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# Generalized Accelerated Hermitian and Skew-Hermitian Splitting Methods for Saddle-Point Problems

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**Abstract.** We generalize the accelerated Hermitian and skew-Hermitian splitting (AHSS) iteration methods for large sparse saddle-point problems. These methods involve four iteration parameters whose special choices can recover the preconditioned HSS and accelerated HSS iteration methods. Also a new efficient case is introduced and we theoretically prove that this new method converges to the unique solution of the saddle-point problem. Numerical experiments are used to further examine the effectiveness and robustness of iterations.

**AMS subject classifications**: 65F10; 65F15

**Key words**: saddle-point problem, Hermitian and skew-Hermitian splitting, preconditioning.

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## 1. Introduction

We consider the iterative solution of large sparse saddle-point problems of the form

$$Ax = \begin{bmatrix} B & E \\ -E^* & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} = b, \tag{1.1}$$

where $B \in \mathbb{C}^{p\times p}$ is Hermitian positive definite, $0 \in \mathbb{C}^{q\times q}$ is zero, $E \in \mathbb{C}^{p\times q}$ has full column rank, $p \geq q$, $f \in \mathbb{C}^p$ and $g \in \mathbb{C}^q$ . These assumptions guarantee the existence and uniqueness of the solution of the system of linear equations (1.1). Therefore, $A \in \mathbb{C}^{n\times n}$, with $n = p + q$, is a nonsingular, non-Hermitian, and positive semidefinite matrix. Linear systems of the form (1.1) arise in a variety of scientific and engineering applications, such as computational fluid dynamics, mixed finite element approximations of elliptic partial differential equations, constrained optimizations and constrained least-squares problems. For more detailed descriptions, see [1, 2, 7, 8] and the references therein. Recent years, there are many effective iterative methods have

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been proposed by researchers, including splitting methods [2–4, 6, 7, 17–19], Uzawa-type schemes [9, 10, 13, 14, 20, 22, 24, 30, 31], and the preconditioned iterative methods [1, 7, 8, 11, 12, 21, 23, 25, 28]. Based on the Hermitian/skew-Hermitian (HS) splitting [15, 17, 29]

$$A = H + S,$$

where

$$H = \frac{1}{2}(A + A^*), \quad S = \frac{1}{2}(A - A^*),$$

Bai *et al.* [5] proposed the Hermitian/skew-Hermitian splitting (HSS) iteration method. Benzi and Golub [11] discussed the convergence and the preconditioning properties of the Hermitian and skew-Hermitian splitting iteration method when it is used to solve the saddle-point problem (1.1) . Bai *et al.* [7] transformed the saddle-point problem (1.1) into an equivalent one, then applied the HSS method directly to the preconditioned block linear system, and established a class of preconditioned Hermitian/skew-Hermitian splitting (PHSS) iteration methods for the non-Hermitian positive semidefinite system of linear equations (1.1). In [2], Bai and Golub presented a class of accelerated Hermitian and skew-Hermitian splitting iteration methods (AHSS) for solving the large sparse saddle-point problem (1.1). These methods are two-parameter generalizations of the PHSS iteration methods studied in [7], and they can recover the PHSS methods as well as yield new ones by suitable choices of the two arbitrary parameters. In this paper we generalize accelerated HSS iteration methods for solving the saddle-point problem (1.1), and introduce an efficient case that is different from the PHSS and AHSS methods.

The paper is organized as follows. In Section 2, we review the PHSS and AHSS iteration methods and present generalized accelerated HSS iteration methods (GAHSS) for solving the saddle-point problem (1.1). In Section 3, we analyze the convergence properties of the new iteration method. In Section 4, numerical experiments are given to demonstrate the feasibility and effectiveness of the new iteration method. Finally, in Section 5, we draw some conclusions.

## 2. The GAHSS iteration

In this section, first we review the PHSS and AHSS iteration methods, see [2, 7] for more details, then present generalized accelerated HSS iteration methods (GAHSS) for solving the saddle-point problem (1.1). Consider matrices

$$P = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \in \mathbb{C}^{n\times n}, \quad \bar{E} = B^{-\frac{1}{2}} E C^{-\frac{1}{2}} \in \mathbb{C}^{p\times q}, \tag{2.1}$$

where $C \in \mathbb{C}^{q\times q}$ is a Hermitian positive definite submatrix, and define

$$\bar{A} = P^{-\frac{1}{2}} A P^{-\frac{1}{2}} = \begin{bmatrix} I_p & \bar{E} \\ -\bar{E}^* & 0 \end{bmatrix},$$

$$\begin{bmatrix}\bar{y}\\ \bar{z}\end{bmatrix} = P^{\frac{1}{2}}\begin{bmatrix}y\\ z\end{bmatrix} = \begin{bmatrix}B^{\frac{1}{2}}y\\ C^{\frac{1}{2}}z\end{bmatrix},$$

$$\bar{b} = \begin{bmatrix}\bar{f}\\ \bar{g}\end{bmatrix} = P^{-\frac{1}{2}}b = \begin{bmatrix}B^{-\frac{1}{2}}f\\ C^{-\frac{1}{2}}g\end{bmatrix}.$$

Then the saddle-point problem (1.1) can be transformed into the following equivalent one:

$$\bar{A}\begin{bmatrix}\bar{y}\\ \bar{z}\end{bmatrix} = \bar{b}. \tag{2.2}$$

As a matter of fact, (2.2) is a preconditioned form of (1.1), with the preconditioning matrix $P$ in (2.1). Also the matrix $C \in \mathbb{C}^{q\times q}$ is a free-of-choice, and the best choice is discussed in [7]. The Hermitian and skew-Hermitian parts of the matrix $\bar{A} \in \mathbb{C}^{n\times n}$ are, respectively

$$\bar{H} = \frac{1}{2}(\bar{A}+\bar{A}^*) = \begin{bmatrix}I_p & \\ & 0\end{bmatrix}, \quad \bar{S} = \frac{1}{2}(\bar{A}-\bar{A}^*) = \begin{bmatrix}0 & \bar{E}\\ -\bar{E}^* & 0\end{bmatrix}.$$

By applying the HSS iteration technique to (2.2), we obtain the iteration scheme

$$\begin{bmatrix}\alpha I_p & \bar{E}\\ -\bar{E}^* & \alpha I_q\end{bmatrix}\begin{bmatrix}\bar{y}^{(k+1)}\\ \bar{z}^{(k+1)}\end{bmatrix} = \begin{bmatrix}\frac{\alpha(\alpha-1)}{\alpha+1}I_p & -\frac{\alpha-1}{\alpha+1}\bar{E}\\ \bar{E}^* & \alpha I_q\end{bmatrix}\begin{bmatrix}\bar{y}^{(k)}\\ \bar{z}^{(k)}\end{bmatrix} + \begin{bmatrix}\frac{2\alpha}{\alpha+1}\bar{f}\\ 2\bar{g}\end{bmatrix}.$$

In the original variable we have

$$\begin{bmatrix}\alpha B & E\\ -E^* & \alpha C\end{bmatrix}\begin{bmatrix}y^{(k+1)}\\ z^{(k+1)}\end{bmatrix} = \begin{bmatrix}\frac{\alpha(\alpha-1)}{\alpha+1}B & -\frac{\alpha-1}{\alpha+1}E\\ E^* & \alpha C\end{bmatrix}\begin{bmatrix}y^{(k)}\\ z^{(k)}\end{bmatrix} + \begin{bmatrix}\frac{2\alpha}{\alpha+1}f\\ 2g\end{bmatrix}. \tag{2.3}$$

The PHSS iteration method [3] is described in the following.

The PHSS iteration method: Given an initial guess $x^{(0)} = (y^{(0)^T}, z^{(0)^T})^T \in \mathbb{C}^n$. For $k = 0, 1, \ldots$ until $\{x^{(k)}\} = \{(y^{(k)^T}, z^{(k)^T})^T\} \in \mathbb{C}^n$ converges, compute the next iterate $x^{(k+1)} = (y^{(k+1)^T}, z^{(k+1)^T})^T$ by solving the 2-by-2 block linear system (2.3), where $\alpha$ is a given positive constant.

Also the PHSS iteration method can be rewritten as

$$\begin{bmatrix}y^{(k+1)}\\ z^{(k+1)}\end{bmatrix} = \mathcal{L}(\alpha)\begin{bmatrix}y^{(k)}\\ z^{(k)}\end{bmatrix} + \mathcal{N}(\alpha)\begin{bmatrix}f\\ g\end{bmatrix}, \tag{2.4}$$

where

$$\mathcal{L}(\alpha) = \begin{bmatrix}\alpha B & E\\ -E^* & \alpha C\end{bmatrix}^{-1}\begin{bmatrix}\frac{\alpha(\alpha-1)}{\alpha+1}B & -\frac{\alpha-1}{\alpha+1}E\\ E^* & \alpha C\end{bmatrix}, \tag{2.5a}$$

$$\mathcal{N}(\alpha) = \begin{bmatrix}\alpha B & E\\ -E^* & \alpha C\end{bmatrix}^{-1}\begin{bmatrix}\frac{2\alpha}{\alpha+1}I_p & \\ & 2I_q\end{bmatrix}. \tag{2.5b}$$

Here, $\mathcal{L}(\alpha)$ is the iteration matrix of the PHSS iteration. In fact, (2.4) may also result from the splitting

$$A = M(\alpha) - N(\alpha) \tag{2.6}$$

of the coefficient matrix $A$, with

$$M(\alpha) = \begin{bmatrix} \frac{\alpha+1}{2}B & \frac{\alpha+1}{2\alpha}E \\ -\frac{1}{2}E^* & \frac{\alpha}{2}C \end{bmatrix}, \quad N(\alpha) = \begin{bmatrix} \frac{\alpha-1}{2}B & -\frac{\alpha-1}{2\alpha}E \\ \frac{1}{2}E^* & \frac{\alpha}{2}C \end{bmatrix}.$$

In actual computations, at each iterate of the PHSS iteration we need to solve a linear system with the coefficient matrix $M(\alpha)$. As this matrix is positive definite, one may solve the aforementioned linear system inexactly by another iteration procedure, e.g., the HSS iteration.

For the AHSS iteration method, Bai and Golub considered in [2] the following HSS iteration

$$\begin{cases} (\Lambda + \bar{H})\bar{x}^{(k+\frac{1}{2})} = (\Lambda - \bar{S})\bar{x}^{(k)} + \bar{b}, \\ (\Lambda + \bar{S})\bar{x}^{(k+1)} = (\Lambda - \bar{H})\bar{x}^{(k+\frac{1}{2})} + \bar{b}, \end{cases} \tag{2.7}$$

where

$$\Lambda = \begin{bmatrix} \alpha I_p & 0 \\ 0 & \beta I_q \end{bmatrix}, \quad \text{with } \alpha \text{ and } \beta \text{ positive constants.}$$

By making use of the definitions of $\bar{H}$ and $\bar{S}$, after straightforward computations, we can rewrite (2.7) in the original variable as

$$\begin{bmatrix} \alpha B & E \\ -E^* & \beta C \end{bmatrix} \begin{bmatrix} y^{(k+1)} \\ z^{(k+1)} \end{bmatrix} = \begin{bmatrix} \frac{\alpha(\alpha-1)}{\alpha+1}B & -\frac{\alpha-1}{\alpha+1}E \\ E^* & \beta C \end{bmatrix} \begin{bmatrix} y^{(k)} \\ z^{(k)} \end{bmatrix} + \begin{bmatrix} \frac{2\alpha}{\alpha+1}f \\ 2g \end{bmatrix}, \tag{2.8}$$

which results in the following AHSS iteration method for solving the saddle-point problem (1.1).

The AHSS iteration method: Given an initial guess $x^{(0)} = (y^{(0)^T}, z^{(0)^T})^T \in \mathbb{C}^n$. For $k = 0, 1, \ldots$ until $\{x^{(k)}\} = \{(y^{(k)^T}, z^{(k)^T})^T\} \in \mathbb{C}^n$ converges, compute the next iterate $x^{(k+1)} = (y^{(k+1)^T}, z^{(k+1)^T})^T$ by solving the 2-by-2 block linear system (2.8), where $\alpha$ and $\beta$ are given positive constants.

Note that when $\alpha = \beta > 0$, the above AHSS iteration method naturally reduces to the PHSS iteration method in [7]. Now we present the generalized accelerated Hermitian and skew-Hermitian splitting (GAHSS) iteration methods.

Consider the following HSS iteration:

$$\begin{cases} (\Lambda_1 + \bar{H})\bar{x}^{(k+\frac{1}{2})} = (\Lambda_1 - \bar{S})\bar{x}^{(k)} + \bar{b}, \\ (\Lambda_2 + \bar{S})\bar{x}^{(k+1)} = (\Lambda_2 - \bar{H})\bar{x}^{(k+\frac{1}{2})} + \bar{b}, \end{cases} \tag{2.9}$$

where

$$\Lambda_1 = \begin{bmatrix} \alpha I_p & 0 \\ 0 & \beta I_q \end{bmatrix}, \quad \text{with } \alpha \text{ and } \beta \text{ positive constants}$$

and

$$\Lambda_2=\begin{bmatrix}\gamma I_p & 0\\ 0 & \delta I_q\end{bmatrix},\quad \text{with } \gamma \text{ and } \delta \text{ positive constants.}$$

We can see when $\alpha=\beta=\gamma=\delta$ the above GAHSS iteration method naturally reduces to the PHSS iteration method and when $\alpha=\gamma$ and $\beta=\delta$ it reduces to the AHSS iteration method. Similar AHSS iteration method, after straightforward computations, we can rewrite (2.9) in the original variable as

$$\begin{bmatrix}\gamma B & E\\ -E^* & \delta C\end{bmatrix}\begin{bmatrix}y^{(k+1)}\\ z^{(k+1)}\end{bmatrix}=\begin{bmatrix}\frac{\alpha(\gamma-1)}{\alpha+1}B & -\frac{\gamma-1}{\alpha+1}E\\ \frac{\delta}{\beta}E^* & \delta C\end{bmatrix}\begin{bmatrix}y^{(k)}\\ z^{(k)}\end{bmatrix}+\begin{bmatrix}(\frac{\gamma-1}{\alpha+1}+1)f\\ (\frac{\delta}{\beta}+1)g\end{bmatrix}. \tag{2.10}$$

Therefore, we obtain the following GAHSS iteration method for solving the saddle-point problem (1.1).

The GAHSS iteration method: Given an initial guess $x^{(0)}=(y^{(0)^T},z^{(0)^T})^T\in\mathbb{C}^n$. For $k=0,1,\dots$ until $\{x^{(k)}\}=\{(y^{(k)^T},z^{(k)^T})^T\}\in\mathbb{C}^n$ converges, compute the next iterate $x^{(k+1)}=(y^{(k+1)^T},z^{(k+1)^T})^T$ by solving the 2-by-2 block linear system (2.10), where $\alpha$, $\beta$, $\gamma$ and $\delta$ are given positive constants.

Unfortunately, the above GAHSS method does not converge to the exact solution of the block system of linear equations (1.1) for any $\alpha,\beta,\gamma$ and $\delta$. In this paper, we consider an efficient case that is different from the PHSS and AHSS methods. We suppose $\alpha=\beta=\gamma$ and $\delta=r\alpha$ with $r$ is a small positive constant, and show this iteration by PHSS(r).

Evidently, the PHSS(r) iteration method can be equivalently rewritten as

$$\begin{bmatrix}y^{(k+1)}\\ z^{(k+1)}\end{bmatrix}=\mathcal{L}(\alpha)\begin{bmatrix}y^{(k)}\\ z^{(k)}\end{bmatrix}+\mathcal{N}(\alpha)\begin{bmatrix}f\\ g\end{bmatrix}, \tag{2.11}$$

where

$$\mathcal{L}(\alpha)=\begin{bmatrix}\alpha B & E\\ -E^* & r\alpha C\end{bmatrix}^{-1}\begin{bmatrix}\frac{\alpha(\alpha-1)}{\alpha+1}B & -\frac{\alpha-1}{\alpha+1}E\\ rE^* & r\alpha C\end{bmatrix} \tag{2.12a}$$

$$\mathcal{N}(\alpha)=\begin{bmatrix}\alpha B & E\\ -E^* & r\alpha C\end{bmatrix}^{-1}\begin{bmatrix}\frac{2\alpha}{\alpha+1}I_p & \\ & (r+1)I_q\end{bmatrix}. \tag{2.12b}$$

In fact, (2.11) may also result from the splitting

$$A'=\begin{bmatrix}B & E\\ -\frac{1+r}{2}E^* & 0\end{bmatrix}=M(\alpha)-N(\alpha), \tag{2.13}$$

of the matrix $A'$, with

$$M(\alpha)=\begin{bmatrix}\frac{\alpha+1}{2}B & \frac{\alpha+1}{2\alpha}E\\ -\frac{1}{2}E^* & \frac{r\alpha}{2}C\end{bmatrix},\quad N(\alpha)=\begin{bmatrix}\frac{\alpha-1}{2}B & -\frac{\alpha-1}{2\alpha}E\\ \frac{r}{2}E^* & \frac{r\alpha}{2}C\end{bmatrix}.$$

In actual computations, at each iterate of the PHSS(r) iteration we need to solve a linear system with the coefficient matrix

$$M'(\alpha)=\begin{bmatrix}\alpha B & E\\ -E^* & r\alpha C\end{bmatrix}, \text{ or equivalently,} \quad M(\alpha). \tag{2.14}$$

**Remark:** If we put $b'=\begin{bmatrix}f\\ \frac{1+r}{2}g\end{bmatrix}$, then the original system $Ax=b$ is equivalent to the system $A'x=b'$, so we may consider $M(\alpha)$ as a preconditioner to the system $Ax=b$.

## 3. Convergence analysis

In this section, we study the convergence properties of the new iteration method. By straightforward computations, we can obtain an explicit expression of the iteration matrix $\mathcal{L}(\alpha)$ in (2.12a).

**Lemma 3.1.** *Consider the system of linear equations* (1.1). *Let $B\in\mathbb{C}^{p\times p}$ be Hermitian positive definite, $E\in\mathbb{C}^{p\times q}$ be of full column rank, and $\alpha>0$ a given constant. Assume that $r$ is a small positive constant and $C\in\mathbb{C}^{q\times q}$ is a Hermitian positive definite matrix. Then we partition $\mathcal{L}(\alpha)$ in* (2.12a) *as*

$$\mathcal{L}(\alpha)=\begin{bmatrix}\mathcal{L}_{11}(\alpha) & \mathcal{L}_{12}(\alpha)\\ \mathcal{L}_{21}(\alpha) & \mathcal{L}_{22}(\alpha)\end{bmatrix},$$

*where*

$$\begin{cases}\mathcal{L}_{11}(\alpha)=\frac{\alpha-1}{\alpha+1}I_p-(\frac{\alpha-1}{\alpha(\alpha+1)}+\frac{r}{\alpha})B^{-1}ES(\alpha)^{-1}E^*,\\ \mathcal{L}_{12}(\alpha)=-(\frac{r(\alpha-1)}{\alpha+1}+r)B^{-1}ES(\alpha)^{-1}C,\\ \mathcal{L}_{21}(\alpha)=(\frac{\alpha-1}{\alpha+1}+r)S(\alpha)^{-1}E^*,\\ \mathcal{L}_{22}(\alpha)=-\frac{\alpha-1}{\alpha+1}I_q+(\frac{r\alpha(\alpha-1)}{\alpha+1}+r\alpha)S(\alpha)^{-1}C,\end{cases}$$

*and*

$$S(\alpha)=r\alpha C+\frac{1}{\alpha}E^*B^{-1}E$$

*is the Schur complement of the matrix $M'(\alpha)$ in* (2.14).

*Proof.* Let

$$\overline{M}(\alpha)=P^{-\frac{1}{2}}M(\alpha)P^{-\frac{1}{2}}=\begin{bmatrix}\frac{\alpha+1}{2}I_p & \frac{\alpha+1}{2\alpha}\bar{E}\\ -\frac{1}{2}\bar{E}^* & \frac{r\alpha}{2}I_q\end{bmatrix},$$

$$\overline{N}(\alpha)=P^{-\frac{1}{2}}N(\alpha)P^{-\frac{1}{2}}=\begin{bmatrix}\frac{\alpha-1}{2}I_p & -\frac{\alpha-1}{2\alpha}\bar{E}\\ \frac{r}{2}\bar{E}^* & \frac{r\alpha}{2}I_q\end{bmatrix},$$

where the matrices $P$ and $\bar{E}$ are defined in (2.1). Then

$$\overline{M}(\alpha)^{-1}=\begin{bmatrix}\frac{2}{\alpha+1}(I_p-\frac{1}{\alpha}\bar{E}\bar{S}(\alpha)^{-1}\bar{E}^*) & \frac{-2}{\alpha}\bar{E}\bar{S}(\alpha)^{-1})\\ \frac{2}{\alpha+1}\bar{S}(\alpha)^{-1}\bar{E}^* & 2\bar{S}(\alpha)^{-1}\end{bmatrix},$$

$$\overline{\mathcal{L}}(\alpha)=\begin{bmatrix}\overline{\mathcal{L}}_{11}(\alpha) & \overline{\mathcal{L}}_{12}(\alpha)\\ \overline{\mathcal{L}}_{21}(\alpha) & \overline{\mathcal{L}}_{22}(\alpha)\end{bmatrix}=\overline{M}(\alpha)^{-1}\overline{N}(\alpha),$$

with

$$\begin{cases} \overline{\mathcal{L}}_{11}(\alpha) = \frac{\alpha-1}{\alpha+1}I_p - (\frac{\alpha-1}{\alpha(\alpha+1)} + \frac{r}{\alpha})\bar{E}\bar{S}(\alpha)^{-1}\bar{E}^*, \\ \overline{\mathcal{L}}_{12}(\alpha) = -(\frac{r(\alpha-1)}{\alpha+1} + r)\bar{E}\bar{S}(\alpha)^{-1}, \\ \overline{\mathcal{L}}_{21}(\alpha) = (\frac{\alpha-1}{\alpha+1} + r)\bar{S}(\alpha)^{-1}\bar{E}^*, \\ \overline{\mathcal{L}}_{22}(\alpha) = -\frac{\alpha-1}{\alpha+1}I_q + (\frac{r\alpha(\alpha-1)}{\alpha+1} + r\alpha)\bar{S}(\alpha)^{-1}, \end{cases}$$

where

$$\bar{S}(\alpha) = r\alpha I_q + \frac{1}{\alpha}\bar{E}^*\bar{E} \tag{3.1}$$

is the Schur complement of the matrix

$$\overline{M'}(\alpha) = \begin{bmatrix} \alpha I_p & \bar{E} \\ -\bar{E}^* & r\alpha I_q \end{bmatrix}.$$

Then we have

$$\mathcal{L}(\alpha) = M(\alpha)^{-1}N(\alpha) = P^{-\frac{1}{2}}\overline{M}(\alpha)^{-1}\overline{N}(\alpha)P^{\frac{1}{2}} = P^{-\frac{1}{2}}\overline{\mathcal{L}}(\alpha)P^{\frac{1}{2}}; \tag{3.2}$$

the result follows immediately. □

Based on Lemma 1.1, we can further obtain the eigenvalues of the iteration matrix $\mathcal{L}(\alpha)$ of the PHSS(r) method.

**Lemma 3.2.** *Let the conditions in Lemma* 3.1 *be satisfied. If* $\bar{\sigma}_k \quad (k = 1, 2, \ldots, q)$ *are the positive singular values of the matrix* $\bar{E} \in \mathbb{C}^{p\times q}$ *in* (2.1) *then the eigenvalues of the iteration matrix* $\mathcal{L}(\alpha)$ *of the PHSS(r) iteration method are* $\frac{\alpha-1}{\alpha+1}$ *with multiplicity* $p - q$*, and*

$$\frac{-b \pm \sqrt{\Delta}}{2(\alpha+1)(r\alpha^2 + \bar{\sigma}_k^2)},$$

*where*

$$b = -2r\alpha^3 + \big(r(\alpha+1) + \alpha - 1\big)\bar{\sigma}_k^2,$$
$$\Delta = b^2 - 4(\alpha^2 - 1)\big(r^2\alpha^4 + (r + r^2)\alpha^2\bar{\sigma}_k^2 + r\bar{\sigma}_k^4\big).$$

*Proof.* From (3.2) we know that $\mathcal{L}(\alpha)$ is similar to $\overline{\mathcal{L}}(\alpha)$. Therefore, we only need to compute the eigenvalues of the matrix $\overline{\mathcal{L}}(\alpha)$. Let $\bar{E} = \overline{U}^*\overline{\Sigma}_1\overline{V}$ be the singular value decomposition [16] of the matrix $\bar{E} \in \mathbb{C}^{p\times q}$, where $\overline{U} \in \mathbb{C}^{p\times p}$ and $\overline{V} \in \mathbb{C}^{q\times q}$ are unitary matrices, and

$$\overline{\Sigma}_1 = \begin{bmatrix} \overline{\Sigma} \\ 0 \end{bmatrix}, \quad \overline{\Sigma} = diag(\overline{\sigma}_1, \overline{\sigma}_2, \ldots, \overline{\sigma}_q) \in \mathbb{C}^{q\times q}.$$

Then after a few computation, we have

$$\overline{S}(\alpha) = \overline{V}^*(r\alpha I_q + \frac{1}{\alpha}\overline{\Sigma}^2)\overline{V},$$

and therefore,

$$\overline{\mathcal{L}}_{11}(\alpha)=\overline{U}^*\begin{bmatrix}\frac{\alpha-1}{\alpha+1}I_q-(\frac{\alpha-1}{\alpha(\alpha+1)}+\frac{r}{\alpha})(r\alpha I_q+\frac{1}{\alpha}\overline{\Sigma}^2)^{-1}\overline{\Sigma}^2 & 0\\ 0 & \frac{\alpha-1}{\alpha+1}I_{p-q}\end{bmatrix}\overline{U},$$

$$\overline{\mathcal{L}}_{12}(\alpha)=\overline{U}^*\begin{bmatrix}-(\frac{r(\alpha-1)}{\alpha+1}+r)\overline{\Sigma}(r\alpha I_q+\frac{1}{\alpha}\overline{\Sigma}^2)^{-1}\\ 0\end{bmatrix}\overline{V},$$

$$\overline{\mathcal{L}}_{21}(\alpha)=\overline{V}^*\begin{bmatrix}-(\frac{\alpha-1}{\alpha+1}+r)\overline{\Sigma}(r\alpha I_q+\frac{1}{\alpha}\overline{\Sigma}^2)^{-1}, & 0\end{bmatrix}\overline{U},$$

$$\overline{\mathcal{L}}_{22}(\alpha)=\overline{V}^*\Big(-\frac{\alpha-1}{\alpha+1}I_q+(\frac{r\alpha(\alpha-1)}{\alpha+1}+r\alpha)(r\alpha I_q+\frac{1}{\alpha}\overline{\Sigma}^2)^{-1}\Big)\overline{V}.$$

Define

$$\overline{Q}=\begin{bmatrix}\overline{U} & 0\\ 0 & \overline{V}\end{bmatrix}\in\mathbb{C}^{n\times n},\quad J_q=\Big(r\alpha I_q+\frac{1}{\alpha}\overline{\Sigma}^2\Big)^{-1}.$$

Then $\overline{Q}$ is a unitary matrix and we have

$$\begin{aligned}&\overline{Q}\,\overline{\mathcal{L}}(\alpha)\,\overline{Q}^*\\ &=\begin{bmatrix}\frac{\alpha-1}{\alpha+1}I_q-(\frac{\alpha-1}{\alpha(\alpha+1)}+\frac{r}{\alpha})J_q\overline{\Sigma}^2 & 0 & -(\frac{r(\alpha-1)}{\alpha+1}+r)\overline{\Sigma}J_q\\ 0 & \frac{\alpha-1}{\alpha+1}I_{p-q} & 0\\ (\frac{\alpha-1}{\alpha+1}+r)\overline{\Sigma}J_q & 0 & -\frac{\alpha-1}{\alpha+1}I_q+\big(\frac{r\alpha(\alpha-1)}{\alpha+1}+r\alpha\big)J_q\end{bmatrix}.\end{aligned}$$

Therefore, the eigenvalues of the matrix $\overline{\mathcal{L}}(\alpha)$ are just $\frac{\alpha-1}{\alpha+1}$ with multiplicity $p-q$, and those of the matrix

$$\begin{bmatrix}\frac{\alpha-1}{\alpha+1}I_q-(\frac{\alpha-1}{\alpha(\alpha+1)}+\frac{r}{\alpha})J_q\overline{\Sigma}^2 & -(\frac{r(\alpha-1)}{\alpha+1}+r)\overline{\Sigma}J_q\\ (\frac{\alpha-1}{\alpha+1}+r)\overline{\Sigma}J_q & -\frac{\alpha-1}{\alpha+1}I_q+\big(\frac{r\alpha(\alpha-1)}{\alpha+1}+r\alpha\big)J_q\end{bmatrix}$$

which are the same as the matrices $\frac{1}{(\alpha+1)(r\alpha^2+\overline{\sigma}_k^2)}\overline{\mathcal{L}}_k(\alpha),\quad k=1,\dots,q$, where

$$\overline{\mathcal{L}}_k(\alpha)=\begin{bmatrix}r\alpha^2(\alpha-1)-r(\alpha+1)\overline{\sigma}_k^2 & -2r\alpha^2\overline{\sigma}_k\\ ((1+r)\alpha^2+(r-1)\alpha)\overline{\sigma}_k & r\alpha^2(\alpha+1)-(\alpha-1)\overline{\sigma}_k^2\end{bmatrix}.$$

The two eigenvalues of the matrix $\overline{\mathcal{L}}_k(\alpha)$ are the two roots of the quadratic equations

$$\lambda^2+\Big(-2r\alpha^3+\big(r(\alpha+1)+\alpha-1\big)\overline{\sigma}_k^2\Big)\lambda+(\alpha^2-1)\big(r^2\alpha^4+(r+r^2)\alpha^2\overline{\sigma}_k^2+r\overline{\sigma}_k^4\big)=0,$$

or in other words,

$$\lambda=\frac{-b\pm\sqrt{\Delta}}{2}.$$

Since the eigenvalues of the matrix $\overline{\mathcal{L}}(\alpha)$ are $\frac{\alpha-1}{\alpha+1}$ with multiplicity $p-q$, and

$$\frac{1}{(\alpha+1)(r\alpha^2+\overline{\sigma}_k^2)}\lambda.$$

This completes our proof. $\square$

**Remark:** The singular values of the matrix $\bar{E} \in \mathbb{C}^{p\times q}$ are the square roots of the eigenvalues of either the matrix $C^{-1}E^*B^{-1}E$, or equivalently, the matrix $E^*B^{-1}EC^{-1}$.

**Theorem 3.1.** *Consider the system of linear equations* (1.1). *Let $B \in \mathbb{C}^{p\times p}$ be Hermitian positive definite, $E \in \mathbb{C}^{p\times q}$ be of full column rank, $\alpha > 0$ a given constant and $r > 0$ a very small constant. Assume that $C \in \mathbb{C}^{q\times q}$ is a Hermitian positive definite matrix and $\bar{\sigma}_k, b$ and $\Delta$ are the same as Lemma* 3.2. *Then*

$$\rho(\mathcal{L}(\alpha)) < 1, \quad \forall \alpha > 0,$$

*i.e., the PHSS(r) iteration converges to the exact solution of the system of linear equations* (1.1).

*Proof.* Obviously, we have

$$\frac{|\alpha - 1|}{\alpha + 1} < 1, \quad \forall \alpha > 0.$$

Because

$$\lim_{r\to 0}\left|\frac{-b \pm \sqrt{\Delta}}{2(\alpha+1)(r\alpha^2 + \bar{\sigma}_k^2)}\right| = \frac{|\alpha-1|}{\alpha+1},$$

by making use of Lemma 3.2 we easily see that $\rho(\mathcal{L}(\alpha)) < 1$ holds for all $\alpha > 0$. $\square$

**Remark:** In actual examples, it is sufficient to consider $r = 0.1, 0.01$ or at most $0.001$. See Section 4, for more details.

The optimal iteration parameter and the corresponding asymptotic convergence factor of the PHSS and PHSS(r) iteration methods are described in the following theorems.

**Theorem 3.2.** ([7]). *Consider the system of linear equations* (1.1). *Let $B \in \mathbb{C}^{p\times p}$ be Hermitian positive definite, $E \in \mathbb{C}^{p\times q}$ be of full column rank, and $\alpha > 0$ a given constant. Assume that $C \in \mathbb{C}^{q\times q}$ is a Hermitian positive definite matrix. If $\sigma_k (k = 1, 2, \ldots, q)$ are the positive singular values of the matrix $B^{-\frac{1}{2}}EC^{-\frac{1}{2}} \in \mathbb{C}^{p\times q}$, and $\sigma_{\min} = min\{\sigma_k\}$ and $\sigma_{\max} = max\{\sigma_k\}$ , then, for the PHSS iteration method of the system of linear equations* (1.1)*, the optimal value of the iteration parameter $\alpha$ is given by*

$$\alpha^* = arg\min_{\alpha} \rho(\mathcal{L}(\alpha)) = \sqrt{\sigma_{\min}\sigma_{\max}},$$

*and correspondingly,*

$$\rho(\mathcal{L}(\alpha^*)) = \frac{\sigma_{\max} - \sigma_{\min}}{\sigma_{\max} + \sigma_{\min}}.$$

For convenience, we use the same notations as in Theorem 3.2.

**Theorem 3.3.** *Consider the system of linear equations* (1.1). *Let* $B \in \mathbb{C}^{p\times p}$ *be Hermitian positive definite,* $E \in \mathbb{C}^{p\times q}$ *be of full column rank, and* $\alpha > 0$ *a given constant. Assume that* $C \in \mathbb{C}^{q\times q}$ *is a Hermitian positive definite matrix. If* $\sigma_k(k = 1, 2, \ldots, q)$ *are the positive singular values of the matrix* $B^{-\frac{1}{2}}EC^{-\frac{1}{2}} \in \mathbb{C}^{p\times q}$, *and* $\sigma_{\min} = min\{\sigma_k\}$ *and* $\sigma_{\max} = max\{\sigma_k\}$ *, then, for the PHSS(r) iteration method of the system of linear equations* (1.1), *the optimal value of the iteration parameter* $\alpha$ *is given by*

$$\alpha^* = arg\min_{\alpha} \rho(\mathcal{L}(\alpha)) = 1,$$

*and correspondingly,*

$$\rho(\mathcal{L}(\alpha^*)) = \max\left\{\frac{r|1-\sigma^2_{\min}|}{r+\sigma^2_{\min}}, \frac{r|1-\sigma^2_{\max}|}{r+\sigma^2_{\max}}\right\}.$$

*Proof.* Based on lemma 3.2, the eigenvalues $\lambda_i$ $(i = 1, \ldots, n)$ of the iteration matrix $\mathcal{L}(\alpha)$ of the PHSS(r) method are $\frac{\alpha-1}{\alpha+1}$ with multiplicity $p-q$, and

$$\frac{-b \pm \sqrt{\Delta}}{2(\alpha+1)(r\alpha^2+\sigma_k^2)}, \tag{3.3}$$

where

$$\begin{aligned} b &= -2r\alpha^3 + \big(r(\alpha+1)+\alpha-1\big)\sigma_k^2, \\ \Delta &= b^2 - 4(\alpha^2-1)\big(r^2\alpha^4 + (r+r^2)\alpha^2\sigma_k^2 + r\sigma_k^4\big). \end{aligned}$$

Since $r$ is a small positive constant, if $r \to 0$ then

$$\left|\frac{-b \pm \sqrt{\Delta}}{2(\alpha+1)(r\alpha^2+\sigma_k^2)}\right| \to \frac{|\alpha-1|}{\alpha+1}.$$

Therefore, when $\alpha \to 1$ it then follows that $\frac{|\alpha-1|}{\alpha+1} \to 0$. Then by substituting $\alpha^*$ in (3.3), we obtain

$$\frac{-b \pm \sqrt{\Delta}}{2(\alpha+1)(r\alpha^2+\sigma_k^2)} = \frac{r(1-\sigma_k^2)}{r+\sigma_k^2}.$$

Finally, we have

$$\max_{\sigma_{\min}\le x\le\sigma_{\max}} \frac{r|1-x^2|}{r+x^2} = \max\left\{\frac{r|1-\sigma^2_{\min}|}{r+\sigma^2_{\min}}, \frac{r|1-\sigma^2_{\max}|}{r+\sigma^2_{\max}}\right\},$$

which can be seen from Fig. 1. □

**Remark:** Recall that in PHSS(r) iteration method $r$ is a small positive constant and similar to PHSS iteration method $\alpha$ is a positive constant. In Theorem 3.3, under taking limit with respect to $r$ and $\alpha$, we compute the optimal parameter. Therefore, $\alpha^*$ is actually not the exact optimal one.

## 4. Numerical experiments

In this section, we use one example to exhibit the superiority of PHSS(r) method to PHSS and AHSS when they are used as solvers as well as preconditioners to GMRES and GMRES($\ell$) see [26, 27], for the saddle-point problem (1.1), from aspects of spectral radius $\rho(.)$ of iteration matrix, number of total iteration steps (denoted by IT) and elapsed CPU time in seconds (denoted by CPU). Here, the integer $\ell$ in GMRES($\ell$) denotes that the algorithm is restarted after every $\ell$ iterations. To this end, we need to choose the matrix C in both PHSS , AHSS and PHSS(r). There are two natural choices of the matrix C: the first is $E^*B^{-1}E$, and the second is $E^*\hat{B}^{-1}E$, where $\hat{B}$ is a good approximation to the matrix block B, see [2, 7] for more details. In actual computations, we choose the right-hand side vector b so that the exact solution of the saddle-point problem (1.1) is $(1,1,\ldots,1)^T \in \mathbb{R}^{n\times n}$ . Besides, all runs are started from an initial vector $x^{(0)} = 0$ , terminated if the current iterations satisfy

$$RES = \frac{\|b - Ax^{(k)}\|_2}{\|b\|_2} \leq 10^{-8}$$

or if the maximum prescribed number of iterations $k_{\max}$ is exceeded, and performed in MATLAB with machine precision $10^{-16}$.

**Example 4.1.** (Bai, Golub and Pan [7]) Consider the saddle-point problem (1.1), in which

$$B = \begin{bmatrix} I\otimes\Upsilon+\Upsilon\otimes I & 0 \\ 0 & I\otimes\Upsilon+\Upsilon\otimes I \end{bmatrix} \in \mathbb{R}^{2m^2\times 2m^2}, \quad E = \begin{bmatrix} I\otimes\Psi \\ \Psi\otimes I \end{bmatrix} \in \mathbb{R}^{2m^2\times m^2}$$

Table 1: Parameter(s) versus spectral radius for Example 4.1.

| Method | $m$ | 8 | 16 | 24 | 32 |
|---|---|---|---|---|---|
| PHSS | $\alpha^*$ | 1.4151 | 1.8718 | 2.2447 | 2.5657 |
| | $\rho(\mathcal{L}(\alpha^*))$ | 0.4146 | 0.5510 | 0.6194 | 0.6626 |
| AHSS | $\alpha^*$ | 1.2278 | 1.5026 | 1.7390 | 1.9482 |
| | $\beta^*$ | 1.6309 | 2.3317 | 2.8974 | 3.3789 |
| | $\rho(\mathcal{L}(\alpha^*,\beta^*))$ | 0.3198 | 0.4481 | 0.5194 | 0.5671 |
| PHSS($r_1$) | $\alpha^*$ | 1 | 1 | 1 | 1 |
| | $\rho(\mathcal{L}(\alpha^*,r_1))$ | **0.0856** | **0.0955** | **0.0978** | **0.0987** |
| PHSS($r_1$) | $\alpha_{exp}$ | 1.01 | 1.01 | 1.01 | 1.01 |
| | $\rho(\mathcal{L}(\alpha_{exp},r_1))$ | **0.0835** | **0.0949** | **0.0975** | **0.0986** |
| PHSS($r_2$) | $\alpha^*$ | 1 | 1 | 1 | 1 |
| | $\rho(\mathcal{L}(\alpha^*,r_2))$ | **0.0087** | **0.0096** | **0.0098** | **0.0099** |
| PHSS($r_2$) | $\alpha_{exp}$ | 1.001 | 1.001 | 1.001 | 1.001 |
| | $\rho(\mathcal{L}(\alpha_{exp},r_2))$ | **0.0085** | **0.0095** | **0.0098** | **0.0099** |

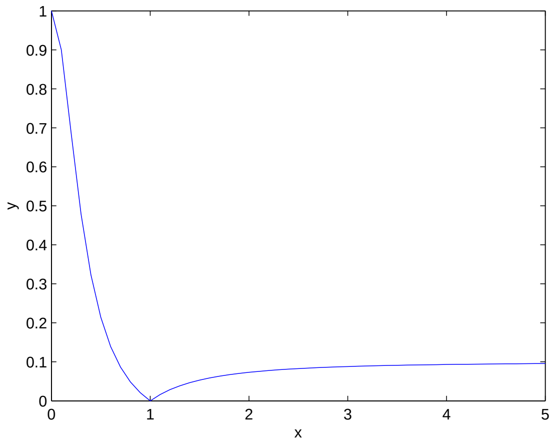


Figure 1: The graph of $y = \frac{r|1-x^2|}{r+x^2}$, $r = 0.1$.

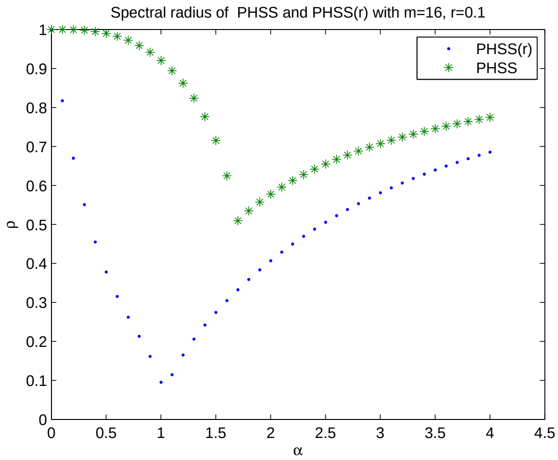


Figure 2: $\alpha$ versus $\rho(\mathcal{L}(\alpha))$ when $\mu = 1, m = 16$ and $r = 0.1$.

and

$$\Upsilon = \frac{\mu}{h^2}.\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}, \quad \Psi = \frac{1}{h^2}.\text{tridiag}(-1, 1, 0) \in \mathbb{R}^{m\times m},$$

with $\otimes$ the Kronecker product symbol, Re$=\frac{h}{2\mu}$, h$=\frac{1}{m+1}$ and $\mu > 0$ the viscosity constant. For this example, we take $k_{\max} = 5m$ , and $\hat{B} = \frac{2\mu}{h^2} I + I \otimes \Upsilon$ the block-diagonal matrix of B. We remark that the number of variables of the corresponding saddle-point problem (1.1) is $n = p + q = 3m^2$.

In Table 1, we list $\alpha^*$ and $(\alpha^*, \beta^*)$, as well as the corresponding $\rho(\mathcal{L}(\alpha^*))$, $\rho(\mathcal{L}(\alpha^*, \beta^*))$ and $\rho(\mathcal{L}(\alpha^*, r))$ for the PHSS, the AHSS and the PHSS(r) iterations, respectively, for various parameters $r (r_1 = 0.1, r_2 = 0.01)$ and problem sizes $m$. We also compute

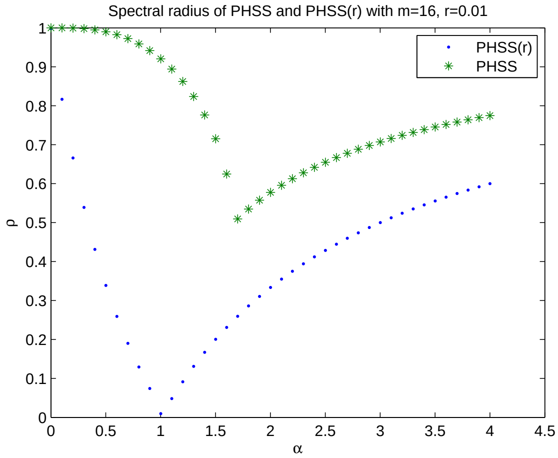


Figure 3: $\alpha$ versus $\rho(\mathcal{L}(\alpha))$ when $\mu = 1, m = 16$ and $r = 0.01$.

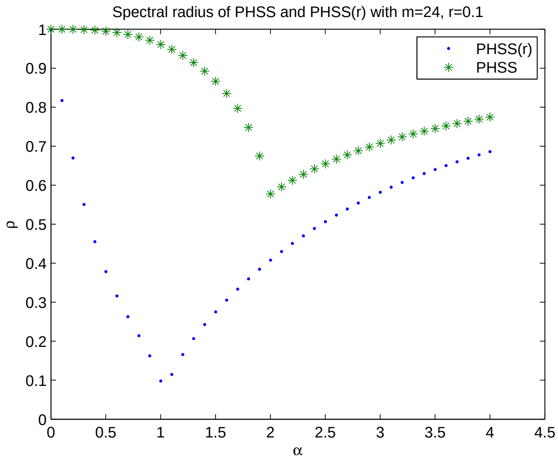


Figure 4: $\alpha$ versus $\rho(\mathcal{L}(\alpha))$ when $\mu = 1, m = 24$ and $r = 0.1$.

experimental optimal parameter $\alpha$ (denoted by $\alpha_{exp}$) and $\rho(\mathcal{L}(\alpha_{exp}, r))$ for PHSS(r) iteration. Figs. 2–5, depict the curves of the spectral radii versus the parameter $\alpha$ of PHSS and PHSS(r) when m = 16, 24 and $\mu = 1$ with $r = 0.1, 0.01$. We see that PHSS(r) possesses faster convergence speed than PHSS.

In Figs. 6–9, we plot the eigenvalue distributions of the original matrix A, the PHSS($\alpha^*$)-preconditioned matrix $M(\alpha^*)^{-1}A$ , the AHSS($\alpha^*, \beta^*$)-preconditioned matrix $M(\alpha^*, \beta^*)^{-1}A$, and the PHSS(r)-preconditioned matrix $M(\alpha^*, r_1)^{-1}A$ respectively, for Example 4.1, when $m = 32$ and $\mu = 1$ as well as $m = 32$ and $\mu = \frac{1}{80}$. Clearly, the matrices A are very ill-conditioned because their spectrums have large ranges along both the real and the imaginary axis and some of the eigenvalues lye close to the origin. However, both $M(\alpha^*)^{-1}A$ , $M(\alpha^*, \beta^*)^{-1}A$ and $M(\alpha^*, r)^{-1}A$ are well-conditioned

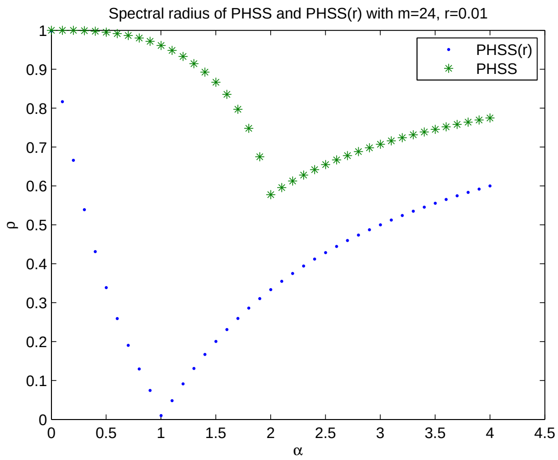


Figure 5: $\alpha$ versus $\rho(\mathcal{L}(\alpha))$ when $\mu = 1, m = 24$ and $r = 0.01$.

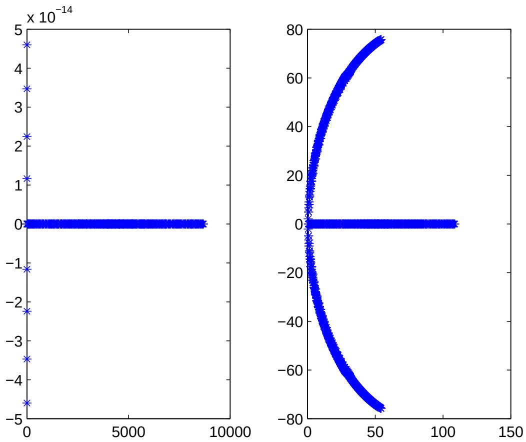


Figure 6: Spectrum of the original matrix $A$ for Example 4.1: $m$=32, $\mu = 1$ (left); and $m$=32, $\mu = \frac{1}{80}$ (right).

because their spectra have considerably smaller ranges along both the real and the imaginary axis and are also clustered and far away from the origin. These observations imply that when PHSS($\alpha^*, r$), PHSS($\alpha^*$) and AHSS($\alpha^*, \beta^*$)are employed to preconditioned GMRES($\ell$) and GMRES, the numerical behaviors of the resulting methods can be improved considerably. In addition, the PHSS($\alpha^*, r$)-preconditioned will perform much better than the AHSS($\alpha^*, \beta^*$)-preconditioned GMRES($\ell$) (or GMRES) and PHSS($\alpha^*$)-preconditioned GMRES($\ell$) (or GMRES). These facts are further confirmed by the numerical results listed in Tables 2 and 3.

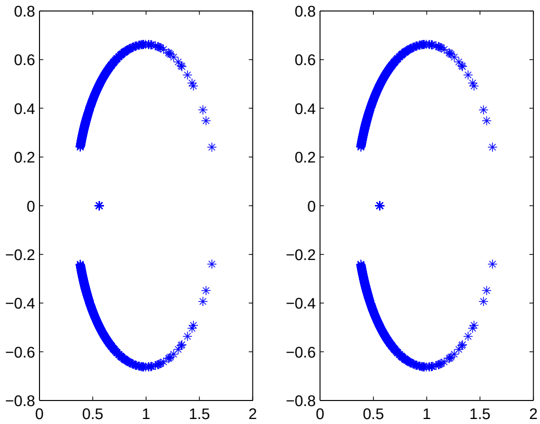


Figure 7: Spectrum of the preconditioned matrix $M(\alpha^*)^{-1}A$ by PHSS for Example 4.1: $m$=32, $\mu = 1$ (left); and $m$=32, $\mu = \frac{1}{80}$ (right).

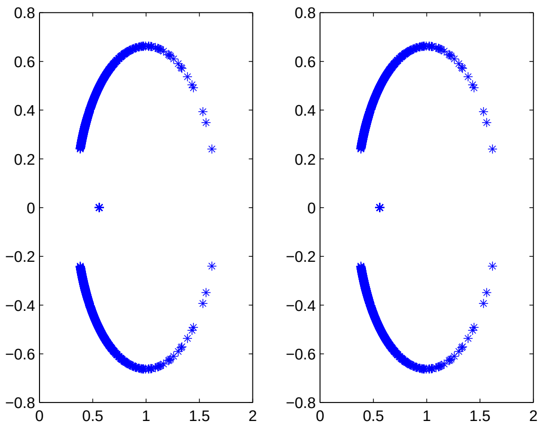


Figure 8: Spectrum of the preconditioned matrix $M(\alpha^*,\beta^*)^{-1}A$ by AHSS for Example 4.1: $m$=32, $\mu = 1$ (left); and $m$=32, $\mu = \frac{1}{80}$ (right).

## 5. Conclusion

For large sparse saddle-point problems, we have presented a class of very effective splitting iteration schemes, called the PHSS(r) iteration methods. Theoretically, we have proved the convergence for this method. We have confirmed numerically that PHSS(r) always performs much better than PHSS and AHSS both as solver and as a preconditioner. Therefore, the PHSS(r) is a very powerful and attractive iterative method for solving large sparse saddle-point problems.

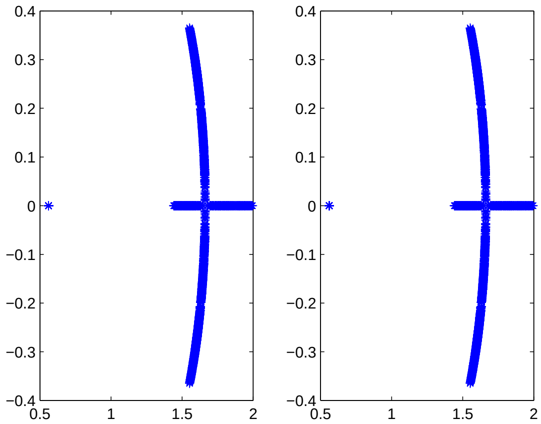


Figure 9: Spectrum of the preconditioned matrix $M(\alpha^*,r_1)^{-1}A$ by PHSS(r) for Example 4.1: $m$=32, $\mu=1$ (left); and $m$=32, $\mu=\frac{1}{80}$ (right).

Table 2: IT and CPU for Example 4.1 ($\mu=1$).

| Method | $m$ | 8 | 16 | 24 | 32 | 48 |
|---|---|---|---|---|---|---|
| PHSS( $\alpha^*$) | IT | 21 | 31 | 38 | 45 | 56 |
| | CPU | 0.071 | 0.634 | 2.309 | 6.633 | 45.929 |
| AHSS($\alpha^*,\beta^*$) | IT | 19 | 26 | 32 | 36 | 44 |
| | CPU | 0.062 | 0.508 | 2.024 | 5.400 | 36.513 |
| PHSS($\alpha^*,r_2$) | IT | 4 | 4 | 4 | 4 | 4 |
| | CPU | **0.024** | **0.088** | **0.283** | **0.763** | **4.131** |
| GMRES | IT | 40 | 80 | 120 | 160 | 240 |
| | CPU | — | — | — | — | — |
| | RES | $3.7\times10^{-04}$ | $3.9\times10^{-04}$ | $4.4\times10^{-04}$ | $5.2\times10^{-03}$ | $2.7\times10^{-03}$ |
| PHSS-GMRES | IT | 22 | 31 | 39 | 45 | 55 |
| | CPU | 0.488 | 3.533 | 13.382 | 43.492 | 136.743 |
| AHSS-GMRES | IT | 19 | 26 | 32 | 36 | 44 |
| | CPU | 0.483 | 1.731 | 5.874 | 15.266 | 107.437 |
| PHSS($r_2$)-GMRES | IT | 4 | 4 | 4 | 4 | 3 |
| | CPU | **0.112** | **0.274** | **0.788** | **2.005** | **8.317** |
| PHSS-GMRES(10) | IT | 22 | 31 | 39 | 45 | 55 |
| | CPU | 0.514 | 3.436 | 19.010 | 62.987 | 145.610 |
| AHSS-GMRES(10) | IT | 19 | 26 | 32 | 36 | 44 |
| | CPU | 0.266 | 1.384 | 5.517 | 15.626 | 107.180 |
| PHSS($r_2$)-GMRES(10) | IT | 4 | 4 | 4 | 4 | 3 |
| | CPU | **0.112** | **0.272** | **0.793** | **2.002** | **8.267** |

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Table 3: IT and CPU for Example 4.1 ($\mu = 1/80$).

| Method | $m$ | 8 | 16 | 24 | 32 | 48 |
|---|---|---|---|---|---|---|
| PHSS( $\alpha^*$) | IT | 21 | 33 | 40 | 46 | 56 |
| | CPU | 0.069 | 0.892 | 3.080 | 6.958 | 46.096 |
| AHSS($\alpha^*, \beta^*$) | IT | 20 | 27 | 32 | 36 | 44 |
| | CPU | 0.0751 | 0.518 | 1.962 | 5.621 | 36.482 |
| PHSS($\alpha^*, r_2$) | IT | 4 | 4 | 4 | 4 | 4 |
| | CPU | **0.027** | **0.097** | **0.296** | **0.906** | **4.144** |
| GMRES | IT | 40 | 80 | 120 | 160 | 240 |
| | CPU | — | — | — | — | — |
| | RES | 1.9$\times 10^{-02}$ | 1.6$\times 10^{-02}$ | 2.2$\times 10^{-02}$ | 5.6$\times 10^{-02}$ | 3.2$\times 10^{-02}$ |
| PHSS-GMRES | IT | 23 | 33 | 40 | 46 | 56 |
| | CPU | 0.296 | 1.747 | 15.230 | 19.690 | 136.643 |
| AHSS-GMRES | IT | 20 | 27 | 32 | 36 | 44 |
| | CPU | 0.272 | 1.452 | 5.547 | 15.473 | 107.388 |
| PHSS($r_2$)-GMRES | IT | 4 | 4 | 4 | 4 | 4 |
| | CPU | **0.113** | **0.276** | **0.796** | **2.020** | **10.867** |
| PHSS-GMRES(10) | IT | 23 | 33 | 40 | 46 | 56 |
| | CPU | 0.360 | 1.860 | 7.508 | 19.579 | 135.810 |
| AHSS-GMRES(10) | IT | 20 | 27 | 32 | 36 | 44 |
| | CPU | 0.277 | 1.403 | 5.523 | 15.294 | 107.267 |
| PHSS($r_2$)-GMRES(10) | IT | 4 | 4 | 4 | 4 | 4 |
| | CPU | **0.111** | **0.273** | **0.792** | **1.996** | **10.640** |

## References


[1] Z.-Z. Bai, *Structured preconditioners for nonsingular matrices of block two-by-two structures*, Math. Comput., 75 (2006), pp. 791–815.

[2] Z.-Z. Bai and G.H. Golub, *Accelerated Hermitian and skew-Hermitian splitting iteration methods for saddle-point problems*, IMA J. Numer. Anal., 27 (2007), pp. 1–23.

[3] Z.-Z. Bai, G.H. Golub and C.-K. Li, *Optimal parameter in Hermitian and skew-Hermitian splitting method for certain two-by-two block matrices*, SIAM J. Sci. Comput., 28 (2006), pp. 583–603.

[4] Z.-Z. Bai, G.H. Golub, L.-Z. Lu and J.-F. Yin, *Block triangular and skew-Hermitian splitting methods for positive-definite linear systems*, SIAM J. Sci. Comput., 26 (2005), pp. 844–863.

[5] Z.-Z. Bai, G.H. Golub and M.K. Ng, *Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems*, SIAM J. Matrix Anal. Appl., 24 (2003), pp. 603–626.

[6] Z.-Z. Bai, G.H. Golub and M.K. Ng, *On successive-overrelaxation acceleration of the Hermitian and skew-Hermitian splitting iterations*, Numer. Linear Algebra Appl., 14 (2007), pp. 319–335.

[7] Z.-Z. Bai, G.H. Golub and J.-Y. Pan, *Preconditioned Hermitian and skew-Hermitian splitting methods for non-Hermitian positive semidefinite linear systems*, Numer. Math., 98 (2004), pp. 1–32.

[8] Z.-Z. Bai, M.K. Ng and Z.-Q. Wang, *Constraint preconditioners for symmetric indefinite*

*matrices*, SIAM J. Matrix Anal. Appl., 31 (2009), pp. 410–433.
[9] Z.-Z. Bai, B.N. Parlett and Z.-Q. Wang, *On generalized successive overrelaxation methods for augmented linear systems*, Numer. Math., 102 (2005), pp. 1–38.
[10] Z.-Z. Bai and Z.-Q. Wang, *On parameterized inexact Uzawa methods for generalized saddle point problems*, Linear Algebra Appl., 428 (2008), pp. 2900–2932.
[11] M. Benzi and G.H. Golub, *A preconditioner for generalized saddle point problems*, SIAM J. Matrix Anal. Appl., 26 (2004), pp. 20–41.
[12] M. Benzi, G.H. Golub and J. Liesen, *Numerical solution of saddle point problems*, Acta Numer., 14 (2005), pp. 1–137.
[13] J.H. Bramble, J.E. Pasciak and A.V. Vassilev, *Analysis of the inexact Uzawa algorithm for saddle point problem*, SIAM J. Numer. Anal., 34 (1997), pp. 1072–1092.
[14] H.C. Elman and G.H. Golub, *Inexact and preconditioned Uzawa algorithms for saddle point problems*, SIAM J. Mumer. Anal., 31 (1994), pp. 1645–1661.
[15] G.H. Golub and D. Vanderstraeten, *On the preconditioning of matrices with skew-symmetric splittings*, Numer. Algor., 25 (2000), pp. 223–239.
[16] G.H. Golub and C.F. Van Loan, *Matrix Computations, 3rd Edition*, The Johns Hopkins University Press, Baltimore, 1996.
[17] G.H. Golub and A.J. Wathen, *An iteration for indefinite systems and its application to the Navier-Stokes equations*, SIAM J. Sci. Comput., 19 (1998), pp. 530–539.
[18] G.H. Golub, X. Wu and J.-Y. Yuan, *SOR-like methods for augmented systems*, BIT, 41 (2001), pp. 71–85.
[19] N. Huang and C.-F. Ma, *The BGS-Uzawa and BJ-Uzawa iterative methods for solving the saddle point problem*, Appl. Math. Comput., 256 (2015), pp. 94–108.
[20] N. Huang and C.-F. Ma, *The nonlinear inexact Uzawa hybrid algorithms based on one-step Newton method for solving nonlinear saddle-point problems*, Appl. Math. Comput., 270 (2015), pp. 291–311.
[21] N. Huang and C.-F. Ma, *An inexact relaxed DPSS preconditioner for saddle point problem*, Appl. Math. Comput., 265 (2015), pp. 431–447.
[22] Q. Hu and J. Zou, *An Iterative Method with Variable Relaxation Parameters for Saddle-Point Problems*, SIAM J. Matrix Anal. Appl., 23 (2001), pp. 317–338.
[23] A. Klawonn, *An optimal preconditioner for a class of saddle point problems with a penalty term*, SIAM J. Sci. Comput., 19 (1998), pp. 540–552.
[24] J.-F. Lu and Z.-Y. Zhang, *A modified nonlinear inexact Uzawa algorithm with a variable relaxation parameter for the stabilized saddle point problem*, SIAM J. Matrix Anal. Appl., 31 (2010), pp. 1934–1957.
[25] W. Queck, *The convergence factor of preconditioned algorithms of the Arrow-Hurwicz type*, SIAM J. Numer. Anal., 26 (1989), pp. 1016–1030.
[26] Y. Saad, *Iterative Methods for Sparse Linear Systems*, PWS Publishing Company, Boston, 1996.
[27] Y. Saad and M.H. Schultz, *GMRES: A generalized minimal residual method for solving nonsymmetric linear systems*, SIAM J. Sci. Statist. Comput., 7 (1986), pp. 856–869.
[28] V. Simoncini and M. Benzi, *Spectral properties of the Hermitian and skew-Hermitian splitting preconditioner for saddle point problems*, SIAM J. Matrix Anal. Appl., 26 (2004), pp. 377–389.
[29] C.-L. Wang and Z.-Z. Bai, *Sufficient conditions for the convergent splittings of non-Hermitian positive definite matrices*, Linear Algebra Appl., 330 (2001), pp. 215–218.
[30] G.-F. Zhang, J.-L. Yang and S.-S. Wang, *On generalized parameterized inexact Uzawa type method for a block two-by-two linear system*, J. Comput. Appl. Math., 255 (2014), pp.

193–207.

[31] W. ZULEHNER, *Analysis of iterative methods for saddle point problems: a unified approach*, Math. Comput., 71 (2002), pp. 479–505.