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On Finite Groups Whose Nilpotentisers Are Nilpotent Subgroups

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Abstract. Let G be a finite group and $x \in G$. The nilpotentiser of x in G is defined to be the subset $Nil_G(x) = \{y \in G : \langle x,y \rangle \text{ is nilpotent}\}$. G is called an $\mathcal{N}$-group ($\mathfrak{n}$-group) if $Nil_G(x)$ is a subgroup (nilpotent subgroup) of G for all $x \in G \setminus Z^*(G)$ where $Z^*(G)$ is the hypercenter of G. In the present paper, we determine finite $\mathcal{N}$-groups in which the centraliser of each noncentral element is abelian. Also we classify all finite $\mathfrak{n}$-groups.

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1 Introduction

Consider $x \in G$. The centraliser, nilpotentiser and engeliser of x in G are

$$C_G(x) = \{y \in G : \langle x,y \rangle \text{ is abelian}\}, \ Nil_G(x) = \{y \in G : \langle x,y \rangle \text{ is nilpotent}\}$$

and

$$E_G(x) = \{y \in G : [y,_n x] = 1 \text{ for some } n\}$$

respectively. Obviously

$$C_G(x) \subseteq Nil_G(x) \subseteq E_G(x) \quad \text{for each } x \in G.$$

Note that $Nil_G(x)$ and $E_G(x)$ are not necessarily subgroups of G. So determining the structure of groups by nilpotentisers (or engelisers) is more complicated than the centralisers. Let G be a finite group. Let $1 \leq Z_1(G) < Z_2(G) < \cdots$ be a series of subgroups of G, where $Z_1(G) = Z(G)$ is the center of G and $Z_{i+1}(G)$, for $i > 1$, is defined by

$$\frac{Z_{i+1}(G)}{Z_i(G)} = Z(\frac{G}{Z_i(G)}).$$

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Let $Z^*(G) = \bigcup_i Z_i(G)$. The subgroup $Z^*(G)$ is called the hypercenter of G. We say a group is $\mathfrak{n}$-group in which $Nil_G(x)$ is a nilpotent subgroup for each $x \in G \setminus Z^*(G)$.

Now a group is $\mathcal{N}$-group in which the nilpotentiser of each element is subgroup and a CA-group is a group in which the centraliser of each noncentral element is abelian (see [16] or [5]). The class of $\mathcal{N}$-groups were defined and investigated by Abdollahi and Zarrin in [1]. In particular they showed that every centerless CA-group is an $\mathcal{N}$-group. In this paper, we shall prove the following generalisation of this result.

Theorem 1.1. *Let G be a nonabelian CA-group. Then G is an $\mathcal{N}$-group if and only if we have one of the following types:*

1. *G has an abelian normal subgroup K of prime index.*
2. *$\frac{G}{Z(G)}$ is a Frobenius group with Frobenius kernel $\frac{K}{Z}$ and Frobenius complement $\frac{L}{Z(G)}$, where K and L are abelian.*
3. *$\frac{G}{Z(G)}$ is a Frobenius group with Frobenius kernel $\frac{K}{Z}$ and Frobenius complement $\frac{L}{Z(G)}$, such that $K = PZ$, where P is a normal Sylow p-subgroup of G for some prime divisor p of $|G|$, P is a CA-group, $Z(P) = P \bigcap Z$ and $L = HZ$, where H is an abelian p'-subgroup of G.*
4. *$\frac{G}{Z(G)} \cong PSL(2,q)$ and $G' \cong SL(2,q)$ where $q > 3$ is a prime-power number and $16 \nmid q^2 - 1$.*
5. *$\frac{G}{Z(G)} \cong PGL(2,q)$ and $G' \cong SL(2,q)$ where $q > 3$ is a prime and $8 \nmid q \pm 3$.*
6. *$G = P \times A$ where A is abelian and P is a nonabelian CA-group of prime-power order.*

A group is said to be an E-group whenever engeliser of each element of such group is subgroup. The class of E-groups was defined and investigated by Peng in [13,14]. Also Heineken and Casolo gave many more results about them (see [3,4,6]). Now recall that an engel group is a group in which the engeliser of every elements is the whole group. If G is an E-group such that the engeliser of every element is engel, G is an $\mathfrak{n}$-group since every finite engel group is nilpotent. This result motivates us to classify all finite $\mathfrak{n}$-groups in following theorem.

But before giving it, recall that the Hughes subgroup of a group G is defined to be the subgroup generated by all the elements of G whose orders are not p and denoted by $H_p(G)$ where p is a prime. Also a group G is said to be of Hughes-Thompson type, if for some prime p it is not a p-group and $H_p(G) \neq G$.

Theorem 1.2. *Let G be a nonnilpotent group. Then G is an $\mathfrak{n}$-group if and only if $\frac{G}{Z^*(G)}$ satisfies one of the following conditions:*

(1) $\frac{G}{Z^*(G)}$ *is a group of Hughes-Thompson type and*

$$\left| Nil_{\frac{G}{Z^*(G)}}(xZ^*(G)) \right| = p$$

for all $xZ^*(G) \in \frac{G}{Z^*(G)} \setminus H_p(\frac{G}{Z^*(G)})$;

(2) $\frac{G}{Z^*(G)}$ *is Frobenius group with Frobenius complement* $\frac{H}{Z^*(G)}$ *and* H *is an* n*-group of* G*;*

(3) $\frac{G}{Z^*(G)} \cong Sz(q)$*;*

(4) $\frac{G}{Z^*(G)} \cong PSL(2,2^m)$*,* $m>1$*.*

Our notations are standard and can be found mainly in [15]. In particular $PSL(2,q)$, $PGL(2,q)$ and $Sz(q)$ are the projective special linear group, projective general linear group of degree 2 over the finite field of size q and the Suzuki simple group over the finite field of size q respectively. Also in this paper G is a finite group and p is a prime.

2 Proofs of the Main Results

To prove our main results, we quote some lemmas that are required in the rest of the paper. Following theorem by schmidt determine all CA-groups. We use improved form of it due to Dolfi et al. ([5]).

Lemma 2.1. *Let* G *be a nonabelian group and write* $Z=Z(G)$*. Then* G *is a* CA*-group if and only if it is of one of the following types:*

(I) G *is nonabelian and has an abelian normal subgroup of prime index.*

(II) $\frac{G}{Z}$ *is a Frobenius group with Frobenius kernel* $\frac{K}{Z}$ *and Frobenius complement* $\frac{L}{Z}$*, where* K *and* L *are abelian.*

(III) $\frac{G}{Z}$ *is a Frobenius group with Frobenius kernel* $\frac{K}{Z}$ *and Frobenius complement* $\frac{L}{Z}$*, such that* $K=PZ$*, where* P *is a normal Sylow* p*-subgroup of* G *for some* $p\in\pi(G)$*,* P *is a* CA*-group (F-group),* $Z(P)=P\bigcap Z$ *and* $L=HZ$*, where* H *is an abelian* p'*-subgroup of* G*.*

(IV) $\frac{G}{Z}\cong S_4$ *and if* $\frac{V}{Z}$ *is the Klein four group in* $\frac{G}{Z}$*, then* V *is nonabelian.*

(V) $G=P\times A$*, where* P *is a nonabelian* CA*-group (F-group) of prime-power order and* A *is abelian.*

(VI) $\frac{G}{Z}\cong PSL(2,p^n)$ *or* $PGL(2,p^n)$ *and* $G'\simeq SL(2,p^n)$ *where* p *is a prime and* $p^n>3$*.*

(VII) $\frac{G}{Z}\cong PSL(2,9)$ *or* $PGL(2,9)$ *and* G' *is isomorphic to the Schur cover of PSL(2, 9).*

Lemma 2.2. *Let* G *be a finite* $\mathcal{N}$*-group. Then all subgroups of* G *are* $\mathcal{N}$*-groups.*

Proof. The proof is clear. □

Lemma 2.3. *Let* G *be a Frobenius group with Frobenius complement* H*. Then* G *is an* $\mathcal{N}$*-group (*$\mathfrak{n}$*-group) if and only if* H *is an* $\mathcal{N}$*-group (n-group).*

Proof. The proof is similar to Proposition 3.1 of [12]. □

Lemma 2.4. *$PGL(2,q)$ is an $\mathcal{N}$-group if and only if $q>3$ is a prime number and $8\nmid(q\pm 3)$.*

Proof. See Proposition 3.4 of [12]. $\square$

Lemma 2.5. *$PSL(2,q)$ is an $\mathcal{N}$-group if and only if $16\nmid q^2-1$.*

Proof. See Lemmas 3.9 and 3.10 of [1]. $\square$

Lemma 2.6. *Let G be a group. Then G is an $\mathcal{N}$-group ($\mathfrak{n}$-group) if and only if $\frac{G}{K}$ is an $\mathcal{N}$-group ($\mathfrak{n}$-group) for some normal subgroup K of G with $K\le Z^*(G)$.*

Proof. See Lemma 2.2 (1)-(4) of [1]. $\square$

Proof of Theorem 1.1. First, note that every centerless CA-group K is an $\mathcal{N}$-group and in particular $Nil_K(x)=C_K(x)$ for each $x\in K$ (see Lemma 3.6 of [1]). Suppose that G is a CA-group. We apply Lemma 2.1 in order to establish our claim.

S_4, the symmetric group of degree 4, is not an $\mathcal{N}$-group since $Nil_{s_4}((12)(34))$ is not a subgroup of S_4. It follows that G does not satisfy (IV) of Lemma 2.1 by Lemmas 2.3 and 2.6. Similarly since $PSL(2,9)$ and $PGL(2,9)$ have some subgroups isomorphic to S_4, G does not satisfy (VII).

Now, assume that G satisfies (I) of Lemma 2.1. Then G has an abelian normal subgroup A of prime index p. If $G=AZ^*(G)$, then G is nilpotent and so G is an $\mathcal{N}$-group. Suppose that $G\neq AZ^*(G)$. Then $\frac{G}{Z^*(G)}$ has a normal abelian subgroup

$$\overline{A}=\frac{AZ^*(G)}{Z^*(G)}$$

of index p. Therefore $\frac{G}{Z^*(G)}$ is a centerless CA-group and so we have the result by Lemma 2.6.

Next, suppose that G satisfies (II) or (III) of Lemma 2.1. Then G is an $\mathcal{N}$-group by Lemmas 2.3 and 2.6.

Now, suppose that G satisfies (V). Then G is nilpotent and so G is an $\mathcal{N}$-group.

Finally, if G satisfies (VI), then we get to parts (4) and (5) of our theorem by Lemmas 2.5, 2.4 and 2.6.

The converse is clear by the previous lemmas and Lemma 2.1. $\square$

Proof of Theorem 1.2. Suppose that G is an $\mathfrak{n}$-group and $Nil(G)=\cap_{g\in G}Nil_G(g)$. Let also $Nil_G(x)$ and $Nil_G(y)$ be two distinct nilpotent subgroups of G for $x,y\in G\setminus Nil(G)$. We claim that

$$Nil_G(x)\bigcap Nil_G(y)=Nil(G).$$

Suppose, for a contradiction, that there exists $t\in (Nil_G(x)\bigcap Nil_G(y))\setminus Nil(G)$. Hence $Nil_G(x)=Nil_G(t)=Nil_G(y)$ which gives a contradiction. Since $Z^*(G)=Nil(G)$ by Proposition 2.2 of [1], we have

$$\Gamma=\left\{\frac{Nil_G(x)}{Z^*(G)}:x\in G\right\}$$

is a partition of $\frac{G}{Z^*(G)}$. Since G is not nilpotent, $\frac{G}{Z^*(G)}$ is one of the followings by page 575 of [17].

a. $\frac{G}{Z^*(G)}$ is a Frobenius group;

b. $\frac{G}{Z^*(G)}$ is a group of Hughes-Thompson type;

c. $\frac{G}{Z^*(G)} \cong PGL(2,p^m)$, p being an odd prime;

d. $\frac{G}{Z^*(G)} \cong PSL(2,p^m)$, p being a prime;

e. $\frac{G}{Z^*(G)} \cong Sz(q)$, $q=2^h$, $h>1$.

To complete the proof in one direction it suffices to prove only two parts (1) and (4) of our theorem. First, we claim that $\frac{G}{Z^*(G)} \ncong PGL(2,p^m)$ for every odd prime p. Since $PGL(2,3) \cong S_4$, G is not an $\mathcal{N}$-group by Lemma 2.6. Suppose, for a contradiction, that

$$\frac{G}{Z^*(G)} \cong PGL(2,q) \text{ and } q=p^m>3.$$

By Lemmas 2.4 and 2.6, G is an $\mathcal{N}$-group if and only if $8 \nmid (q\pm 3)$ ($q>3$ is prime). We choose an element $xZ^*(G) \in \frac{G}{Z^*(G)}$ of order two. By page 575 of [17], $C_{\frac{G}{Z^*(G)}}(xZ^*(G))$ is not nilpotent and therefore $Nil_{\frac{G}{Z^*(G)}}(xZ^*(G))$ is not so. Since

$$\frac{Nil_G(x)}{Z^*(G)} = Nil_{\frac{G}{Z^*(G)}}(xZ^*(G))$$

by Lemma 2.2 (3) of [1], we deduce that $Nil_G(x)$ is not nilpotent which establishes the claim.

Now, we claim that $\frac{G}{Z^*(G)} \cong PSL(2,p^m)$ for $p^m \in \{5,2^m : m>1\}$. Suppose, for a contradiction, that $\frac{G}{Z^*(G)} \cong PSL(2,q)$ where $q=p^m \neq 5$ is odd. By Lemmas 2.5 and 2.6, we have $16 \nmid q^2-1$. It follows from Lemma 2.5 that $C_{PSL(2,q)}(x) = Nil_{PSL(2,q)}(x)$. Consequently $Nil_{PSL(2,q)}(x)$ is either abelian or generalised dihedral group by Proposition 3.21 of [2]. Next by Satz 8.10 of [8], all Sylow p-subgroups of $PSL(2,q)$ are abelian in this case. Now if $C_G(x)$ is a centraliser of $PSL(2,q)$ isomorphic to generalised dihedral group D, then D must be nilpotent and so it must be 2-group. This implies that $C_G(x)$ is abelian. Therefore $q=3$ or 5 , a contradiction.

Now, let $\frac{G}{Z^*(G)}$ be a group of Hughes-Thompson type. By Theorem 1 in [7], $H_p(\frac{G}{Z^*(G)})$ has index p in $\frac{G}{Z^*(G)}$ for some prime p. Also it was proved by Kegel in [10] that $H_p(\frac{G}{Z^*(G)})$ is nilpotent and in Satz 3 of [11], $H_p(\frac{G}{Z^*(G)})$ is a component of partition of $\frac{G}{Z^*(G)}$ and so $H_p(\frac{G}{Z^*(G)})$ is a nilpotentiser of index p of $\frac{G}{Z^*(G)}$. It follows that $|Nil_{\frac{G}{Z^*(G)}}(xZ^*(G))| = p$ for all $xZ^*(G) \in \frac{G}{Z^*(G)} \setminus H_p(\frac{G}{Z^*(G)})$. This completes the proof in one direction.

Conversely assume that $\frac{G}{Z(G)} \cong PSL(2,2^m)$ for an integer $m>1$ or $Sz(q)$. Then it is enough to show that $PSL(2,2^m)$ and $Sz(q)$ are $\mathfrak{n}$-groups by Lemma 2.6.

First, note that $SZ(q)$ is an $\mathfrak{n}$-group by the proof of Theorem 3.8 of [1]. Since $PSL(2,2^m)$ is a CA-group by Lemma 2.1, we have the result by the second part of Lemma 3.6 of [1] and Lemma 2.6.

Now, suppose that $\frac{G}{Z^*(G)}$ is a Frobenius group such that its Frobenius complement is an $\mathfrak{n}$-group. By Lemma 2.3, $\frac{G}{Z^*(G)}$ is an $\mathfrak{n}$-group and so G is an $\mathfrak{n}$-group by Lemma 2.6.

Now let $\frac{G}{Z^*(G)}$ be a group of Hughes-Thompson type. Then

$$\Gamma = \left\{ H_p(\frac{G}{Z^*(G)}), \frac{H_i}{Z^*(G)} : 1 \leq \cdots \leq r \right\}$$

is a partition of $\frac{G}{Z^*(G)}$ for some prime p such that $|\frac{H_i}{Z^*(G)}| = p$ for each i. Then we claim that

$$H_p(\frac{G}{Z^*(G)}) = Nil_{\frac{G}{Z^*(G)}}(yZ^*(G))$$

for each $yZ^*(G) \in H_p(\frac{G}{Z^*(G)})$.

If the equality does not occur, then there is some element $xZ^*(G)$ of order p such that $xZ^*(G) \in Nil_{\frac{G}{Z^*(G)}}(yZ^*(G)) \setminus H_p(\frac{G}{Z^*(G)})$. Now since $xZ^*(G)$ belongs to some component of partition of $\frac{G}{Z^*(G)}$, say $\frac{H_j}{Z^*(G)}$ and $|Nil_{\frac{G}{Z^*(G)}}(xZ^*(G))| = p$ by assumption, we have

$$\frac{H_j}{Z^*(G)} = Nil_{\frac{G}{Z^*(G)}}(xZ^*(G)).$$

On the other hand $yZ^*(G) \in Nil_{\frac{G}{Z^*(G)}}(xZ^*(G))$, which implies that

$$|Nil_{\frac{G}{Z^*(G)}}(xZ^*(G))| > p,$$

a contradiction. This proves the claim.

Now, if $tZ^*(G) \notin H_p(\frac{G}{Z^*(G)})$, then $Nil_{\frac{G}{Z^*(G)}}(tZ^*(G))$ is a component of partition $\frac{G}{Z^*(G)}$ by hypothesis. Thus $\frac{G}{Z^*(G)}$ is an $\mathfrak{n}$-group and so G is an $\mathfrak{n}$-group by Lemma 2.6. This completes the proof. $\square$

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References

[1] A. Abdollahi, M. Zarrin. Non-nilpoten graph of a group. Comm. Algebra, 2010, 38(12): 4390-4403.

[2] A. Abdollahi, S. Akbari, H. R. Maimani. Non-commuting graph of a group. J. Algbera, 2006, 298(2): 468-492.

[3] C. Casolo. Finite groups in which subnormalizers are subgroups. Rend. Semin. Mat. Univ. Padova, 1989, 82: 25-53.

[4] C. Casolo. Subnormalizers in finite groups. Comm. Algebra, 1990, 18011: 3791-3818.

[5] S. Dolfi, M. Herzog, E. Jabara. Finite groups whose noncentral commuting elements have centralizers of equal size. Bull. Aust. Math Soc., 2010, 82: 293-304.

[6] H. Heineken. On E-groups in the sence of Peng. Glasg. Math. J, 1989, 31(2): 231-242.

[7] D. Hughes, J. G. Thompson. The Hp-problem and the structure of the Hp-groups. Pacific J. Math, 1963, 9: 1097-1102.

[8] B. Huppert. Endlich gruppen I. Springer-Verlag, Berlin, 1967.

[9] B. Huppert, N.Blackburn. Finite groups, III, Springer-Verlag, Berlin, 1982.

[10] O. H. Kegel. Die nilpotenz der Hp-gruppen, Math. Z, 1960/1961, 71: 373-376.

[11] O. H. Kegel. Nicht-einfache partitionen endlicher Gruppen. Arch. Math., 1961, 12(1): 170-175.

[12] S. M. Jafarian Amiri, H. Madadi, H. Rostami. On the probability of generating nilpotent subgroups in a finite group. Bull. Aust. Math Soc., 2015.

[13] T. A. Peng. On groups with nilpotent derived groups. Arch. Math, 1969, 20(3): 251-253.

[14] T. A. Peng. Finite soluble groups with an Engel condition. J. Algebra, 1969, 11(3): 319-330.

[15] D. J. S. Robinson. A course in the theory of groups. Springer-Verlag New York, 1996.

[16] R. Schmidt. Zentralisatorverb*ä*nde endlicher gruppen. Rend. Sem. Mat. Univ. Padova, 1970, 44: 97-131.

[17] G. Zappa. Partitions and other coverings of finite groups. Illinois J. Math, 2003, 47(1-2): 571-580.