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On Quasi-Chebyshevity Subsets of Unital Banach Algebras

M. Iranmanesh and F. Soleimany*

Department of Mathematical Sciences, Shahrood University of Technology, P.O. Box 3619995161-316, Shahrood, Iran

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Abstract. In this paper, first, we consider closed convex and bounded subsets of infinite-dimensional unital Banach algebras and show with regard to the general conditions that these sets are not quasi-Chebyshev and pseudo-Chebyshev. Examples of those algebras are given including the algebras of continuous functions on compact sets. We also see some results in C^*-algebras and Hilbert C^*-modules. Next, by considering some conditions, we study Chebyshev of subalgebras in C^*-algebras.

Key Words: Best approximation, Quasi-Chebyshev sets, Pseudo-Chebyshev, C^*-algebras, Hilbert C^*-modules.

AMS Subject Classifications: 41A50, 41A65, 41A99, 46L05

1 Introduction

The subject of approximation theory is an old branch of analysis and has attracted the attention of several mathematicians during last years. This theory which has many important applications in mathematics and some other sciences has been studied by many authors, e.g., [4, 15]. A basic problem in the theory is "Given a point x and a set W in normed space X, determine a point w_0 of W which is at a minimum distance from x" i.e. to find $w_0 \in W$ such that

$$\|x-w_0\| = d_W(x) = \inf_{w\in W}\|x-w\|. \tag{1.1}$$

The set of all best approximations to x from W is denoted by $\mathscr{P}_W(x)$. Thus

$$\mathscr{P}_W(x) := \{w \in W \mid \|x-w\| = d_W(x)\}. \tag{1.2}$$

*Corresponding author. *Email addresses:* `m.iranmanesh2012@gmail.com` (M. Iranmanesh), `enfazh.bmaam@gmail.com` (F. Soleimany)

If each $x\in X$ has at least one best approximation in W, then W is called a proximinal set and W is said to be non-proximinal if $\mathscr{P}_W(x)=\varnothing$ for some $x\in X\backslash W$. A problem which has been intensively studied is to check whether a Banach space X does or does not contain bounded closed non-proximinal sets. The results in general Banach spaces can be found in [1,5,6]. A subset W of a Banach space X is called quasi-Chebyshev if $\mathscr{P}_W(x)$ is a non-empty and compact set in X for every $x\in X$ (see [10]). Some results on characterizations of quasi-Chebyshev subspaces in Banach spaces can be found in [9,10]. In the paper, we introduce the problem exist non-quasi-Chebyshev and non-pseudo -Chebyshev sets in unital Banach algebras. All this works done by applying the related fixed point and approximation theory results. We give characterizations of quasi-Chebyshev subalgebras in C^*-algebras in terms of substate function. The structure of this paper is as follows. In Section 2 we records some facts about Banach algebras, spectral properties of C^*-algebras $\mathbb{A}$ and Hilbert C^*-modules. In Section 3, we approach the question on the existence of non-quasi-Chebyshev sets in unital abelian Banach algebras by using the related fixed points and invariant approximation results. As a consequence, we obtain some results on the algebra of continuous functions $C(S)$, where S is a compact set. We show that every closed bounded convex set in a C^*-algebra $\mathbb{A}$ is quasi-Chebyshev if and only if $\mathbb{A}$ be finite dimensional. Similarly, we get some results for Hilbert C^*-modules. Best approximation and quasi-Chebyshev of subalgebra in C^*-algebras, is discussed and characterized in Section 4.

2 Preliminaries

Let us start with some basic definitions, which will be used later. Consider $\mathbb{A}$ as a unital algebra with the unit e. If $\mathbb{A}$ is a Banach space with respect to a norm which satisfies the multiplicative inequality

$$\|xy\|\le\|x\|\|y\|\;(x,y\in\mathbb{A}),\tag{2.1}$$

then the pair $(\mathbb{A},\|\cdot\|)$ is called a normed algebra. A complete unital normed algebra is called unital Banach algebra. $a\in\mathbb{A}$ is said to be invertible if there is an element b in $\mathbb{A}$ such that $ab=ba=e$. The fields of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. The symbol F denotes a field that can be either $\mathbb{R}$ or $\mathbb{C}$. The spectrum of an element x of a unital algebra $\mathbb{A}$ over F is the set

$$\sigma(x)=\{\lambda\in F: x-\lambda \text{ is non-invertible}\}.\tag{2.2}$$

The spectral radius of x is defined by

$$r(x)=\sup_{\lambda\in\sigma(x)}|\lambda|.\tag{2.3}$$

A nonzero homomorphism $\tau:\mathbb{A}\to F$, where $\mathbb{A}$ is a unital algebra over F, is called a character. We denote by $\Omega(\mathbb{A})$ the set of all characters on $\mathbb{A}$. If $\mathbb{A}$ is a unital abelian

Banach algebra and $x\in\mathbb{A}$, we define a continuous function $\widehat{x}$ by

$$\widehat{x}:\Omega(\mathbb{A})\to F, \quad \tau\longmapsto\tau(x). \tag{2.4}$$

We call $\widehat{x}$ the Gelfand transform of x. We denote by $\widehat{\mathbb{A}}$ the set $\{\widehat{x}:x\in\mathbb{A}\}$. It is easy to see that $\widehat{\mathbb{A}}$ is self-adjoint if and only if for each $x\in\mathbb{A}$, there exists an element $y\in\mathbb{A}$ such that $\tau(x)=\overline{\tau(y)}$ for each $\tau\in\Omega(\mathbb{A})$. If $\mathbb{A}$ is a unital abelian complex Banach algebra, $\Omega(\mathbb{A})\neq\emptyset$ and $\sigma(a)=\{\tau(a):\tau\in\Omega(\mathbb{A})\}$ for all $a\in\mathbb{A}$. If $\mathbb{A}$ is a unital Banach algebra, then $\Omega(\mathbb{A})$ is compact (see [11]). We follwed with the concept of C^*-algebras. An involution $*$ on an algebra $\mathbb{A}$ is a mapping $x\to x^*$ from $\mathbb{A}$ onto $\mathbb{A}$ such that $(\lambda x+y)^*=\overline{\lambda}x^*+y^*,(xy)^*=y^*x^*$ and $(x^*)^*=x$, for all $x,y\in\mathbb{A}$ and $\lambda\in\mathbb{C}$. An involutive Banach algebra is called a Banach $*$-algebra. A Banach $*$-algebra $\mathbb{A}$ is said to be a C^*-algebra if $\|xx^*\|=\|x\|^2$, for each $x\in\mathbb{A}$. Clearly under the norm topology on $B(H)$, the set of all bounded linear operators on a Hilbert space H is a C^*-algebra relative to involution $T\to T^*$, which is defined by

$$\langle x,Ty\rangle=\langle T^*x,y\rangle, \quad \forall x,y\in H.$$

Let $\mathbb{A}$ be an algebra then $\mathbb{M}_n(\mathbb{A})$ denotes the algebra of all $n\times n$-matrices $a=[a_{ij}]$ with entries a_{ij} in $\mathbb{A}$. If $\mathbb{A}$ is a C^*-algebra, so $\mathbb{M}_n(\mathbb{A})$, where the involution is given by $[a_{ij}]^*=[a^*_{ji}]$. If $\mathbb{A}$ and $\mathbb{B}$ be two C^*-algebras, we denote by $\mathbb{A}\otimes\mathbb{B}$ their algebraic tensor product. Note that for any C^*-algebra one can identify the space $\mathbb{M}_n(\mathbb{A})$ with the tensor product $\mathbb{M}_n(\mathbb{C})\otimes\mathbb{A}$.

A (right) Hilbert C^*-module V over a C^*-algebra $\mathcal{A}$ is a linear space which is a right $\mathcal{A}$-module equipped with an $\mathcal{A}$-valued inner-product $\langle\cdot,\cdot\rangle:V\times V\to\mathbb{A}$ that is sesquilinear, positive definite and respects the module action, i.e.,

(1) $\langle x,\alpha y+\beta z\rangle=\alpha\langle x,y\rangle+\beta\langle x,z\rangle$ for $x,y,z\in V$, $\alpha,\beta\in\mathbb{C}$,

(2) $\langle x,ya\rangle=\langle x,y\rangle a$ for $x,y\in V$, $a\in\mathbb{A}$,

(3) $\langle x,y\rangle^*=\langle y,x\rangle$ for $x,y\in V$,

(4) $\langle x,x\rangle\geq 0$ for $x\in V$; if $\langle x,x\rangle=0$ then $x=0$,

and V is complete with respect to the norm defined by $\|x\|=\|\langle x,x\rangle\|^{\frac{1}{2}}$, for each $x\in V$.

Lemma 2.1 (see [3])**.** *Let E be a C^*-module over a C^*-algebra $\mathbb{A}$. Then E can be isometrically embedded in $B(H,K)$, where H,K are Hilbert spaces.*

In the following, we recall some useful lemmas in fixed point theory that will be needed in the sequel. Let $(X,\cdot\|)$ be a Banach space, A mapping $T:E\subseteq X\to X$ is non-expansive if $\|Tx-Ty\|\leq\|x-y\|$ for each $x,y\in E$. The fixed point set of mapping T is denoted by $\mathscr{F}(T)=\{x\in X:T(x)=x\}$.

Lemma 2.2 (see [14])**.** *Let X be a Banach space, C a compact convex subset of X and $T:C\to C$ a continuous map. Then T has at least one fixed point in C.*

Definition 2.1 (see [8])**.** Let K be a convex subset of a Banach space X. A map $V:K\to X$ is called convex if $\|V(\frac{x+y}{2})\|\le\frac{1}{2}[\|V(x)\|+\|V(y)\|]$ for $x,y\in K$.

Lemma 2.3 (see [8])**.** *Let K be a non-empty weakly compact convex subset of a Banach space X and let $T:K\to X$ be non-expansive and suppose $I-T$ is convex on K. Then T has a fixed point.*

3 A non-quasi-Chebyshev sets of Banach algebras

In this section, we consider closed, convex and bounded subsets of infinite dimensional unital Banach algebras and show with regard to the general conditions that these sets are not quasi-Chebyshev and pseudo- Chebyshev.

Definition 3.1. A closed subset W of a normed linear space X is called non-quasi-Chebyshev if it is not quasi-Chebyshev.

Definition 3.2. Let U be a closed subset of a normed linear space X. Then U is called ω-quasi-Chebyshev if the set $\mathscr{P}_U(x)$ is non-empty and weakly compact for all $x\in X\backslash U$. And U is ω-non-quasi-Chebyshev if it is not ω-quasi-Chebyshev.

Definition 3.3. Let X be a Banach algebra, we say X has property (N) if $\Omega(X)\neq\varnothing$ and for $x,y\in X$, $|\tau(x)|\le|\tau(y)|$ for each $\tau\in\Omega(X)$ implies that $\|x\|\le\|y\|$.

Fupinwong and Dhompongsa studied the fixed-point property of unital commutative Banach algebras over a field F (see [7]). They obtained the following results.

Consider X as an infinite-dimensional unital abelian Banach algebra, which has property (N) such that $\widehat{X}$ is self-adjoint and

$$\inf\{r(x):x\in\mathbb{A},\ \|x\|=1\}>0, \tag{3.1}$$

then

(1) there exists a sequence $\{x_n\}\in X$ such that

$$\{\tau(x_n):\tau\in\Omega(X)\}\subseteq[0,1].$$

For each $n\in N$, and $A_n=\{\widehat{(x_n)}^{-1}\{1\}$ is a sequence of non-empty pairwise disjoint subsets of $\Omega(X)$.

(2) Consider the mapping T_n on X by

$$T_n:X\longrightarrow X,\quad x\longrightarrow x_nx, \tag{3.2}$$

and sets

$$E_n=\{x\in X:0\le\tau(x)\le1,\tau\in\Omega(X)\text{ and }\tau(x)=1\text{ for }\tau\in A_n\}. \tag{3.3}$$

Then T_n hasn't any fixed-point in E_n.

Theorem 3.1. *Let X be an infinite dimensional unital abelian Banach algebra with property (N) such that $\widehat{X}$ is self-adjoint and satisfying* (3.1)*, then X contains a non-quasi-Chebyshev subset (Moreover, X contains a non-w-quasi-Chebyshev set).*

Proof. Assume on the contrary X and every of its closed subsets be quasi-Chebyshev (ω-quasi-Chebyshev). Let T_n and E_n defined by (3.2) and (3.3). Since X has property (N) and $|\tau(x_n)|\le 1=\tau(e)$ then $\|x_n\|\le 1$ for $n\in N$. For $x,y\in X$, we have

$$\|T_n(x)-T_n(y)\|=\|x_n(x-y)\|\le\|x_n\|\|x-y\|\le\|x-y\|.$$

Then T_n is non-expansive on X. For $n\in N$, E_n is a T_n-invariant convex set of X, then for $x\in X$, $\mathscr{P}_{E_n}(x)$ is convex and by assumption is non-empty compact (weakly compact) set. We show that for $x=0$, $\mathscr{P}_{E_n}(0)$ is T_n-invariant so. Let $y\in\mathscr{P}_{E_n}(0)$, further $y\in E_n$ and then T_ny is in E_n since $T_n(E_n)\subseteq E_n$. As T_n is non-expansive, it follows that for $g\in E_n$,

$$\|T_ny-0\|=\|T_ny-T_n0\|\le\|y-0\|\le\|g-0\|,$$

and therefore T_ny is in $\mathscr{P}_{E_n}(0)$. Thus T_n maps $\mathscr{P}_{E_n}(0)$ into itself. On the other hand, since the multiplication operation $(a,b)\to ab$ is jointly continuous in Banach algebras. Then T_n is a continuous map. As $I-T_n$ is convex by applying Lemma 2.2 and Lemma 2.3 there exist $p_n\in\mathscr{P}_{E_n}(0)\cap\mathscr{F}(T_n)$ for each $n\in N$. Therefore $p_n\in E_n\cap\mathscr{F}(T_n)$. But this is a contradiction by part (2) of the results of Fupinwong and Dhompongsa. $\square$

Definition 3.4. A closed subset W of a normed linear space X is called pseudo-Chebyshev, if the set $\mathscr{P}_W(x)$ be a non-empty and finite-dimensional subset of X for all $x\in X\setminus W$.

A closed subset W of a normed linear space X is non-pseudo-Chebyshev if it is not pseudo-Chebyshev.

Theorem 3.2. *Let X be an infinite dimensional unital abelian Banach algebra with property (N) such that $\widehat{X}$ is self-adjoint and satisfying* (3.1)*. Then X has an non-pseudo-Chebyshev subset.*

Proof. Assume on the contrary that X and every of its closed subsets are pseudo-Chebyshev. Let T_n and E_n be in such as (3.2) and (3.3). First we show that E_n is a bounded set for each $n\in N$. For $x\in E_n$,

$$\begin{aligned}\frac{1}{\|x\|}\sup_{\tau\in\Omega(X)}|\tau(x)|=&\frac{1}{\|x\|}\sup_{\tau\in\Omega(X)}|\tau(x)|\\ =&\sup|\tau(\frac{x}{||x||})|=r\Big(\frac{x}{||x||}\Big)\\ \ge&\inf\{r(y):y\in X,\|y\|=1\}=\beta.\end{aligned}$$

Since $\Omega(X)$ is compact there is a character τ_0 on X such that $\sup_{\tau\in\Omega(X)}|\tau(x)|=|\tau_0(x)|$. Thus

$$\|x\|\le\frac{|\tau_0(x)|}{\beta}.$$

Hence E_n is bounded. Moreover, $\mathscr{P}_{E_n}(x)$ is bounded for $x \in X$. Since a closed, bounded and finite dimensional subset of a normed space is compact, so by Bolzano Weierstrass theorem $\mathscr{P}_{E_n}(x)$ is a compact set. Similar to the proof of Theorem 3.1 we can show a contradiction under this assumption. $\square$

We denoted by $C_F(S)$ the Banach algebra of continuous functions from a topological space S to F, with the supremom norm. By results in [7], for each $x \in C_F(S)$, $\sigma(x) = x(S)$ and

$$r(x) = \sup_{\lambda \in \sigma(x)} |\lambda| = \sup_{s \in S} |x(s)| = \|x\|, \tag{3.4}$$

therefore if $x, y \in C_F(S)$ and $|x(s)| \le |y(s)|$ for each $s \in S$, then $\|x\| \le \|y\|$.

Corollary 3.1. Let S be a compact Hausdorff topological space. If $C_F(S)$ is infinite-dimensional then $C_F(S)$ has a closed convex non-quasi and a closed convex non-pseudo-Chebyshev subset.

Proof. Clearly $C_F(S)$ has property (N). By (3.4) we have $\inf\{r(x) : x \in X, \|x\| = 1\} = 1 > 0$. Since $C_F(S)$ satisfies the conditions of Theorem 3.1, will get that $C_F(S)$ has a closed convex non-quasi- Chebyshev subset.

Also as a consequence of Theorem 3.2, we conclude that $C_F(S)$ has a closed convex non-pseudo-Chebyshev subset. $\square$

In the following, we prove some results on C^*-algebras. This will done by using Gelfand theorem.

Theorem 3.3. *Let $\mathbb{A}$ be a C^*-algebra, then $dim(\mathbb{A}) = \infty$ if and only if $\mathbb{A}$ has a bounded non-quasi-Chebyshev subset.*

Proof. If $dim(\mathbb{A}) = \infty$ then by a result of Ogasawara theorem (Theorem 1 in [12]) $\mathcal{A}$ contains an infinite dimensional commutative C^*-subalgebra $\mathbb{B}$. Then by Gelfand Theorem (see [11]) $\mathbb{B} \simeq C(\Omega(\mathbb{A}))$ hence as a consequence of Corollary 3.1, $\mathbb{B}$ has an non-quasi-Chebyshev subset This is true also for $\mathbb{A}$.

For the inverse let $dim(\mathbb{A}) < \infty$ and W be a closed and bounded subset of $\mathbb{A}$. By Bolzano Weierstrass Theorem, W is compact. Thus for each $a \in \mathbb{A}$, $\mathscr{P}_W(a)$ is non-empty and compact, which make a contradiction. Thus $dim(\mathbb{A}) = \infty$. $\square$

Theorem 3.4. *Let $\mathbb{A}$ be a C^*-algebra then $dim(\mathbb{A}) = \infty$ if and only if $\mathbb{A}$ has a non-pseudo-Chebyshev subset.*

Proof. The proof is similar to the that of Theorem 3.3. $\square$

Corollary 3.2. Let E be a C^*-module over a C^*-algebra $\mathbb{A}$, then $dim(E) = \infty$ if and only if E has an non-quasi-Chebyshev subset.

Proof. By Lemma 2.1, E is isometrically embedded in C^*-subalgebra $B(H,K)$, where H, K are Hilbert spaces. Now, it is a consequence of Theorem 3.3. □

Corollary 3.3. Let E be a C^*-module over a C^*-algebra $\mathbb{A}$ then $dim(E)=\infty$ if and only if E has a non-pseudo-Chebyshev subset.

Proof. It is a consequence of Theorem 3.4. □

4 Characterizations of quasi-Chebyshev in C^*-algebras

In this section, we give some characterizations of best approximations and quasi-Chebyshev subalgebras in C^*-algebras.

Definition 4.1. Let A be a Banach space and B be a proper closed subspace of A. An element $Z\in A$ is called B-minimal if 0 is the best approximation to Z in B.

Definition 4.2. The mapping $p:A\to\mathbb{R}$ is substate function if for $h,g\in A$ and $\alpha\in\mathbb{R}^+$,

i) $\|p\|=1$.

ii) $p(h+g)\geq p(h)+p(g)$ and $p(\alpha h)=\alpha p(h)$.

iii) p be a positive function i.e., $p(h)\geq 0$, for $h\geq 0$.

In Definition 4.2 if P be a linear function then p is called a state.

Let $\mathbb{A}$ be a C^*-algebra, an element $x\in\mathbb{A}$ is hermitian if $x=x^*$, we denote by $\mathbb{A}_h$, the set of all hermitian element of $\mathbb{A}$. in fact

$$\mathbb{A}_h=\{x\in\mathbb{A}:x=x^*\}. \tag{4.1}$$

By the above assumptions, we have the following lemmas.

Lemma 4.1 (see [13]). *Let $\mathbb{A}$ be a unital C^*-algebra, B a unital C^*-subalgebra of $\mathbb{A}$ and $a\in\mathbb{A}_h$. If a is B-minimal, then there exists a state ϕ of $\mathcal{A}$ such that $\phi(a^2)=\|a\|^2$ and $\phi(ab+b^*a)=0$ for all $b\in B$.*

Let x, y be two elements of a normed linear space X, then x is orthogonal to y in the Birkhoff-James sense [2] if $\|x\|\leq\|x+\lambda y\|$ for all $\lambda\in\mathbb{C}$. If x be Birkhoff-James orthogonal to y, we write $x\perp_B y$. Let G be a subset of X and $x\in X$. Then x is said to be orthogonal to G ($x\perp_B G$) whenever $x\perp_B g$, $\forall g\in G$.

Lemma 4.2 (see [2]). *Let $\mathbb{A}$ be a C^*-algebra, $a,b\in\mathbb{A}$. Then $a\perp_B b$ if and only if there exists a state φ on $\mathbb{A}$ that $\varphi(a^*a)=\|a\|^2$ and $\varphi(a^*b)=0$.*

Lemma 4.3. *Let M be a subalgebra of C^*-algebra $\mathbb{A}$ and $a\in\mathbb{A}_h$. If $|\mathscr{P}_M(a)|\geq 1$ then there exist $g_0\in\mathscr{P}_M(a)$ is hermitian.*

Proof. Suppose $m_0 \in \mathscr{P}_M(a)$, we show the real part of representation $m_0 = \frac{m_0+m_0^*}{2} + i\frac{m_0-m_0^*}{2}$ is sightly element. We have for $m \in M$,

$$\begin{aligned} \|a - \frac{m_0+m_0^*}{2}\| &= \|\frac{a+a^*}{2} - \frac{m_0+m_0^*}{2}\| \\ &\le \|\frac{a-m_0}{2}\| + \|\frac{a^*-m_0^*}{2}\| \\ &\le \|a-m\|. \end{aligned}$$

Hence $\frac{m_0+m_0^*}{2} \in \mathscr{P}_M(a) \cap \mathbb{A}_h$. $\square$

Theorem 4.1. *Let M be a subalgebra of $\mathbb{A}$, $m_0 \in M$ and $a \in \mathbb{A} \setminus M$. Then the following statements are equivalent.*

i) $m_0 \in \mathscr{P}_M(a)$.

ii) There exists a substate φ on $\mathbb{A}$ such that for $m \in M$

$$\varphi((a-m_0)^*(a-m_0)) = \|a-m_0\|^2, \tag{4.2a}$$
$$\varphi(m^*(a-m_0)) \le 0. \tag{4.2b}$$

Proof. $i) \to ii)$. Let $m_0 \in \mathscr{P}_M(a)$ so $a - m_0 \perp_B m$, for $m \in M$ then by Lemma 4.2 there exists a state φ^m such that $\varphi^m((a-m_0)^*(a-m_0)) = \|a-m_0\|^2$ and $\varphi^m(m^*(a-m_0)) = 0$.

Define $\varphi : \mathbb{A} \longrightarrow \mathbb{R}$, by $\varphi(h) = \inf_{m \in M} Re\, \varphi^m(h)$. We show that it is a substate. For $\alpha \in \mathbb{R}^+$ and $h \in \mathbb{A}$

$$\varphi(\alpha h) = \inf_{m \in M} Re \varphi^m(\alpha h) = \alpha \inf_{m \in M} Re \varphi^g(h) = \alpha \varphi(h),$$

and for each $h, k \in \mathbb{A}$ we have

$$\begin{aligned} \varphi(h+k) &= \inf_{m \in M} Re\varphi^m(h+k) \\ &= \inf_{m \in M} Re\varphi^m(h) + \inf_{m \in M} Re\varphi^g(k) \\ &\ge \inf_{m \in M} Re\varphi^m(h) + \inf_{m \in U} Re\varphi^m(k) = \varphi(h) + \varphi(k). \end{aligned}$$

Since φ^m is positive function and $\|\varphi^m\| = 1 = \varphi^m(e)$ (see Corollary 3.3.4 [11]) then φ is positive and $\|\varphi\| = 1$. Also

$$\varphi((a-m_0)^*(a-m_0)) = \|a-m_0\|^2,$$

and for $m \in M$, we have

$$\varphi(m^*(a-m_0)) \le \varphi^m(m^*(a-m_0)) = 0.$$

This completes the proof of this part.

$ii) \to i)$. Suppose that such a substate exists. By the Cauchy-Schwartz inequality for $m \in M$ we get,

$$\begin{aligned}\|a-m_0\|^2 &= \varphi((a-m_0)^*(a-m_0))\\ &\leq -\varphi((m-m_0)^*(a-m_0))+\varphi((a-m_0)^*(a-m_0))\\ &\leq -\varphi((m-a)^*(a-m_0))-\varphi((a-m_0)^*(a-m_0))\\ &\quad +\varphi(a-m_0)^*(a-m_0))\\ &= -\varphi((m-a)^*(a-m_0))\\ &\leq |-\varphi((m-a)^*(a-m_0))| \leq \|\varphi\|\|(m-a)^*\|\|a-m_0)\|\\ &\leq \|a-m\|\|a-m_0\|.\end{aligned}$$

Hence $\|a-m_0\| \leq \|a-m\|$, i.e., $m_0 \in \mathscr{P}_M(a)$, which completes the proof. ☐

Theorem 4.2. *Let M be a unital proximinal *-subalgebra of $\mathbb{A}$. Then the following statements are equivalent.*

i) M is a quasi-Chebyshev subalgebra.

ii) There do not exist substate φ on $\mathbb{A}$, $x_0 \in \mathbb{A}$ and a sequence $x_n \in \mathbb{A}$ without a convergent subsequence with $x_0 - x_n \in M$ $(n=1,2,\cdots)$ such that for $m \in M$,

$$\varphi(x_n^* x_n) = \|x_n\|^2, \tag{4.3a}$$
$$\varphi(m^* x_n) \leq 0. \tag{4.3b}$$

Proof. $i) \Rightarrow ii)$ Suppose that (ii) does not hold, then there is ψ on $\mathbb{A}$, $x_0 \in \mathbb{A}$ and a sequence $x_n \in \mathbb{A}$ without a convergent subsequence and $x_0 - x_n \in M$, sataisfid conditions (4.3a), (4.3b). Put $g_n = x_0 - x_n$ by Theorem 4.1, $g_n \in \mathscr{P}_M(x_0)$, without a convergent subsequence, this is a contradiction.

$ii) \Rightarrow i)$. Assume if possible that M is not quasi-Chebyshev in $\mathbb{A}$. Since M is proximinal in $\mathbb{A}$, for $x \in \mathbb{A}$, $\mathscr{P}_M(x) \neq \emptyset$, let $a \in \mathbb{A}$ such that $g_n \in \mathscr{P}_M(a)$ without a convergent subsequence. We assume that a is a hermitian element of $\mathbb{A}$, also by Lemma 4.3, $\{g_n\}$ is hermitian (If $a \neq a^*$, then we can consider the Hermitian element $X = \begin{bmatrix} 0 & a^* \\ a & 0 \end{bmatrix}$ in $\mathbb{M}_2(\mathbb{A})$. If this element has a best approximation in $\mathbb{M}_2(M)$ then it is easy to see that it has a best approximation in $\mathbb{M}_2(M)$ with the form $M_0 = \begin{bmatrix} 0 & m^* \\ m & 0 \end{bmatrix}$ where $m \in M$). By applying Lemma 4.1 there exist state function ϕ_n such that

$$\begin{aligned}&\phi_n((a-g_n)^2) = \|a-g_n\|^2,\\ &\phi_n((a-g_n)(g)^* + g(a-g_n)) = 0.\end{aligned}$$

Now we define $\varphi(h)=\inf_{n\in N} Re\varphi_n(h)$, similar to the proof of the pervious theorem we can show that it is a substate. Also for $i,n\in\mathbb{N}$, $\phi_n((a-g_i)^2)=\|a-g_i\|^2$ because

$$\begin{aligned}\phi_n((a-g_i)^2 =&\phi_n((a-g_n+g_n-g_i)(a-g_n+g_n-g_i)^*)\\ =&\phi_n((a-g_n)(a-g_n)^*)+\phi_n((a-g_n)(g_n-g_i)^*\\ &+(g_n-g_i)(a-g_n)^*)+\phi_n((g_n-g_i)(g_n-g_i)^*)\\ \geq&\phi_n((a-g_n)(a-g_n)^*)=\|a-g_n\|^2=\|a-g_i\|^2.\end{aligned}$$

On the other hand, we have

$$\phi_n((a-g_i)^2)\leq\|\phi_n\|\|a-g_i\|^2=\|a-g_i\|^2.$$

Hence $\phi_n((a-g_i)^2)=\|a-g_i\|^2$. Therefore for $n\in\mathbb{N}$ we have

$$\varphi((a-g_n)(a-g_n)^*)=\varphi((a-g_n)^2)=\|a-g_n\|^2.$$

But $\varphi(g^*(a-g_n))\leq Re\varphi_n(g^*(a-g_n))=\frac{1}{2}(\varphi_n((a-g_n)g^*+g(a-g_n))=0$. Hence

$$\varphi(g^*(a-g_n))\leq 0,$$

which is a contraction by part (ii). $\square$

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