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On Weighted L^p- Approximation by Weighted Bernstein-Durrmeyer Operators

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Abstract. In the present paper, we establish direct and converse theorems for weighted Bernstein-Durrmeyer operators under weighted L^p-norm with Jacobi weight $w(x)=x^\alpha(1-x)^\beta$. All the results involved have no restriction $\alpha,\beta<1-\frac{1}{p}$, which indicates that the weighted Bernstein-Durrmeyer operators have some better approximation properties than the usual Bernstein-Durrmeyer operators.

Key Words: Weighted L^p-approximation, weighted Bernstein-Durrmeyer operators, direct and converse theorems.

AMS Subject Classifications: 41A10, 41A25

1 Introduction

Let

$$w(x)=x^\alpha(1-x)^\beta,\quad \alpha,\beta>-1,\quad 0\le x\le 1,$$

be the classical Jacobi weights. Let

$$L_w^p:=\begin{cases}\{f:wf\in L^p(0,1)\}, & 1\le p<\infty,\\ \Big\{f:f\in C(0,1),\ \lim_{x(1-x)\to 0}(wf)(x)=0\Big\}, & p=\infty.\end{cases}$$

Set

$$\|f\|_{p,w,I}=\begin{cases}\left(\int_I|(wf)(x)|^p dx\right)^{1/p}, & 1\le p<\infty,\\ \sup_{x\in I}|(wf)(x)|, & p=\infty.\end{cases}$$

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When $I=[0,1]$, we briefly write $\|f\|_{p,w}$ instead of $\|f\|_{p,w,[0,1]}$. Obviously, $\|f\|_{p,w}$ is the norm of L^p_w spaces.

For any $f\in L^p([0,1])$, $1\le p\le\infty$, the corresponding Bernstein-Durrmeyer operators $M_n(f,x)$ are defined as follows:

$$M_n(f,x)=(n+1)\sum_{k=0}^{n}p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)dt,\quad x\in[0,1],$$

where

$$p_{n,k}(x)=\left(\begin{array}{c}k\\ n\end{array}\right)x^k(1-x)^{n-k},\quad x\in[0,1],\quad k=0,1,\cdots,n.$$

The approximation properties of $M_n(f,x)$ in L^p_w were also studied by Zhang (see [9]). Some approximation results were given under the restrictions

$$-\frac{1}{p}<\alpha,\beta<1-\frac{1}{p}$$

on the weight parameters. Generally speaking, the restrictions can not be eliminated for the approximation by $M_n(f,x)$. For the weighted approximation by Kantorovich-Bernstein operators defined by

$$K_n(f,x):=\sum_{k=0}^{n}(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}f(u)du\,p_{nk}(x),$$

the situation is similar (see [5]). Recently, Della Vecchia, Mastroianni and Szabados (see [2]) introduced a weighted generalization of the $K_n(f,x)$ as follows:

$$K_n^{\#}(f,x):=\sum_{k=0}^{n}\frac{\int_{I_k}(wf)(t)dt}{\int_{I_k}w(t)dt}p_{nk}(x),\quad x\in[0,1]. \tag{1.1}$$

When $\alpha=\beta=0$, $K_n^{\#}(f,x)$ reduces to the classical Kantorovich-Bernstein operator $K_n(f,x)$. Della Vecchia, Mastroianni and Szabados obtained the direct and converse theorems and a Voronovskaya-type relation in [2], and solved the saturation problem of the operator in [3]. Their results showed that $K_n^{\#}(f,x)$ allows a wider class of functions than the operator $K_n(f,x)$. In fact, they dropped the restrictions $\alpha,\beta<1-\frac{1}{p}$ on the weight parameters. Later, Yu (see [8]) introduced another kind of modified Bernstein-Kantorvich operators, and established direct and converse results on weighted approximation which also have no restrictions $\alpha,\beta<1-\frac{1}{p}$.

Then, a natural question is: can we modified the Bernstein-Durrmeyer operators properly such that the restrictions $\alpha,\beta<1-\frac{1}{p}$ on weighted approximation can be dropped? In the present paper, we will show that the weighted Bernstein-Durrmeyer operator introduced by Berens and Xu (see [1]) is the one we need. The weighted Bernstein-Durrmeyer operator is defined as follows:

$$M_n^*(f,x)=\sum_{k=0}^{n} p_{n,k}(x)C_{n,k}\int_0^1 p_{n,k}(t)w(t)f(t)dt, \quad x\in[0,1], \quad f\in L_w^1,$$

where

$$C_{n,k}=\left(\int_0^1 p_{n,k}(t)w(t)dt\right)^{-1}, \quad k=0,1,\cdots,n.$$

Define

$$W_w^{2,p}:=\left\{f\in L_w^p: f'\in AC(0,1),\ \|\varphi^2 f''\|_{p,w}<\infty\right\},$$

where $\varphi(x)=\sqrt{x(1-x)}$, and $AC(I)$ is the set of all absolutely continuous functions on I. For $f\in L_w^p$, define the weighted modulus of smoothness by

$$\omega_\varphi^2(f,t)_{p,w}:=\sup_{0<h\le t}\left\{\left\|\Delta_{h\varphi}^2 f\right\|_{p,w,[Ch^2,1-Ch^2]}+\left\|\overrightarrow{\Delta}_h^2 f\right\|_{p,w,[0,Ch^2]}+\left\|\overleftarrow{\Delta}_h^2 f\right\|_{p,w,[1-Ch^2,1]}\right\},$$

with

$$\begin{aligned}
&\Delta_{h\varphi}^2 f(x)=f\left(x+h\frac{\varphi(x)}{2}\right)-2f(x)+f\left(x-h\frac{\varphi(x)}{2}\right),\\
&\overrightarrow{\Delta}_h^2 f(x)=f(x+2h)-2f(x+h)+f(x),\\
&\overleftarrow{\Delta}_h^2 f(x)=f(x)-2f(x-h)+f(x-2h).
\end{aligned}$$

Define

$$E_0(f)_{p,w}:=\inf_{C\in\mathbf{R}}\|f-C\|_{p,w}$$

to be the best approximation of f in weighted L_w^p spaces by constants.

The main results of the present paper are the following:

Theorem 1.1. *If $f\in L_w^p$, $1\le p\le\infty$, then*

$$\|f-M_n^*(f)\|_{p,w}\le C\left(\omega_\varphi^2\left(f,\frac{1}{\sqrt{n}}\right)_{p,w}+\frac{E_0(f)_{p,w}}{n}\right). \tag{1.2}$$

Theorem 1.2. *If $f\in L_w^p$, $1\le p\le\infty$, then*

$$\|f-M_n^*(f)\|_{p,w}=\mathcal{O}\left(n^{-\gamma/2}\right)\Leftrightarrow\omega_\varphi^2(f,h)_{p,w}=\mathcal{O}(h^\gamma), \quad 0<\gamma<2. \tag{1.3}$$

2 Auxiliary lemmas

We need the following inequalities:

$$\int_0^1 p_{n,k}(x)dx=\frac{1}{n+1}, \quad \text{(see [7])}, \tag{2.1a}$$

$$\sum_{k=0}^{n}\left(\frac{k^*}{n}\right)^{-u}\left(1-\frac{k^*}{n}\right)^{-v} p_{n,k}(x)\le Cx^{-u}(1-x)^{-v}, \quad u,v\ge 0, \quad \text{(see [8])}, \tag{2.1b}$$

$$\sum_{k=0}^{n} p_{n,k}(x)\left|\frac{k}{n}-x\right|^{\gamma}\le Cn^{-\frac{1}{2}}\varphi^{\gamma}(x), \quad \gamma\ge 0, \quad \text{(see [7])}, \tag{2.1c}$$

$$\int_0^1 w^p(x)\varphi^{-2}(x)p_{n,k}(x)|k-nx|^2dx\le Cw^p\left(\frac{k^*}{n}\right), \quad \text{(see [2])}, \tag{2.1d}$$

where

$$k^*:=\begin{cases}1, & k=0,\\ k, & 1\le k\le n-1,\\ n-1, & k=n.\end{cases} \tag{2.2}$$

It should be noted that (2.1d) is contained in the first inequality of [2, pp. 9].

Lemma 2.1. *For $1\le p<\infty$, $0\le k\le n$ and $n\ge 3$, we have*

$$\int_0^1 w^p(x)p_{n,k}(x)dx\sim n^{-1}w^p\left(\frac{k^*}{n}\right), \tag{2.3}$$

where k^* is defined by (2.2).

Proof. By the fact that (see [3])

$$\frac{\Gamma(n+\alpha)}{n^{\alpha}\Gamma(n)}=1+\mathcal{O}\left(\frac{1}{n}\right), \quad \alpha>-1,$$

we deduce that

$$\begin{aligned}&\int_0^1 w^p(x)p_{n,k}(x)dx\\ =&\begin{pmatrix} n\\ k\end{pmatrix}\int_0^1 x^{k+\alpha p}(1-x)^{n-k+\beta p}dx\\ =&\begin{pmatrix} n\\ k\end{pmatrix}\frac{\Gamma(k+\alpha p+1)\Gamma(n-k+\beta p+1)}{\Gamma(n+\alpha p+\beta p+2)}\\ =&\frac{(n+1)^{\alpha p+\beta p+1}\Gamma(n+1)}{\Gamma(n+\alpha p+\beta p+2)}\frac{\Gamma(k+\alpha p+1)}{(k+1)^{\alpha p}\Gamma(k+1)}\end{aligned}$$

$$
\begin{aligned}
&\times\frac{\Gamma(n-k+\beta p+1)}{(n-k+1)^{\beta p}\Gamma(n-k+1)}\frac{(k+1)^{\alpha p}(n-k+1)^{\beta p}}{(n+1)^{\alpha p+\beta p+1}}\\
=&\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\frac{(k+1)^{\alpha p}(n-k+1)^{\beta p}}{(n+1)^{\alpha p+\beta p+1}}\\
\sim&n^{-1}w^p\left(\frac{k^*}{n}\right).
\end{aligned}
$$

Thus, we complete the proof. □

Especially, by taking $p=1$ in (2.3), we get

$$C_{n,k}^{-1}=\int_0^1 w(x)p_{n,k}(x)dx\sim n^{-1}w\left(\frac{k^*}{n}\right),\quad k=0,1,\cdots,n. \tag{2.4}$$

Lemma 2.2. *For any $f\in L_w^p$, $1\le p\le\infty$, we have*

$$\|M_n^*(f)\|_{p,w}\le C\|f\|_{p,w}.$$

Proof. When $p=\infty$, by (2.1a), (2.4) and (2.1b), we get

$$
\begin{aligned}
|w(x)M_n^*(f,x)|\le&w(x)\sum_{k=0}^n p_{n,k}(x)C_{n,k}\int_0^1 p_{n,k}(t)|w(t)f(t)|dt\\
\le&\|f\|_{\infty,w}w(x)\sum_{k=0}^n p_{n,k}(x)C_{n,k}\int_0^1 p_{n,k}(t)dt\\
\le&\|f\|_{\infty,w}w(x)\sum_{k=0}^n p_{n,k}(x)w^{-1}\left(\frac{k^*}{n}\right)\\
=&C\|f\|_{\infty,w}.
\end{aligned}\tag{2.5}
$$

When $1<p<\infty$, by using Hölder's inequality, (2.4) and (2.3), we have

$$
\begin{aligned}
\|M_n^*(f)\|_{p,w}^p\le&\int_0^1 w^p(x)\left(\sum_{k=0}^n p_{n,k}(x)C_{n,k}^p\left|\int_0^1 p_{n,k}(t)w(t)f(t)dt\right|^p\right)\left(\sum_{k=0}^n p_{n,k}(x)\right)^{p-1}dx\\
\le&\frac{1}{(n+1)^{p-1}}\int_0^1 w^p(x)\sum_{k=0}^n p_{n,k}(x)C_{n,k}^p\int_0^1 p_{n,k}(t)|w(t)f(t)|^p dtdx\\
\le&Cn\sum_{k=0}^n\left(\int_0^1 w^p(x)p_{n,k}(x)dx\right)w^{-p}\left(\frac{k^*}{n}\right)\int_0^1 p_{n,k}(t)|w(t)f(t)|^p dt\\
\le&C\sum_{k=0}^n\int_0^1 p_{n,k}(t)|w(t)f(t)|^p dt\\
=&C\|f\|_{p,w}^p.
\end{aligned}\tag{2.6}
$$

By a similar and more simpler deduction, we see that (2.6) also holds for $p=1$.

Combining (2.5) and (2.6), Lemma 2.2 is proved. □

Lemma 2.3. *If $f\in W_w^{2,p}$, then*

$$\|\varphi^2 M_n^{*\prime\prime}(f)\|_{p,w}\leq C\|\varphi^2 f''\|_{p,w},\quad 1\leq p\leq\infty.$$

Proof. Direct calculations yield that (see [4, pp. 331-332]),

$$\begin{aligned}
&M_n^{*\prime\prime}(f,x)\\
=&\frac{n!}{(n-2)!}\sum_{j=0}^{2}\left(\begin{array}{c}2\\ j\end{array}\right)\sum_{k=j}^{n-2+j}p_{n-2,k-j}(x)(-1)^{2-j}C_{n,k}\int_0^1 p_{n,k}(t)w(t)f(t)dt\\
=&\frac{n!}{(n-2)!}\sum_{k=0}^{n-2}p_{n-2,k}(x)\int_0^1\sum_{j=0}^{2}\left(\begin{array}{c}2\\ j\end{array}\right)(-1)^{2-j}C_{n,k+j}p_{n,k+j}(t)w(t)f(t)dt\\
=&\frac{n!\Gamma(n+\alpha+\beta+1)}{(n-2)!\Gamma(n+\alpha+\beta+3)}\sum_{k=0}^{n-2}p_{n-2,k}(x)C_{n+2,k+2}\int_0^1\frac{d^2p_{n+2,k+2}(t)}{dt^2}w(t)f(t)dt. \qquad (2.7)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\varphi^2M_n^{*\prime\prime}(f)\|_{p,w}\leq& C\left\|\sum_{k=0}^{n-2}p_{n,k+1}(x)C_{n+2,k+2}\int_0^1 p_{n,k+1}(t)w(t)\varphi^2(t)f''(t)dt\right\|_{p,w}\\
\leq& C\left\|\sum_{k=0}^{n-2}p_{n,k+1}(x)C_{n,k+1}\int_0^1 p_{n,k+1}(t)w(t)|\varphi^2(t)f''(t)|dt\right\|_{p,w}\\
\leq& C\left\|\sum_{k=0}^{n}p_{n,k}(x)C_{n,k}\int_0^1 p_{n,k}(t)w(t)|\varphi^2(t)f''(t)|dt\right\|_{p,w}.
\end{aligned}$$

For $p=\infty$, by (2.1a), (2.1b) and (2.1c), we have

$$\begin{aligned}
|w(x)\varphi^2(x)M_n^{*\prime\prime}(f,x)|\leq& w(x)\sum_{k=0}^{n}p_{n,k}(x)C_{n,k}\int_0^1 p_{n,k}(t)w(t)\left|\varphi^2(t)f''(t)\right|dt\\
\leq&\|w\varphi^2f''\|w(x)\sum_{k=0}^{n}p_{n,k}(x)w^{-1}\left(\frac{k^*}{n}\right)\\
\leq& C\|\varphi^2f''\|_{\infty,w}. \qquad (2.8)
\end{aligned}$$

For $1<p<\infty$ (for $p=1$, it can be treated similarly and more simpler), by using Hölder's inequality, (2.1a), (2.3) and (2.4), we have

$$\begin{aligned}
&\|\varphi^2M_n^{*\prime\prime}f\|_{p,w}^p\\
\leq& C\int_0^1\sum_{k=0}^{n}p_{n,k}(x)C_{n,k}^p\left|\int_0^1 p_{n,k}(t)w(t)\left|\varphi^2(t)f''(t)\right|dt\right|^p w^p(x)dx\\
\leq& C\int_0^1\sum_{k=0}^{n}p_{n,k}(x)C_{n,k}^p\int_0^1 p_{n,k}(t)\left|w(t)\varphi^2(t)f''(t)\right|^p dt\left(\int_0^1 p_{n,k}(t)dt\right)^{p-1}w^p(x)dx
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{C}{(n+1)^{p-1}}\sum_{k=0}^{n}\left(\int_0^1 p_{n,k}(x)w^p(x)dx\right)C_{n,k}^{p}\int_0^1 p_{n,k}(t)\left|w(t)\varphi^2(t)f''(t)\right|^p dt\\
&\le C\sum_{k=0}^{n}\int_0^1 p_{n,k}(t)\left|w(t)\varphi^2(t)f''(t)\right|^p dt\\
&= C\|\varphi^2 f''\|_{p,w}^p. \qquad (2.9)
\end{aligned}
$$

We finish Lemma 2.3 by combining (2.8) and (2.9). □

Lemma 2.4. *If $f\in L_w^p$, then*

$$
\|\varphi^2 M_n^{*\prime\prime}(f)\|_{p,w}\le Cn\|f\|_{p,w},\quad 1\le p\le\infty.
$$

Proof. We prove the result by estimating the integral on two intervals $E_n=\left[\frac{1}{n},1-\frac{1}{n}\right]$ and $E_n^c=[0,1]\ E_n$ respectively.

Simple calculation leads to

$$
\begin{aligned}
&\varphi^2(x)M_n^{*\prime\prime}(f,x)\\
=&\frac{n^2}{\varphi^2(x)}\sum_{k=0}^{n}p_{n,k}(x)\left(\frac{k}{n}-x\right)^2 C_{n,k}\int_0^1 p_{n,k}(t)w(t)f(t)dt-nM_n^*(f,x)\\
&-\frac{d}{dx}\left(\varphi^2(x)\right)\frac{n}{\varphi^2(x)}\sum_{k=0}^{n}p_{n,k}(x)\left(\frac{k}{n}-x\right)C_{n,k}\int_0^1 p_{n,k}(t)w(t)f(t)dt\\
=:&I_1(n,x)+I_2(n,x)+I_3(n,x). \qquad (2.10)
\end{aligned}
$$

For $I_1(n,x)$, when $p=\infty$, by applying (2.1a)-(2.1c), (2.4) and Cauchy's inequality, we have

$$
\begin{aligned}
|w(x)I_1(n,x)|&\le n^2\frac{w(x)}{\varphi^2(x)}\sum_{k=0}^{n}p_{n,k}(x)\left(\frac{k}{n}-x\right)^2 C_{n,k}\left|\int_0^1 p_{n,k}(t)w(t)f(t)dt\right|\\
&\le n^2\|f\|_{\infty,w}\frac{w(x)}{\varphi^2(x)}\sum_{k=0}^{n}p_{n,k}(x)\left(\frac{k}{n}-x\right)^2 C_{n,k}\int_0^1 p_{n,k}(t)dt\\
&\le Cn^2\|f\|_{\infty,w}\frac{w(x)}{\varphi^2(x)}\sum_{k=0}^{n}p_{n,k}(x)\left(\frac{k}{n}-x\right)^2 w^{-1}\left(\frac{k^*}{n}\right)\\
&\le Cn^2\|f\|_{\infty,w}\frac{w(x)}{\varphi^2(x)}\left(\sum_{k=0}^{n}p_{n,k}(x)\left(\frac{k}{n}-x\right)^4\right)^{\frac12}\left(\sum_{k=0}^{n}p_{n,k}(x)w^{-2}\left(\frac{k^*}{n}\right)\right)^{\frac12}\\
&\le Cn^2\|f\|_{\infty,w}\frac{w(x)}{\varphi^2(x)}\left(n^{-2}\varphi^4(x)\right)^{\frac12}\left(w^{-2}(x)\right)^{\frac12}\\
&= Cn\|f\|_{\infty,w}. \qquad (2.11)
\end{aligned}
$$

When $1\le p<\infty$, by using Hölder's inequality twice for $p>1$ ($p=1$ is more direct), (2.1c), (2.1d), (2.4) and (2.1a),

$$\begin{aligned}
&\int_0^1 |w(x)I_1(n,x)|^p dx\\
\le&\int_0^1 n^{2p}\frac{w^p(x)}{\varphi^{2p}(x)}\left(\sum_{k=0}^n p_{n,k}(x)\left(\frac{k}{n}-x\right)^2\right)^{p-1}\sum_{k=0}^n p_{n,k}(x)\left(\frac{k}{n}-x\right)^2 C_{n,k}^p\\
&\quad\times\left(\int_0^1 p_{n,k}(t)|w(t)f(t)|dt\right)^p dx\\
\le& Cn^{p+1}\int_0^1 w^p(x)\varphi^{-2}(x)\sum_{k=0}^n p_{n,k}(x)\left(\frac{k}{n}-x\right)^2 C_{n,k}^p\left(\int_0^1 p_{n,k}(t)w(t)f(t)dt\right)^p dx\\
\le& Cn^{p+1}\sum_{k=0}^n\int_0^1 w^p(x)\varphi^{-2}(x)p_{n,k}(x)\left(\frac{k}{n}-x\right)^2 dx C_{n,k}^p\left(\int_0^1 p_{n,k}(t)w(t)f(t)dt\right)^p\\
\le& Cn^{p+1}\sum_{k=0}^n n^{-2}w^p\left(\frac{k^*}{n}\right)C_{n,k}^p\left(\int_0^1 p_{n,k}(t)w(t)f(t)dt\right)^p\\
\le& Cn^{2p-1}\sum_{k=0}^n\int_0^1 p_{n,k}(t)|w(t)f(t)|^p dt\left(\int_0^1 p_{n,k}(t)dt\right)^{p-1}\\
\le& Cn^p\|f\|_{p,w}^p.
\end{aligned}\tag{2.12}$$

For $I_2(n,x)$, by Lemma 2.2, we have

$$|I_2\|_{p,w}\le Cn\|f\|_{p,w},\quad 1\le p\le\infty.\tag{2.13}$$

For $I_3(n,x)$, when $p=\infty$, by (2.1a)-(2.1d) and (2.4), we have

$$\begin{aligned}
|w(x)I_3(n,x)|\le& n\frac{w(x)}{\varphi^2(x)}\sum_{k=0}^n p_{n,k}(x)\left|\frac{k}{n}-x\right|C_{n,k}\int_0^1 p_{n,k}(t)|w(t)f(t)|dt\\
\le&\|f\|_{\infty,w}\frac{w(x)}{\varphi^2(x)}\sum_{k=0}^n p_{n,k}(x)\left|\frac{k}{n}-x\right|C_{n,k}\\
\le&\|f\|_{\infty,w}\frac{w(x)}{\varphi^2(x)}\left(\sum_{k=0}^n p_{n,k}(x)\left(\frac{k}{n}-x\right)^2\right)^{\frac12}\left(\sum_{k=0}^n p_{n,k}(x)C_{n,k}^2\right)^{\frac12}\\
\le& C\|f\|_{\infty,w}\frac{n^{\frac12}}{\varphi(x)}\le Cn\|f\|_{\infty,w},
\end{aligned}\tag{2.14}$$

where in the last inequality, we used the fact $1/n^{\frac12}\varphi(x)\le C$, $x\in E_n$.

When $1\le p<\infty$, by using Hölder's inequality, (2.1a), (2.1c), (2.1d), (2.4), and the fact

$1/n^{\frac{1}{2}}\varphi(x)\le C$, $x\in E_n$ again, we deduce that

$$\begin{aligned}
&\int_{E_n}|w(x)I_3(n,x)|^p dx\\
\le&\int_{E_n}w^p(x)\frac{n^p}{\varphi^{2p}(x)}\left|\sum_{k=0}^n p_{n,k}(x)\left(\frac{k}{n}-x\right)C_{n,k}\int_0^1 p_{n,k}(t)w(t)f(t)dt\right|^p dx \qquad (2.15)\\
\le&\int_{E_n}w^p(x)\frac{n^p}{\varphi^{2p}(x)}\left(\sum_{k=0}^n p_{n,k}(x)\left|\frac{k}{n}-x\right|\right)^{p-1}\\
&\times\sum_{k=0}^n p_{n,k}(x)\left|\frac{k}{n}-x\right|C_{n,k}^p\left|\int_0^1 P_{n,k}(t)w(t)f(t)dt\right|^p dx\\
\le&C\int_{E_n}w^p(x)\frac{n^p}{\varphi^{2p}(x)}n^{-\frac{1}{2}(p-1)}\varphi^{p-1}(x)\sum_{k=0}^n p_{n,k}(x)\left|\frac{k}{n}-x\right|\\
&\times C_{n,k}^p\int_0^1 p_{n,k}(t)|w(t)f(t)|^p dt\left(\int_0^1 p_{n,k}(t)\right)^{p-1}dx\\
\le&C\int_{E_n}w^p(x)\frac{n^{-\frac{1}{2}p+\frac{3}{2}}}{\varphi^{p+1}(x)}\sum_{k=0}^n p_{n,k}(x)\left|\frac{k}{n}-x\right|C_{n,k}^p\int_0^1 p_{n,k}(t)|w(t)f(t)|^p dtdx\\
\le&Cn\sum_{k=0}^n C_{n,k}^p\left(\int_{E_n}w^p(x)\varphi^{-2}(x)p_{n,k}(x)\left(\frac{k}{n}-x\right)^2dx\right)^{\frac{1}{2}}\\
&\times\left(\int_{E_n}w^p(x)\varphi^{-2}(x)p_{n,k}(x)dx\right)^{\frac{1}{2}}\int_0^1 p_{n,k}(t)|w(t)f(t)|^p dt\\
\le&Cn^{3/2}\sum_{k=0}^n C_{n,k}^p\left(n^{-2}w^p\left(\frac{k^*}{n}\right)\right)^{\frac{1}{2}}\left(\int_{E_n}w^p(x)p_{n,k}(x)dx\right)^{\frac{1}{2}}\int_0^1 p_{n,k}(t)|w(t)f(t)|^p dt\\
\le&Cn^p\sum_{k=0}^n\int_0^1 p_{n,k}(t)|w(t)f(t)|^p dt\\
=&Cn^p\|f\|_{p,w}^p. \qquad (2.16)
\end{aligned}$$

By combining (2.10)-(2.15), we already have

$$\|\varphi^2 M_n^{*\prime\prime}(f)\|_{p,w,E_n}\le Cn\|f\|_{p,w},\quad 1\le p\le\infty. \tag{2.17}$$

Now, we estimate the integral on E_n^c. By (2.7), we have

$$M_n^{*\prime\prime}(f,x)=\frac{n!}{(n-2)!}\sum_{k=0}^{n-2}p_{n-2,k}(x)\int_0^1\sum_{j=0}^2\binom{2}{j}(-1)^{2-j}C_{n,k+j}p_{n,k+j}(t)w(t)f(t)dt.$$

When $1\le p<\infty$, noting that $n\varphi^2(x)\le C$ for $x\in E_n^c$, by Hölder's inequality, (2.3) and (2.4), we have

$$
\begin{aligned}
&\|\varphi^2 M_n^{*\prime\prime} f\|_{p,w,E_n^c}^p \\
\le&\int_{E_n^c}\left|w(x)\varphi^2(x)n^2\sum_{k=0}^{n-2}p_{n-2,k}(x)\int_0^1\sum_{j=0}^{2}\tbinom{2}{j}(-1)^{2-j}C_{n,k+j}p_{n,k+j}(t)w(t)f(t)dt\right|^p dx\\
\le& Cn^{2p}\sum_{j=0}^{2}\int_{E_n^c}\left|w(x)\varphi^2(x)\sum_{k=0}^{n-2}p_{n-2,k}(x)C_{n,k+j}\int_0^1 p_{n,k+j}(t)w(t)f(t)dt\right|^p dx\\
\le& Cn^{2p}\sum_{j=0}^{2}\int_{E_n^c}w^p(x)\varphi^{2p}(x)\sum_{k=0}^{n-2}p_{n-2,k}(x)C_{n,k+j}^p\left|\int_0^1 p_{n,k+j}(t)w(t)f(t)dt\right|^p dx\\
\le& Cn^{2p}\sum_{j=0}^{2}\int_{E_n^c}w^p(x)n^{-p}\sum_{k=0}^{n-2}p_{n-2,k}(x)n^p w^{-p}\left(\frac{k^*}{n}\right)\\
&\quad\times\int_0^1 p_{n,k+j}(t)|w(t)f(t)|^p dt\left(\int_0^1 p_{n,k+j}(t)dt\right)^{p-1}dx\\
\le& Cn^{p+1}\sum_{j=0}^{2}\sum_{k=0}^{n-2}w^{-p}\left(\frac{k^*}{n}\right)\int_{E_n^c}w^p(x)p_{n-2,k}(x)dx\int_0^1 p_{n,k+j}(t)|w(t)f(t)|^p dt\\
\le& Cn^{p+1}\sum_{j=0}^{2}\sum_{k=0}^{n-2}n^{-1}w^p\left(\frac{k^*}{n-2}\right)w^{-p}\left(\frac{k^*}{n}\right)\int_0^1 p_{n,k+j}(t)|w(t)f(t)|^p dt\\
\le& Cn^p\int_0^1\sum_{j=0}^{2}\sum_{k=0}^{n-2}p_{n,k+j}(t)|w(t)f(t)|^p dt\\
\le& Cn^p\|f\|_{p,w}^p.
\end{aligned}
\tag{2.18}
$$

When $p=\infty$, for $x\in E_n^c$, by (2.1a) and (2.1b),

$$
\begin{aligned}
&|w(x)\varphi^2(x)M_n^{*\prime\prime}(f,x)|\\
\le& Cn^2\|f\|_{\infty,w}\sum_{j=0}^{2}\sum_{k=0}^{n-2}w(x)\varphi^2(x)p_{n-2,k}(x)C_{n,k+j}\int_0^1 p_{n,k+j}(t)dt\\
\le& C\|f\|_{\infty,w}\sum_{j=0}^{2}\sum_{k=0}^{n-2}w(x)p_{n-2,k}(x)C_{n,k+j}\\
\le& C\|f\|_{\infty,w}w(x)\sum_{k=0}^{n-2}p_{n-2,k}(x)nw^{-1}\left(\frac{k^*}{n-2}\right)\\
\le& Cn\|f\|_{\infty,w}.
\end{aligned}
\tag{2.19}
$$

By (2.18) and (2.19), we see that

$$
\|\varphi^2 M_n^{*\prime\prime}(f)\|_{p,w,E_n^c}\le Cn\|f\|_{p,w},\quad 1\le p\le\infty. \tag{2.20}
$$

By (2.17) and (2.20), we complete the proof of Lemma 2.4. $\square$

Lemma 2.5. *For any nonnegative integer m, set*

$$T_{n,m}(x):=M_n^*\left((x-t)^m,x\right)=\sum_{k=0}^{n}p_{nk}(x)C_{n,k}\int_0^1 p_{nk}(t)w(t)(x-t)^m dt.$$

Then

$$T_{n,2m}=\sum_{i=0}^{m}p_{i,m,n,\alpha,\beta}(x)\left(\frac{\varphi^2(x)}{n}\right)^{m-i}n^{-2i} \tag{2.21}$$

and

$$T_{n,2m-1}=\sum_{i=0}^{m-1}q_{i,m,n,\alpha,\beta}(x)\left(\frac{\varphi^2(x)}{n}\right)^{m-i-1}n^{-2i+1}, \tag{2.22}$$

where $p_{i,m,n,\alpha,\beta}(x)$ are polynomials in x of fixed degree with coefficients that are bounded uniformly for all n.

Proof. Analogue to [4], we have the recursion relation:

$$\begin{aligned}(n+m+2)T_{n,m+1}(x)=&x(1-x)\left(2mT_{n,m-1}(x)-T'_{nm}(x)\right)\\&-(1-2x)(m+1)T_{nm}(x).\end{aligned} \tag{2.23}$$

Direct calculations yield that

$$T_{n0}(x)=1, \tag{2.24}$$

and

$$\begin{aligned}T_{n1}(x)&=\sum_{k=0}^{n}p_{nk}(x)C_{n,k}\int_0^1 w(t)p_{nk}(t)tdt-x\\&=\sum_{k=0}^{n}\frac{B((k+\alpha+1)+1,n-k+\beta+1)}{B(k+\alpha+1,n-k+\beta+1)}p_{nk}(x)-x\\&=\sum_{k=0}^{n}\frac{k+\alpha+1}{n+\alpha+\beta+2}p_{nk}(x)-x\\&=\frac{nx+\alpha+1}{n+\alpha+\beta+2}-x\\&=\frac{\alpha+1-(\alpha+\beta+2)x}{n+\alpha+\beta+2},\end{aligned} \tag{2.25}$$

where $B(p,q):=\int_0^1 x^{p-1}(1-x)^{q-1}dx$, $p,q>0$.

By (2.23)-(2.25) and a simple induction process, we obtain (2.21) and (2.22). $\square$

By (2.21), we have

Lemma 2.6. *For any given m, it holds that*

$$T_{n,2m}(x)\le Cn^{-m}\left(\varphi^2(x)+\frac{1}{n}\right)^m. \tag{2.26}$$

Lemma 2.7. *For $1\le p<\infty$, $f\in W_w^{2,p}$, there is a positive constant C such that*

$$\|M_n^*(R_2(f,t,x),x)\|_{p,w,E_n}\le\frac{C}{n}\|\varphi^2(x)f''\|_{p,w}, \tag{2.27}$$

where

$$R_2(f,t,x),x):=\int_x^t(t-v)f''(v)dv.$$

Proof. Firstly, we consider the case $p=1$. Set $g(v)=w(v)\varphi^2(v)f''(v)$. By the inequality (see [5]):

$$\frac{|t-u|}{\varphi^2(u)}\le\frac{|t-x|}{\varphi^2(x)}\quad\text{for any } u \text{ between } x \text{ and } t,$$

we have

$$\begin{aligned}
&\int_{E_n}w(x)|M_n^*(R_2(f,t,x),x)|dx\\
\le&\int_{E_n}w(x)\sum_{k=0}^n p_{n,k}(x)C_{n,k}\int_0^1 p_{n,k}(t)w(t)\left|\int_x^t|t-v||f''(v)|dv\right|dtdx\\
\le&\int_{E_n}\frac{w(x)}{\varphi^2(x)}\sum_{k=0}^n p_{n,k}(x)C_{n,k}\int_0^1 p_{n,k}(t)w(t)\left|\int_x^t g(v)dv\right||t-x|\left(\frac{1}{w(x)}+\frac{1}{w(t)}\right)dtdx\\
\le&\int_{E_n}\varphi^{-2}(x)\sum_{k=0}^n p_{n,k}(x)C_{n,k}\int_0^1 p_{n,k}(t)w(t)\left|\int_x^t g(v)dv\right||t-x|dtdx\\
&\quad+\int_{E_n}w(x)\varphi^{-2}(x)\sum_{k=0}^n p_{n,k}(x)C_{n,k}\int_0^1 p_{n,k}(t)\left|\int_x^t g(v)dv\right||t-x|dtdx\\
=:&I_1+I_2.
\end{aligned} \tag{2.28}$$

Set

$$\begin{aligned}
&D(l,n,x):=\{t:\ln^{-\frac12}\varphi(x)\le|t-x|\le(l+1)n^{-\frac12}\varphi(x)\},\\
&F(l,x):=\{v:v\in(0,1),\ |v-x|\le(l+1)n^{-\frac12}\varphi(x)\},\\
&G(l,v):=\{x:x\in E_n,\ v\in F(l,x)\}.
\end{aligned}$$

For $l\ge 1$, by (2.1b), (2.4), and (2.26) with $w\equiv 1$, we deduce that

$$\begin{aligned}&\sum_{k=0}^{n}p_{n,k}(x)C_{n,k}\int_{D(l,n,x)}p_{n,k}(t)|t-x|dt\\ \le&\frac{Cn^2}{l^4\varphi^4(x)}\sum_{k=0}^{n}p_{n,k}(x)nw^{-1}\left(\frac{k^*}{n}\right)\int_{D(l,n,x)}p_{n,k}(t)|t-x|^5dt\\ \le&\frac{Cn^3}{l^4\varphi^4(x)}\left(\sum_{k=0}^{n}p_{n,k}(x)w^{-2}(\frac{k^*}{n})\right)^{\frac{1}{2}}\left(\sum_{k=0}^{n}p_{n,k}(x)\left(\int_0^1p_{n,k}(t)|t-x|^5dt\right)^2\right)^{\frac{1}{2}}\\ \le&\frac{Cn^2}{l^4\varphi^4(x)w(x)}\left(\sum_{k=0}^{n}p_{n,k}(x)(n+1)\int_0^1p_{n,k}(t)|t-x|^{10}dt\right)^{\frac{1}{2}}\\ \le&\frac{Cn^2}{l^4\varphi^4(x)w(x)}\left(n^{-\frac{1}{2}}\varphi(x)\right)^5\le\frac{Cn^{-\frac{1}{2}}\varphi(x)}{(l+1)^4w(x)}.\end{aligned}$$

For $l=0$, by (2.1b) and (2.4),

$$\sum_{k=0}^{n}p_{n,k}(x)C_{n,k}\int_{D(l,n,x)}p_{n,k}(t)|t-x|dt\le Cn^{-\frac{1}{2}}\varphi(x)\sum_{k=0}^{n}p_{n,k}(x)w^{-1}\left(\frac{k^*}{n}\right)\le C\frac{n^{-\frac{1}{2}}\varphi(x)}{w(x)}.$$

Therefore,

$$\begin{aligned}I_2\le&\int_{E_n}w(x)\varphi^{-2}(x)\sum_{l=0}^{\infty}\sum_{k=0}^{n}p_{n,k}(x)C_{n,k}\int_{D(l,n,x)}p_{n,k}(t)\left|\int_x^tg(v)dv\right||t-x|dtdx\\ \le&Cn^{-1}\sum_{l=0}^{\infty}\frac{1}{(l+1)^4}\int_{E_n}\frac{n^{\frac{1}{2}}}{\varphi(x)}\int_{F(l,x)}g(v)dvdx\\ \le&Cn^{-1}\sum_{l=0}^{\infty}\frac{1}{(l+1)^4}n^{\frac{1}{2}}\int_0^1g(v)\left\{\int_{G(l,v)}\varphi^{-1}(x)dx\right\}dv.\end{aligned}$$

Noting that $G(l,v)\subset E_n\subset[0,1]$ for $l\ge n^{\frac{1}{4}}$ and $\int_0^1\varphi^{-1}(x)dx\le C$, we have

$$n^{\frac{1}{2}}\sum_{l\ge n^{\frac{1}{4}}}\frac{1}{(l+1)^4}\int_0^1g(v)\left\{\int_{G(l,v)}\varphi^{-1}(x)dx\right\}dv\le C\|\varphi^2f''\|_{1,w}.$$

Since (see [5])

$$\int_{\{x:|v-x|\le h\varphi(x)\}}\varphi^{-1}(x)dx\le Ch,$$

then (by taking $h=(l+1)n^{\frac{1}{2}}$)

$$\int_{G(l,v)}\varphi^{-1}(x)dx\le C(l+1)n^{\frac{1}{2}}.$$

Therefore,

$$n^{\frac{1}{2}}\sum_{0\le l\le n^{\frac{1}{4}}}\frac{1}{(l+1)^4}\int_0^1 g(v)\left\{\int_{G(l,v)}\varphi^{-1}(x)dx\right\}dv\le C\|\varphi^2 f''\|_{1,w}.$$

Thus, we can conclude that

$$I_2\le\frac{C}{n}\|\varphi^2 f''\|_{1,w}. \tag{2.29}$$

Now, we begin to prove the following

$$I_1\le\frac{C}{n}\|\varphi^2 f''\|_{1,w}. \tag{2.30}$$

For $l\ge 1$, by (2.4), and (2.26), we deduce that

$$\begin{aligned}&\sum_{k=0}^{n}p_{n,k}(x)C_{n,k}\int_{D(l,n,x)}p_{n,k}(t)w(t)|t-x|dt\\ \le& C\frac{n^2}{l^4\varphi^4(x)}\sum_{k=0}^{n}p_{n,k}(x)C_{n,k}\int_0^1 p_{n,k}(t)w(t)|t-x|^5dt\\ \le& C\frac{n^2}{l^4\varphi^4(x)}\left[\sum_{k=0}^{n}p_{n,k}(x)C_{n,k}^2\left(\int_0^1 p_{n,k}(t)w(t)|t-x|^5dt\right)^2\right]^{\frac{1}{2}}\\ \le& C\frac{n^2}{l^4\varphi^4(x)}\left[\sum_{k=0}^{n}p_{n,k}(x)C_{n,k}\int_0^1 p_{n,k}(t)w(t)|t-x|^{10}dt\right]^{\frac{1}{2}}\\ \le& C\frac{n^2}{l^4\varphi^4(x)}\left(n^{-5}\varphi^{10}(x)\right)^{\frac{1}{2}}\le\frac{n^{-\frac{1}{2}}\varphi(x)}{(l+1)^4}.\end{aligned}$$

For $l=0$, by (2.1b), (2.3) and (2.4), we have

$$\begin{aligned}&\sum_{k=0}^{n}p_{n,k}(x)C_{n,k}\int_{D(l,n,x)}p_{n,k}(t)w(t)|t-x|dt\\ \le& Cn^{-\frac{1}{2}}\varphi(x)\sum_{k=0}^{n}p_{n,k}nw^{-1}\left(\frac{k^*}{n}\right)\int_0^1 p_{n,k}(t)w(t)dt\\ \le& Cn^{-\frac{1}{2}}\varphi(x).\end{aligned}$$

Then, we can derive (2.30) in a similar way to the proof of (2.29).

By combining (2.28)-(2.30), we obtain Lemma 2.7 for $p=1$.

Finally, we prove Lemma 2.7 for $1<p\le\infty$.

Set

$$G(g,x)=\sup_t\left|\frac{1}{t-x}\int_x^t g(v)dv\right|.$$

The following maximal function inequality are well known

$$\|G(g)\|_p \le C\|g\|_p.$$

Since $1/w(v) \le C(1/w(t)+1/w(x))$ for any v between x and t, by the maximal function inequality, we have

$$\begin{aligned}
&\|M_n^*(R_2(f,t,x),x)\|_{p,w,E_n}\\
\le & C\left\|\varphi^{-2}(x)M_n^*\left(|t-x|\left(\frac{1}{w(t)}+\frac{1}{w(x)}\right)\int_x^t w(v)\varphi^2(v)f''(v)dv,x\right)\right\|_{p,w,E_n}\\
\le & C\left\|\varphi^{-2}(x)M_n^*\left((t-x)^2\left(\frac{1}{w(t)}+\frac{1}{w(x)}\right)G(g,x),x\right)\right\|_{p,w,E_n}\\
\le & C\|g\|_p\left\|\varphi^{-2}(x)M_n^*\left((t-x)^2\left(\frac{1}{w(t)}+\frac{1}{w(x)}\right),x\right)\right\|_{\infty,w,E_n}.
\end{aligned}$$

Therefore, we only need to prove that

$$\|K\|_{\infty,E_n} = \left\|\frac{w(x)}{\varphi^2(x)}M_n^*\left((t-x)^2\left(\frac{1}{w(t)}+\frac{1}{w(x)}\right),x\right)\right\|_{\infty,E_n} \le \frac{C}{n}, \tag{2.31}$$

where $\|K\|_{\infty,E_n}$ is the usual supremum norm of K on E_n, and

$$K = \frac{1}{\varphi^2(x)}M_n^*((t-x)^2,x) + \frac{w(x)}{\varphi^2(x)}M_n^*\left(\frac{(t-x)^2}{w(t)},x\right). \tag{2.32}$$

For the first part of K, by (2.26),

$$\frac{1}{\varphi^2(x)}M_n^*((t-x)^2,x) \le \frac{C}{n}, \quad x\in E_n. \tag{2.33}$$

For the second part of K, by (2.4), (2.1b) and (2.26) (with $w=1$),

$$\begin{aligned}
&\frac{w(x)}{\varphi^2(x)}M_n^*\left(\frac{(t-x)^2}{w(t)},x\right)\\
=&\frac{w(x)}{\varphi^2(x)}\sum_{k=0}^n p_{n,k}(x)C_{n,k}\int_0^1 p_{n,k}(t)(t-x)^2dt\\
\le&\frac{w(x)}{\varphi^2(x)}\left[\sum_{k=0}^n p_{n,k}(x)w^{-2}\left(\frac{k^*}{n}\right)\sum_{k=0}^n p_{n,k}(x)n^2\left(\int_0^1 p_{n,k}(t)(t-x)^2dt\right)^2\right]^{\frac{1}{2}}\\
\le&\frac{1}{\varphi^2(x)}\left[\sum_{k=0}^n p_{n,k}(x)(n+1)\int_0^1 p_{n,k}(t)(t-x)^4dt\right]^{\frac{1}{2}}\\
\le&\frac{1}{\varphi^2(x)}\left(n^{-\frac{1}{2}}\varphi(x)\right)^2 \le \frac{C}{n}.
\end{aligned} \tag{2.34}$$

By (2.33) and (2.34), we get (2.31), and thus Lemma 2.7 is valid for $1<p\le\infty$. □

3 Proofs of theorems

Proof of Theorem 1.1. It is sufficient to prove that

$$\|M_n^*(f)-f\|_{p,w}\leq \frac{C}{n}\left(\|\varphi^2 f''\|_{p,w}+\|f'\|_{p,w}\right) \tag{3.1}$$

for $\varphi^2 f''\in L_w^p$. By the Taylor's formula

$$f(t)=f(x)+f'(x)(t-x)+\int_x^t (t-v)f''(v)dv,$$

we have

$$w(x)(M_n^*(f,x)-f(x))=w(x)f'(x)M_n^*((t-x),x)+w(x)M_n^*(R_2(f,t,x),x).$$

Then, by (2.25) and (2.27), we get (3.1) immediately. $\square$

Proof of Theorem 1.2. The "$\Longleftarrow$" part follows from Theorem 1.1. The "$\Longrightarrow$" part can be done by using the argument of proof of Theorem 9.3.2 in [5], we omit the details here. $\square$

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