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# Non-Relativistic and Low Mach Number Limits of a Compressible Full MHD-$P1$ Approximate Model Arising in Radiation Magnetohydrodynamics

Xiuhui Yang*

*Department of Mathematics, College of Science, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, P.R. China.*

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**Abstract.** In this paper we study the non-relativistic and low Mach number limits of strong solutions to a full compressible MHD-$P1$ approximate model arising in radiation magnetohydrodynamics. We prove that, as the parameters go to zero, the solutions of the primitive system converge to that of the classical incompressible magnetohydrodynamic equations.

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**Key words**: Full MHD-$P1$ approximate model, non-relativistic and low Mach number limit.

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## 1 Introduction

In this paper we consider the following full compressible MHD-$P1$ approximate model arising in radiation magnetohydrodynamics [2,4,7]:

$$\partial_t\rho+\operatorname{div}(\rho u)=0, \tag{1.1a}$$

$$\partial_t(\rho u)+\operatorname{div}(\rho u\otimes u)+\frac{1}{\epsilon_1^2}\nabla p-\mu\Delta u-(\lambda+\mu)\nabla\operatorname{div}u=I_1+\operatorname{rot}b\times b, \tag{1.1b}$$

$$\begin{aligned}&\partial_t(\rho e)+\operatorname{div}(\rho u e)+p\operatorname{div}u-\Delta\mathcal{T}\\&\qquad=\epsilon_1^2\left(\frac{\mu}{2}|\nabla u+\nabla u^t|^2+\lambda(\operatorname{div}u)^2+|\operatorname{rot}b|^2\right)+I_0-\mathcal{T}^4,\end{aligned} \tag{1.1c}$$

$$\partial_t b+\operatorname{rot}(b\times u)-\Delta b=0,\quad \operatorname{div}b=0, \tag{1.1d}$$

$$\epsilon_2\partial_t I_0+\operatorname{div}I_1=\mathcal{T}^4-I_0, \tag{1.1e}$$

$$\epsilon_2\partial_t I_1+\nabla I_0=-I_1\ \text{ in }\ \mathbb{T}^3\times(0,\infty), \tag{1.1f}$$

*Corresponding author. *Email address:* `xhyang@nuaa.edu.cn` (X. Yang)

 

where $\rho,u,\mathcal{T},b$ and $I:=I_0+I_1\cdot\omega$ denote the density, velocity, temperature, magnetic field, and the radiation intensity of the fluid, respectively. $\omega\in\mathbb{S}^2$ is the direction vector. The viscosity coefficients $\mu$ and $\lambda$ of the fluid satisfy $\mu>0$ and $\lambda+\frac{2}{3}\mu\geq 0$. $\epsilon_1>0$ is the (scaled) Mach number and $\epsilon_2>0$ is the (scaled) light speed. $\mathbb{T}^3$ is a periodic domain in $\mathbb{R}^3$.

When the magnetic field in (1.1) is neglected, i.e. $b=0$, the system is reduced to the Navier-Stokes-Fourier-$P1$ model, and the papers [4,8,14] established non-relativistic and low Mach number limits of the problem.

If we ignore the radiation effort in (1.1), the system is reduced to full compressible magnetohydrodynamic equations and has received many studies, for example, see [3,6, 9–13]. The local strong solution was obtained by Fan-Yu [9]. The global weak solutions were obtained by Fan-Yu [10], Ducomet-Feireisl [6] and Hu-Wang [11] respectively. The low Mach number limit problems were studied by Jiang-Ju-Li [12] in $\mathbb{T}^3$ for well-prepared initial data, Jiang-Ju-Li-Xin [13] in $\mathbb{R}^3$ for ill-prepared initial data, and Cui-Ou-Ren [3] in a bounded domain for well-prepared initial data.

Very recently, Xie and Klingenberg [16] studied the non-relativistic limit for the three-dimensional ideal compressible radiation magnetohydrodynamics and obtained the limiting problem which is a widely used macroscopic model in radiation magnetohydrodynamics.

In this paper, we study the non-relativistic and low Mach number limits to the full compressible MHD-$P1$ approximate model (1.1) and hence extend the result in [4] to more general models. For simplicity, we will consider the case that the fluid is a polytropic ideal gas, i.e., the internal energy $e$ and the the pressure $p$ satisfy

$$e:=C_V\mathcal{T},\qquad p:=R\rho\mathcal{T}$$

with positive constants $C_V$ and $R$.

In the following, we introduce the new unknowns $\sigma$ and $\theta$ with

$$\rho:=1+\epsilon_1\sigma,\quad \mathcal{T}:=1+\epsilon_1\theta. \tag{1.2}$$

Then the system (1.1) can be rewritten as

$$\partial_t\sigma+\mathrm{div}(\sigma u)+\frac{1}{\epsilon_1}\mathrm{div}\,u=0, \tag{1.3a}$$

$$\begin{aligned}&\rho\partial_t u+\rho u\cdot\nabla u+\frac{R}{\epsilon_1}(\nabla\sigma+\nabla\theta)+R\nabla(\sigma\theta)\\&\qquad-\mu\Delta u-(\lambda+\mu)\nabla\mathrm{div}\,u=I_1+\mathrm{rot}b\times b,\end{aligned} \tag{1.3b}$$

$$\begin{aligned}&C_V\rho(\partial_t\theta+u\cdot\nabla\theta)+R(\rho\theta+\sigma)\mathrm{div}\,u+\frac{R}{\epsilon_1}\mathrm{div}\,u-\Delta\theta\\&\qquad=\epsilon_1\left(\frac{\mu}{2}|\nabla u+\nabla u^t|^2+\lambda(\mathrm{div}u)^2+|\mathrm{rot}b|^2\right)+I_0-(1+\epsilon_1\theta)^4,\end{aligned} \tag{1.3c}$$

$$\partial_t b+\mathrm{rot}(b\times u)-\Delta b=0,\quad \mathrm{div}\,b=0, \tag{1.3d}$$

$$\epsilon_2\partial_t I_0+\mathrm{div}\,I_1=(1+\epsilon_1\theta)^4-I_0, \tag{1.3e}$$

$$\epsilon_2\partial_t I_1+\nabla I_0=-I_1\quad\text{in }\ \mathbb{T}^3\times(0,\infty). \tag{1.3f}$$

We impose the initial conditions to the system (1.3) as following:

$$(\sigma,u,\theta,b,I_0,I_1)(\cdot,0)=(\sigma_0,u_0,\theta_0,b_0,I_{00},I_{10}) \text{ in } \mathbb{T}^3. \tag{1.4}$$

A local existence result for (1.3)-(1.4) in the following sense can be shown in a similar way as in [17]. Thus we omit the details of the proof.

**Proposition 1.1** (Local existence)**.** Let $0<\epsilon_1,\epsilon_2<1$ and $\epsilon:=(\epsilon_1,\epsilon_2)$. Suppose that the initial data $(\sigma_0^\epsilon,u_0^\epsilon,\theta_0^\epsilon,b_0^\epsilon,I_{00}^\epsilon,I_{10}^\epsilon)$ satisfy that $1+\epsilon_1\sigma_0^\epsilon\geq m>0$ for some positive constant $m$, and

$$\partial_t^k\sigma^\epsilon(0),\ \partial_t^k u^\epsilon(0),\ \partial_t^k\theta^\epsilon(0),\ \partial_t^k b^\epsilon(0),\ \partial_t^k I_0^\epsilon(0),\ \partial_t^k I_1^\epsilon(0)\in H^{2-k}(\mathbb{T}^3),\ \ k=0,1,2.$$

Then there exists a positive constant $T^\epsilon>0$ such that the problem (1.3)–(1.4) has a unique solution $(\sigma^\epsilon,u^\epsilon,\theta^\epsilon,b^\epsilon,I_0^\epsilon,I_1^\epsilon)$, satisfying that $1+\epsilon_1\sigma^\epsilon>0$ in $\mathbb{T}^3\times(0,T^\epsilon)$, and for $k=0,1,2$,

$$\begin{aligned}&\partial_t^k\sigma^\epsilon,\ \partial_t^k I_0^\epsilon,\ \partial_t^k I_1^\epsilon\in C([0,T^\epsilon];H^{2-k}),\\ &\partial_t^k u^\epsilon,\ \partial_t^k\theta^\epsilon,\ \partial_t^k b^\epsilon\in C([0,T^\epsilon];H^{2-k})\cap L^2(0,T^\epsilon,H^{3-k}).\end{aligned}$$

**Remark 1.1.** To simplify the statement, we have used $\partial_t u(0)$ to signify the quantity $\partial_t u|_{t=0}$ which is obtained through equation (1.3b), and $\partial_t^2 u(0)$ is given recursively by using $\partial_t$(1.3b) in the same manner. Similarly, we can define $\partial_t\sigma(0)$, $\partial_t\theta(0)$, $\partial_t b(0)$, $\partial_t I_0(0)$, $\partial_t I_1(0)$, $\partial_t^2\sigma(0)$, $\partial_t^2\theta(0)$, $\partial_t^2 b(0)$, $\partial_t^2 I(0)$ and $\partial_t^2 I_1(0)$.

Denote

$$|||u|||_{k,j}:=\sum_{i=0}^{j}\|\partial_t^i u\|_{H^{k-i}(\mathbb{T}^3)},\quad |||u|||_{k,j}(0):=\sum_{i=0}^{j}\|\partial_t^i u(0)\|_{H^{k-i}(\mathbb{T}^3)}.$$

The main result of this paper reads as follows.

**Theorem 1.1.** *Assume that* $(\sigma^\epsilon,u^\epsilon,\theta^\epsilon,b^\epsilon,I_0^\epsilon,I_1^\epsilon)$ *is the unique solution obtained in Proposition* 1.1. *Assume further that the initial data* $(\sigma_0^\epsilon,u_0^\epsilon,\theta_0^\epsilon,b_0^\epsilon,I_{00}^\epsilon,I_{10}^\epsilon)$ *satisfy*

$$|||(\sigma^\epsilon,u^\epsilon,\theta^\epsilon,b^\epsilon,I_{00}^\epsilon,I_{10}^\epsilon)|||_{2,2}(0)+\|(1+\epsilon_1\sigma_0^\epsilon)^{-1}\|_{L^\infty}\leq D_0. \tag{1.5}$$

*Then there exist positive constants* $T_0$ *and* $D$*, independent of* $\epsilon$*, such that* $(\sigma^\epsilon,u^\epsilon,\theta^\epsilon,b^\epsilon,I_0^\epsilon,I_1^\epsilon)$ *satisfy the uniform estimates:*

$$\begin{aligned}&\sup_{0\leq t\leq T_0}\left(|||(\sigma^\epsilon,u^\epsilon,\theta^\epsilon,b^\epsilon)|||_{2,2}+\|(1+\epsilon_1\sigma^\epsilon)^{-1}\|_{L^\infty}\right)(t)+\left(\int_0^{T_0}|||(u^\epsilon,\theta^\epsilon,b^\epsilon)|||_{3,2}^2dt\right)^{\frac{1}{2}}\\ &\quad+\sup_{0\leq t\leq T_0}\left(\sqrt{\epsilon_2}\|(I_0^\epsilon,I_1^\epsilon)\|_{H^2}+\|(I_0^\epsilon,I_1^\epsilon)\|_{H^1}+\|\partial_t(I_0^\epsilon,I_1^\epsilon)\|_{L^2}\right)(t)\\ &\quad+\left(\int_0^{T_0}\left(\|(I_0^\epsilon,I_1^\epsilon)\|_{H^2}^2+\|\partial_t(I_0^\epsilon,I_1^\epsilon)\|_{H^1}^2+\|\partial_t^2(I_0^\epsilon,I_1^\epsilon)\|_{L^2}^2\right)dt\right)^{\frac{1}{2}}\leq D.\end{aligned} \tag{1.6}$$

*Furthermore,* $(\sigma^\epsilon,u^\epsilon,\theta^\epsilon,b^\epsilon,I_0^\epsilon,I_1^\epsilon)$ *converge to* $(\sigma,u,\theta,b,I_0=1,I_1=0)$ *in certain Sobolev space as* $\epsilon\to 0$*, and there exists a function* $\pi(x,t)$ *such that* $(u,b,\pi)$ *in* $C([0,T_0];H^2)$ *solves the following problem of the incompressible magnetohydrodynamic system*

$$\begin{cases} \partial_t u+u\cdot\nabla u+\nabla\pi-\mu\Delta u=0,\ \mathrm{div}\,u=0, \\ \partial_t b+\mathrm{rot}(b\times u)-\Delta b=0,\ \mathrm{div}\,b=0, \\ u(\cdot,0)=u_0,\ b(\cdot,0)=b_0 \ \ in\ \ \mathbb{T}^3, \end{cases} \tag{1.7}$$

*where* $(u_0,b_0)$ *is the weak limit of* $(u_0^\epsilon,b_0^\epsilon)$ *in* $H^2$ *with* $\mathrm{div}\,u_0=0$ *in* $\mathbb{T}^3$.

Denote

$$\begin{aligned} M^\epsilon(t):=&\sup_{0\le s\le t}\Big(|||(\sigma^\epsilon,u^\epsilon,\theta^\epsilon,b^\epsilon)|||_{2,2}(s)+\|(1+\epsilon_1\sigma^\epsilon)^{-1}\|_{L^\infty}(s)\Big) \\ &+\sup_{0\le s\le t}\Big(\sqrt{\epsilon_2}\|(I_0^\epsilon,I_1^\epsilon)\|_{H^2}+\|(I_0^\epsilon,I_1^\epsilon)\|_{H^1}+\|\partial_t(I_0^\epsilon,I_1^\epsilon)\|_{L^2}\Big)(s) \\ &+\left(\int_0^t|||(u^\epsilon,\theta^\epsilon,b^\epsilon)|||_{3,2}^2ds\right)^{\frac12}+\left(\int_0^t|||(I_0^\epsilon,I_1^\epsilon)|||_{2,2}^2ds\right)^{\frac12}, \end{aligned} \tag{1.8a}$$

$$M_0^\epsilon:=M^\epsilon(t=0). \tag{1.8b}$$

Our next result is

**Theorem 1.2.** *Let* $T^\epsilon$ *be the maximal time of existence for the problem* (1.3)–(1.4) *in the sense of Proposition* 1.1*. Then for any* $t\in[0,T^\epsilon)$*, we have*

$$M^\epsilon(t)\le C_0(M_0^\epsilon)\exp\Big[t^{\frac14}C(M^\epsilon(t))\Big] \tag{1.9}$$

*for some given nondecreasing continuous function* $C_0(\cdot)$ *and* $C(\cdot)$.

In the remainder of this this paper we will give the proofs of Theorems 1.1 and 1.2.

## 2 Proofs of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. Below we shall drop the super script "$\epsilon$" of $\rho^\epsilon,\sigma^\epsilon,u^\epsilon,\theta^\epsilon$, etc. for the sake of simplicity; moreover, we write $M^\epsilon(t)$ and $M^\epsilon(0)$ as $M$ and $M_0$, respectively. Since the physical constants $C_V$, and $R$ do not bring any essential difficulties in our arguments, we shall take $C_V=R=1$.

First, by the same calculations as that in [3], we get

$$(|||\rho|||_{2,2}+\|\rho^{-1}\|_{L^\infty})(t)\le C_0(M_0)\exp(\sqrt{t}C(M)), \tag{2.1}$$

$$\|(\sigma,u,\theta,b)(t)\|_{L^2}^2+\|(u,\theta,b)\|_{L^2(0,t;H^1)}^2\le C_0(M_0)\exp(\sqrt{t}C(M)), \tag{2.2}$$

$$\|(\nabla\sigma,\mathrm{div}\,u,\nabla\theta,\nabla b)(t)\|_{L^2}^2+\|(\nabla\mathrm{div}\,u,\Delta\theta,\Delta b)\|_{L^2(0,t;L^2)}^2\le C_0(M_0)\exp(\sqrt{t}C(M)). \tag{2.3}$$

**Lemma 2.1.** *For any* $0\le t\le\min\{T^\epsilon,1\}$*, we have*

$$\|\mathrm{rot}u(t)\|_{L^2}^2+\|\mathrm{rot}^2u\|_{L^2(0,t;L^2)}^2\le C_0(M_0)\exp(\sqrt{t}C(M)).$$

*Proof.* Let $\omega:=\mathrm{rot}u$. From (1.3b) we easily derive

$$\rho(\partial_t\omega+u\cdot\nabla\omega)-\mu\Delta\omega=K+\mathrm{rot}I_1, \tag{2.4}$$

where $K:=-(\partial_j\rho\partial_tu_i-\partial_i\rho\partial_tu_j)-[\partial_j(\rho u_k)\partial_ku_i-\partial_i(\rho u_k)\partial_ku_j]+\mathrm{rot}(\mathrm{rot}b\times b)$.

Testing (2.4) by $\omega$ and using $(1.1\text{a})_1$, we see that

$$\begin{aligned}
&\frac{1}{2}\|\sqrt{\rho}\omega(t)\|_{L^2}^2+\mu\|\mathrm{rot}\omega\|_{L^2(0,t;L^2)}^2\\
=&C_0(M_0)+\int_0^t\int K\omega dxds+\int_0^t\int\mathrm{rot}I_1\omega dxds\\
\le&C_0(M_0)\exp(\sqrt{t}C(M))+\int_0^t\|I_1\|_{H^1}\|\omega\|_{L^2}ds\\
\le&C_0(M_0)\exp(\sqrt{t}C(M)),
\end{aligned}$$

which leads to the lemma, where we used the estimate in [3]:

$$\int_0^t\int K\omega dxds\le C_0(M_0)\exp(\sqrt{t}C(M)).$$

□

**Lemma 2.2.** *For any* $0\le t\le\min(T^\epsilon,1)$*, we have that*

$$\begin{aligned}
&\|\partial_t(\sigma,u,\theta,b)(t)\|_{L^2}^2+\|(\mathrm{rot}\partial_tu,\mathrm{div}\partial_tu,\nabla\partial_t\theta,\nabla\partial_tb)\|_{L^2(0,t;L^2)}^2\\
\le&C_0(M_0)\exp(\sqrt{t}C(M)).
\end{aligned}$$

*Proof.* Applying the operator $\partial_t$ to (1.3), we find that

$$\partial_t^2\sigma+\frac{1}{\epsilon_1}\mathrm{div}\partial_tu=-\mathrm{div}\partial_t(\sigma u), \tag{2.5}$$

$$\begin{aligned}
&\rho(\partial_t^2u+u\cdot\nabla\partial_tu)+\frac{R}{\epsilon_1}(\nabla\partial_t\sigma+\nabla\partial_t\theta)-\mu\Delta\partial_tu-(\lambda+\mu)\nabla\mathrm{div}\partial_tu\\
&\quad=-\partial_t\rho\partial_tu-\partial_t(\rho u)\cdot\nabla u-R\nabla\partial_t(\sigma\theta)+\partial_tI_1,
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
&C_V\rho(\partial_t^2\theta+u\cdot\nabla\partial_t\theta)+\frac{R}{\epsilon_1}\mathrm{div}\partial_tu-\kappa\Delta\partial_t\theta\\
&\quad=\epsilon_1\partial_t\left(\frac{\mu}{2}|\nabla u+\nabla u^t|^2+\lambda(\mathrm{div}u)^2+|\mathrm{rot}b|^2\right)-C_V\partial_t\rho\partial_t\theta\\
&\qquad-C_V\partial_t(\rho u)\cdot\nabla\theta-R\partial_t((\rho\theta+\sigma)\mathrm{div}u)+\partial_t(I_0-(1+\epsilon_1\theta)^4),
\end{aligned} \tag{2.7}$$

$$\partial_t^2b+\partial_t\mathrm{rot}(b\times u)-\Delta b_t=0. \tag{2.8}$$

Testing (2.5)–(2.8) by $R\partial_t\sigma,\partial_tu,\partial_t\theta$ and $\partial_tb$ respectively, then doing as same as that in [3], we reach the lemma. □

By carrying out the very similar calculations to that in [3], we get

$$\|(\nabla \mathrm{div} u, \Delta\theta, \Delta b)(t)\|_{L^2}^2 + \|\partial_t \nabla(\sigma,\theta,b)\|_{L^2(0,t;L^2)}^2 \le C_0(M_0)\exp(\sqrt{t}C(M)). \tag{2.9}$$

**Lemma 2.3.** *For any* $0\le t\le \min\{T^\epsilon,1\}$, *we have*

$$\|\mathrm{rot}^2 u(t)\|_{L^2}^2 + \|\Delta \mathrm{rot} u\|_{L^2(0,t;L^2)}^2 \le C_0(M_0)\exp(\sqrt{t}C(M)).$$

*Proof.* Testing (2.4) by $-\Delta\omega$, we obtain that

$$\begin{aligned}
&\frac{1}{2}\|\sqrt{\rho}\mathrm{rot}\omega\|_{L^2}^2(t) + \mu\|\Delta\omega\|_{L^2(0,t;L^2)}^2 \\
\le& C_0(M_0) - \int_0^t\int K\Delta\omega dxds + \int_0^t\int \rho u\cdot(\nabla|\mathrm{rot}\omega|^2 + \Delta\omega\nabla\omega)dxds \\
&-\int_0^t\int \mathrm{rot} I_1 \Delta\omega dxds =: K_1+K_2+K_3+K_4.
\end{aligned} \tag{2.10}$$

It has been proven in [3] that

$$K_1+K_2+K_3 \le C_0(M_0)\exp(\sqrt{t}C(M)).$$

We bound $K_4$ as follows.

$$K_4 \le \int_0^t \|\mathrm{rot} I_1\|_{L^2}\|\Delta\omega\|_{L^2} ds \le \sqrt{t}C(M).$$

Inserting the above estimates into (2.10) gives the lemma. □

**Lemma 2.4.** *For any* $0\le t\le \min\{T^\epsilon,1\}$, *we have*

$$\int_0^t \|\mathrm{rot}\partial_t\omega\|_{H^1} ds \le C_0(M_0)\exp(\sqrt{t}C(M)).$$

*Proof.* Applying the operator $\partial_t$ to Eq. (2.4), we deduce that

$$\rho(\partial_t^2\omega + u\cdot\nabla\partial_t\omega) - \mu\Delta\partial_t\omega = Q + \partial_t \mathrm{rot} I_1, \tag{2.11}$$

where

$$Q := \partial_t K - \partial_t(\rho u)\cdot\nabla\omega - \partial_t\rho\partial_t\omega.$$

Then by the very similar calculations as that in [3], we reach the lemma. □

**Lemma 2.5.** *For any* $0\le t\le \min\{T^\epsilon,1\}$, *we have*

$$\|\partial_t\omega(t)\|_{L^2}^2 + \|\partial_t\nabla\omega\|_{L^2(0,t;L^2)}^2 \le C_0(M_0)\exp(\sqrt{t}C(M)).$$

*Proof.* Testing Eq. (2.11) by $\partial_t\omega$ and using Eq. (1.1a), we observe that

$$\begin{aligned}&\frac{1}{2}\|\sqrt{\rho}\partial_t\omega\|_{L^2}^2(t)+\mu\|\nabla\partial_t\omega\|_{L^2(0,t;L^2)}^2\\=&C_0(M_0)+\int_0^t\int Q\partial_t\omega dxds+\int_0^t\int\partial_t\mathrm{rot}I_1\partial_t\omega dxds.\end{aligned}\tag{2.12}$$

It has been proved in [3] that

$$\int_0^t\int Q\partial_t\omega dxds\le C_0(M_0)\exp(\sqrt{t}C(M)).$$

We bound the third term of right hand side of Eq. (2.12) as

$$\begin{aligned}&\int_0^t\int\partial_t\mathrm{rot}I_1\partial_t\omega dxds\le\int_0^t\|\partial_t\mathrm{rot}I_1\|_{L^2}\|\partial_t\omega\|_{L^2}ds\\\le&C(M)\int_0^t\|\partial_t\mathrm{rot}I_1\|_{L^2}ds\le\sqrt{t}C(M).\end{aligned}$$

Substituting the above estimates into Eq. (2.12) yields the lemma. □

Now, by the very similar calculations to that in [3], we conclude that

$$\begin{aligned}&\|\partial_t(\nabla\sigma,\mathrm{div}u,\nabla\theta,\nabla b)(t)\|_{L^2}^2+\|\partial_t(\nabla\mathrm{div}u,\Delta\theta,\Delta b)\|_{L^2(0,t;L^2)}^2\\&\qquad\le C_0(M_0)\exp\left(t^{\frac{1}{4}}C(M)\right),\\&\|\nabla^2\sigma(t)\|_{L^2}^2+\|\nabla^2\mathrm{div}u\|_{L^2(0,t;L^2)}^2\le C_0(M_0)\exp(\sqrt{t}C(M)),\\&\|(\Delta\theta,\Delta b)\|_{L^2(0,t;H^1)}\le C_0(M_0)\exp\left(t^{\frac{1}{4}}C(M)\right),\\&\|\partial_t^2(\sigma,u,\theta,b)(t)\|_{L^2}^2+\|\partial_t^2(u,\theta,b)\|_{L^2(0,t;H^1)}^2\le C_0(M_0)\exp\left(t^{\frac{1}{4}}C(M)\right).\end{aligned}$$

Finally, we estimate $I_0$ and $I_1$ in order to close the energy estimate.

**Lemma 2.6.** *For any* $0\le t\le\min\{T^\epsilon,1\}$, *we have*

$$\begin{aligned}&\sqrt{\epsilon_2}\|(I_0,I_1)(t)\|_{H^2}^2+\|(I_0,I_1)(t)\|_{H^1}^2+\|\partial_t(I_0,I_1)(t)\|_{L^2}^2\\&\quad+\int_0^t|||(I_0,I_1)|||_{2,2}^2ds\le C_0(M_0)\exp(\sqrt{t}C(M)).\end{aligned}$$

*Proof.* By the very same calculations as that in [8], we can prove the lemma and thus we omit the details here. □

By collecting all the above results together, we completes the proof of (1.9).

# 3 Proof of Theorem 1.1

The proof of Theorem 1.1 is based on some ideas developed in [1, 5, 15]. Assume that Theorem 1.2 holds and $T_\epsilon < \infty$ is the maximal life time of existence for the solution obtained in Proposition 1.1. Then for any $0 \le t \le \min\{T_\epsilon, 1\}$, we have (1.9), where $M_0 \le D_0$ for $0 < \epsilon \le 1$. In the sequence, we choose $D > C_0(D_0)$ and next $T_1 \le 1$ such that

$$C_0(D_0)\exp(T_1^{\frac{1}{4}} C(D)) < D.$$

Let $t < \min\{T_\epsilon, T_1\}$. By combining the inequalities (1.9) and the above inequality, we have that $M(t) \neq D$. Besides, we can assume without restriction that $D_0 \le D$, so that $M \le D$. Since the function $M(t)$ is continuous, we obtain

$$M(t) \le D \quad \text{for} \quad t < \min\{T_\epsilon, T_1\} \quad \text{and} \quad 0 < \epsilon \le 1.$$

Then $T_\epsilon > T_1$ for $0 < \epsilon \le 1$. Otherwise, by using the above uniform estimates and applying Proposition 1.1 repeatedly, one can extend the time interval of existence to $[0, T_1]$, which contradicts to the maximality of $T_\epsilon$. Therefore, $M(t) \le D$ for any $t \in [0, T_1]$ where $T_1$ is independent of $0 < \epsilon \le 1$. Clearly, the conclusion is also true for $T_\epsilon = \infty$ by applying the same argument. Thus we obtain (1.6). The convergence part is just an easy application of the uniform estimates and Arzelá-Ascoli's theorem. Hence we omit the details here. Thus the proof is completed. □

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