min} \le a(\omega,x) \le a_{max} \;\; \forall\, x \in D\,) = 1\,. \tag{2.1}$$

The random process $f : \Omega \times [0,T] \times D \to \mathbb{R}$ has bounded second moment — i.e.

$$\mathbb{E}\left[\int_0^\infty \int_D |f|^2 dx dt\right] < \infty\,.$$

Let $Y_n : (\Omega, \mathscr{F}, P) \to (\Gamma_n, \mathscr{B}(\Gamma))$ for $n = 1, 2, \cdots d$ be random variables and let $\Gamma_n = Y_n(\Omega) \subset \mathbb{R}$ be the image of Y_n. For a parameter space $\Gamma = \prod_{n=1}^d \Gamma_n \subset \mathbb{R}^d$, we assume that the random variables $(Y_1, \cdots Y_d)$ are independent and have a joint probability density function $\rho : \Gamma \to \mathbb{R}_+$ expressed in terms of the probability density functions ρ_n of Y_n — viz. $\rho = \prod_{n=1}^d \rho_n$ [11]. Thus for any Borel measurable function g, we may define the expected value $\mathbb{E}[g(Y_1,\cdots,Y_d)] = \int_\Gamma g(y_1,\cdots,y_n)\,\rho\, dy$.

Assumption 2.2. The coefficients a and the force f have the forms

$$a(\omega,x) = a(Y_1(\omega),\cdots,Y_d(\omega),x) \quad \text{on} \quad \Omega \times D\,, \tag{2.2}$$

and

$$f(\omega,t,x)=f(Y_1(\omega),\cdots,Y_d(\omega),t,x)\quad\text{on}\quad\Omega\times[0,\infty)\times D\,, \tag{2.3}$$

where $d\in\mathbb{N}_+$ and $\{Y_n\}_{n=1}^d$ are real random variables with zero mean.

Now we introduce a stochastic Hilbert space

$$L^2_{\mathbb{P}}(\Omega;H^1_0(D))=\left\{v:\Omega\to H^1_0(D)\mid\mathbb{E}\left[||v||^2_{H^1_0(D)}\right]<\infty\right\}$$

with the norm $||v||_{L^2_{\mathbb{P}}(\Omega;H^1_0(D))}=\mathbb{E}[||v||^2_{H^1_0(D)}]$, and similarly another stochastic Hilbert space

$$L^2_{\mathbb{P}}(\Omega;L^2(0,\infty;H^1_0(D)))=\left\{w:\Omega\to L^2(0,\infty;H^1_0(D))\mid\mathbb{E}\left[||w||^2_{L^2(0,\infty;H^1_0(D))}\right]<\infty\right\}$$

with the norm $||w||^2_{L^2_{\mathbb{P}}(\Omega;L^2(0,\infty;H^1_0(D)))}=\mathbb{E}[||w||^2_{L^2(0,\infty;H^1_0(D))}]$. If we consider the tensor product space $L^2_{\mathbb{P}}(\Omega)\otimes H^1_0(D)$ with tensor inner product

$$(v,\hat{v})_{L^2_{\mathbb{P}}(\Omega)\otimes H^1_0(D)}=\int_\Omega\int_D v_x(\omega,x)\hat{v}_x(\omega,x)\,dx\,d\mathbb{P},$$

then we have the isomorphism $L^2_{\mathbb{P}}(\Omega)\otimes H^1_0(D)\simeq L^2_{\mathbb{P}}(\Omega;H^1_0(D))$; refer to [7]. Moreover, the same argument with tensor inner product

$$(w,\hat{w})_{L^2_{\mathbb{P}}(\Omega)\otimes L^2(0,\infty;H^1_0(D))}=\int_\Omega\int_0^\infty\int_D w_x(\omega,t,x)\hat{w}_x(\omega,t,x)\,dx\,dt\,d\mathbb{P}$$

applies to $L^2_{\mathbb{P}}(\Omega)\otimes L^2(0,\infty;H^1_0(D))\simeq L^2_{\mathbb{P}}(\Omega;L^2(0,\infty;H^1_0(D)))$.

3. Finite Element Approximation

A variational formulation of Eq. (1.2) is as follows.
Find $u\in L^2_{\mathbb{P}}(\Omega)\otimes L^2(0,\infty;H^1_0(D))$ such that

$$\begin{cases}\mathbb{E}\left[\int_D u_t v\,dx\right]+\mathbb{E}\left[\int_D au_x v'\,dx\right]+\mathbb{E}\left[\int_D uu_x v\,dx\right]\\ \qquad\qquad=\mathbb{E}\left[\int_D fv\,dx\right]\quad\forall\,v\in L^2_{\mathbb{P}}(\Omega)\otimes H^1_0(D)\,,\\ u(0,x)=u_0(x)\quad\text{in } D\,,\end{cases}$$

where the derivative notations mean differentiation with respect to $(t,x)\in(0,\infty)\times D$.

We restrict our attention to the case that random variables Y_n are bounded, or equivalently Γ_n is a bounded set. Since we have a finite set of random variables, the problem has a deterministic equivalent as follows.

Find $u \in L^2_\rho(\Gamma) \otimes L^2(0,\infty; H^1_0(D))$ such that

$$\begin{cases} \int_\Gamma \rho \int_D u_t v dx dy + \int_\Gamma \rho \int_D a u_x v' dx dy + \int_\Gamma \rho \int_D u u_x v dx dy \\ \qquad\qquad = \int_\Gamma \rho \int_D f v dx dy \quad \forall\, v \in L^2_\rho(\Gamma) \otimes H^1_0(D), \\ u(0,x) = u_0(x) \quad \text{in } D, \end{cases} \tag{3.1}$$

where the spaces $L^2_\rho(\Gamma) \otimes H^1_0(D)$ and $L^2_\rho(\Gamma) \otimes L^2(0,\infty; H^1_0(D))$ are analogues of $L^2_{\mathbb{P}}(\Omega) \otimes H^1_0(D)$ and $L^2_{\mathbb{P}}(\Omega) \otimes L^2(0,\infty; H^1_0(D))$ replaced by $(\Gamma, \mathscr{B}(\Gamma), \rho\, dy)$.

For convenience, we consider the solution u as a function of $u : \Gamma \to L^2(0,\infty; H^1_0(D))$ and use the notation $u(y)$ whenever we would like to emphasise the dependence on the parameter y. Then problem (3.1) is equivalent to the following setting.

Find $u(y) \in L^2(0,\infty; H^1_0(D))$ such that

$$\begin{cases} \int_D u_t(y)\phi dx dy + \int_D a(y) u_x(y) \phi' dx dy + \int_D u(y) u_x(y) \phi\, dx\, dy \\ \qquad\qquad = \int_D f(y)\phi\, dx\, dy \;\; \forall\, \phi \in H^1_0(D),\ \rho\text{-a.e. in } \Gamma, \\ u(0,x) = u_0(x) \;\; \text{in } D. \end{cases} \tag{3.2}$$

If we fix the point $y \in \Gamma$, then Eq. (3.2) becomes a deterministic partial differential equation, so that some usual relevant approximation method can be applied — e.g. the finite element method.

Given $y \in \Gamma$ (so the problem turns into a deterministic problem), a typical finite element approximation of (3.2) is as follows. First choose an N-dimensional conforming finite element subspace $V^h \subset H^1_0(D)$, and then seek $u^h(t,\cdot) \in V^h$ such that

$$\begin{cases} \int_D u^h_t(y)\phi^h dx dy + \int_D a(y) u^h_x(y)(\phi^h)' dx dy + \int_D u^h(y) u^h_x(y) \phi^h dx dy \\ \qquad\qquad = \int_D f(y)\phi^h dx dy \quad \forall\, \phi^h \in V^h,\ \rho\text{-a.e. in } \Gamma, \\ u(0,x) = u_0(x) \;\; \text{in } D, \end{cases} \tag{3.3}$$

where $u^h_0(x) \in V^h_0$ is an approximation (e.g. a projection) of $u_0(x)$. If $\{\phi_i(x)\}_{i=1}^N$ is a basis of the finite dimensional space V^h, we write

$$u^h(y,t,x) = \sum_{i=1}^N u_i(y,t)\phi_i(x) \tag{3.4}$$

and take $\phi^h(x) \in V^h$ in (3.3) to be each of the basis functions ϕ_i. According to the Galerkin method, on substituting (3.4) into (3.3) we obtain the related matrices and nonlinear tensor as follows.

If $(\cdot,\cdot)$ denotes the inner product of $L^2(D)$-space and $a_y(u,v)$ the bilinear form $\int_D a(y)u'v'dx$ for $u,v \in H_0^1(D)$, then we write $m_{ij} = (\phi_i,\phi_j)$, $s_{ij,y} = a_y(\phi_i^{'},\phi_j^{'})$, $r_{ijk} = (\phi_i\phi_k^{'},\phi_j)$, $f_j = (f,\phi_j)$, and $u_0^j = (u_0,\phi_j)$ for $i,j = 1,\cdots N$. We then set the matrices $M = (m_{ij})$ and $S_y = (s_{ij,y})$, the nonlinear tensor $\mathscr{R} = (r_{ijk})$, the vector $\vec{f} = (f_1,\cdots f_N)$, and an initial condition $\vec{u}_0 = (u_0^1,\cdots,u_0^N)^T$. With $\vec{u}(y,t) = (u_1(y,t),\cdots,u_N(y,t))^T$ and $\vec{u}(0) = (u_1(y,0),\cdots,u_N(y,0))^T$ unknown vectors, the system (3.3) can then be written in the matrix form

$$\begin{cases} M\dfrac{d\vec{u}}{dt} + S_y\vec{u} + (\vec{u})^T\mathscr{R}\vec{u} = \vec{f}\,, \\ M\vec{u}(0) = \vec{u}_0\,. \end{cases} \qquad \rho\text{-a.e. } y \text{ in } \Gamma, \tag{3.5}$$

Recall that S_y and $\vec{f}$ are random, hence (3.5) is a system of stochastic nonlinear ordinary differential equations that consists of N equations and N unknowns. Since M is an invertible matrix, under the assumptions (2.1), (2.2) and (2.3) this can be rewritten as a system of first order stochastic nonlinear ordinary differential equations subject to the initial condition — viz.

$$\frac{d\vec{u}}{dt} = M^{-1}\left(-S_y\vec{u} - (\vec{u})^T\mathscr{R}\vec{u} + \vec{f}\,\right), \quad \vec{u}_0 = M^{-1}\vec{u}_0\,,\ \rho\text{-a.e. } y \text{ in } \Gamma\,. \tag{3.6}$$

Given y in Γ, the terms on the right-hand side are continuously differentiable with respect to t, so there exists only one solution to the system for a given realisation of the random variables. (The system has a zero equilibrium solution if the boundary conditions take zeros.) Thus we can obtain the numerical solution of Eq. (1.2) via ODE schemes; and in Section 5 the system (3.6) is applied to the feedback control design, where the approximate solution obtained is used to compare with a control solution combined with the Smolyak method introduced in the next section.

4. Stochastic Collocation and the Smolyak Formula

In practice, we are usually interested in the r-th statistical moment, involving multi-dimensional integration in a random parameter space. For instance, as mentioned in Section 2 the rth statistical moment of any Borel measurable function g must be

$$\mathbb{E}[g^r(Y_1,\cdots,Y_d)] = \int_{\Gamma_1}\cdots\int_{\Gamma_d} g^r(y_1,\cdots,y_d)\rho(y_1,\cdots,y_d)dy_1,\cdots,dy_d\,,$$

where each $\Gamma_n \subset \mathbb{R}$ for $n = 1,2,\cdots,d$ and ρ is a joint probability density function. The Monte Carlo (MC) method is very simple and easy to implement but converges very slowly, so we prefer to take advantage of a quadrature rule in multi-dimensional space combined by a tensor product. However, a disadvantage of simple full tensor product quadrature is that the number of points required to construct interpolation increases exponentially as the dimension of stochastic space is increased, called the *curse of dimensionality*. Smolyak introduced an impressive algorithm, which provides much fewer points than the full tensor

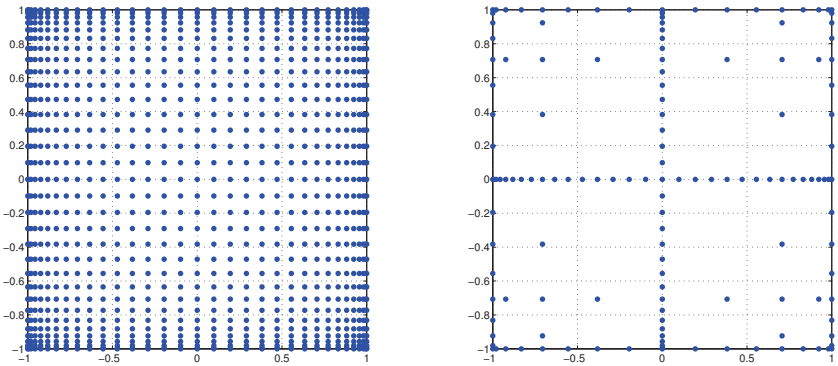

Figure 1: Full tensor product grids using Clenshaw-Curtis abscissas (left). Isotropic Smolyak sparse grids using Clenshaw-Curtis abscissas with level 5 (right).

product formula in multi-dimensional space, imparting an interpolation strategy with a reduced number of abscissas required while maintaining the approximation quality of the interpolation up to a logarithmic scale. Fig. 1 shows full tensor prouduct grids and isotropic Smolyak sparse grids using Clenshaw-Curtis abscissas. This algorithm provides a linear combination of tensor products chosen such that the interpolation property is conserved for higher dimensions. Detailed explanations of the stochastic collocation idea and the Smolyak procedure can be found in Refs. [2,5,16,18,22–25,28,29].

4.1. Stochastic collocation method

The idea of the collocation method is to approximate the function $u(y;t,x)$ for all $y \in \Gamma$ and for all $(t,x) \in (0,T] \times [0,1]$. Let $\mathscr{P}_{\mathbf{p}}(\Gamma) \subset L^2_\rho(\Gamma)$ denote the span of tensor product polynomials with degree at most $\mathbf{p} = (p_1 \cdots p_d)$ — i.e. $\mathscr{P}_{\mathbf{p}}(\Gamma) = \bigotimes_{n=1}^{d} \mathscr{P}_{p_n}(\Gamma_n)$ with

$$\mathscr{P}_{p_n}(\Gamma_n) = \text{span}(y_n^m, m = 0 \cdots p_n), \quad n = 1, \cdots d,$$

where $L^2_\rho(\Gamma) = \left\{v : \int_\Gamma |v(y)|^2 \rho(y) dy < \infty\right\}$. We write the dimension of $\mathscr{P}_{\mathbf{p}}(\Gamma)$ as $d_p = \prod_{n=1}^{d}(p_n+1)$. Stochastic collocation entails the sampling of the solution $u(y_k)$ on a suitable set of points $y_k \in \Gamma$. An interpolation of the u is then

$$u^{\mathbf{p}}(y;\cdot,\cdot) = \sum_k u(y_k,\cdot,\cdot) l_k^{\mathbf{p}}(y),$$

and the approximated rth statistical moment is

$$\mathbb{E}[u^r](t,x) \approx \sum_k u^r(y_k,t,x) \int_\Gamma l_k^{\mathbf{p}}(y)\rho(y)dy. \tag{4.1}$$

In practice, we use the fully discrete solution by some numerical schemes rather than the exact solution.

Remark 4.1. We only discretise a solution in the random parameter space here by the collocation method. This is possible since we already have an information about a probability density function of random input data of the stochastic PDE. It is still left to decide numerical schemes for the domain $(0,T]\times[0,1]$. We will choose the finite element method for spatial discretisation and the backward Euler method for temporal discretisation later.

In the next subsection, we show the construction of the interpolation through a full tensor product, regardless of the interpolation nodes.

4.2. Smolyak formula

We introduce an index $i \in \mathbb{N}_+$, and for each value of i let $\{y_1^i, \cdots, y_{m_i}^i\} \subset [-1,1]$ be a sequence of abscissas for Lagrange interpolation on $[-1,1]$. Let $W = W(0,\infty;D)$ be a Banach space of functions $v : [0,\infty)\times D \to \mathbb{R}$. For $u \in C^0(\Gamma_n;W)$ with $d=1$, we introduce a sequence of one-dimensional Lagrange interpolation operators $\mathscr{U}^i : C^0(\Gamma;W) \to V_{m_i}(\Gamma;W)$ such that

$$\mathscr{U}^i(u)(y) = \sum_{j=1}^{m_i} u(y_j^i) l_j^i(y) \qquad \forall\, u \in C^0(\Gamma;W)\,, \tag{4.2}$$

where $l_j^i \in \mathscr{P}_{m_i-1}(\Gamma)$ are Lagrange polynomials of degree $p_i = m_1 - 1$ — i.e.

$$l_j^i(y) = \prod_{k=1,k\neq l}^{m_i} \frac{(y-y_k^i)}{(y_j^i - y_k^i)}$$

and

$$V_{m_i}(\Gamma;W) = \left\{ v \in C^0(\Gamma_n;W) : v(y,x,t) = \sum_{k=1}^{m_i} \tilde{v}_k(x,t) l_k^i(y),\ \{\tilde{v}_k\}_{k=1}^{m_i} \in W \right\}.$$

The formula (4.2) exactly reproduces all polynomials of degree less than m_i. In the multivariate case $d>1$, for each $u \in C^0(\Gamma_n;W)$ and multi-index $\mathbf{i} = (i_1,\cdots,i_d) \in \mathbb{N}_+^d$ we define the full tensor product interpolation formulas

$$\begin{aligned} \mathscr{I}_{\mathbf{i}}^d u(y) &= \left(\mathscr{U}^{i_1} \otimes \cdots \otimes \mathscr{U}^{i_d}\right)(u)(y) \\ &= \sum_{j_1=1}^{m_{i_1}} \cdots \sum_{j_d=1}^{m_{i_d}} u\left(y_{j_1=1}^{m_{i_1}} \cdots y_{j_d=1}^{m_{i_d}}\right)\left(l_{j_1=1}^{m_{i_1}} \otimes \cdots \otimes l_{j_d=1}^{m_{i_d}}\right). \end{aligned} \tag{4.3}$$

Clearly, the above product needs $\prod_{n=1}^N m_{i_n}$ function evaluations.

Let us now describe the Smolyak isotropic formulas $\mathscr{A}(q,d)$, for which detailed explanation and analysis is found in Refs. [5, 22, 23]. The question is how to render tensor products with a relatively small number of nodes, and choose the linear combination in order to preserve an interpolation property in going from one dimension to many dimensions. The Smolyak formulas that provide this are just linear combinations of the product formulas (4.3). With $\mathscr{U}^0 = 0$, for $i \in \mathbb{N}_+$ we define

$$\Delta^i = \mathscr{U}^i - \mathscr{U}^{i-1}\,.$$

Moreover, given an integer $q \in \mathbb{N}_+$ that is hereafter called the *level*, for $\mathbf{i} \in \mathbb{N}_+^d$ with $|\mathbf{i}| = i_1 + \cdots + i_d$ the Smolyak algorithm is

$$\mathscr{A}(q,d) = \sum_{|\mathbf{i}|\le q} \left(\Delta^{i_1} \otimes \cdots \otimes \Delta^{i_d}\right). \tag{4.4}$$

Equivalently, the algorithm (4.4) can be written as [29]

$$\mathscr{A}(q,d) = \sum_{q-d+1\le|\mathbf{i}|\le q} (-1)^{q-|\mathbf{i}|} \binom{d-1}{q-|\mathbf{i}|} \left(\mathscr{U}^{i_1} \otimes \cdots \otimes \mathscr{U}^{i_d}\right).$$

To compute $\mathscr{A}(q,d)(u)$, we only need to know function values on the sparse grid

$$\mathscr{H}(q,d) = \bigcup_{q-d+1\le|\mathbf{i}|\le q} \left(\mathscr{Y}^{i_1} \otimes \cdots \otimes \mathscr{Y}^{i_d}\right),$$

where $\mathscr{Y}^i = \{y_1^i, \cdots, y_{m_i}^i\} \subset [-1,1]$ denotes the set of nodes used by $\mathscr{U}^i$. If the sets are nested (i.e. $\mathscr{Y}^i \subset \mathscr{Y}^{i+1}$), then $\mathscr{H}(q,d) \subset \mathscr{H}(q+1,d)$ and

$$\mathscr{H}(q,d) = \bigcup_{|\mathbf{i}|=q} \left(\mathscr{Y}^{i_1} \otimes \cdots \otimes \mathscr{Y}^{i_d}\right).$$

It is notable that the Smolyak algorithm, as presented in this Section, is isotropic since all directions are treated equally.

By comparing (4.2) and (4.2), we observe that the Smolyak approximation employing nested points requires fewer function evaluations than the corresponding formula for non-nested points. In the next subsection, we introduce three particular sets of abscissa, nested and non-nested.

4.3. Interpolation abscissas

Clenshaw-Curtis abscissas. We first consider the Smolyak algorithm based on polynomial interpolation at the extrema of Chebyshev polynomials. For any choice of $m_i > 1$, the nodes are given by

$$y_j^i = -\cos\left(\frac{\pi(j-1)}{m_i-1}\right), \quad j = 1, \cdots, m_i,$$

and in addition we define $y_i^i = 0$ if $m_1 = 1$. With this choice the set of points is nested, and thereby the numbers m_i of points which are used in the formulas $\mathscr{U}^i$ are

$$m_1 = 1 \quad \text{and} \quad m_i = 2^{i-1} + 1 \quad \text{for } i > 1.$$

It is important to choose $m_1 = 1$ if we are interested in optimal approximation in relatively large d, because in all other cases the number of points used by $\mathscr{A}(q,d)$ increases too fast with d.

Gauss-Legendre abscissas. We also consider the Smolyak formulas based on polynomial interpolation at the zeros of the orthogonal polynomials with respect to a weight ρ. If the random variable follows $U(-1,1)$, we choose a uniform density in $[-1,1]$ as a weight ρ. Since the Legendre polynomials are orthogonal to the uniform density, the zeros of the Legendre polynomials naturally lead to the Gauss-Legendre abscissas that have a maximum degree of precision of $2m_i - 1$. However, the Gauss-Legendre abscissas are not nested.

Remark 4.2. The convergence properties of the stochastic collocation techniques require the regularity of the solution with respect to the parameter space Γ. One can find the regularity condition of the random input data for the elliptic problem in Refs. [24] and [25]. In general, such a condition should be verified for each particular application of the stochastic PDE, and here we probe the convergence of our problem through computational experiments.

5. Distributed Feedback Control of the Stochastic Burgers Equation

5.1. Linear quadratic regulator design

We now design distributed feedback control of the problem (1.1) subject to (1.2). Using the argument in Section 2, we can re-state our problem as follows. Find an optimal control $f^*(t) \in L^2_\rho(\Gamma) \otimes L^2(0,\infty; L^2(D))$ which minimises the cost functional

$$J(f) = \int_\Gamma \rho \int_0^\infty \left(||u(t)||^2_{L^2(D)} + \beta ||f(t)||^2_{L^2(D)} \right) dt\, dy \tag{5.1}$$

subject to (3.2). Under the assumption (2.3), we replace the forcing term f with the special form $b(x)z(y,t)$ in the system (5.1) and (3.2), where $z(y,t)$ is a control input and $b(x)$ is a given function used to distribute the control over the domain. Given ρ-a.e. y in Γ, we may however consider the cost functional

$$J_y(z(y)) = \int_0^\infty \left(||u(y,t)||^2_{L^2(D)} + R|z(y,t)|^2 \right) dt\,, \tag{5.2}$$

where $R = \beta ||b||^2_{L^2(D)}$ and it is notable that $J(z) = \int_\Gamma J_y(z(y))\,\rho\, dy$. This observation implies that if we find an optimal minimiser $z^*(y,t)$ to J_y subject to (3.2) at each parameter value y, which then becomes a deterministic problem, we can minimise (5.1) too. Subsequently, we prefer to focus on solving (5.2) rather than (5.1), subject to (3.2).

Discretising (5.2) and (3.2) with the techniques developed in Section 3 and 4, we arrive at the following problem. Find an optimal control $z^*(t)$ which minimises the cost functional

$$J_k(z) = \int_0^\infty \left(\vec{u}(y_k,t)^T Q \vec{u}(y_k,t) + R|z(y_k,t)|^2 \right) dt \tag{5.3}$$

subject to

$$\frac{d}{dt}\vec{u}(t) = A_k \vec{u}(t) + G(\vec{u}(t)) + Bz(t), \quad \vec{u}(0) = \vec{u}_0\,, \quad t > 0, \quad y_k \in \Gamma, \tag{5.4}$$

where the $y_k \in \Gamma$ are determined by the Smolyak formula. The original cost functional is then approximated by

$$J(z) = \int_\Gamma J_y(z(y))\,\rho\,dy \approx \sum_k w_k J_k(z(y_k)),$$

where $w_k = \int_\Gamma l_k^{\mathbf{P}}(y)\rho(y)dy$.

According to (3.6), we say $Q = M$, $A_k = -M^{-1}S_{y_k}$ and $G(\vec{u}) = -M^{-1}(\vec{u})^T\mathscr{R}\vec{u}$. For simplicity, we take $\vec{u}_0 = M^{-1}\vec{u}_0$. Since we assume that $f(y,x,t) = b(x)z(y,t)$, for each finite element basis function $f_j = (f,\phi_j) = z(y,t)(b(x),\phi_j)$ such that $B_j = (b(x),\phi_j)$, $j = 1,\cdots,N$ and $B = (B_1,\cdots,B_N)^T$ we bring out $R = \beta B^T B$. It is notable that S_{y_k} depends on y_k, so A_k is also a function on Γ.

Now, we turn to closed-loop control in state feedback form, where we utilize suboptimal control strategies to construct a suboptimal feedback synthesis. Assuming that the nonlinear term in the Burgers equation is small, a suboptimal feedback control z^* can be obtained by using well-known linear quadratic regulator theory [3,8,21,26,27]. The optimal control $z^*(t)$ can be found by

$$z^*(t) = -R^{-1}B^T P_k\vec{u}(t) = -K_k\vec{u}(t)\,, \tag{5.5}$$

where K_k is called the feedback operator and P_k is the symmetric positive definite solution of the algebraic Riccati equation

$$P_k A_k + A_k^T P_k - P_k B R^{-1} B^T P_k + Q = 0\,.$$

5.2. Linear feedback controllers with state estimate feedback

A simple classical feedback control design is the linear quadratic regulator (LQR), which assumes that the full state is 'feedback' into the system through the control, bur knowledge of the full state is not possible for many complicated physical systems. As a realistic alternative, a compensator design provides a state estimate based on state measurements to be used in the feedback control law.

We may not assume that we know the full state, but instead we can assume a state measurement of the form

$$w(t) = \mathscr{C}u(t)\,,$$

where $\mathscr{C} \in \mathscr{L}(L^2(D),\mathbb{R}^m)$. Taking advantage of the same discretising schemes in Section 3, the matrix $C \in \mathscr{L}(\mathbb{R}^N,\mathbb{R}^m)$ approximating $\mathscr{C}$ is obtained. Then we reformulate the problem — viz. find an optimal control $z^*(y_k,t)$ which minimises the cost functional (5.3) subject to

$$\begin{cases} \dfrac{d}{dt}\vec{u}(t) = A_k\vec{u}(t) + G(\vec{u}(t)) + Bz(t)\,, \quad \vec{u}(0) = \vec{u}_0\,, \\ \vec{w}(t) = C\vec{u}(t)\,, \end{cases} \qquad t > 0\,, \quad y_k \in \Gamma\,. \tag{5.6}$$

We can apply theory and results to show that a stabilising compensator based controller can be applied to the system [3]. The observer design is mainly needed in order to provide

the feedback control law with estimated state variables, so the control law and the observer are combined into a complete system called a compensator. This technique needs a limited measurement of the state as a condition. According to the given state measurement in (5.6), a state estimate denoted by $\vec{u}_c(t)$ is computed by solving the observer equation

$$\frac{d}{dt}\vec{u}_c(t) = A_k\vec{u}_c(t) + G(\vec{u}_c(t)) + Bz(t) + L\,[\,\vec{w}(t) - C\vec{u}_c(t)\,]\,, \qquad \vec{u}_c(0) = \vec{u}_{c_0}\,. \tag{5.7}$$

In the usual manner, he functional gain operator K_k and estimator gain operator L_k can be a linear quadratic regulator (LQR) and a Kalman estimator (LQE), respectively, The feedback control law is again given by the same formula in (5.5):

$$z^*(t) = -R^{-1}B^T P_k\vec{u}_c(t) = -K_k\vec{u}_c(t)\,,$$

where K_k is the functional gain operator. To describe how to obtain L_k, let P_k be the non-negative definite solution of

$$A_kP + PA_k^T - P_kC^TR^{-1}CP_k + Q = 0\,.$$

If the solution P_k exists, we can define

$$L_k = P_kC^T, \tag{5.8}$$

so from (5.7) and (5.8) we obtain the closed loop compensator as

$$\begin{cases} \dfrac{d}{dt}\vec{u}(t) = A_k\vec{u}(t) - BK_k\vec{u}_c(t) + G(\vec{u}(t))\,, \\ \dfrac{d}{dt}\vec{u}_c(t) = L_kC\vec{u}(t) + (A_k - L_kC - BK_k)\vec{u}_c(t) + G(\vec{u}_c(t))\,, \\ \vec{u}(0) = \vec{u}_0\,, \quad \vec{u}_c(0) = \vec{u}_{c_0}\,, \end{cases} \tag{5.9}$$

for a given parameter value $y_k \in \Gamma$ derived from the Smolyak formula.

6. Computational Results

We set the stochastic coefficient $a(\omega, x)$ where $x \in D = [0,1]$ in problem (1.2) as

$$\begin{aligned} a(\omega,x) =& a_{min} + \delta\exp\big\{(Y_1(\omega)\cos(\pi x) + Y_2(\omega)\sin(\pi x))e^{-1/8} \\ &+ (Y_3(\omega)\cos(2\pi x) + Y_4(\omega)\sin(2\pi x))e^{-1/4}\big\}\,, \end{aligned} \tag{6.1}$$

where $a_{min} = 1/100$, $\delta = 0.05$ and the real random variables $Y_1, \cdots, Y_4$ are independent and identically distributed with $\mathbb{E}[Y_i] = 0$ and $\mathbb{E}[Y_iY_j] = \delta_{ij}$ for $i, j = 1, \cdots, 4$. The random variables $Y_1, \cdots, Y_4$ are also uniformly distributed in the interval $[-\sqrt{3}, \sqrt{3}]$. Then we obtain the joint probability density function ρ of $(Y_1, \cdots, Y_4)$ as $(2\sqrt{3})^{-4}$ in this case. Using the sparse grid collocation method with the Clenshaw-Curtis or the Gauss-Legendre points, we implement our computation. Fig. 2 shows the mean and variance of $a(\omega, x)$.

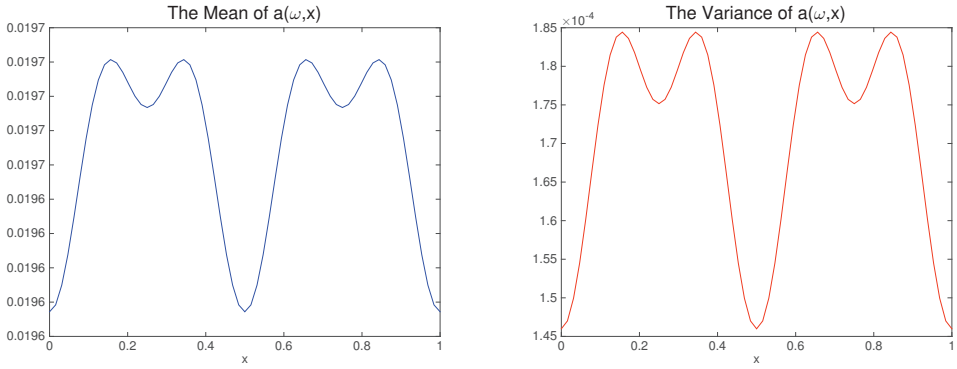

Figure 2: The mean (left) and the variance (right) of $a(\omega, x)$.

Table 1: Numbers of points for CC and GL abscissas with dimension 4.

Level	level 0	level 1	level 2	level 3	level 4	level 5	level 6
♯ of points for CC	1	9	41	137	401	1105	2929
♯ of points for GL	1	9	57	289	1268	4994	

For the spatial discretisation, we choose a typical continuous piecewise linear basis consisting of a conforming finite element subspace. In all computations, we implemented the same spatial discretisation with grid size $h = 1/64$, and the temporal discretisation in the backward Euler method with $\Delta t = 1/100$.

The mean and the variance of the solutions were investigated at the final time $T = 1$, computed over the Clenshaw-Curtis (CC) and the Gauss-Legendre (GL) sparse grid abscissas. The number of points of each grids are shown in Table 1. We observe that the number of points in the CC abscissas is relatively smaller than those of the GL abscissas at the same level. For comparison, we also solved (1.2) by the Monte Carlo (MC) method. To compute the statistical data of the random solutions by the MC simulation, we need to make an average of many realisations — e.g. the rth statistical moments $\mathbb{E}[u^r]$ of the random solution was estimated by the MC ensemble average

$$\mathbb{E}[u^r](t,x) \approx u^r_{MC}(t,x) = \frac{1}{M}\sum_{m=1}^{M} \left|u(y_m,t,x)\right|^r,$$

where $u(y_m,t,x)$ is a realisation from random sampling and M is the total number of realisations.

Fig. 3 presents the means (first column) and the variances (second column) of the solutions for the stochastic Burgers equation. The first row shows the results from the MC method with 12000 realisations, and In the second row and the third row from the CC gird and the GL grid approximations with Smolyak rule of 4 dimension with level 4, respectively

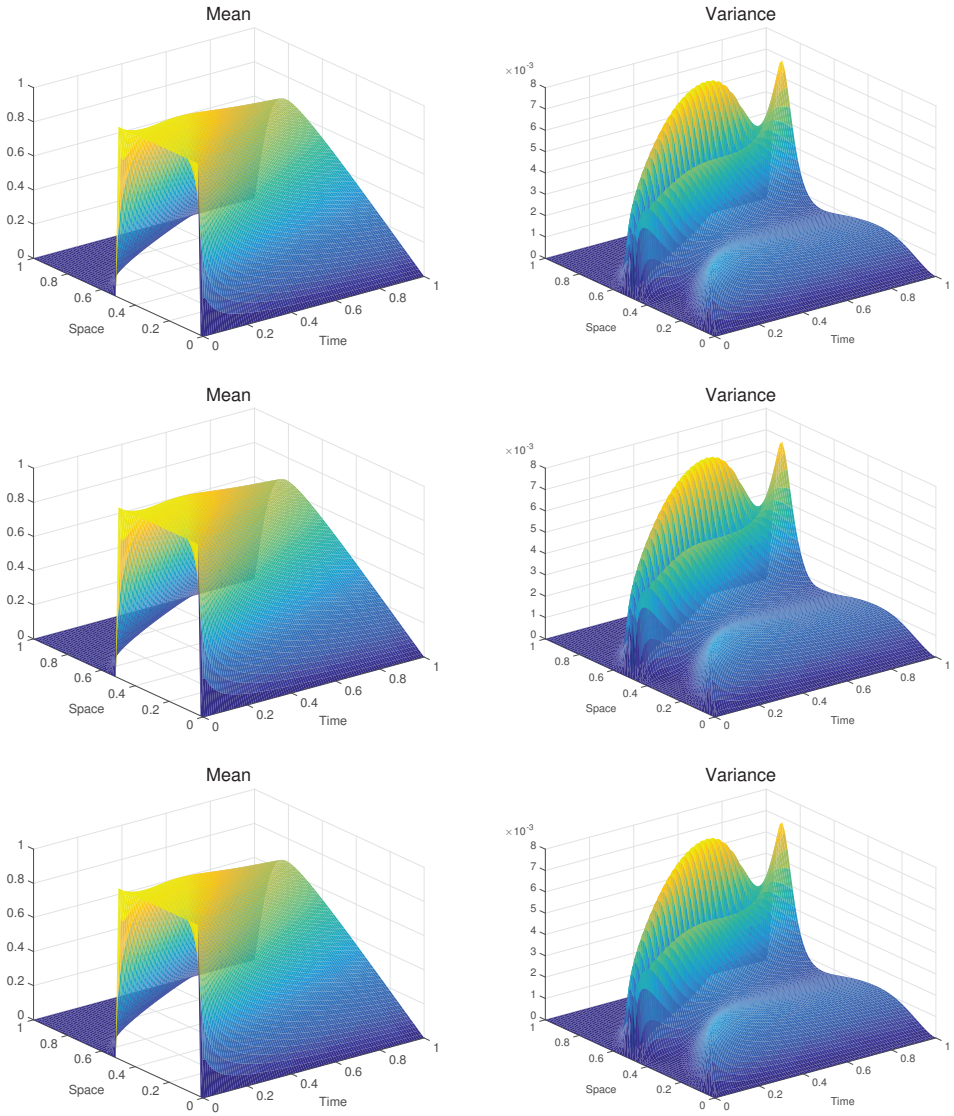

Figure 3: The means (first column) and variances (second column) of the solutions of the stochastic Burgers equation. The MC method with 12,000 realisations is used in the first row. Both the CC grid in the second row and the GL grid in the third row are used with level 4.

— i.e.

$$\mathbb{E}[u(t,x)] \approx \sum_k w_k u(y_k,t,x) \quad \text{and} \quad \mathrm{Var}[u(t,x)] \approx \sum_k w_k \left|u(y_k,t,x) - \mathbb{E}[u(t,x)]\right|^2,$$

where y_k and w_k are points and weights in the CC gird and the GL grid respectively. We also compared the convergence of the rth statistical moments obtained by the MC method and

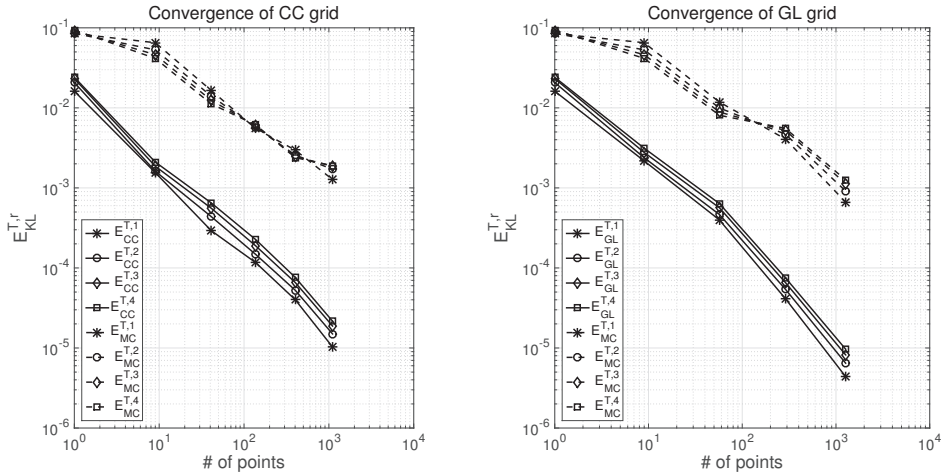

Figure 4: Convergences of the solutions computed over the CC gird (left) and over the GL gird (right). The dashed lines in both graphs represent the MC simulations.

the sparse grid collocation method over the CC or GL grid, by defining the error measures

$$E^{T,r}_{MC\setminus CC \text{ or } GL} := \int_0^T \left|\left| \mathbb{E}\left[u^r_{MC\setminus KL}\right] - \mathbb{E}\left[u^r\right] \right|\right|_{L^2(0,1)} dt\,,$$

where $\mathbb{E}[u^r_{MC}]$ and $\mathbb{E}[u^r_{CC \text{ or } GL}]$ denote the rth statistical moments obtained by the MC method and the CC or GL grid approximation, respectively. Again the final time $T = 1$ was chosen. Since we cannot find the explicit exact moments of the solution $\mathbb{E}[u^r]$, we set the moments of the solutions with Smolyak quadrature for 4 dimension with level 5 approximation over the CC or GL grid as the benchmark solutions at each comparison.

In Fig. 4, one can see the convergent rates of the CC grid (left) and the GL grid (right) approximations versus the number of points of each grids represented in Table 1. As a reference, the convergent rates of the MC simulations were computed and provided for both graphs. It is well known that the MC moments tend to the exact stochastic moments as the sample size increases, and its convergent rate is $\mathcal{O}(M^{-1/2})$ for the sample size M. Fig. 4 confirms that the convergent rate of the MC method again. From this convergence comparison between the MC method and the CC grid approximation or the MC method and the GL grid approximation in the figures, we see that the sparse grid collocation method converges faster than the MC method as the sample sizes increases.

So far, we have demonstrated the result from the uncontrolled solution of the stochastic Burgers equation. We next tested the numerical tools for the optimal control problem, as suggested in Section 5. As mentioned, we have to solve (5.3) subject to (5.4) at each random parameter values repeatedly. The control input operator is $b(x) = x$, $B_j = \int_D b(x)\phi_j(x)dx$, $j = 1,\cdots,N$, and $B = (B_1,\cdots,B_N)^T$, where $\phi_j(x)$ are continuous piecewise linear functions. We set a control weight $\beta = 64/13335$ at (5.1), which induces $R = 0.1$ at (5.2) and (5.3). Finally, we took the measurement operator $\mathscr{C}$ to be

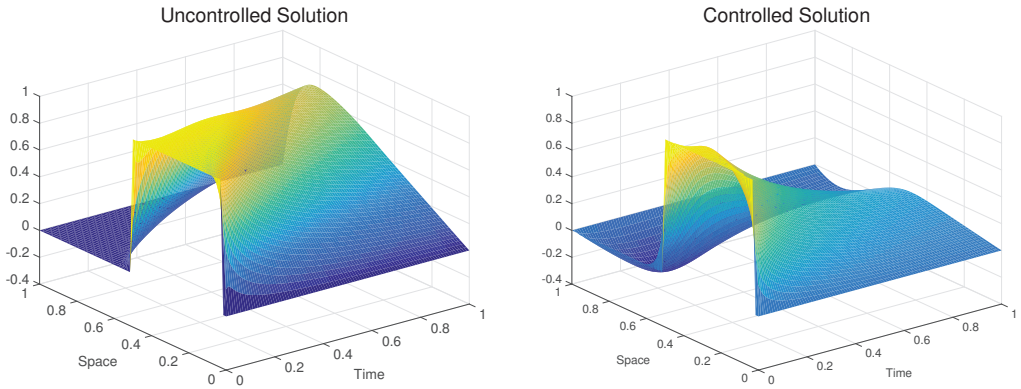

Figure 5: Realisations of the uncontrolled solution (left) and the controlled solution (right).

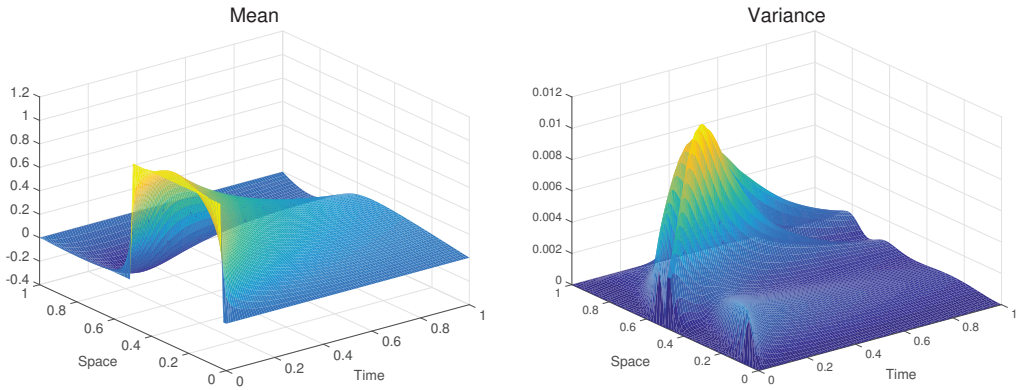

Figure 6: The mean (left) and the variance (right) of the controlled solution computed over CC grid.

$\mathscr{C}u(t,x) = 8\int_{3/4}^{5/6} u(t,x)dx$ for the state estimate feedback controller, so that the discretised operator C becomes $C_j = 8\int_{3/4}^{5/6} \phi_j(x)dx$, $j = 1,\cdots,N$ and $C = (C_1,\cdots,C_N)^T$ where $\phi_j(x)$ are also finite element basis functions, and for the control system an initial condition is needed for the state estimate. We used $u_{c_0}(x) = u_c(0,x) = u_0(x)$, which implies that the initial condition for the state has no error. The parameter space Γ had dimension 4 in our example, and grid nodes in Γ were selected under the CC grid rule of level 5.

Realisations of the uncontrolled solution and the controlled solution from one sample point in the CC grid are depicted in Fig. 5. Fig. 6 shows the mean and the variance of the controlled solutions approximated by the CC grid points and weights. We recall that $u(y_k,t,x)$ is the solution of system (5.9) that minimises the functional J_k in (5.3). Fig. 7 shows how much the realised controlled solutions disperse from their means at $x = 0.25$ (the first column) and $x = 0.75$ (the second column), obtained from the MC method (the first row, 1105 sample points) and the CC grid method with level 5 (the second row, 1105 points). The black lines of all the graphs represent the means of the controlled solutions.

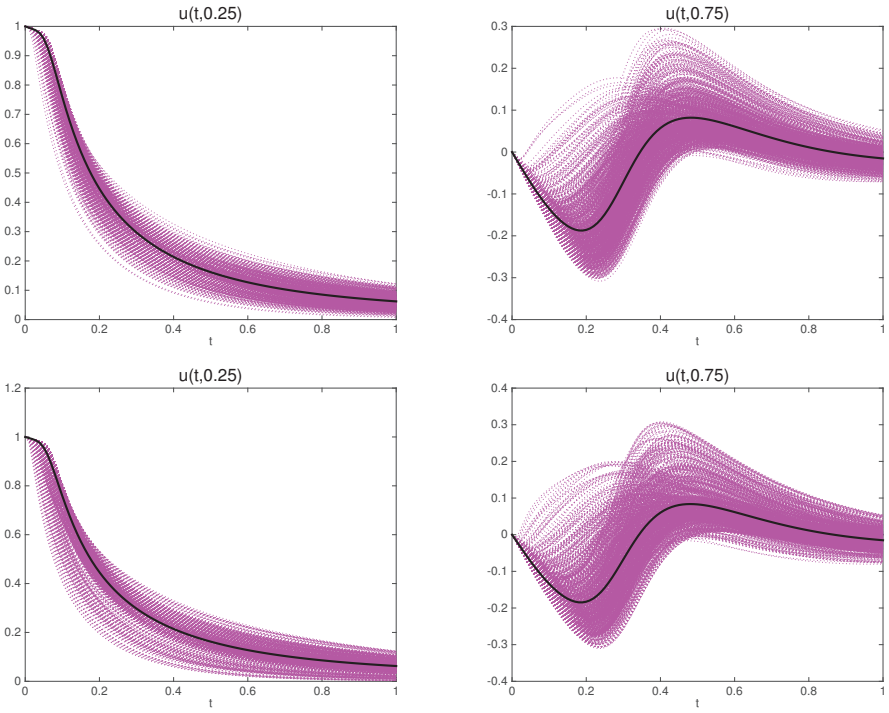

Figure 7: The realisations of the controlled solutions at $x = 0.25$ (the first column) and $x = 0.75$ (the second column), which are obtained by the MC method (the first row, 1105 sample points) and the CC grid method with level 5 (the second row, 1105 points). The black lines of all the graphs presents the means of the controlled solutions.

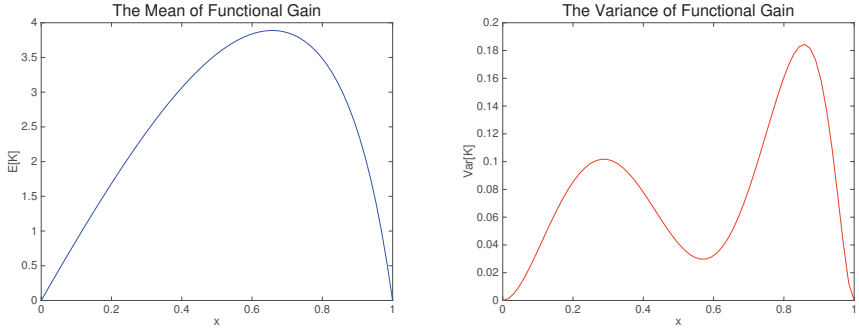

Figure 8: The mean (left) and the variance (right) of the functional gain K_k.

We also computed the means and variances of the functional gain operator K and the estimator gain operator L (see Fig. 8 and Fig. 9), denoted by $\mathbb{E}[K]$, $\mathrm{Var}[K]$, $\mathbb{E}[L]$ and $\mathrm{Var}[L]$, respectively. Lastly, we defined a norm

$$||u(t)||_{L^2_{\mathbb{P}}(\Omega;L^2(D))} = \left(\mathbb{E}\left[\int_D |u(t)|^2 dx\right]\right)^{1/2},$$

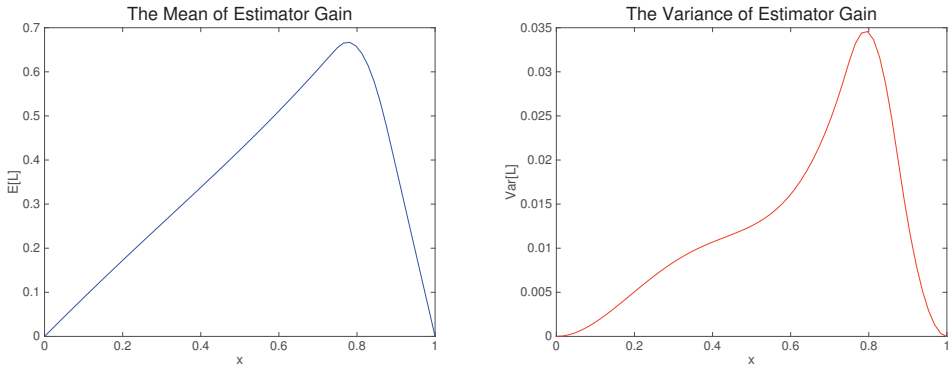

Figure 9: The mean (left) and the variance (right) of the estimator gain L_k.

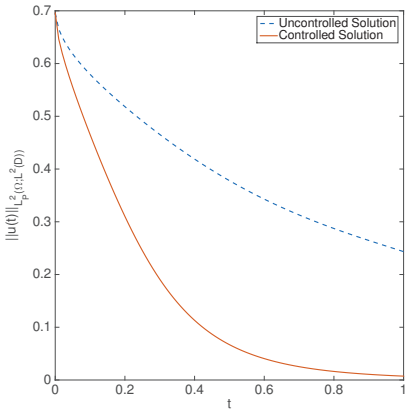

Figure 10: $L^2_{\mathbb{P}}(\Omega; L^2(D))$-norms for the solutions of the stochastic Burgers equation and its control problem.

which is also the norm of $L^2_{\mathbb{P}}(\Omega) \otimes L^2(D)$ as considered in Section 2, and related that norm to both the uncontrolled solutions and the controlled solutions along the time axis. The expectation $\mathbb{E}$ was of course approximated here by the CC grid with level 5. Fig. 10 presents the results, showing that the average norm of the controlled solutions evidently decays to zero more quickly than the average norm of the uncontrolled solutions.

7. Conclusions

An efficient and practical method has been presented for the derivation of finite-dimensional approximation of the stochastic Burgers equation with a random coefficient, and its distributed control problem with the synthesis of a linear feedback controller for the nonlinear parabolic PDE system. When solving a stochastic problem, numerical schemes for not only the temporal-spatial domain but also random parameter space were provided — viz. the finite element method combined with the backward Euler method and the sparse grid

stochastic collocation method, respectively. It is notable that the dimension of the random parameter space is usually greater than in the temporal-spatial domain. A sparse grid reduces the computational cost to obtain statistical moments in high dimensional parameter space, and works satisfactorily for our nonlinear parabolic equation.

For the application of feedback control procedures to the stochastic Burgers equation, we presented a suboptimal control and feedback procedure using the well-developed LQR and LQE theories to build an uncoupled bundle of closed-loop systems. This strategy can be extended to fairly general cost functions and time-dependent equations, including in particular stochastic equations. Although not strictly justified even in the deterministic equations, our approach has shown good numerical performance. The associated application of the linear feedback controller to the stochastic Burgers equation with random coefficient was also very successful, and the sparse grid stochastic collocation method helps in computing statistical moments and drawing a confidence interval with low cost.

Acknowledgments

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References

[1] J.A. Atwell, J.T. Borggaard and B.B. King, *Reduced order controller for Burgers' equation with a nonlinear observer*, Int. J. Appl. Math. Comp. Sci. **11**, 1311-1330 (2001).
[2] H.J. Bungartz and M. Griebel, *Sparse grids*, Acta Num. **13**, 1-123 (2004).
[3] J.A. Burns and S. Kang, *A control problem for Burgers' equation with bounded input/output*, Nonlinear Dynamics **2**, 235-262 (1991).
[4] A. Balogh and M, Krstic, *Burgers equation with nonlinear boundary feedback: H^1 stability, well-posedness and simulation*, Machematical Problems in Engineering **6**, 189-200 (2000).
[5] V. Barthelmann, E. Novak and K. Ritter, *High dimensional polynomial interpolation on sparse grids*, Adv. Comp. Math. **12**, 273-288 (2000).
[6] I. Babuska, F. Nobile and R. Tempone, *A stochastic collocation method for elliptic partial differential equations with random input data*, SIAM J. Num. Anal. **45**, 1005-1034 (2007).
[7] I. Babuska, R. Tempone and G.E. Zouraris, *Galerkin Finite Element Approximations of Stochastic Elliptic Partial Differential Equations*, SIAM J. Num. Anal. **42**, 800-825 (2005).
[8] C.T. Chen, *Linear System Theory and Design*, Holt, Rinehart and Winston, New York (1984).
[9] H. Choi, R. Temam, P. Moin and J. Kim, *Feedback control for unsteady flow and its application to the stochastic Burgers equation*, J. Fluid Mech. **253**, 509-543 (1993).
[10] P. Chen, A. Quarteroni and G. Rozza, *A weighted reduced basis method for elliptic partial differential equations with random input data*, SIAM J. Num. Anal. **51**, 3163-3185 (2013).
[11] R. Durrett, *Probability: Theory and Examples*, 2nd ed., Duxbury Press, Belmont CA (1964).
[12] G. Da Prato and A Debussche, *Control of the stochastic Burgers model of turbulence*, SIAM J. Control Optim. **37**, 1123-1149 (1999).
[13] M. El-Beltagy, M. Wafa and O. Galal, *Upwind finite-volume solution of the stochastic Burgers equation*, Appl. Math. **3**, 1818-1825 (2012).
[14] M. El-Beltagy and M. Wafa, *Stochastic 2D incompressible Navier-Stokes solver using the vorticity stream function formulation*, J. Appl, Math. **2013**, 1-14 (2013).

[15] C.A.J. Fletcher, *Burgers Equation: A model for all reasons*, Numerical Solutions of Partial Differential Equations, J. Noye (Ed.), North-Holland (1982).
[16] M.D. Gunzburger and A. Labovsky, *Effects of approximate deconvolution models on the solution of the stochastic Navier-Stokes equations*, J. Comp. Math. **29** 131-140 (2011).
[17] M. Krstic, *On global stabilization of Burgers equation by boundary control*, Systems Control Letts. **37**, 123-141 (1999).
[18] N. Kim and H.-C. Lee, *Sparse grid collocation method for an optimal control problem involving a stochastic partial differential equation with random inputs*, East Asian J. Appl. Math. **4**, 166-188 (2014).
[19] K. Kunisch and S. Volkwein, *Control of the Burgers equation by a reduced-order approach using proper orthogonal decomposition*, J. Optim. Theory Appl. **102**, 345-371 (1999).
[20] M. Loève, *Probability Theory*, 4th ed., Springer, New York (1977).
[21] H.-C. Lee and G.-R. Piao, *Boundary feedback control of the Burgers equations by a reduced-order approach using centroidal Voronoi tessellations*, J. Sci. Comp. **43** 369-387 (2010).
[22] E. Novak and K. Ritter, *High dimensional integration of smooth functions over cubes*, Num. Math. **75**, 79-97 (1996).
[23] E. Novak, K. Ritter, R. Schmitt and A. Steinbauer, *On an interpolatory method for high dimensional integration*, J. Comp. Appl. Math. **112**, 215-228 (1999).
[24] F. Nobile, R. Tempone and C.G. Webster, *A sparse grid stochastic collocation method for elliptic partial differential equations with random input data*, SIAM J. Num. Anal. **46**, 2309-2345 (2008).
[25] F. Nobile, R. Tempone and C.G. Webster, *An anisotropic sparse grid stochastic collocation method for elliptic partial differential equations with random input data*, SIAM J. Num. Anal. **46**, 2411-2442 (2008).
[26] G.-R. Piao and H.-C. Lee, *Distributed feedback control of the Benjamin-Bona-Mahony-Burgers equation by a reduced-order model*, East Asian J. Appl. Math. **5**, 61-74 (2015).
[27] G.-R. Piao, H.-C. Lee and J.-Y. Lee, *Distributed feedback control of the Burgers equation by a reduced-order approach using weighted centroidal Voronoi tessellation*, J. KSIAM **13**, 293-305 (2009).
[28] S.A. Smolyak, *Quadrature and interpolation formulas for tensor products of certain classes of functions*, Dokl. Akad. Nauk SSSR **4**, 240-243 (1963).
[29] G.W. Wasilkowski, H. Woźniakowski, *Explicit cost bounds of algorithms for multivariate tensor product problems*, J. Complexity **11** 1-56 (1995).