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An Analysis of Complex-valued Periodic Solution of a Delayed Discontinuous Neural Networks

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Abstract. In this paper, we investigate global stability of complex-valued periodic solution of a delayed discontinuous neural networks. By employing discontinuous, nondecreasing and bounded properties of activation, we analyzed exponential stability of state trajectory and L^1-exponential convergence of output solution for complex-valued delayed networks. Meanwhile, we applied to complex-valued discontinuous neural networks with periodic coefficients. The new results depend on $M-$matrices of real and imaginary parts and hence can include ones of real-valued neural networks. An illustrative example is given to show the effectiveness of our theoretical results.

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1 Introduction

The global exponential stability of neural networks have been extensively studied because of their wide range of applications, such as image and signal processing, pattern recognition, optimization and automatic control, and so on (see [1-3]). An equilibrium point can be viewed as a special periodic solution of neural networks with arbitrary periods. In this sense, the analysis of periodic solutions of neural networks can be considered to be more general than that of equilibrium point. Therefore, the global exponential stability of the periodic solution received extensive concerns. In [4-7], the authors investigated the stability of periodic solutions of neural networks, where the assumptions on neuron activation functions include Lipschitz conditions, bounded and monotonic increasing properties. In [8], Du and Xu discussed the global robust exponential stability and periodic solutions for interval Cohen-Grossberg neural networks with mixed delays. The stability analysis for periodicity of BAM neural networks with discontinuous neuron activations and impulses have been studied in [9]. As shown by [10], the authors studied

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finite time stability of periodic solution for Hopfield neural networks with discontinuous activations. We usually consider the properties of real-valued neural networks (RVNNs) in most of the papers. However, complex-valued neural networks can solve the problem that real neural networks can not solve, we can refer to [11] and the references therein. In [12], Rao and Murthy have studied global activation dynamics of a discrete CVNNs and have obtained easily verifiable sufficient conditions for global exponential stability of the unique equilibrium pattern. Due to importance of time delay in the finite speeds of the switching and transmissions of signal of neural network [14], stability criteria of real-valued or complex-valued neural networks with time delay have been reported in [13,15-18].

However, most of the results concerning the neural networks are based on the assumption that the activations are continuous or even Lipschitzian. Forti and Nistri [19] is the first one to discuss global stability of the equilibrium points for the neural networks with discontinuous neuron activations. They pointed out that a brief review of some common neural networks with discontinuous activation is important. In [20], Forti et al. introduced some new sufficient conditions for the global exponential stability of recurrently connected neural networks with (possibily) discontinuous and unbounded activation functions. Recently, there have been extensive results on the dynamical behaviors of neural networks with discontinuous activations [21-23].

Based on the previous scholar's research, we will study the stability of the periodic solution of a delayed neural network with discontinuous activations. In particular, we drop the assumption of Lipschitz continuity on the activation functions, which is usually required in most of the papers. Meanwhile, since the complex-valued neural network is more general than the real-valued neural network, it can solve the problem that the real neural network can not solve. Therefore, the stability of the periodic solution of a delayed complex-valued neural network with discontinuous activations has theoretical valued and practical application valued.

The organization of this paper is as follow. In Section 2, we introduce some definitions and preliminary lemmas. In Section 3, under suitable assumptions, we prove a result on the continuability and the uniqueness of the solution of any associated initial output problems (IOP) and give an estimation on the difference between the states and the outputs of the solutions of two different IOP. Our main results are contained in section 4, some sufficient conditions are given to guarantee the existence and exponential stability of a unique complex-valued periodic solution. Finally, our results are illustrated by an example.

2 Preliminaries

In this paper, we extend the work in [24] to the following model of a delayed complex-valued neural network with periodic coefficients

$$\dot{Z}(t) = -C(t)Z(t) + D(t)f(Z(t)) + E(t)f(Z(t-\tau)) + H(t), \quad t \geq 0, \tag{2.1}$$

where $Z(t)=(z_1(t),z_2(t),...,z_n(t))^T\in\mathbb{C}^n$ is the state vector of the neural network with n neurons. $f(\cdot)$ is the activation function defined by

$$f(z)=f_R(Re(z))+if_I(Im(z)),$$

and $f(Z(t))=(f(z_1(t)),f(z_2(t)),...,f(z_n(t)))^T\in\mathbb{C}^n$ is a diagonal function representing the neuron input-output activations; $C(t)=\mathrm{diag}(c_1(t),c_2(t),...,c_n(t))\in\mathbb{R}^{n\times n}$ is the periodic complex diagonal matrix function of the neuron self-inhibitions; $D(t)=(d_{ij}(t))_{n\times n}\in\mathbb{C}^{n\times n}$ is the periodic complex matrix function representing the neuron interconnection matrix; $E(t)=(e_{ij}(t))_{n\times n}\in\mathbb{C}^{n\times n}$ is the periodic complex matrix function representing the delayed connection weight matrix; $H(t)=(h_1(t),h_2(t),..,h_n(t))^T\in\mathbb{C}^n$ is a complex vector function representing the neuron inputs; $\tau>0$ is the delay in the neuron response. This is a quite general equation of delayed complex-valued neural networks models with periodic coefficients. However, we point out that differential systems modelling periodic delayed complex-valued neural networks with discontinuous activations functions were almost not studied.

Let $x(t)=(x_1(t),x_2(t),...,x_n(t))^T$, $x_i(t)=Re(z_i(t))$, $y(t)=(y_1(t),y_2(t),...,y_n(t))^T$, $y_i(t)=Im(z_i(t))$; $D(t)=D^R(t)+iD^I(t)$, $D^R(t)=(d^R_{ij}(t))_{n\times n}$, $d^R_{ij}(t)=Re(d_{ij}(t))$, $D^I(t)=(d^I_{ij}(t))_{n\times n}$, $d^I_{ij}(t)=Im(d_{ij}(t))$; $E(t)=E^R(t)+iE^I(t)$, $E^R(t)=(e^R_{ij}(t))_{n\times n}$, $e^R_{ij}(t)=Re(e_{ij}(t))$, $E^I(t)=(e^I_{ij}(t))_{n\times n}$, $e^I_{ij}(t)=Im(e_{ij}(t))$; $H(t)=H^R(t)+iH^I(t)$, $H^R(t)=(h^R_1(t),h^R_2(t),...,h^R_n(t))^T$, $h^R_i(t)=Re(h_i(t))$, $H^I(t)=(h^I_1(t),h^I_2(t),...,h^I_n(t))^T$, $h^I_i(t)=Im(h_i(t))$. Then system (2.1) can be rewritten as

$$\begin{aligned}\dot{x}(t)=&-C(t)x(t)+D^R(t)f_R(x(t))-D^I(t)f_I(y(t))\\&+E^R(t)f_R(x(t-\tau))-E^I(t)f_I(y(t-\tau))+H^R(t),\end{aligned}\tag{2.2a}$$

$$\begin{aligned}\dot{y}(t)=&-C(t)y(t)+D^I(t)f_R(x(t))+D^R(t)f_I(y(t))\\&+E^I(t)f_R(x(t-\tau))+E^R(t)f_I(y(t-\tau))+H^I(t),\end{aligned}\tag{2.2b}$$

or

$$\begin{aligned}\dot{x}_i(t)=&-c_i(t)x_i(t)+\sum_{j=1}^n d^R_{ij}(t)f_R(x_j(t))-\sum_{j=1}^n d^I_{ij}(t)f_I(y_j(t))\\&+\sum_{j=1}^n e^R_{ij}(t)f_R(x_j(t-\tau))-\sum_{j=1}^n e^I_{ij}(t)f_I(y_j(t-\tau))+h^R_i(t),\\ \dot{y}_i(t)=&-c_i(t)y_i(t)+\sum_{j=1}^n d^I_{ij}(t)f_R(x_j(t))+\sum_{j=1}^n d^R_{ij}(t)f_I(y_j(t))\\&+\sum_{j=1}^n e^I_{ij}(t)f_R(x_j(t-\tau))+\sum_{j=1}^n e^R_{ij}(t)f_I(y_j(t-\tau))+h^I_i(t),\end{aligned}$$

for all $i\in\mathcal{N}:=\{1,2,...,n\}$. For later discussion, we introduce the following notations.

We say a vector $x=(x_1,x_2,...,x_n)^T\in\mathbb{R}^n$ is positive if $x_i>0$ for all $i\in\mathcal{N}$. Let $\beta=(\beta_1,\beta_2,...,\beta_n)^T\in\mathbb{R}^n$ be positive. We define the norm

$$||x||_\beta=\sum_{i=1}^n\beta_i|x_i|,\quad \forall x\in\mathbb{R}^n.$$

μ represents the Lebesgue measure in $\mathbb{R}^n$. Given a set $Q\subset\mathbb{R}^n$, $K[Q]$ denotes the closure of the convex hull of Q. Notice that, since a monotone function can only have jump discontinuities, for any f monotone non-decreasing and $x\in\mathbb{R}$, it holds

$$K[f(x)]=[f(x^-),f(x^+)],$$

where $f(x^-)$ and $f(x^+)$ denote, respectively, the left and right limit of f at x. Given a function $f:J\subset\mathbb{R}\to\mathbb{R}$, $\operatorname{esssup}_{t\in J}f(t)=\min\{M\in\mathbb{R}:f(t)\le M \text{ for a.a. } t\in J\}$ and $\operatorname{essinf}_{t\in J}f(t)=\max\{M\in\mathbb{R}:f(t)\ge M \text{ for a.a. } t\in J\}$. Given $\tau>0$ and $u:[-\tau,+\infty)\to\mathbb{R}^n$, for every $t\ge 0$ we will denote by $u_t:[-\tau,0]\to\mathbb{R}^n$ the function $s\mapsto u(t+s)$. For all $i,j\in\mathcal{N}$, we define

$$\begin{aligned}&\bar{d}_{ii}^R=\operatorname*{esssup}_{t\ge0}d_{ii}^R(t),\quad \bar{d}_{ij}^R=\operatorname*{esssup}_{t\ge0}|d_{ij}^R(t)|,\quad \bar{d}_{ii}^I=\operatorname*{esssup}_{t\ge0}d_{ii}^I(t),\quad \bar{d}_{ij}^I=\operatorname*{esssup}_{t\ge0}|d_{ij}^I(t)|,\\ &\bar{e}_{ij}^R=\operatorname*{esssup}_{t\ge0}|e_{ij}^R(t)|,\quad \bar{e}_{ij}^I=\operatorname*{esssup}_{t\ge0}|e_{ij}^I(t)|,\quad \underline{c}_i=\operatorname*{essinf}_{t\ge0}c_i(t).\end{aligned}$$

An $n\times n$ matrix A is said to be an $M-$matrix, if: (1) $a_{ii}>0,i\in\mathcal{N}$; (2) $a_{ij}\le 0,i\ne j$ and $i,j\in\mathcal{N}$; (3) all successive principal minors of A are positive. As A is an $M-$matrix, there exists a positive vector $\beta\in\mathbb{R}^n$ such that $\beta^TA>0$.

For each $i\in\mathcal{N}$, we make the following basic assumptions.

▲$c_i(t)>0$ for $t\ge 0$;

▲f_i is monotone non-decreasing and bounded. $\forall x,\tilde{x},y,\tilde{y}\in\mathbb{R}$, $\forall\gamma_i(t)\in K[f_R(x_i(t))]$, $\forall\tilde{\gamma}_i(t)\in K[f_R(\tilde{x}_i(t))]$, $\forall\eta_i(t)\in K[f_I(y_i(t))]$, $\forall\tilde{\eta}_i(t)\in K[f_I(\tilde{y}_i(t))]$, $\exists l_i^R,l_i^I\in(0,1]$ such that

$$|\gamma_i(t)-\tilde{\gamma}_i(t)|<l_i^R|x_i(t)-\tilde{x}_i(t)|,\quad |\eta_i(t)-\tilde{\eta}_i(t)|<l_i^I|y_i(t)-\tilde{y}_i(t)|.$$

▲ C, D, E and H are measurable and locally bounded.

Definition 2.1. System (2.1) is said to be exponential stable if the real and imaginary parts of its each trajectory are global exponentially stable.

Definition 2.2. Functions $x,y:[-\tau,T)\to\mathbb{R}^n,T\in(0,+\infty]$, are state solutions of system (2.2) on $[0,T)$ if

(i) x,y are continuous on $[-\tau,T)$, absolutely continuous on $[0,T)$;

(ii) there exist measurable functions $\gamma=(\gamma_1,\gamma_2,...,\gamma_n)^T$, $\eta=(\eta_1,\eta_2,...,\eta_n)^T:[-\tau,T)\to\mathbb{R}^n$ such that $\gamma(t)\in K[f_R(x(t))]$ and $\eta(t)\in K[f_I(y(t))]$ for a.a. $t\in[-\tau,T)$ and

$$\begin{aligned}\dot{x}(t)&=-C(t)x(t)+D^R(t)\gamma(t)-D^I(t)\eta(t)+E^R(t)\gamma(t-\tau)-E^I(t)\eta(t-\tau)+H^R(t),\\ \dot{y}(t)&=-C(t)y(t)+D^I(t)\gamma(t)+D^R(t)\eta(t)+E^I(t)\gamma(t-\tau)+E^R(t)\eta(t-\tau)+H^I(t),\end{aligned}$$

for a.a. $t\in[0,T)$.

Any functions γ and η satisfying (ii) are called output solutions associated to the states x and y, and the couple $[x,y;\gamma,\eta]$ will be called simply a solution of system (2.2).

With this definition it turns out that states x and y are solutions of system (2.2) since it satisfies

$$\begin{aligned}\dot{x}(t)\in &-C(t)x(t)+D^R(t)K[f_R(x(t))]-D^I(t)K[f_I(y(t))]\\ &+E^R(t)K[f_R(x(t-\tau))]-E^I(t)K[f_I(y(t-\tau))]+H^R(t), \qquad (2.3a)\\ \dot{y}(t)\in &-C(t)y(t)+D^I(t)K[f_R(x(t))]+D^R(t)K[f_I(y(t))]\\ &+E^I(t)K[f_R(x(t-\tau))]+E^R(t)K[f_I(y(t-\tau))]+H^I(t), \qquad (2.3b)\end{aligned}$$

for a.a. $t\in[0,T)$.

Lemma 2.1. ([19]) *If $x,y:J\to\mathbb{R}^n$ are state solutions of system* (2.2)*, then functions $t\mapsto||x||_\beta$ and $t\mapsto||y||_\beta$ are locally Lipschitz continuous and*

$$\frac{d}{dt}||x(t)||_\beta=v(t)^T\dot{x}(t)=\sum_{i=1}^n v_i(t)\dot{x}_i(t),\qquad \frac{d}{dt}||y(t)||_\beta=w(t)^T\dot{y}(t)=\sum_{i=1}^n w_i(t)\dot{y}_i(t),$$

for a.a. $t\in J$, where $v_i(t)=\beta_i sign(x_i(t))$, if $x_i(t)\neq 0$, while $v_i(t)$ can be arbitrarily chosen in $[-\beta_i,\beta_i]$, if $x_i(t)=0$, and $w_i(t)=\beta_i sign(y_i(t))$, if $y_i(t)\neq 0$, while $w_i(t)$ can be arbitrarily chosen in $[-\beta_i,\beta_i]$, if $y_i(t)=0$.

Definition 2.3. A solution $[x,y;\gamma,\eta]$ of system (2.2) on $[0,+\infty)$ is a periodic solution of period ω if

$$\begin{aligned}&x(t+\omega)=x(t),\quad y(t+\omega)=y(t),\quad \text{for all } t\geq-\tau,\\ &\gamma(t+\omega)=\gamma(t),\quad \eta(t+\omega)=\eta(t),\quad \text{for a.a. } t\geq-\tau.\end{aligned}$$

The following concept of convergence in measure was employed [19].

Definition 2.4. If $\gamma,\eta:[0,+\infty)\to\mathbb{R}^n$ are measurable functions, then $\xi,\zeta\in\mathbb{R}^n$ are limits in measure of $\gamma(t)$ and $\eta(t)$, respectively, as $t\to+\infty$ if $\forall\epsilon>0,\exists t_\epsilon>0$ such that

$$\mu\{t\in[t_\epsilon,+\infty):||\gamma(t)-\xi||>\epsilon\}<\epsilon,\qquad \mu\{t\in[t_\epsilon,+\infty):||\eta(t)-\zeta||>\epsilon\}<\epsilon,$$

and in this case we write $\mu-\lim_{t\to+\infty}\gamma(t)=\xi$ and $\mu-\lim_{t\to+\infty}\eta(t)=\zeta$.

Definition 2.5. Let $\tau>0$ be fixed and $\gamma,\eta,\xi,\zeta:[-\tau,+\infty)\to\mathbb{R}^n$ be locally integrable functions. We say ξ and ζ L^1-exponentially converge to γ and η, respectively, provided that there are positive constants $I_1,I_2,\delta>0$ such that

$$\int_{-\tau}^0||\xi_t(s)-\gamma_t(s)||ds\leq I_1e^{-\delta t},\qquad \int_{-\tau}^0||\zeta_t(s)-\eta_t(s)||ds\leq I_2e^{-\delta t},$$

where $\forall t\geq 0$ and $||\cdot||\in\mathbb{R}^n$ stands for any norm.

Proposition 2.1. ([24]) If ξ and ζ L^1-exponentially converge to γ and η, respectively. Then $\xi-\gamma,\zeta-\eta\in L^1([-\tau,+\infty),\mathbb{R}^n)$ and $\mu-\lim_{t\to+\infty}[\xi(t)-\gamma(t)]=0$, $\mu-\lim_{t\to+\infty}[\zeta(t)-\eta(t)]=0$.

3 Exponential stability of CVNN

First of all, we introduce the definition of associated initial output problem.

Definition 3.1. For any given measurable functions $\tilde{\gamma},\tilde{\eta}:[-\tau,0)\to\mathbb{R}^n$ and any vectors $x_0,y_0\in\mathbb{R}^n$, the initial output problem (IOP) associated to system (2.2) with initial data $[x_0,y_0;\tilde{\gamma},\tilde{\eta}]$ consists in finding a couple of functions $[x,y;\gamma,\eta]$ such that $x,y:[0,T)\to\mathbb{R}^n$ are absolutely continuous, and $\gamma,\eta:[-\tau,T)\to\mathbb{R}^n$ are measurable functions and, moreover, we have

$$\begin{cases} \dot{x}(t) = -C(t)x(t)+D^R(t)\gamma(t)-D^I(t)\eta(t)+E^R(t)\gamma(t-\tau) \\ \qquad\quad -E^I(t)\eta(t-\tau)+H^R(t), \quad \text{for } a.a.\ t\in[0,T), \\ \dot{y}(t) = -C(t)y(t)+D^I(t)\gamma(t)+D^R(t)\eta(t)+E^I(t)\gamma(t-\tau) \\ \qquad\quad +E^R(t)\eta(t-\tau)+H^I(t), \quad \text{for } a.a.\ t\in[0,T), \\ \gamma(t) \in K[f_R(x(t))], \quad \text{for } a.a.\ t\in[0,T), \\ \eta(t) \in K[f_I(y(t))], \quad \text{for } a.a.\ t\in[0,T), \\ x(0)=x_0, \quad y(0)=y_0, \\ \gamma(s)=\tilde{\gamma}(s), \quad \eta(s)=\tilde{\eta}(s), \quad \text{for } a.a.\ t\in[-\tau,0]. \end{cases} \tag{3.1}$$

Now, similarly as [24], we can prove the existence of solution of IOP for system (2.2).

Lemma 3.1. *Under our assumptions, any IOP has at least a maximal solution $[x,y;\gamma,\eta]$ on $[0,T)$ for some $T\in(0,+\infty]$.*

Remark 3.1. A $\omega-$periodic solution of system (2.2) clearly satisfies

$$x(0)=x(\omega), \quad \gamma(\omega+s)=\gamma(s)=\tilde{\gamma}(s), \tag{3.2a}$$
$$y(0)=y(\omega), \quad \eta(\omega+s)=\eta(s)=\tilde{\eta}(s), \tag{3.2b}$$

for a.a. $s\in[-\tau,0]$. On the other hand, when C,D,E and H are $\omega-$periodic functions, it is easy to see that a solution $[x,y;\gamma,\eta]$ of an IOP gives rise to a $\omega-$periodic solution of system (2.2) provided that it is defined at least up to ω and satisfies (3.2).

Assumption 3.1. For all $i\in\mathcal{N}$ we have that $\underline{c}_i>0$ and the matrice $A=(a_{ij})\in\mathbb{R}^{n\times n}$, $B=(b_{ij})\in\mathbb{R}^{n\times n}$ given by

$$a_{ij}=\begin{cases} -\bar{d}_{ii}^R-\bar{e}_{ii}^R, & \text{for } i=j, \\ -\bar{d}_{ij}^R-\bar{e}_{ij}^R, & \text{for } i\neq j, \end{cases} \qquad b_{ij}=\begin{cases} -\bar{d}_{ii}^I-\bar{e}_{ii}^I, & \text{for } i=j, \\ -\bar{d}_{ij}^I-\bar{e}_{ij}^I, & \text{for } i\neq j, \end{cases}$$

are M-matrices and $\sum\limits_{i=1}^n \bar{e}_{ij}^R\neq 0$, $\sum\limits_{i=1}^n \bar{e}_{ij}^I\neq 0$.

Assumption 3.1 implies that there is a positive vector $\beta\in\mathbb{R}^n$ such that $\beta^T A>0$, $\beta^T B>0$. Moreover, for every $\delta\geq 0$ let us consider the $n\times n$ matrice $A_\delta=(a_{ij}^\delta)$ and $B_\delta=(b_{ij}^\delta)$ given by

$$a_{ij}^{\delta}=\begin{cases}-\bar{d}_{ii}^R-\bar{e}_{ii}^R e^{\delta\tau}, & \text{for } i=j,\\ -\bar{d}_{ij}^R-\bar{e}_{ij}^R e^{\delta\tau}, & \text{for } i\neq j,\end{cases}\qquad b_{ij}^{\delta}=\begin{cases}-\bar{d}_{ii}^I-\bar{e}_{ii}^I e^{\delta\tau}, & \text{for } i=j,\\ -\bar{d}_{ij}^I-\bar{e}_{ij}^I e^{\delta\tau}, & \text{for } i\neq j,\end{cases}$$

Since $A_0=A$ and $B_0=B$ by a continuity argument we can fix $\delta\in(0,\min\{\underline{c}_1,...,\underline{c}_n\})$ such that $\beta^T A_\delta>0$ and $\beta^T B_\delta>0$. Actually it is easy to see that

$$\beta^T A_\delta>0,\quad \forall\delta\in[0,\bar{\delta}),\qquad \beta^T B_\delta>0,\quad \forall\delta\in[0,\bar{\bar{\delta}}),$$

where

$$\bar{\delta}=-\frac{1}{\tau}\min_{i\in\mathcal{N}}\log\frac{\beta_i\bar{d}_{ii}^R-\sum_{j\neq i}\beta_j\bar{d}_{ji}^R}{\sum_{j=1}^n\beta_j\bar{e}_{ji}^R},\qquad \bar{\bar{\delta}}=-\frac{1}{\tau}\min_{i\in\mathcal{N}}\log\frac{\beta_i\bar{d}_{ii}^I-\sum_{j\neq i}\beta_j\bar{d}_{ji}^I}{\sum_{j=1}^n\beta_j\bar{e}_{ji}^I},\tag{3.3}$$

in such a way that the argument we developed will be valid for all $\delta\in(0,\min\{\bar{\delta},\bar{\bar{\delta}},\underline{c}_1,...,\underline{c}_n\})$.

Theorem 3.1. *If Assumption* 3.1 *is satisfied, then*

S_1: every solution of any IOP is defined on $[0,+\infty)$;

S_2: any IOP has a unique solution $[x,y;\gamma,\eta]$ in the sense that, if $[x,y;\gamma,\eta]$ and $[\tilde{x},\tilde{y};\tilde{\gamma},\tilde{\eta}]$ are two solutions with the same initial data, then $x(t)=\tilde{x}(t)$ and $y(t)=\tilde{y}(t)$ for all $t\geq 0$, $\gamma(t)=\tilde{\gamma}(t)$ and $\eta(t)=\tilde{\eta}(t)$ for almost all $t\geq 0$;

S_3: if $[x,y;\gamma,\eta]$ and $[\tilde{x},\tilde{y};\tilde{\gamma},\tilde{\eta}]$ are solutions of any IOP with different initial data, for any $\delta\in(0,\min\{\bar{\delta},\bar{\bar{\delta}},\underline{c}_1,...,\underline{c}_n\})$

$$||x(t)-\tilde{x}(t)||_\beta\leq ve^{-\delta t},\quad \int_{t-\tau}^t||\gamma(s)-\tilde{\gamma}(s)||_1ds\leq ve^{-\delta t}\tag{3.4a}$$

$$||y(t)-\tilde{y}(t)||_\beta\leq ve^{-\delta t},\quad \int_{t-\tau}^t||\eta(s)-\tilde{\eta}(s)||_1ds\leq ve^{-\delta t}\tag{3.4b}$$

$\forall t\geq 0$, for some $v>0$.

Proof. Without loss of generality, for $\forall s\in\mathbb{R}$ we can assume that

$$\left.\begin{array}{l}\forall\sigma\in K[f_R(s)],\\ \forall\sigma\in K[f_I(s)],\end{array}\right\}\ \Rightarrow\ \sigma\cdot s\geq 0.\tag{3.5}$$

Indeed, if (3.5) does not hold, it is sufficient to rewrite system (2.2) in the following way

$$\begin{aligned}\dot{x}(t)=&-C(t)x(t)+D^R\tilde{f}_R(x(t))-D^I(t)\tilde{f}_I(y(t))\\&+E^R(t)\tilde{f}_R(x(t-\tau))-E^I(t)\tilde{f}_I(y(t-\tau))+\tilde{H}^R(t),\end{aligned}\tag{3.6a}$$

$$\begin{aligned}\dot{y}(t)=&-C(t)y(t)+D^I\tilde{f}_R(x(t))+D^R(t)\tilde{f}_I(y(t))\\&+E^I(t)\tilde{f}_R(x(t-\tau))+E^R(t)\tilde{f}_I(y(t-\tau))+\tilde{H}^I(t),\end{aligned}\tag{3.6b}$$

where $\tilde{f}_R(s)=f_R(s)-\rho_1$, $\tilde{f}_I(s)=f_I(s)-\rho_2$, $\tilde{H}^R(t)=H^R(t)+(D^R(t)+E^R(t))\rho_1-(D^I(t)+E^I(t))\rho_2$ and $\tilde{H}^I(t)=H^I(t)+(D^I(t)+E^I(t))\rho_1+(D^R(t)+E^R(t))\rho_2$ for some $\rho_1\in K[f_R(0)]$ and $\rho_2\in K[f_I(0)]$.

Given $\delta \in (0,\min\{\bar{\delta},\bar{\bar{\delta}},\underline{c}_1,...,\underline{c}_n\})$, we define the mapping $V[x,y;\gamma,\eta](\cdot):[0,T)\to\mathbb{R}$ as follows

$$V[x,y;\gamma,\eta](t)=V_1[x;\gamma,\eta](t)+V_2[y;\gamma,\eta](t),$$

$$\begin{cases} V_1[x;\gamma,\eta](t)=e^{\delta t}||x(t)||_\beta+\sum_{i,j=1}^n\beta_i\bar{e}_{ij}^R\int_{t-\tau}^t|\gamma_j(s)|e^{\delta(s+\tau)}ds+\sum_{i,j=1}^n\beta_i\bar{e}_{ij}^I\int_{t-\tau}^t|\eta_j(s)|e^{\delta(s+\tau)}ds,\\ V_2[y;\gamma,\eta](t)=e^{\delta t}||y(t)||_\beta+\sum_{i,j=1}^n\beta_i\bar{e}_{ij}^I\int_{t-\tau}^t|\gamma_j(s)|e^{\delta(s+\tau)}ds+\sum_{i,j=1}^n\beta_i\bar{e}_{ij}^R\int_{t-\tau}^t|\eta_j(s)|e^{\delta(s+\tau)}ds,\end{cases}$$

defined on the couples $[x,y;\gamma,\eta]$, where $x,y:[0,T)\to\mathbb{R}^n$ are locally Lipschitz continuous and $\gamma,\eta:[-\tau,T)\to\mathbb{R}^n$ are locally integrable (we do not exclude the case $T=+\infty$).

$V_1[x;\gamma,\eta](\cdot)$ and $V_2[y;\gamma,\eta](\cdot)$ are absolutely continuous functions and their derivatives can be evaluated by means of Lemma 2.1. In particular, we have

$$\frac{d}{dt}||x(t)||_\beta=\sum_{i=1}^n v_i(t)\dot{x}_i(t),\quad \text{for } a.a.\ t\in J, \tag{3.7a}$$

$$\frac{d}{dt}||y(t)||_\beta=\sum_{i=1}^n w_i(t)\dot{y}_i(t),\quad \text{for } a.a.\ t\in J, \tag{3.7b}$$

with

$$v_i(t)=\begin{cases}\beta_i sign(x_i(t)), & \text{if } x_i(t)\neq 0,\\ \beta_i sign(\gamma_i(t)), & \text{if } x_i(t)=0,\gamma_i(t)\neq 0,\\ 0, & \text{if } x_i(t)=\gamma_i(t)=0,\end{cases} \tag{3.8a}$$

$$w_i(t)=\begin{cases}\beta_i sign(y_i(t)), & \text{if } y_i(t)\neq 0,\\ \beta_i sign(\eta_i(t)), & \text{if } y_i(t)=0,\eta_i(t)\neq 0,\\ 0, & \text{if } y_i(t)=\eta_i(t)=0.\end{cases} \tag{3.8b}$$

We remark that with this choice we have

$$\begin{cases} v_i(t)x_i(t)=\beta_i|x_i(t)|, & v_i(t)\gamma_i(t)=\beta_i|\gamma_i(t)|,\\ w_i(t)y_i(t)=\beta_i|y_i(t)|, & w_i(t)\eta_i(t)=\beta_i|\eta_i(t)|,\end{cases}$$

if $[x,y;\gamma,\eta]$ is a solution of system (2.2).

S_1: Let us consider a solution $[x,y;\gamma,\eta]$ of any IOP in its maximal interval of existence $[0,T)$, for some $T\in(0,+\infty]$ and let us evaluate the derivative of $V_1[x;\gamma,\eta](\cdot)$ and $V_2[y;\gamma,\eta](\cdot)$. For V_1, we have

$$\begin{aligned}\dot{V}_1[x;\gamma,\eta](t)=&\ \delta e^{\delta t}||x(t)||_\beta+e^{\delta t}\sum_{i=1}^n v_i(t)\dot{x}_i(t)\\ &+e^{\delta t}\sum_{i,j=1}^n\beta_i\bar{e}_{ij}^R\Big[e^{\delta\tau}|\gamma_j(t)|-|\gamma_j(t-\tau)|\Big]+e^{\delta t}\sum_{i,j=1}^n\beta_i\bar{e}_{ij}^I\Big[e^{\delta\tau}|\eta_j(t)|-|\eta_j(t-\tau)|\Big]\\ =&\ e^{\delta t}\Bigg\{-\sum_{i=1}^n\beta_i(c_i(t)-\delta)|x_i(t)|+\sum_{i=1}^n v_i(t)h_i^R(t)+\sum_{i,j=1}^n v_i(t)d_{ij}^R(t)\gamma_j(t)-\sum_{i,j=1}^n v_i(t)d_{ij}^I(t)\eta_j(t)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i,j=1}^{n} v_i(t)e_{ij}^R(t)\gamma_j(t-\tau)-\sum_{i,j=1}^{n} v_i(t)e_{ij}^I(t)\eta_j(t-\tau)+\sum_{i,j=1}^{n}\beta_i\bar{e}_{ij}^R(e^{\delta\tau}|\gamma_j(t)|-|\gamma_j(t-\tau)|)\\
&+\sum_{i,j=1}^{n}\beta_i\bar{e}_{ij}^I(e^{\delta\tau}|\eta_j(t)|-\eta_j(t-\tau)|)\Bigg\}\\
=\;& e^{\delta t}\Bigg\{-\sum_{i=1}^{n}\beta_i(c_i(t)-\delta)|x_i(t)|+||H^R(t)||_\beta+\sum_{i,j=1}^{n} v_i(t)e_{ij}^R(t)\gamma_j(t-\tau)-\sum_{i,j=1}^{n}\beta_i\bar{e}_{ij}^R|\gamma_j(t-\tau)|\\
&-\sum_{i,j=1}^{n} v_i(t)e_{ij}^I(t)\eta_j(t-\tau)-\sum_{i,j=1}^{n}\beta_i\bar{e}_{ij}^I|\eta_j(t-\tau)|+\sum_{i,j=1}^{n} v_i(t)d_{ij}^R(t)\gamma_j(t)+\sum_{i,j=1}^{n}\beta_i\bar{e}_{ij}^R e^{\delta\tau}|\gamma_j(t)|\\
&-\sum_{i,j=1}^{n} v_i(t)d_{ij}^I(t)\eta_j(t)+\sum_{i,j=1}^{n}\beta_i\bar{e}_{ij}^I e^{\delta\tau}|\eta_j(t)|\Bigg\}\\
\le\;& e^{\delta t}\Bigg\{-\sum_{i=1}^{n}\beta_i(\underline{c}_i-\delta)|x_i(t)|+||H^R(t)||_\beta-\sum_{i=1}^{n}\Big[\beta_i(-\bar{d}_{ii}^R-\bar{e}_{ii}^R e^{\delta\tau})|\gamma_i(t)|\\
&+\sum_{j\neq i}\beta_i(-\bar{d}_{ij}^R-\bar{e}_{ij}^R e^{\delta\tau})|\gamma_j(t)|\Big]+\sum_{i,j=1}^{n}\beta_i(\bar{d}_{ij}^I+\bar{e}_{ij}^I e^{\delta\tau})|\eta_j(t)|\Bigg\}\\
=\;& e^{\delta t}\Bigg\{-\sum_{i=1}^{n}\beta_i(\underline{c}_i-\delta)|x_i(t)|+||H^R(t)||_\beta-\beta^T A_\delta(|\gamma_1(t)|,\ldots,|\gamma_n(t)|)^T\\
&+\sum_{i,j=1}^{n}\beta_i(\bar{d}_{ij}^I+\bar{e}_{ij}^I e^{\delta\tau})|\eta_j(t)|\Bigg\}\\
\le\;& e^{\delta t}\Bigg\{-\sum_{i=1}^{n}\beta_i(\underline{c}_i-\delta)|x_i(t)|+R_1(t)\Bigg\}\\
\le\;& e^{\delta t}R_1(t),\qquad \text{for } a.a.\ t<T,
\end{aligned}
\tag{3.9}
$$

and for

$$
\begin{aligned}
&\dot{V}_2[y;\gamma,\eta](t)=\;\delta e^{\delta t}||y(t)||_\beta+e^{\delta t}\sum_{i=1}^{n} w_i(t)\dot{y}_i(t)\\
&+e^{\delta t}\sum_{i,j=1}^{n}\beta_i\bar{e}_{ij}^I\Big[e^{\delta\tau}|\gamma_j(t)|-|\gamma_j(t-\tau)|\Big]+e^{\delta t}\sum_{i,j=1}^{n}\beta_i\bar{e}_{ij}^R\Big[e^{\delta\tau}|\eta_j(t)|-|\eta_j(t-\tau)|\Big]\\
=\;& e^{\delta t}\Bigg\{-\sum_{i=1}^{n}\beta_i(c_i(t)-\delta)|y_i(t)|+\sum_{i=1}^{n} w_i(t)h_i^I(t)+\sum_{i,j=1}^{n} w_i(t)d_{ij}^I(t)\gamma_j(t)\\
&+\sum_{i,j=1}^{n} w_i(t)d_{ij}^R(t)\eta_j(t-\tau)+\sum_{i,j=1}^{n} w_i(t)e_{ij}^I(t)\gamma_j(t-\tau)+\sum_{i,j=1}^{n} w_i(t)e_{ij}^R(t)\eta_j(t-\tau)
\end{aligned}
$$

$$
\begin{aligned}
&\quad +\sum_{i,j=1}^{n}\beta_i\bar{e}_{ij}^{I}(e^{\delta\tau}|\gamma_j(t)|-|\gamma_j(t-\tau)|)+\sum_{i,j=1}^{n}\beta_i\bar{e}_{ij}^{R}(e^{\delta\tau}|\eta_j(t)|-\eta_j(t-\tau)|)\Bigg\}\\
&= e^{\delta t}\Bigg\{-\sum_{i=1}^{n}\beta_i(c_i(t)-\delta)|y_i(t)|+||H^I(t)||_\beta+\sum_{i,j=1}^{n}w_i(t)e_{ij}^{I}(t)\gamma_j(t-\tau)-\sum_{i,j=1}^{n}\beta_i\bar{e}_{ij}^{I}|\gamma_j(t-\tau)|\\
&\quad +\sum_{i,j=1}^{n}w_i(t)e_{ij}^{R}(t)\eta_j(t-\tau)-\sum_{i,j=1}^{n}\beta_i\bar{e}_{ij}^{R}|\eta_j(t-\tau)|+\sum_{i,j=1}^{n}w_i(t)d_{ij}^{I}(t)\gamma_j(t)+\sum_{i,j=1}^{n}\beta_i\bar{e}_{ij}^{I}e^{\delta\tau}|\gamma_j(t)|\\
&\quad +\sum_{i,j=1}^{n}w_i(t)d_{ij}^{R}(t)\eta_j(t)+\sum_{i,j=1}^{n}\beta_i\bar{e}_{ij}^{R}e^{\delta\tau}|\eta_j(t)|\Bigg\}\\
&\le e^{\delta t}\Bigg\{-\sum_{i=1}^{n}\beta_i(\underline{c}_i-\delta)|y_i(t)|+||H^I(t)||_\beta-\sum_{i=1}^{n}\Big[\beta_i(-\bar{d}_{ii}^{R}-\bar{e}_{ii}^{R}e^{\delta\tau})|\eta_i(t)|\\
&\quad +\sum_{j\neq i}\beta_i(-\bar{d}_{ij}^{R}-\bar{e}_{ij}^{R}e^{\delta\tau})|\eta_j(t)|\Big]+\sum_{i,j=1}^{n}\beta_i(\bar{d}_{ij}^{I}+\bar{e}_{ij}^{I}e^{\delta\tau})|\gamma_j(t)|\Bigg\}\\
&= e^{\delta t}\Bigg\{-\sum_{i=1}^{n}\beta_i(\underline{c}_i-\delta)|y_i(t)|+||H^I(t)||_\beta-\beta^T A_\delta(|\eta_1(t)|,\ldots,|\eta_n(t)|)^T\\
&\quad +\sum_{i,j=1}^{n}\beta_i(\bar{d}_{ij}^{I}+\bar{e}_{ij}^{I}e^{\delta\tau})|\gamma_j(t)|\Bigg\}\\
&\le e^{\delta t}\Bigg\{-\sum_{i=1}^{n}\beta_i(\underline{c}_i-\delta)|y_i(t)|+R_2(t)\Bigg\}\\
&\le e^{\delta t}R_2(t),\qquad \text{for } a.a.\ t<T,
\end{aligned}\tag{3.10}
$$

where

$$R_1(t)=||H^R(t)||_\beta+\sum_{i,j=1}^{n}\beta_i(\bar{d}_{ij}^{I}+\bar{e}_{ij}^{I}e^{\delta\tau})|\eta_j(t)|,\tag{3.11a}$$

$$R_2(t)=||H^I(t)||_\beta+\sum_{i,j=1}^{n}\beta_i(\bar{d}_{ij}^{I}+\bar{e}_{ij}^{I}e^{\delta\tau})|\gamma_j(t)|,\tag{3.11b}$$

an integration between 0 and t leads to

$$||x(t)||_\beta\le V_1[x,\gamma,\eta](t)e^{-\delta t}\le V_1[x,\gamma,\eta](0)e^{-\delta t}+e^{-\delta t}\int_0^T R_1(s)e^{\delta s}ds,\tag{3.12a}$$

$$||y(t)||_\beta\le V_2[y,\gamma,\eta](t)e^{-\delta t}\le V_2[y,\gamma,\eta](0)e^{-\delta t}+e^{-\delta t}\int_0^T R_2(s)e^{\delta s}ds,\tag{3.12b}$$

$\forall t<T$, that is, x and y remain bounded on bounded intervals. Therefore, we have $T=+\infty$.

S_2: Let $[x,y;\gamma,\eta]$ and $[\tilde{x},\tilde{y};\tilde{\gamma},\tilde{\eta}]$ be two solutions of any IOP with the same initial data $[x_0,y_0;\gamma_0,\eta_0]$. In particular, we have that differences $p(t)=x(t)-\tilde{x}(t)$ and $q(t)=y(t)-\tilde{y}(t)$ satisfy

$$\begin{aligned}\dot{p}(t)=&-C(t)p(t)+D^R(\gamma(t)-\tilde{\gamma}(t))-D^I(\eta(t)-\tilde{\eta}(t))\\&+E^R(t)[\gamma(t-\tau)-\tilde{\gamma}(t-\tau)]-E^I(t)[\eta(t-\tau)-\tilde{\eta}(t-\tau)],\ \text{for } a.a.\ t\geq 0,\end{aligned}\tag{3.13a}$$

$$\begin{aligned}\dot{q}(t)=&-C(t)q(t)+D^I(\gamma(t)-\tilde{\gamma}(t))+D^R(\eta(t)-\tilde{\eta}(t))\\&+E^I(t)[\gamma(t-\tau)-\tilde{\gamma}(t-\tau)]+E^R(t)[\eta(t-\tau)-\tilde{\eta}(t-\tau)],\ \text{for } a.a.\ t\geq 0,\end{aligned}\tag{3.13b}$$

$$p(0)=0,\quad q(0)=0,\tag{3.13c}$$

$$\gamma(s)-\tilde{\gamma}(s)=0,\ \eta(s)-\tilde{\eta}(s)=0,\ \text{for } a.a.\ s\in[-\tau,0],\tag{3.13d}$$

we obtain that

$$\begin{aligned}&\dot{V}[x-\tilde{x},y-\tilde{y},\gamma-\tilde{\gamma},\eta-\tilde{\eta}](t)\leq\ e^{\delta t}\Bigg\{-\sum_{i=1}^{n}\beta_i(\underline{c}_i-\delta)|x_i(t)-\tilde{x}_i(t)|-\sum_{i=1}^{n}\beta_i(\underline{c}_i-\delta)|y_i(t)\\&-\tilde{y}_i(t)|+\sum_{i,j=1}^{n}\beta_i(\bar{d}_{ij}^I+\bar{e}_{ij}^Ie^{\delta\tau})|\gamma_j(t)-\tilde{\gamma}_j(t)|+\sum_{i,j=1}^{n}\beta_i(\bar{d}_{ij}^I+\bar{e}_{ij}^Ie^{\delta\tau})|\eta_j(t)-\tilde{\eta}_j(t)|\Bigg\}\\\leq\ &e^{\delta t}\Bigg\{-\sum_{i=1}^{n}\beta_i(\underline{c}_i-\delta)|x_i(t)-\tilde{x}_i(t)|-\sum_{i=1}^{n}\beta_i(\underline{c}_i-\delta)|y_i(t)-\tilde{y}_i(t)|\\&-\sum_{i=1}^{n}\Big[\beta_i(-\bar{d}_{ii}^I-\bar{e}_{ii}^Ie^{\delta\tau})|x_i(t)-\tilde{x}_i(t)|+\sum_{j\neq i}\beta_i(-\bar{d}_{ij}^I-\bar{e}_{ij}^Ie^{\delta\tau})|x_j(t)-\tilde{x}_j(t)|\Big]\\&-\sum_{i=1}^{n}\Big[\beta_i(-\bar{d}_{ii}^I-\bar{e}_{ii}^Ie^{\delta\tau})|y_i(t)-\tilde{y}_i(t)|+\sum_{j\neq i}\beta_i(-\bar{d}_{ij}^I-\bar{e}_{ij}^Ie^{\delta\tau})|y_j(t)-\tilde{y}_j(t)|\Big]\Bigg\}\\=\ &e^{\delta t}\Bigg\{-\sum_{i=1}^{n}\beta_i(\underline{c}_i-\delta)|x_i(t)-\tilde{x}_i(t)|-\sum_{i=1}^{n}\beta_i(\underline{c}_i-\delta)|y_i(t)-\tilde{y}_i(t)|\\&-\beta^TB_\delta(|x_1(t)-\tilde{x}_1(t)|,\ldots,|x_n(t)-\tilde{x}_n(t)|)^T-\beta^TB_\delta(|y_1(t)-\tilde{y}_1(t)|,\ldots,|y_n(t)-\tilde{y}_n(t)|)^T\Bigg\}\\\leq\ &0,\end{aligned}\tag{3.14}$$

while the initial condition implies that $V[x-\tilde{x},y-\tilde{y},\gamma-\tilde{\gamma},\eta-\tilde{\eta}](0)=0$, since $V[x-\tilde{x},y-\tilde{y},\gamma-\tilde{\gamma},\eta-\tilde{\eta}](t)\geq 0$ and for all t, we deduce that actually $V[x-\tilde{x},y-\tilde{y},\gamma-\tilde{\gamma},\eta-\tilde{\eta}](t)\equiv 0$ and, hence, $x(t)=\tilde{x}(t)$ and $y(t)=\tilde{y}(t)$ for all $t\geq 0$, $\gamma(t)=\tilde{\gamma}(t)$ and $\eta(t)=\tilde{\eta}(t)$ for a.a. $t\geq 0$.

S_3: Let $[x,y;\gamma,\eta]$ and $[\tilde{x},\tilde{y};\tilde{\gamma},\tilde{\eta}]$ be the solutions of IOP (3.1) with initial data $[x_0,y_0;\gamma_0,\eta_0]$, $[\tilde{x}_0,\tilde{y}_0;\tilde{\gamma}_0,\tilde{\eta}_0]$, respectively. The differences $p(t)=x(t)-\tilde{x}(t)$ and $q(t)=y(t)-\tilde{y}(t)$ satisfy

$$\begin{aligned}\dot{p}(t)=&-C(t)p(t)+D^R(\gamma(t)-\tilde{\gamma}(t))-D^I(\eta(t)-\tilde{\eta}(t))\\&+E^R(t)[\gamma(t-\tau)-\tilde{\gamma}(t-\tau)]-E^I(t)[\eta(t-\tau)-\tilde{\eta}(t-\tau)],\ \text{for } a.a.\ t\geq 0, &\text{(3.15a)}\\ \dot{q}(t)=&-C(t)q(t)+D^I(\gamma(t)-\tilde{\gamma}(t))+D^R(\eta(t)-\tilde{\eta}(t))\\&+E^I(t)[\gamma(t-\tau)-\tilde{\gamma}(t-\tau)]+E^R(t)[\eta(t-\tau)-\tilde{\eta}(t-\tau)],\ \text{for } a.a.\ t\geq 0. &\text{(3.15b)}\end{aligned}$$

The same kind of estimates employed in the proof of (3.14) leads to

$$\dot{V}[x-\tilde{x},y-\tilde{y},\gamma-\tilde{\gamma},\eta-\tilde{\eta}](t)\leq 0,\ \text{for } a.a.\ t\geq 0. \tag{3.16}$$

Therefore, we have

$$\begin{aligned}||x(t)-\tilde{x}(t)||_\beta\leq e^{-\delta t}V[x-\tilde{x},y-\tilde{y},\gamma-\tilde{\gamma},\eta-\tilde{\eta}](t)\leq e^{-\delta t}V[x-\tilde{x},y-\tilde{y},\gamma-\tilde{\gamma},\eta-\tilde{\eta}](0),\\ ||y(t)-\tilde{y}(t)||_\beta\leq e^{-\delta t}V[x-\tilde{x},y-\tilde{y},\gamma-\tilde{\gamma},\eta-\tilde{\eta}](t)\leq e^{-\delta t}V[x-\tilde{x},y-\tilde{y},\gamma-\tilde{\gamma},\eta-\tilde{\eta}](0),\end{aligned}$$

for all $t\geq 0$. On the other hand, we can estimate

$$\sum_{j=1}^n\int_{t-\tau}^t|\gamma_j(s)-\tilde{\gamma}_j(s)|ds\leq\frac{V[x-\tilde{x},y-\tilde{y},\gamma-\tilde{\gamma},\eta-\tilde{\eta}](0)}{\min_{j\in\mathcal{N}}\sum_{i=1}^n\beta_i\bar{e}_{ij}^R}e^{-\delta t}, \tag{3.17a}$$

$$\sum_{j=1}^n\int_{t-\tau}^t|\eta_j(s)-\tilde{\eta}_j(s)|ds\leq\frac{V[x-\tilde{x},y-\tilde{y},\gamma-\tilde{\gamma},\eta-\tilde{\eta}](0)}{\min_{j\in\mathcal{N}}\sum_{i=1}^n\beta_i\bar{e}_{ij}^I}e^{-\delta t}. \tag{3.17b}$$

The proof is complete. □

4 Exponential stability and convergence of periodic solutions

Let us consider the vector space $\Omega=\mathbb{R}^n\times\mathbb{R}^n\times L^1([-\tau,0],\mathbb{R}^n)\times L^1([-\tau,0],\mathbb{R}^n)$, where we identify L^1-functions which coincide up to a set of zero measure. Define

$$\begin{aligned}|||[x_0,y_0;\gamma_0,\eta_0]|||_\Omega=&||x_0||_\beta+\sum_{i,j=1}^n\beta_i\bar{e}_{ij}^R\int_{-\tau}^0|\gamma_{0j}(s)|ds+\sum_{i,j=1}^n\beta_i\bar{e}_{ij}^I\int_{-\tau}^0|\eta_{0j}(s)|ds\\&+||y_0||_\beta+\sum_{i,j=1}^n\beta_i\bar{e}_{ij}^I\int_{-\tau}^0|\gamma_{0j}(s)|ds+\sum_{i,j=1}^n\beta_i\bar{e}_{ij}^R\int_{-\tau}^0|\eta_{0j}(s)|ds,\end{aligned}$$

$\forall[x_0,y_0;\gamma_0,\eta_0]\in\Omega$, gives rise to a norm which makes Ω a Banach space.

Theorem 4.1. *If Assumption* 1 *holds and C,D,E,H are $\omega-$ periodic functions then system* (2.2) *has a unique $\omega-$periodic solution $[x,y;\gamma,\eta]$.*

Proof. S_1 and S_2 of Theorem 1 imply that system (2.2) defines a semi-flow $\mathscr{F}^t:\Omega\to\Omega, t\geq 0$ in the following way

$$\begin{cases}\mathscr{F}^t[x_0,y_0;\gamma_0,\eta_0]=[\mathscr{F}_1^t[x_0;\gamma_0,\eta_0],\ \mathscr{F}_2^t[y_0;\gamma_0,\eta_0]],\\ \mathscr{F}_1^t[x_0;\gamma_0,\eta_0]=[x(t);\gamma_t,\eta_t],\quad \mathscr{F}_2^t[y_0;\gamma_0,\eta_0]=[y(t);\gamma_t,\eta_t],\end{cases}$$

where $\forall[x_0,y_0;\gamma_0,\eta_0]\in\Omega$, $[x,y;\gamma,\eta]$ is the solution of IOP with initial data $[x_0,y_0;\gamma_0,\eta_0]$. Moreover, the semigroup relations $\mathscr{F}^{s+t}=\mathscr{F}^s\circ\mathscr{F}^t=\mathscr{F}^t\circ\mathscr{F}^s$ hold for all $s,t\geq 0$ and the $\omega-$periodic solutions of system (2.1) are in 1-to-1 correspondence with the fixed points of the map $\mathscr{F}^\omega$.

Let us fix any $[x_0,y_0;\gamma_0,\eta_0]$, $[\tilde{x}_0,\tilde{y}_0;\tilde{\gamma}_0,\tilde{\eta}_0]\in\Omega$ and consider the solutions $[x,y;\gamma,\eta]$ and $[\tilde{x},\tilde{y};\tilde{\gamma},\tilde{\eta}]$ of IOP with initial data $[x_0,y_0;\gamma_0,\eta_0]$ and $[\tilde{x}_0,\tilde{y}_0;\tilde{\gamma}_0,\tilde{\eta}_0]$, respectively. Recalling the definition of V and the proof of S_3 of Theorem 1, we obtain that

$$\begin{aligned}&\quad ||\mathscr{F}_1^t[x_0;\gamma_0,\eta_0]-\mathscr{F}_1^t[\tilde{x}_0;\tilde{\gamma}_0,\tilde{\eta}_0]||_\Omega=\ ||[x(t)-\tilde{x}(t);\gamma_t-\tilde{\gamma}_t,\eta_t-\tilde{\eta}_t]||_\Omega\\ &\leq\ e^{-\delta t}V_1[x-\tilde{x};\gamma-\tilde{\gamma},\eta-\tilde{\eta}](t)\leq\ e^{-\delta t}V_1[x-\tilde{x};\gamma-\tilde{\gamma},\eta-\tilde{\eta}](0)\\ &\leq\ e^{\delta(\tau-t)}||[x_0-\tilde{x}_0;\gamma_0-\tilde{\gamma}_0,\eta_0-\tilde{\eta}_0]||_\Omega, \qquad (4.1)\\ &\quad ||\mathscr{F}_2^t[y_0;\gamma_0,\eta_0]-\mathscr{F}_2^t[\tilde{y}_0;\tilde{\gamma}_0,\tilde{\eta}_0]||_\Omega=\ ||[y(t)-\tilde{y}(t);\gamma_t-\tilde{\gamma}_t,\eta_t-\tilde{\eta}_t]||_\Omega\\ &\leq\ e^{-\delta t}V_2[y-\tilde{y};\gamma-\tilde{\gamma},\eta-\tilde{\eta}](t)\leq\ e^{-\delta t}V_2[y-\tilde{y};\gamma-\tilde{\gamma},\eta-\tilde{\eta}](0)\\ &\leq\ e^{\delta(\tau-t)}||[y_0-\tilde{y}_0;\gamma_0-\tilde{\gamma}_0,\eta_0-\tilde{\eta}_0]||_\Omega, \qquad (4.2)\end{aligned}$$

$\forall t\geq 0$. In particular, setting $t=m\omega$ where $m\in\mathbb{N}$ such that $m\omega-\tau>0$, we have that $(\mathscr{F}^\omega)^m=\mathscr{F}^{m\omega}$ is a contraction mapping and has a unique fixed point which is, therefore, the unique fixed point of $\mathscr{F}^\omega$. ☐

An immediate consequence of Proposition 2.1, S_3 of Theorem 3.1 and Theorem 4.1 is the following result which states in a precise way the globally exponential stability of the state and convergence in measure of the output of the $\omega-$periodic solution of system (2.1). Therefore, we have the following main theorem.

Theorem 4.2. *Under the assumptions of Theorem* 4.1, *let* $[x,y;\gamma,\eta]$ *be the unique* $\omega-$*periodic solution of system* (2.2) *and* $[\tilde{x},\tilde{y};\tilde{\gamma},\tilde{\eta}]$ *be any other solution of system* (2.2). *Then* $[x,y]$ *converges exponentially to* $[\tilde{x},\tilde{y}]$, $[\gamma,\eta]$ L^1*-exponentially converges to* $[\tilde{\gamma},\tilde{\eta}]$, *respectively. The convergence rate is* $\delta\in(0,\min\{\bar{\delta},\bar{\bar{\delta}},\underline{c}_1,\ldots,\underline{c}_n\})$, *where* $\bar{\delta},\bar{\bar{\delta}}$ *are defined in* (3.3).

Remark 4.1. From Assumption 1, we can see that exponential stability of state trajectory and L^1-exponentially convergence of output solution of system (2.1) depend on M-matrices of real and imaginary parts. As system (2.1) reduce to real-valued neural networks, i.e., all the imaginary parts are zeros, our result can include ones in [24].

5 An example

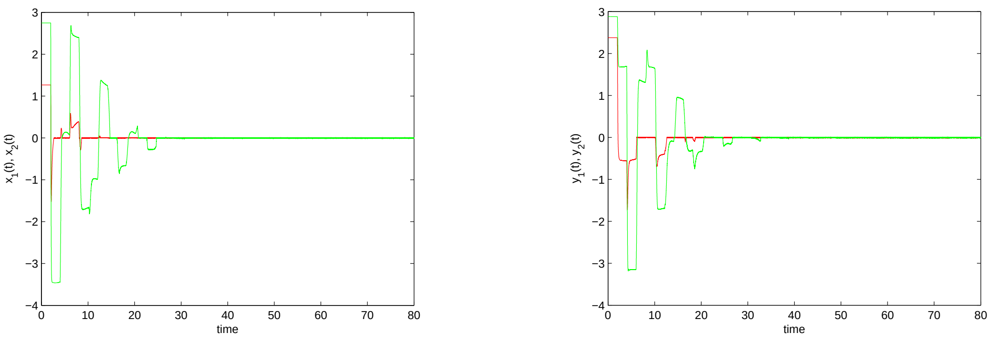
Let us consider system (2.2) with $\tau=1$

$$\begin{aligned}\dot{x}(t)+i\dot{y}(t)=&-C(t)x(t)+D^R(t)f_R(x(t))-D^I(t)f_I(y(t))+E^R(t)f_R(x(t-\tau))\\&-E^I(t)f_I(y(t-\tau))+H^R(t)+i[-C(t)y(t)+D^I(t)f_R(x(t))\\&+D^R(t)f_I(y(t))+E^I(t)f_R(x(t-\tau))+E^R(t)f_I(y(t-\tau))+H^I(t)].\end{aligned}$$

Choose the function

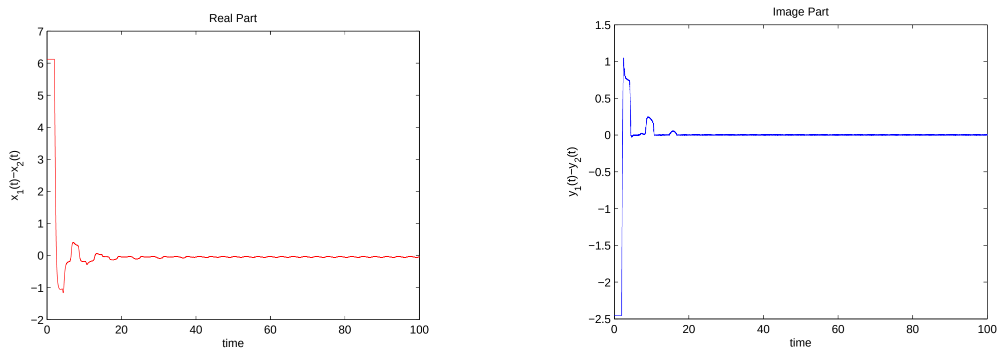
$$\begin{cases} f_R(x)=\begin{cases} x^2+1, & x>0,\\ -x^2-1, & x<0,\end{cases} \qquad f_I(y)=\begin{cases} y^2+2, & y>0,\\ -y^2, & y<0,\end{cases}\\ D^R(t)=\begin{pmatrix} -3-\sin^2 t & -1-\cos^2 t\\ -\cos^2 t & -2-\sin^2 t\end{pmatrix}, \quad D^I(t)=\begin{pmatrix} \frac{1}{2}\sin t & \frac{1}{3}\cos t\\ \frac{1}{3}\cos(2t) & \frac{1}{4}\sin(2t)\end{pmatrix},\\ E^R(t)=\begin{pmatrix} -\frac{1}{2}\cos(3t) & \frac{1}{4}\sin^2(3t)\\ \frac{1}{2}\sin^2 t & frac13\cos^2 t\end{pmatrix}, \quad E^I(t)=\begin{pmatrix} \sin t & \frac{1}{2}\cos t\\ \frac{1}{2}\cos^2 t & \frac{1}{3}\sin^2(2t)\end{pmatrix},\\ H^R(t)=\begin{pmatrix} 3\sin t\\ 3\cos(2t)\end{pmatrix}, \quad H^I(t)=\begin{pmatrix} 3\cos(2t)\\ 3\sin(3t)\end{pmatrix}.\end{cases}$$

It is straightforward to check that A_δ,B_δ are $M-$matrice. Therefore, it is easy to verify that the assumptions of Theorems 1, 2 and 3 are fulfilled. Then it has the unique $2\pi-$periodic solution $[x,y;\gamma,\eta]$, which is globally exponentially stable. With 20 pairs of random initial values, we obtained the rapidity of the convergence of the state is illustrated by the pictures in Fig. 1 and Fig. 2 and the phase plane portrait of the state $z_i,i=1,2$ in Fig. 3.



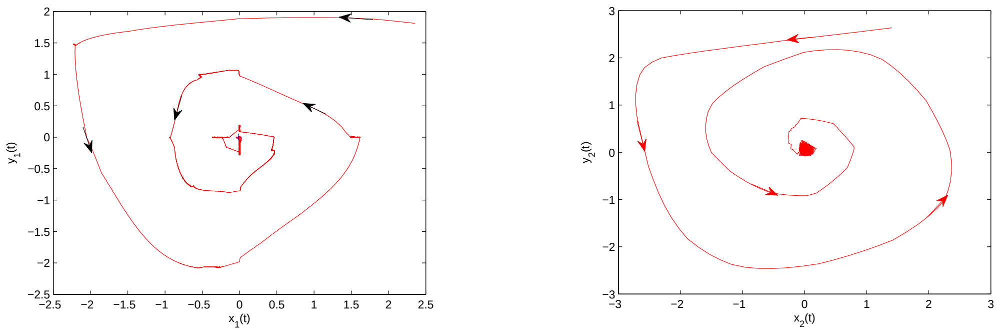
(a) The state of the $x_1(t),x_2(t)$. (b) The state of the $y_1(t),y_2(t)$.

Figure 1: The states of $x_j(t)$ and $y_j(t)$.



(a) The state of the $x_1(t)-x_2(t)$. (b) The state of the $y_1(t)-y_2(t)$.

Figure 2: The states of $x_1(t)-x_2(t)$ and $y_1(t)-y_2(t)$.



(a)Phase plane behavior of the state variables $(x_1(t),y_1(t))$ (b)Phase plane behavior of the state variables $(x_2(t),y_2(t))$

Figure 3: The phase plane behaviors.

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