J. Math. Study
doi: 10.4208/jms.v51n4.18.01

Vol. **51**, No. 4, pp. 337-376
December 2018

# Attractors for a Caginalp Phase-field Model with Singular Potential

Alain Miranville[1,2,*] and Charbel Wehbe[1]

[1] *Laboratoire de Mathématiques et Applications, UMR CNRS 7348, SP2MI, Boulevard Marie et Pierre Curie - Téléport 2, F-86962 Chasseneuil Futuroscope Cedex, France.*

[2] *Xiamen University, School of Mathematical Sciences, Xiamen, Fujian, P.R. China.*

Received February 2, 2018; Accepted November 1, 2018

**Abstract.** We consider a phase field model based on a generalization of the Maxwell Cattaneo heat conduction law, with a logarithmic nonlinearity, associated with Neumann boundary conditions. The originality here, compared with previous works, is that we obtain global in time and dissipative estimates, so that, in particular, we prove, in one and two space dimensions, the existence of a unique solution which is strictly separated from the singularities of the nonlinear term, as well as the existence of the finite-dimensional global attractor and of exponential attractors. In three space dimensions, we prove the existence of a solution.

**AMS subject classifications**: 35B40, 35B41, 35K51, 80A22, 80A20, 35Q53, 45K05, 35K55, 35G30, 92D50

**Key words**: Caginalp phase-field system, Maxwell-Cattaneo law, logarithmic potential, Neumann boundary conditions, well-posedness, global attractor, exponential attractor.

## 1 Introduction

The Caginalp phase-field model

$$\frac{\partial u}{\partial t}-\Delta u+g(u)=\theta, \tag{1.1}$$

$$\frac{\partial \theta}{\partial t}-\Delta\theta=-\frac{\partial u}{\partial t}, \tag{1.2}$$

has been proposed to model phase transition phenomena, for example melting-solidification phenomena, in certain classes of materials. Caginalp considered the Ginzburg-Landau free energy and the classical Fourier law to derive his system, see, e.g., [1,2].

*Corresponding author. *Email addresses:* alain.miranville@math.univ-poitiers.fr (A. Miranville), charbel_wehbe83@hotmail.com (C. Wehbe)

Here, $u$ denotes the order parameter and $\theta$ the (relative) temperature. Furthermore, all physical constants have been set equal to one. For more details and references we refer the reader to [2–4]. This model has been extensively studied (see, e.g., [5] and the references therein). Now, a drawback of the Fourier law is the so-called "paradox of heat conduction", namely, it predicts that thermal signals propagate with infinite speed, which, in particular, violates causality (see, e.g., [5]). One possible modification, in order to correct this unrealistic feature, is the Maxwell-Cattaneo law. We refer the reader to [3,5,6] for more discussions on the subject.

In this paper, we consider the following model

$$\frac{\partial u}{\partial t}-\Delta u+g(u)=\frac{\partial \alpha}{\partial t}, \tag{1.3}$$

$$\frac{\partial^2 \alpha}{\partial t^2}+\frac{\partial \alpha}{\partial t}-\Delta \alpha=-\frac{\partial u}{\partial t}-u, \tag{1.4}$$

which is a generalization of the original Caginalp system (see [2]). In this context $\alpha$ is the thermal displacement variable, defined by

$$\alpha=\int_0^t \theta d\tau+\alpha_0. \tag{1.5}$$

As mentioned above the Caginalp system can be obtained by considering the Ginzburg-Landau free energy

$$\Psi(u,\theta)=\int_\Omega \left(\frac{1}{2}|\nabla u|^2+G(u)-\theta u\right)dx, \tag{1.6}$$

the enthalpy $H=u+\theta$ and by writing

$$\frac{1}{d}\frac{\partial u}{\partial t}=-\partial_u \Psi, \tag{1.7}$$

$$\frac{\partial H}{\partial t}=-\mathrm{div} q, \tag{1.8}$$

where $d>0$ is a relaxation parameter, $\partial_u$ denotes a variational derivative and $q$ is the thermal flux vector. Setting $d=1$ and taking the usual Fourier law

$$q=-\nabla\theta, \tag{1.9}$$

we find (1.1)-(1.2).

The Maxwell-Cattaneo law reads

$$(1+\eta\frac{\partial}{\partial t})q=-\nabla\theta, \tag{1.10}$$

where $\eta$ is a relaxation parameter; when $\eta=0$, one recovers the Fourier law. Taking for simplicity $\eta=1$, it follows from (1.8) that

$$\left(1+\frac{\partial}{\partial t}\right)\frac{\partial H}{\partial t}-\Delta\theta=0,$$

hence the following second-order (in time) equation for the relative temperature

$$\frac{\partial^2\theta}{\partial t^2}+\frac{\partial\theta}{\partial t}-\Delta\theta=-\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial t^2}. \tag{1.11}$$

Integrating finally (1.11) between 0 and $t$, we obtain the equation

$$\frac{\partial^2\alpha}{\partial t^2}+\frac{\partial\alpha}{\partial t}-\Delta\alpha=-\frac{\partial u}{\partial t}-u+f, \tag{1.12}$$

where $f$ depends on the initial data (for $u$ and $\theta$), which reduces to (1.4) when $f$ vanishes. Furthermore, noting that $\theta=\frac{\partial\alpha}{\partial t}$, (1.1) can be rewritten in the equivalent form (1.3).

We endow this model with Neumann boundary conditions and initial conditions. Then, we are led to the following initial and boundary value problem $(P)$:

$$\frac{\partial u}{\partial t}-\Delta u+g(u)=\frac{\partial\alpha}{\partial t} \quad \text{in} \quad \Omega, \tag{1.13}$$

$$\frac{\partial^2\alpha}{\partial t^2}+\frac{\partial\alpha}{\partial t}-\Delta\alpha=-\frac{\partial u}{\partial t}-u \quad \text{in} \quad \Omega, \tag{1.14}$$

$$\frac{\partial u}{\partial\nu}=\frac{\partial\alpha}{\partial\nu}=0 \quad \text{on} \quad \partial\Omega, \tag{1.15}$$

$$u(0)=u_0, \quad \alpha(0)=\alpha_0, \quad \frac{\partial\alpha}{\partial t}(0)=\alpha_1, \tag{1.16}$$

in a bounded and regular domain $\Omega\subset\mathbb{R}^n$ ($n$ is to be specified later), with boundary $\partial\Omega$.

We assume here that $g=G'$, where

$$\begin{aligned} G(s)=-\kappa_0 s^2+\kappa_1[(1+s)\ln(1+s)+(1-s)\ln(1-s)],\\ s\in(-1,1), \quad 0<\kappa_1<\kappa_0, \end{aligned} \tag{1.17}$$

i.e.,

$$g(s)=-2\kappa_0 s+\kappa_1\ln\left(\frac{1+s}{1-s}\right), \quad s\in(-1,1). \tag{1.18}$$

In particular, it follows from (1.18) that

$$g'(s)\geq -2\kappa_0, \quad s\in(-1,1), \tag{1.19}$$

$$-c_0\leq G(s)\leq g(s)s+c_0, \quad s\in(-1,1). \tag{1.20}$$

Concerning the mathematical setting, we introduce the following Hilbert spaces

$$
\begin{aligned}
&F=H^1(\Omega)\times H^1(\Omega)\times L^2(\Omega), \quad F_1=H^2(\Omega)\times H^2(\Omega)\times H^1(\Omega),\\
&F_2=H^3(\Omega)\times H^3(\Omega)\times H^2(\Omega), \quad F_3=(H^4(\Omega))^2\times H^3(\Omega),\\
&\overline{\Phi_0}=\Big\{(u,\alpha,\frac{\partial\alpha}{\partial t})\in\overline{\Phi},\langle u+\frac{\partial\alpha}{\partial t}\rangle=0\Big\},\\
&\overline{\Phi}=\Big\{(u,\overline{\alpha},\frac{\partial\alpha}{\partial t})\in\overline{F_1},\ \|u\|_{L^\infty}<1\Big\}, \quad \overline{F_1}=H^2(\Omega)\times\overline{H^2}(\Omega)\times H^1(\Omega),\\
&\overline{\Psi_0}=\Big\{(u,\alpha,\frac{\partial\alpha}{\partial t})\in\overline{\Psi},\langle u+\frac{\partial\alpha}{\partial t}\rangle=0\Big\},\\
&\overline{\Psi}=\Big\{(u,\overline{\alpha},\frac{\partial\alpha}{\partial t})\in\overline{F_2},\ \|u\|_{L^\infty}<1\Big\}, \quad \overline{F_2}=H^3(\Omega)\times\overline{H^3}(\Omega)\times H^2(\Omega),
\end{aligned}
$$

and $\|.\|_Y$ denotes the norm on the Banach space $Y$. Throughout this paper, the $L^2$-inner product and the associated norm will be denoted by $((.,.))$ and $\|.\|$ respectively.

Our aim in this paper is to prove the existence of a solution in the case of the logarithmic nonlinearity (1.18). The main difficulty is to prove that the order parameter is separated from the singularities of $g$. In particular, we are only able to prove such a property in one and two space dimensions. In three space dimensions, we prove the existence of a solution.

Throughout the paper, the same letter $c$ (and, sometimes, $c'$) denotes constants which may change from line to line.

## 2 A priori estimates

The singularities of the potential $g$ lead us to define the quantity

$$D(v)=\frac{1}{1-\|v\|_{L^\infty}}, \quad v\in L^\infty(\Omega), \quad \|v\|_{L^\infty}\neq 1.$$

We a priori assume that $\|u_0\|_{L^\infty}<1$ and $\|u\|_{L^\infty(\mathbb{R}^+\times\Omega)}<1$.

We set

$$Bu:=-\Delta u+u, \quad \text{with} \quad D(B)=H^2_N(\Omega)=\Big\{u\in H^2(\Omega),\ \frac{\partial u}{\partial\nu}=0 \quad \text{on} \quad \partial\Omega\Big\}.$$

Otherwise, we set $\|u\|_1=((B^{\frac12}u,B^{\frac12}u))^{\frac12}$ for all $u\in H^1(\Omega)$ and we have that this norm is equivalent to the usual norm of $H^1(\Omega)$.

We rewrite (1.13) in the form

$$\frac{\partial u}{\partial t}+Bu+f(u)=\frac{\partial\alpha}{\partial t}, \tag{2.1}$$

where $f(s)=g(s)-s$. Note that $f$ and $g$ satisfy the same properties. We set $F(s)=\int_0^s f(\tau)d\tau$.

We multiply (2.1) by $u+\frac{\partial u}{\partial t}$, (1.14) by $\frac{\partial \alpha}{\partial t}$. Integrating over $\Omega$ and summing the two resulting equations, by (1.20) we obtain

$$\begin{aligned}&\frac{d}{dt}\Big(\|u\|^2+\|u\|_1^2+2\int_\Omega F(u)dx+\|\frac{\partial \alpha}{\partial t}\|^2+\|\nabla\alpha\|^2\Big)\\&\quad+c(\|u\|_1^2+2\int_\Omega F(u)dx+\|\frac{\partial \alpha}{\partial t}\|^2+\|\frac{\partial u}{\partial t}\|^2)\le c', \qquad c>0.\end{aligned} \tag{2.2}$$

We then rewrite (1.14) in the form

$$\frac{\partial H}{\partial t}+H-\Delta\alpha=0, \tag{2.3}$$

where

$$H=u+\frac{\partial \alpha}{\partial t}.$$

Integrating (2.3) over $\Omega$, we obtain

$$\frac{d\langle H\rangle}{dt}+\langle H\rangle=0. \tag{2.4}$$

In particular, we deduce from (2.4) that

$$\langle H(t)\rangle=e^{-t}\langle H(0)\rangle, \tag{2.5}$$

hence

$$\lim_{t\to+\infty}H(t)=0. \tag{2.6}$$

Furthermore, if $\langle H(0)\rangle=0$, i.e., $\langle u_0+\alpha_1\rangle=0$, we have conservation of the enthalpy,

$$\langle H(t)\rangle=0, \qquad \forall t\ge 0. \tag{2.7}$$

Setting $\overline{\phi}=\phi-\langle\phi\rangle$, we then have

$$\frac{\partial \overline{H}}{\partial t}+\overline{H}-\Delta\overline{\alpha}=0. \tag{2.8}$$

We multiply (2.8) by $\frac{\partial \overline{\alpha}}{\partial t}$ to find

$$\frac{d}{dt}\Big(\|\frac{\partial \overline{\alpha}}{\partial t}\|^2+\|\nabla\alpha\|^2\Big)+\|\frac{\partial \overline{\alpha}}{\partial t}\|^2\le c\Big(\|u\|^2+\|\frac{\partial u}{\partial t}\|^2\Big). \tag{2.9}$$

We also multiply (2.8) by $\overline{\alpha}$ and have

$$\frac{1}{2}\frac{d}{dt}\Big(\|\overline{\alpha}\|^2+2(\frac{\partial \overline{\alpha}}{\partial t},\overline{\alpha})\Big)+\|\nabla\alpha\|^2=-(\overline{u},\overline{\alpha})-(\frac{\partial \overline{u}}{\partial t},\overline{\alpha})+\|\frac{\partial \overline{\alpha}}{\partial t}\|^2,$$

which yields, noting that $\|\overline{\phi}\|\leq c\|\nabla\phi\|,\quad \phi\in H^1(\Omega),\quad c>0,$

$$\frac{d}{dt}\Big(\|\overline{\alpha}\|^2+2(\frac{\partial\overline{\alpha}}{\partial t},\overline{\alpha})\Big)+\|\nabla\alpha\|^2\leq c\Big(\|u\|^2+\|\frac{\partial u}{\partial t}\|^2+\|\frac{\partial\overline{\alpha}}{\partial t}\|^2\Big). \tag{2.10}$$

We first sum (2.9) and $\delta_1\times$(2.10) to have

$$\begin{aligned}&\frac{d}{dt}\Big(\|\nabla\alpha\|^2+\|\frac{\partial\overline{\alpha}}{\partial t}\|^2+\delta_1\|\overline{\alpha}\|^2+2\delta_1(\frac{\partial\overline{\alpha}}{\partial t},\overline{\alpha})\Big)+c\Big(\|\frac{\partial\overline{\alpha}}{\partial t}\|^2+\|\nabla\alpha\|^2\Big)\\&\leq c'\Big(\|u\|^2+\|\frac{\partial u}{\partial t}\|^2\Big),\qquad c>0,\end{aligned} \tag{2.11}$$

where $\delta_1>0$ is small enough so that, in particular,

$$\|\frac{\partial\overline{\alpha}}{\partial t}\|^2+\delta_1\|\overline{\alpha}\|^2+2\delta_1(\frac{\partial\overline{\alpha}}{\partial t},\overline{\alpha})\geq c\Big(\|\overline{\alpha}\|^2+\|\frac{\partial\overline{\alpha}}{\partial t}\|^2\Big),\quad c>0, \tag{2.12}$$

and then sum (2.2) and $\delta_2\times$(2.11), where $\delta_2>0$ is small enough, to obtain

$$\frac{dE_1}{dt}+c\Big(E_1+\|\frac{\partial u}{\partial t}\|^2\Big)\leq c',\quad c>0, \tag{2.13}$$

where

$$\begin{aligned}E_1=&\|u\|^2+\|u\|_1^2+2\int_\Omega F(u)dx+\|\nabla\alpha\|^2+\|\frac{\partial\alpha}{\partial t}\|^2\\&+\delta_2\Big(\|\nabla\alpha\|^2+\|\frac{\partial\overline{\alpha}}{\partial t}\|^2+\delta_1\|\overline{\alpha}\|^2+2\delta_1(\frac{\partial\overline{\alpha}}{\partial t},\overline{\alpha})\Big).\end{aligned} \tag{2.14}$$

We now multiply (1.13) by $-\Delta u$, and have, owing to (1.19),

$$\frac{d}{dt}\|\nabla u\|^2+\|\Delta u\|^2\leq c\Big(\|\nabla u\|^2+\|\frac{\partial\alpha}{\partial t}\|^2\Big). \tag{2.15}$$

Summing (2.13) and $\delta_3\times$(2.15), where $\delta_3>0$ is small enough, we finally obtain

$$\frac{dE_2}{dt}+c\Big(E_2+\|\Delta u\|^2+\|\frac{\partial u}{\partial t}\|^2\Big)\leq c',\quad c>0, \tag{2.16}$$

where

$$E_2=E_1+\delta_3\|\nabla u\|^2, \tag{2.17}$$

satisfies

$$\begin{aligned}&c\Big(\|u\|^2_{H^1(\Omega)}+\int_\Omega F(u)dx+\|\overline{\alpha}\|^2_{H^1(\Omega)}+\|\frac{\partial\alpha}{\partial t}\|^2\Big)-c'\leq E_2\\&\leq c''\Big(\|u\|^2_{H^1(\Omega)}+\int_\Omega F(u)dx+\|\overline{\alpha}\|^2_{H^1(\Omega)}+\|\frac{\partial\alpha}{\partial t}\|^2\Big)+c''',\quad c,c''>0 \text{ and } c',c'''\geq 0.\end{aligned} \tag{2.18}$$

We differentiate (1.13) with respect to time to find, owing to (1.14),

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial t})-\Delta\frac{\partial u}{\partial t}+g'(u)\frac{\partial u}{\partial t}=-\frac{\partial\alpha}{\partial t}+\Delta\alpha-u-\frac{\partial u}{\partial t}. \tag{2.19}$$

We then multiply (2.19) by $\frac{\partial u}{\partial t}$. Hölder's inequality and (1.19) yield

$$\frac{d}{dt}\|\frac{\partial u}{\partial t}\|^2+\|\nabla\frac{\partial u}{\partial t}\|^2\le c\Big(\|u\|^2+\|\overline{\alpha}\|^2_{H^1(\Omega)}+\|\frac{\partial u}{\partial t}\|^2+\|\frac{\partial \alpha}{\partial t}\|^2\Big). \tag{2.20}$$

We multiply (1.14) by $-\Delta\frac{\partial \alpha}{\partial t}$. Integrating over $\Omega$, we get

$$\frac{d}{dt}\Big(\|\Delta\alpha\|^2+\|\nabla\frac{\partial \alpha}{\partial t}\|^2\Big)+\|\nabla\frac{\partial \alpha}{\partial t}\|^2\le c\Big(\|\nabla u\|^2+\|\nabla\frac{\partial u}{\partial t}\|^2\Big). \tag{2.21}$$

Finally, we multiply (1.14) by $-\Delta\alpha$ and we integrate over $\Omega$ to have

$$\frac{1}{2}\frac{d}{dt}\Big(\|\nabla\alpha\|^2+2(\nabla\frac{\partial \alpha}{\partial t},\nabla\alpha)\Big)+\|\Delta\alpha\|^2=(u,\Delta\alpha)+(\frac{\partial u}{\partial t},\Delta\alpha)+\|\nabla\frac{\partial \alpha}{\partial t}\|^2,$$

which implies

$$\frac{d}{dt}\Big(\|\nabla\alpha\|^2+2(\nabla\frac{\partial \alpha}{\partial t},\nabla\alpha)\Big)+\|\Delta\alpha\|^2\le c\Big(\|u\|^2+\|\frac{\partial u}{\partial t}\|^2+\|\nabla\frac{\partial \alpha}{\partial t}\|^2\Big). \tag{2.22}$$

We sum (2.21) and $\delta_4\times$(2.22), where $\delta_4>0$ is small enough, to get

$$\|\nabla\frac{\partial \alpha}{\partial t}\|^2+\delta_4\Big(\|\nabla\alpha\|^2+2(\nabla\frac{\partial \alpha}{\partial t},\nabla\alpha)\Big)\ge c\Big(\|\nabla\frac{\partial \alpha}{\partial t}\|^2+\|\nabla\alpha\|^2\Big),\quad c>0. \tag{2.23}$$

We then have

$$\frac{dE_3}{dt}+c\Big(\|\nabla\frac{\partial \alpha}{\partial t}\|^2+\|\Delta\alpha\|^2\Big)\le c'\Big(\|u\|^2_{H^1(\Omega)}+\|\frac{\partial u}{\partial t}\|^2_{H^1(\Omega)}\Big),\quad c>0, \tag{2.24}$$

where

$$E_3=\|\Delta\alpha\|^2+\|\nabla\frac{\partial \alpha}{\partial t}\|^2+\delta_4\Big(\|\nabla\alpha\|^2+2(\nabla\frac{\partial \alpha}{\partial t},\nabla\alpha)\Big). \tag{2.25}$$

Now we sum (2.20) and $\delta_5\times$(2.24), where $\delta_5>0$ is small enough, to get

$$\begin{aligned}&\frac{dE_4}{dt}+c\Big(\|\nabla\frac{\partial u}{\partial t}\|^2+\|\nabla\frac{\partial \alpha}{\partial t}\|^2+\|\Delta\alpha\|^2\Big)\\ &\le c'\Big(\|u\|^2_{H^1(\Omega)}+\|\frac{\partial u}{\partial t}\|^2+\|\nabla\alpha\|^2+\|\frac{\partial \alpha}{\partial t}\|^2\Big),\quad c>0,\end{aligned} \tag{2.26}$$

where

$$E_4=\|\frac{\partial u}{\partial t}\|^2+\delta_5 E_3. \tag{2.27}$$

Finally, we sum(2.16) and $\delta_6\times$(2.26), where $\delta_6>0$ is small enough, to get

$$\frac{dE_5}{dt}+c\Big(E_5+\|\nabla\frac{\partial u}{\partial t}\|^2\Big)\le c',\quad c>0, \tag{2.28}$$

where
$$E_5=E_2+\delta_6E_4, \tag{2.29}$$
satisfies
$$\begin{aligned}&c\Big(\|\frac{\partial\alpha}{\partial t}\|^2_{H^1(\Omega)}+\|\Delta\alpha\|^2+\|\frac{\partial u}{\partial t}\|^2+\|u\|^2_{H^1(\Omega)}+\int_\Omega G(u)dx+\|\overline{\alpha}\|^2_{H^1(\Omega)}+\|\frac{\partial\overline{\alpha}}{\partial t}\|^2\Big)-c^{'}\le E_5\\ \le& c^{''}\Big(\|\frac{\partial\alpha}{\partial t}\|^2_{H^1(\Omega)}+\|\Delta\alpha\|^2+\|\frac{\partial u}{\partial t}\|^2+\|u\|^2_{H^1(\Omega)}+\int_\Omega G(u)dx+\|\overline{\alpha}\|^2_{H^1(\Omega)}+\|\frac{\partial\overline{\alpha}}{\partial t}\|^2\Big)+c^{'''}.\end{aligned} \tag{2.30}$$
Using (2.16), (2.20) and Gronwall's lemma, we deduce
$$\begin{aligned}&\|u(t)\|^2_{H^1}+\|\overline{\alpha}(t)\|^2_{H^1}+\|\frac{\partial\alpha}{\partial t}(t)\|^2+\int_0^t e^{-c(t-s)}\Big(\|\Delta u\|^2+\|\frac{\partial u}{\partial t}\|^2\Big)ds\\ \le& Q\Big(D(u_0)+\|u_0\|^2_{H^1}+\|\overline{\alpha_0}\|^2_{H^1}+\|\alpha_1\|^2\Big)e^{-ct}+c^{'},\qquad c>0,\end{aligned} \tag{2.31}$$
and
$$\begin{aligned}&\|\frac{\partial u}{\partial t}(t)\|^2+\int_0^t e^{-c(t-s)}\|\nabla\frac{\partial u}{\partial t}\|^2ds\\ \le& c^{'}\int_0^t e^{-c(t-s)}\Big(\|\overline{\alpha}\|^2_{H^1}+\|\frac{\partial u}{\partial t}\|^2+\|\frac{\partial\alpha}{\partial t}\|^2+\|u\|^2\Big)ds+e^{-ct}\|\frac{\partial u}{\partial t}(0)\|^2.\end{aligned} \tag{2.32}$$
Note that
$$\|\frac{\partial u}{\partial t}(0)\|^2\le Q\Big(D(u_0)+\|u_0\|^2_{H^2}+\|\alpha_1\|^2\Big). \tag{2.33}$$
Using (2.31) and (2.33), we deduce from (2.32) the following inequality
$$\begin{aligned}&\|\frac{\partial u}{\partial t}(t)\|^2+\int_0^t e^{-c(t-s)}\|\frac{\partial u}{\partial t}\|^2_{H^1}ds\\ \le& Q\Big(D(u_0)+\|u_0\|^2_{H^2}+\|\overline{\alpha_0}\|^2_{H^1}+\|\alpha_1\|^2\Big)e^{-ct}+c^{'},\quad c>0.\end{aligned} \tag{2.34}$$
We rewrite (1.13) in an elliptic form for $t\ge 0$ fixed,
$$-\Delta u+g(u)=-\frac{\partial u}{\partial t}+\frac{\partial\alpha}{\partial t}, \tag{2.35}$$
$$\frac{\partial u}{\partial\nu}=0\quad\text{on}\quad\partial\Omega. \tag{2.36}$$
We multiply (2.35) by $-\Delta u$. Using (1.19), Hölder and Young's inequalities, we obtain
$$\|\Delta u\|^2\le c\Big(\|\nabla u\|^2+\|\frac{\partial u}{\partial t}\|^2+\|\frac{\partial\alpha}{\partial t}\|^2\Big). \tag{2.37}$$
Using now (2.31), (2.34) and (2.37) we find
$$\|u(t)\|^2_{H^2}\le Q(D(u_0)+\|u_0\|^2_{H^2}+\|\overline{\alpha_0}\|^2_{H^1}+\|\alpha_1\|^2)e^{-ct}+c^{'},\quad c>0. \tag{2.38}$$

Applying Gronwall's lemma to (2.28) and using (2.30) we have

$$\|\overline{\alpha}(t)\|_{H^2}^2+\|\frac{\partial\alpha}{\partial t}(t)\|_{H^1}^2\leq Q(D(u_0)+\|u_0\|_{H^2}^2+\|\overline{\alpha_0}\|_{H^2}^2+\|\alpha_1\|_{H^1}^2)e^{-ct}+c', \quad c>0. \tag{2.39}$$

By (2.34), (2.38) and (2.39) we get

$$\begin{aligned}&\|u(t)\|_{H^2}^2+\|\overline{\alpha}(t)\|_{H^2}^2+\|\frac{\partial\alpha}{\partial t}(t)\|_{H^1}^2+\|\frac{\partial u}{\partial t}(t)\|^2+\int_0^t e^{-c(t-s)}\|\frac{\partial u}{\partial t}\|_{H^1}^2ds\\ \leq& Q\Big(D(u_0)+\|u_0\|_{H^2}^2+\|\overline{\alpha_0}\|_{H^2}^2+\|\alpha_1\|_{H^1}^2\Big)e^{-ct}+c', \qquad c>0. \end{aligned}\tag{2.40}$$

Our aim now is to prove that $u$ a priori satisfies

$$\|u(t)\|_{L^\infty(\Omega)}\leq 1-\delta, \quad \forall t\geq 0, \tag{2.41}$$

where $\delta>0$ is to be specified later.

**In one space dimension,** we have, owing to the embedding $H^1(\Omega)\subset C(\overline{\Omega})$, an estimate on $\frac{\partial\alpha}{\partial t}$ in $L^\infty(\mathbb{R}^+\times\Omega)$. It is then not difficult to prove the separation property (2.41) for solutions to the parabolic equation

$$\frac{\partial u}{\partial t}-\Delta u+g(u)=h, \tag{2.42}$$

with right-hand side $h\in L^\infty(\mathbb{R}^+\times\Omega)$.

Indeed, by (2.40), $h$ satisfies

$$\|\frac{\partial\alpha}{\partial t}\|_{L^\infty}^2\leq Q\big(D(u_0)+\|u_0\|_{H^2(\Omega)}^2+\|\overline{\alpha_0}\|_{H^2(\Omega)}^2+\|\alpha_1\|_{H^1(\Omega)}^2\big)e^{-ct}+c', \quad c>0. \tag{2.43}$$

Let $y_\pm(t)$ be the solutions of the following ODEs : $y_\pm'+g(y_\pm)=l_\pm(t); \quad y_\pm(0)=\pm\|u_0\|_{L^\infty}$. where $l_\pm(t)=\pm\|h(t)\|_{L^\infty}$.

We prove (see [16] and [20]):

$$|y_\pm(t)|\leq 1-\delta(D(u_0)+\|l_\pm\|_{L^\infty(0,1)}), \qquad 0\leq t\leq 1, \tag{2.44}$$

$$|y_\pm(t+1)|\leq 1-\delta(\|l_\pm\|_{L^\infty(t,t+1)}), \qquad t\geq 0. \tag{2.45}$$

Thus, due to the comparison principle, we deduce the following inequalities :

$$y_-(t)\leq u(x,t)\leq y_+(t), \quad (x,t)\in\Omega\times\mathbb{R}^+. \tag{2.46}$$

Estimates (2.43)–(2.46) imply that

$$D(u(t))\leq Q(D(u_0)+\|u_0\|_{H^2(\Omega)}^2+\|\overline{\alpha_0}\|_{H^2(\Omega)}^2+\|\alpha_1\|_{H^1(\Omega)}^2)e^{-ct}+c'. \tag{2.47}$$

Combining (2.40) and (2.47), we obtain

$$\begin{aligned}&D(u(t))+\|u(t)\|^2_{H^2}+\|\overline{\alpha}(t)\|^2_{H^2}+\|\frac{\partial\alpha}{\partial t}(t)\|^2_{H^1}+\|\frac{\partial u}{\partial t}\|^2+\int_0^t e^{-c(t-s)}\|\frac{\partial u}{\partial t}\|^2_{H^1}ds\\ &\leq Q(D(u_0)+\|u_0\|^2_{H^2(\Omega)}+\|\overline{\alpha_0}\|^2_{H^2(\Omega)}+\|\alpha_1\|^2_{H^1(\Omega)})e^{-ct}+c'. \end{aligned}\tag{2.48}$$

In particular

$$\|u(t)\|_{L^\infty}\leq 1-\delta, \quad \forall t\geq 0, \tag{2.49}$$

where $\delta>0$ depends on $D(u_0),\|u_0\|_{H^2(\Omega)},\|\overline{\alpha_0}\|_{H^2(\Omega)}$ and $\|\alpha_1\|_{H^1(\Omega)}$.

We now turn to the **two-dimensional case**. To this end, we derive further a priori estimates.

We multiply (1.14) by $\Delta^2\frac{\partial\alpha}{\partial t}$. Using the Hölder and Young inequalities, we have

$$\frac{d}{dt}(\|\Delta\frac{\partial\alpha}{\partial t}\|^2+\|\nabla\Delta\alpha\|^2)+2\|\Delta\frac{\partial\alpha}{\partial t}\|^2\leq\frac{1}{\epsilon}(\|\Delta u\|^2+\|\Delta\frac{\partial u}{\partial t}\|^2)+2\epsilon\|\Delta\frac{\partial\alpha}{\partial t}\|^2. \tag{2.50}$$

We then multiply (1.14) by $\Delta^2\alpha$ to get

$$\begin{aligned}&\frac{d}{dt}\Big(2(\Delta\frac{\partial\alpha}{\partial t},\Delta\alpha)+\|\Delta\alpha\|^2\Big)+2\|\nabla\Delta\alpha\|^2\\ &\leq\frac{1}{\epsilon}\Big(\|\nabla u\|^2+\|\nabla\frac{\partial u}{\partial t}\|^2\Big)+2\|\Delta\frac{\partial\alpha}{\partial t}\|^2+2\epsilon\|\nabla\Delta\alpha\|^2. \end{aligned}\tag{2.51}$$

Summing (2.50) and $\epsilon\times$(2.51), where $\epsilon>0$ is small enough such that $1-2\epsilon>0$ and $1-\epsilon>0$, we have

$$\frac{dE_6(t)}{dt}+cE_6(t)\leq c'\Big(\|\Delta u\|^2+\|\Delta\frac{\partial u}{\partial t}\|^2\Big), \tag{2.52}$$

where

$$E_6(t)=\|\Delta\frac{\partial\alpha}{\partial t}\|^2+\|\nabla\Delta\alpha\|^2+\epsilon\Big(2(\Delta\frac{\partial\alpha}{\partial t},\Delta\alpha)+\|\Delta\alpha\|^2\Big)$$

and

$$c\Big(\|\Delta\alpha\|^2+\|\nabla\Delta\alpha\|^2+\|\Delta\frac{\partial\alpha}{\partial t}\|^2\Big)\leq E_6\leq c'\Big(\|\Delta\alpha\|^2+\|\nabla\Delta\alpha\|^2+\|\Delta\frac{\partial\alpha}{\partial t}\|^2\Big). \tag{2.53}$$

Applying Gronwall's lemma to (2.52), we have

$$E_6(t)\leq c'\int_0^t e^{-c(t-s)}\Big[\|\Delta u\|^2+\|\Delta\frac{\partial u}{\partial t}\|^2\Big]ds+E_6(0)e^{-ct}. \tag{2.54}$$

Furthermore, by (2.40) we get

$$\int_0^t e^{-c(t-s)}\|\Delta u\|^2ds\leq Q\Big(D(u_0)+\|u_0\|^2_{H^2(\Omega)}+\|\overline{\alpha_0}\|^2_{H^2(\Omega)}+\|\alpha_1\|^2_{H^1(\Omega)}\Big)e^{-ct}+c'. \tag{2.55}$$

Now, we differentiate (1.13) with respect to time to have, owing to (1.14),

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial t})-\Delta\frac{\partial u}{\partial t}=\varphi, \tag{2.56}$$

where

$$\varphi=-\frac{\partial\alpha}{\partial t}+\Delta\alpha-u-\frac{\partial u}{\partial t}-g^{'}(u)\frac{\partial u}{\partial t}. \tag{2.57}$$

Multiplying (2.56) by $-\Delta\frac{\partial u}{\partial t}$, using Young and Hölder's inequalities, we obtain

$$\frac{d}{dt}\|\nabla\frac{\partial u}{\partial t}\|^2+\|\Delta\frac{\partial u}{\partial t}\|^2\le\|\varphi\|^2. \tag{2.58}$$

We apply Gronwall's lemma to (2.58), to have

$$\begin{aligned}&\|\nabla\frac{\partial u}{\partial t}\|^2+\int_0^t e^{-c(t-s)}\|\Delta\frac{\partial u}{\partial t}\|^2ds\\ \le& c^{'}\int_0^t e^{-c(t-s)}(\|\varphi\|^2+\|\nabla\frac{\partial u}{\partial t}\|^2)ds+e^{-ct}\|\nabla\frac{\partial u}{\partial t}(0)\|^2.\end{aligned} \tag{2.59}$$

Hence we have to estimate the term $\int_0^t e^{-c(t-s)}\|\varphi\|^2ds$. To do so, we first prove the following lemma.

**Lemma 2.1.** $\forall M>0$:

$$\begin{aligned}&\int_{(t,t+1)\times\Omega}e^{M|g(u(x,t))|}dxdt\\ \le& Q(D(u_0)+\|u_0\|^2_{H^2(\Omega)}+\|\overline{\alpha_0}\|^2_{H^2(\Omega)}+\|\alpha_1\|^2_{H^1(\Omega)})e^{-ct}+c^{'},\end{aligned} \tag{2.60}$$

*where $c^{'}$ only depends on $M$.*

*Proof.* We can assume, without loss of generality, that

$$g'(s)\ge 0,\quad s\in(-1,1). \tag{2.61}$$

We fix $M>0$ and multiply (2.42) by $g(u)e^{M|g(u)|}$ to have :

$$\begin{aligned}&\frac{d}{dt}\int_\Omega G_M(u)dx+\int_\Omega|\nabla u|^2g^{'}(u)(1+M|g(u)|)e^{M|g(u)|}dx\\ &+\int_\Omega|g(u)|^2e^{M|g(u)|}dx=\int_\Omega h\cdot g(u)e^{M|g(u)|}dx,\end{aligned} \tag{2.62}$$

where $G_M(s)=\int_0^s\tau e^{M|\tau|}d\tau$, which yields, integrating between $t$ and $t+1$, and using (2.48) and (2.61) the following estimate

$$\begin{aligned}&\int_{(t,t+1)\times\Omega}|g(u)|^2e^{M|g(u)|}dxdt\\ \le& Q\Big(D(u_0)+\|u_0\|^2_{H^2}+\|\overline{\alpha_0}\|^2_{H^2}+\|\alpha_1\|^2_{H^1}\Big)e^{-ct}+c^{'}+\int_{(t,t+1)\times\Omega}|h|\cdot|g(u)|e^{M|g(u)|}dxdt.\end{aligned} \tag{2.63}$$

In order to estimate the second term in the right-hand side of (2.63), we use the following Young's inequality

$$ab \le \phi(a)+\psi(b), \quad a,b \ge 0, \tag{2.64}$$

where

$$\phi(s)=e^s-s-1, \quad \psi(s)=(1+s)\ln(1+s)-s, \quad s\ge 0. \tag{2.65}$$

Taking $a=N|h|$ and $b=N^{-1}|g(u)|e^{M|g(u)|}$, where $N>0$ is to be fixed later, in (2.64), we obtain

$$|h|\cdot|g(u)|e^{M|g(u)|} \le e^{N|h|}+\Big(1+N^{-1}|g(u)|e^{M|g(u)|}\Big)\ln\Big(1+N^{-1}|g(u)|e^{M|g(u)|}\Big).$$

Now, if $|g(u)|\le 1$, then

$$|h|\cdot|g(u)|e^{M|g(u)|} \le e^{N|f|}+\Big(1+N^{-1}e^{M}\Big)\ln\Big(1+N^{-1}e^{M}\Big).$$

Furthermore, if $|g(u)|\ge 1$, then $|g(u)|e^{M|g(u)|}\ge 1$ and

$$\begin{aligned}
&|h|\cdot|g(u)|e^{M|g(u)|} \le e^{N|h|}+\Big(1+N^{-1}|g(u)|e^{M|g(u)|}\Big)\ln\Big((1+N^{-1})|g(u)|e^{M|g(u)|}\Big)\\
&= e^{N|h|}+MN^{-1}|g(u)|^2e^{M|g(u)|}+N^{-1}\ln(1+N^{-1})|g(u)|e^{M|g(u)|}\\
&\qquad +N^{-1}|g(u)|\ln(|g(u)|)e^{M|g(u)|}+M|g(u)|+\ln(|g(u)|)+\ln(1+N^{-1})\\
&\le e^{N|h|}+N^{-1}\Big(M+1+\ln(1+N^{-1})\Big)|g(u)|^2e^{M|g(u)|}+(1+M)|g(u)|+\ln(1+N^{-1})\\
&\le e^{N|h|}+N^{-1}\Big(M+1+\ln(1+N^{-1})\Big)|g(u)|^2e^{M|g(u)|}+\frac{1}{4}|g(u)|^2e^{M|g(u)|}+c,
\end{aligned}$$

because $(1+M)|g(u)|\le\frac{1}{4}|g(u)|^2+(1+M)^2\le\frac{1}{4}|g(u)|^2e^{M|g(u)|}+(1+M)^2$, where $c$ depends on $N$ and $M$. Choosing finally $N=N(M)$ large enough, we find, in both cases

$$|h|\cdot|g(u)|e^{M|g(u)|}\le e^{N|h|}+\frac{1}{2}|g(u)|^2e^{M|g(u)|}+c, \tag{2.66}$$

where $c$ only depends on $M$. We thus deduce from (2.63) and (2.66) the following inequality

$$\begin{aligned}
\int_{(t,t+1)\times\Omega}|g(u)|^2e^{M|g(u)|}dxdt \le\; & Q(D(u_0)+\|u_0\|^2_{H^2(\Omega)}+\|\overline{\alpha_0}\|^2_{H^2(\Omega)}\\
&+\|\alpha_1\|^2_{H^1(\Omega)})e^{-ct}+c'+2\int_{(t,t+1)\times\Omega}\exp(N|h|)dxdt,
\end{aligned} \tag{2.67}$$

where $c'$ only depends on $M$.

To conclude, we use the following Orlicz's embedding inequality

$$\int_\Omega e^{N|v|}dx\le e^{c(\|v\|^2_{H^1(\Omega)}+1)}, \qquad \forall v\in H^1(\Omega), \tag{2.68}$$

where $c$ only depends on $\Omega$ and $N$. It then follows from (2.43), (2.67) and (2.68) that

$$\begin{aligned}&\int_{(t,t+1)\times\Omega}|g(u)|^2e^{M|g(u)|}dxdt\\ &\leq Q(D(u_0)+\|u_0\|^2_{H^2(\Omega)}+\|\overline{\alpha_0}\|^2_{H^2(\Omega)}+\|\alpha_1\|^2_{H^1(\Omega)})e^{-ct}+c'.\end{aligned}\tag{2.69}$$

Noting finally that

$$\begin{aligned}\int_{(t,t+1)\times\Omega}e^{M|g(u)|}dx\leq&\int_{|g(u)|\leq 1}e^{M|g(u)|}dx+\int_{|g(u)|\geq 1}e^{M|g(u)|}dx\\ &\leq c+\int_{|g(u)|\geq 1}|g(u)|^2e^{M|g(u)|}dx\leq c+\int_{(t,t+1)\times\Omega}|g(u)|^2e^{M|g(u)|}dx,\end{aligned}\tag{2.70}$$

where $c$ only depends on $M$, (2.69) yields the desired inequality (2.60). □

It is not difficult to show, by comparing growths, that the logarithmic function $g$ satisfies

$$|g'(s)|\leq e^{c|g(s)|+c'},\quad s\in(-1,1),\quad c,c'\geq 0.\tag{2.71}$$

Therefore,

$$\int_{(t,t+1)\times\Omega}|g'(s)|^pdxdt\leq\int_{(t,t+1)\times\Omega}e^{cp|g(u)|+c'p}dxdt,\tag{2.72}$$

whence, owing to (2.60),

$$\begin{aligned}\|g'(u)\|_{L^p((t,t+1)\times\Omega)}\leq&Q(D(u_0)+\|u_0\|^2_{H^2(\Omega)}+\|\overline{\alpha_0}\|^2_{H^2(\Omega)}\\ &+\|\alpha_1\|^2_{H^1(\Omega)})e^{-ct}+c',\quad\forall p\geq 1.\end{aligned}\tag{2.73}$$

Thus $\varphi$ in (2.57) satisfies, owing to (2.60) (for $p=4$) and the above a priori estimates (which imply that $\frac{\partial u}{\partial t}\in L^\infty(t,t+1,L^2(\Omega))\cap L^2(t,t+1,H^1(\Omega))\subset L^4(t,t+1,H^{\frac{1}{2}}(\Omega))\subset L^4((t,t+1)\times\Omega))$,

$$\|\varphi\|_{L^2((t,t+1)\times\Omega)}\leq Q\Big(D(u_0)+\|u_0\|^2_{H^2(\Omega)}+\|\overline{\alpha_0}\|^2_{H^2(\Omega)}+\|\alpha_1\|^2_{H^1(\Omega)}\Big)e^{-ct}+c',\tag{2.74}$$

hence,

$$\begin{aligned}&\int_0^t e^{c(s-t)}\|\varphi\|^2ds\leq\|\varphi\|_{L^2((t,t+1)\times\Omega)}\\ &\leq Q(D(u_0)+\|u_0\|^2_{H^2(\Omega)}+\|\overline{\alpha_0}\|^2_{H^2(\Omega)}+\|\alpha_1\|^2_{H^1(\Omega)})e^{-ct}+c')\\ &\leq Q(D(u_0)+\|u_0\|^2_{H^2(\Omega)}+\|\overline{\alpha_0}\|^2_{H^2(\Omega)}+\|\alpha_1\|^2_{H^1(\Omega)})e^{-ct}+c'.\end{aligned}\tag{2.75}$$

Furthemore, we have

$$\|\nabla\frac{\partial u}{\partial t}(0)\|^2\leq Q\Big(D(u_0)+\|u_0\|^2_{H^3(\Omega)}+\|\alpha_1\|^2_{H^1(\Omega)}\Big).\tag{2.76}$$

Using (2.75) and (2.76) in (2.59) and by (2.48), we deduce

$$\begin{aligned}&\|\frac{\partial u}{\partial t}\|_{H^1}^2+\int_0^t e^{-c(t-s)}\|\frac{\partial u}{\partial t}\|_{H^2}^2 ds\\ \leq& Q\Big(D(u_0)+\|u_0\|_{H^3(\Omega)}^2+\|\overline{\alpha_0}\|_{H^2(\Omega)}^2+\|\alpha_1\|_{H^1(\Omega)}^2\Big)e^{-ct}+c'.\end{aligned}\tag{2.77}$$

By (2.48), (2.53), (2.55) and (2.77), we deduce from (2.54)

$$\begin{aligned}&\|\overline{\alpha}(t)\|_{H^3(\Omega)}^2+\|\frac{\partial \alpha}{\partial t}(t)\|_{H^2}^2\\ \leq& Q(D(u_0)+\|u_0\|_{H^3(\Omega)}^2+\|\overline{\alpha_0}\|_{H^3(\Omega)}^2+\|\alpha_1\|_{H^2(\Omega)}^2)e^{-ct}+c'.\end{aligned}\tag{2.78}$$

Rewriting again (1.13) in the form

$$\frac{\partial u}{\partial t}-\Delta u+g(u)=h,\tag{2.79}$$

we have, owing to the above estimates,

$$h\in L^\infty((t,t+1)\times\Omega)\tag{2.80}$$

and the separation property follows as in the one-dimensional case.

# 3 Existence of solutions

We have the

**Theorem 3.1.** *(i) In one space dimension, we assume that*

$$D(u_0)+\|u_0\|_{H^2}^2+\|\overline{\alpha_0}\|_{H^2}^2+\|\alpha_1\|_{H^1}^2<+\infty, \qquad D(u_0)>0.\tag{3.1}$$

*Then,* (1.13)-(1.16) *possesses a unique solution* $(u,\alpha,\frac{\partial \alpha}{\partial t})$ *such that*

$$\begin{aligned}&D(u(t))+\|u(t)\|_{H^2}^2+\|\overline{\alpha}(t)\|_{H^2}^2+\|\frac{\partial \alpha}{\partial t}(t)\|_{H^1}^2+\|\frac{\partial u}{\partial t}\|^2+\int_0^t e^{-c(t-s)}\|\frac{\partial u}{\partial t}\|_{H^1}^2 ds\\ \leq& Q(D(u_0)+\|u_0\|_{H^2(\Omega)}^2+\|\overline{\alpha_0}\|_{H^2(\Omega)}^2+\|\alpha_1\|_{H^1(\Omega)}^2)e^{-ct}+c', \quad c>0,t\geq 0.\end{aligned}\tag{3.2}$$

*(ii) In two space dimension, we assume that*

$$D(u_0)+\|u_0\|_{H^3}^2+\|\overline{\alpha_0}\|_{H^3}^2+\|\alpha_1\|_{H^2}^2<+\infty, \qquad D(u_0)>0.\tag{3.3}$$

*Then,* (1.13)-(1.16) *possesses a unique solution* $(u,\alpha,\frac{\partial \alpha}{\partial t})$ *such that*

$$\begin{aligned}&D(u(t))+\|u(t)\|_{H^3}^2+\|\overline{\alpha}(t)\|_{H^3}^2+\|\frac{\partial \alpha}{\partial t}(t)\|_{H^2}^2+\|\frac{\partial u}{\partial t}(t)\|_{H^1}^2+\int_0^t e^{-c(t-s)}\|\frac{\partial u}{\partial t}\|_{H^2}^2 ds\\ \leq& Q(D(u_0)+\|u_0\|_{H^3(\Omega)}^2+\|\overline{\alpha_0}\|_{H^3(\Omega)}^2+\|\alpha_1\|_{H^2(\Omega)}^2)e^{-ct}+c', \quad c>0,t\geq 0.\end{aligned}\tag{3.4}$$

*(iii) In three space dimension, we consider the set* $K=\{\varphi\in L^2(\Omega),\ -1\le\varphi\le 1,\quad a.e.\ in\ \Omega\}$ *and we assume that* $(u_0,\alpha_0,\alpha_1)\in F_K=(K\cap H^1(\Omega))\times H^1(\Omega)\times L^2(\Omega)$. *Then,* (1.13)-(1.16) *possesses a unique solution* $(u,\alpha,\frac{\partial\alpha}{\partial t})$ *such that*

$$(u,\overline{\alpha})\in L^\infty(\mathbb{R}^+,H^1(\Omega)^2),\qquad (u,\frac{\partial u}{\partial t})\in L^2(0,T,H^2(\Omega)\times L^2(\Omega)),$$
$$\alpha\in L^\infty(0,T,H^1(\Omega))\qquad \text{and}\qquad \frac{\partial\alpha}{\partial t}\in L^\infty(\mathbb{R}^+,L^2(\Omega)),\qquad \forall T>0.$$

*Moreover for all* $t>0$, $\|u(t)\|_{L^\infty(\Omega)}\le 1$ *and the set* $\{x\in\Omega/|u(x,t)|=1\}$ *has measure zero.*

*Proof.* In one and two space dimensions, the proof of existence is standard, once we have the separation property (2.41), since the problem then reduces to one with a regular nonlinearity. Indeed, we consider the same problem, in which the logarithmic function $g$ is replaced by the $C^1$ function

$$g_\delta(s)=\begin{cases} g(-\delta)+g'(-\delta)(s+\delta), & s\in(-\infty,-\delta[\,,\\ g(s), & s\in[-\delta,\delta],\\ g(\delta)+g'(\delta)(s-\delta), & s\in]\delta,+\infty),\end{cases}$$

where $\delta$ is the same constant as in (2.41).

This function meets all the requirements of [25] to have the existence of a regular solution $(u_\delta,\alpha_\delta,\frac{\partial\alpha_\delta}{\partial t})$.

Furthermore, It is not difficult to see that $g$ and $g_\delta$ satisfy (1.19), (1.20) and (2.71), for the same constants. We can thus derive the same estimates as above, with the very same constants.

Since $g$ and $g_\delta$ coincide on $[-\delta,\delta]$, we finally deduce that $u_\delta$ is solution to the original problem.

In three space dimension, following an idea of Debussche and Dettori [7] we consider the approximation of the function $g$ by a polynomial of odd degree $g_N$, and the boundary value problem $(P_N)$ that one obtains by replacing $g$ by $g_N$ in problem $(P)$

$$\frac{\partial u_N}{\partial t}-\Delta u_N+g_N(u_N)=\frac{\partial\alpha_N}{\partial t},\tag{3.5}$$
$$\frac{\partial^2\alpha_N}{\partial t^2}+\frac{\partial\alpha_N}{\partial t}-\Delta\alpha_N=-\frac{\partial u_N}{\partial t}-u_N,\tag{3.6}$$
$$\frac{\partial u_N}{\partial\nu}=\frac{\partial\alpha_N}{\partial\nu}=0,\tag{3.7}$$
$$u_N(0)=u_0,\quad \alpha_N(0)=\alpha_0,\quad \frac{\partial\alpha_N}{\partial t}(0)=\alpha_1.\tag{3.8}$$

The existence and uniqueness of a solution $(u_N,\alpha_N,\frac{\partial\alpha_N}{\partial t})$ to problem (3.5)-(3.8) have been proved in [25]. We then construct the solution of problem (1.13)-(1.16) as the limit of $(u_N,\alpha_N,\frac{\partial\alpha_N}{\partial t})$ as $N\to+\infty$. Indeed, we first derive uniform estimates with respect to $N$ for problem (3.5)-(3.8). Replacing $(u,\alpha)$ in (2.16) by $(u_N,\alpha_N)$, we write

$$\frac{dE_{2N}}{dt}+c\Big(E_{2N}+\|\Delta u_N\|^2+\|\frac{\partial u_N}{\partial t}\|^2\Big)\le c^{'},\quad c>0, \tag{3.9}$$

where

$$\begin{aligned}E_{2N}=&\|u_N\|^2+\|u_N\|_1^2+2\int_\Omega F_N(u_N)dx+\|\nabla\alpha_N\|^2+\|\frac{\partial\alpha_N}{\partial t}\|^2\\&+\delta_2\Big(\|\nabla\alpha_N\|^2+\|\frac{\partial\overline{\alpha_N}}{\partial t}\|^2+\delta_1\|\overline{\alpha_N}\|^2+2\delta_1(\frac{\partial\overline{\alpha_N}}{\partial t},\overline{\alpha_N})\Big)+\delta_3\|\nabla u_N\|^2,\end{aligned} \tag{3.10}$$

satisfies

$$\begin{aligned}&c\Big(\|u_N\|^2_{H^1(\Omega)}+\int_\Omega F_N(u_N)dx+\|\overline{\alpha_N}\|^2_{H^1(\Omega)}+\|\frac{\partial\alpha_N}{\partial t}\|^2\Big)-c^{'}\le E_{2N}\\&\le c^{''}\Big(\|u_N\|^2_{H^1(\Omega)}+\int_\Omega F_N(u_N)dx+\|\overline{\alpha_N}\|^2_{H^1(\Omega)}+\|\frac{\partial\alpha_N}{\partial t}\|^2\Big)+c^{'''}\\&\quad c,c^{''}>0\quad\text{and}\quad c^{'},c^{'''}\ge0.\end{aligned} \tag{3.11}$$

Using Gronwall's Lemma we have

$$\sup_{t\in[0,T]}\Big\{\|u_N(t)\|^2_{H^1(\Omega)}+\|\overline{\alpha_N}(t)\|^2_{H^1(\Omega)}+\|\frac{\partial\alpha_N}{\partial t}(t)\|^2\Big\}\le c, \tag{3.12}$$

where $c$ is independent of $N$. Hence there exists a subsequence of $(u_N,\overline{\alpha_N},\frac{\partial\alpha_N}{\partial t})$ that we denote again by $(u_N,\overline{\alpha_N},\frac{\partial\alpha_N}{\partial t})$ which satisfies as $N\to+\infty$

$$u_N\overset{*}{\rightharpoonup}u\quad\text{weakly star in}\quad L^\infty(0,T,H^1(\Omega)), \tag{3.13}$$

$$\overline{\alpha_N}\overset{*}{\rightharpoonup}\overline{\alpha}\quad\text{weakly star in}\quad L^\infty(0,T,H^1(\Omega)), \tag{3.14}$$

$$\frac{\partial\alpha_N}{\partial t}\overset{*}{\rightharpoonup}\frac{\partial\alpha}{\partial t}\quad\text{weakly star in}\quad L^\infty(0,T,L^2(\Omega)). \tag{3.15}$$

Moreover, integrating (3.9) over $(0,t)$, we obtain

$$E_{2N}(t)+\int_0^t\Big(\|\Delta u_N\|^2+\|\frac{\partial u_N}{\partial t}\|^2\Big)ds\le c,\quad\forall t\in[0,T],\quad c\ge0, \tag{3.16}$$

where $c$ is independent of $N$. We then deduce

$$\frac{\partial u_N}{\partial t}\rightharpoonup\frac{\partial u}{\partial t}\quad\text{weakly in}\quad L^2(0,T,L^2(\Omega)), \tag{3.17}$$

$$\Delta u_N\rightharpoonup\Delta u\quad\text{weakly in}\quad L^2(0,T,L^2(\Omega)). \tag{3.18}$$

Replacing $H$ by $H_N$ in (2.5), we write

$$\langle H_N(t)\rangle = e^{-t}\langle H_N(0)\rangle, \tag{3.19}$$

which can be written as

$$\frac{d\langle \alpha_N\rangle}{dt}(t) = e^{-t}\langle H_N(0)\rangle - \langle u_N(t)\rangle. \tag{3.20}$$

Integrating (3.20) over $(0,t)$ we obtain

$$\langle \alpha_N(t)\rangle = \langle \alpha_0\rangle + \langle H_N(0)\rangle(1-e^{-t}) - \int_0^t \langle u_N(s)\rangle ds. \tag{3.21}$$

We then deduce

$$\langle \alpha_N(t)\rangle^2 \le c(E_{2N}(0)+\|\alpha_0\|^2)+c^{'}t^2. \tag{3.22}$$

Using the equivalent norm in $H^1(\Omega)$ we get

$$\sup_{t\in[0,T]} \|\alpha_N\|^2_{H^1(\Omega)} \le c, \tag{3.23}$$

where $c$ is independent of $N$. We deduce that

$$\alpha_N \overset{*}{\rightharpoonup} \alpha \quad \text{weakly star in} \quad L^\infty(0,T,H^1(\Omega)). \tag{3.24}$$

We now multiply (3.5) by $g_N(u_N)$ and integrate over $\Omega$ using $g_N^{'}(s)\ge -c$ to have

$$\|g_N(u_N)\|^2 \le c\Big(\|\nabla u_N\|^2 + \|\frac{\partial u_N}{\partial t}\|^2 + \|\frac{\partial \alpha_N}{\partial t}\|^2\Big). \tag{3.25}$$

Integrating (3.25) over $(0,t)$, we deduce

$$\|g_N(u_N)\|^2_{L^2(Q)} \le c, \tag{3.26}$$

where $c$ is independent of $N$ and $Q=\Omega\times(0,T)$.

By (3.26) and for a subsequence we obtain

$$g_N(u_N) \rightharpoonup g^* \quad \text{weakly in} \quad L^2(Q). \tag{3.27}$$

Letting $N\to+\infty$ in the equation (3.5), we deduce from (3.15), (3.17), (3.18) and (3.27) that $(u,\alpha,\frac{\partial\alpha}{\partial t})$ satisfies

$$\frac{\partial u}{\partial t} - \Delta u + g^* = \frac{\partial \alpha}{\partial t} \quad \text{in} \quad L^2(Q). \tag{3.28}$$

From (3.6) we deduce (setting $\beta_N = \frac{\partial \alpha_N}{\partial t}$) :

$$\Big\langle \frac{\partial \beta_N}{\partial t} + \frac{\partial \alpha_N}{\partial t}, \chi\Big\rangle + \langle \nabla\alpha_N, \nabla\chi\rangle = -\Big\langle \frac{\partial u_N}{\partial t} + u_N, \chi\Big\rangle, \quad \forall \chi \in \mathcal{D}((0,T)\times\Omega),$$

where $\langle .,. \rangle$ denote the duality product between $\mathcal{D}'((0,T)\times\Omega)$ and $\mathcal{D}((0,T)\times\Omega)$.

Then letting $N \longrightarrow +\infty$, using (3.13), (3.15), (3.17) and (3.24) we deduce

$$\frac{\partial^2\alpha}{\partial t^2}+\frac{\partial\alpha}{\partial t}-\Delta\alpha=-\frac{\partial u}{\partial t}-u \quad \text{in} \quad L^2(Q). \tag{3.29}$$

Moreover using [12], (3.13), (3.17) on the one hand and (3.15), (3.24) on the other hand implies that as $N \longrightarrow +\infty$

$$u_N \longrightarrow u \qquad \text{in} \qquad C([0,T],L^2(\Omega)), \tag{3.30}$$

$$\alpha_N \longrightarrow \alpha \qquad \text{in} \qquad C([0,T],L^2(\Omega)), \tag{3.31}$$

so that in particular $u(x,0)=u_0$ and $\alpha(x,0)=\alpha_0$ in $\Omega$.

Furthermore we deduce from (3.15) that $\dfrac{\partial\alpha_N}{\partial t}\in L^2(0,T;H^{-1}(\Omega))$. On the other hand we have $\dfrac{\partial\beta_N}{\partial t}\in L^2(0,T;H^{-1}(\Omega))$, so that

$$\frac{\partial\alpha_N}{\partial t}\in C([0,T],H^{-1}(\Omega)).$$

Using Strauss Theorem, we get $\dfrac{\partial\alpha_N}{\partial t}\in C_W([0,T],L^2(\Omega))$ and there exist a subsequence $\dfrac{\partial\alpha_\mu}{\partial t}\in C_W([0,T],L^2(\Omega))$ such that in particular as $\mu\rightarrow+\infty$ we have

$$\frac{\partial\alpha_\mu}{\partial t}(0)\rightharpoonup\frac{\partial\alpha}{\partial t}(0) \quad \text{weakly in} \quad L^2(\Omega).$$

Since $\dfrac{\partial\alpha_\mu}{\partial t}(0)\rightarrow\alpha_1$ in $L^2(\Omega)$, we deduce that $\dfrac{\partial\alpha}{\partial t}(x,0)=\alpha_1$.

Note also that using Lions' Theorem and (3.13)–(3.15), (3.17) and (3.24), we get

$$u\in L^\infty(0,T,H^1(\Omega)), \quad \frac{\partial u}{\partial t}\in L^2(0,T,L^2(\Omega)), \quad \alpha\in L^\infty(0,T,H^1(\Omega)),$$
$$\overline{\alpha}\in L^\infty(0,T,H^1(\Omega)), \quad \frac{\partial\alpha}{\partial t}\in L^\infty(0,T,L^2(\Omega)), \quad u,\alpha\in C([0,T],L^2(\Omega)).$$

Hence $(u,\alpha,\dfrac{\partial\alpha}{\partial t})$ satisfy

$$\frac{\partial u}{\partial t}-\Delta u+g^*=\frac{\partial\alpha}{\partial t} \qquad \text{in } L^2(Q), \tag{3.32}$$

$$\frac{\partial^2\alpha}{\partial t^2}+\frac{\partial\alpha}{\partial t}-\Delta\alpha=-\frac{\partial u}{\partial t}-u \qquad \text{in } L^2(Q), \tag{3.33}$$

$$\frac{\partial u}{\partial\nu}=\frac{\partial\alpha}{\partial\nu}=0 \qquad \text{on } \partial\Omega\times(0,T), \tag{3.34}$$

$$u(0)=u_0, \quad \alpha(0)=\alpha_0, \quad \frac{\partial\alpha}{\partial t}(0)=\alpha_1 \qquad \text{in } \Omega. \tag{3.35}$$

We now prove that $g^\star = g(u)$ and the set $\{x\in\Omega,\ |u(x,t)|=1\}$ has measure zero. To do so we adapt a method introduced by Debussche and Dettori [7]. For an arbitrary small $\eta\in(0,1)$ and for all $t\in(0,T)$, we set

$$E_\eta^N(t)=\Big\{x\in\Omega/|u_N(x,t)|>1-\eta\Big\},$$

and we denote by $|E_\eta^N(t)|$ its measure namely $|E_\eta^N(t)|=meas(E_\eta^N(t))=\int_{E_\eta^N(t)}dx$ and by $\chi_\eta^N(t)$ its characteristic function :

$$\chi_\eta^N(x,t)=\begin{cases}1, & \text{if}\quad x\in E_\eta^N(t),\\ 0, & \text{elsewhere.}\end{cases}$$

Integrating (3.9) over $(t,t+r)$, we obtain

$$E_{2N}(t+r)+\int_t^{t+r}\Big(E_{2N}+\|\frac{\partial u_N}{\partial t}\|^2\Big)ds\le c^{'}(r)+E_{2N}(t),\quad \forall t\ge 0,\quad \forall r>0. \tag{3.36}$$

To continue the proof of the theorem we state the following two lemmas.

**Lemma 3.1.** *There exists a constant c such that for all $r>0$*

$$\|\frac{\partial u_N}{\partial t}(t)\|^2\le c(\frac{1}{r}+1)(E_{2N}(0)+1),\quad \forall t\ge r>0. \tag{3.37}$$

*Proof.* Replacing $(u,\alpha)$ in (2.20) by $(u_N,\alpha_N)$, we write

$$\frac{d}{dt}\|\frac{\partial u_N}{\partial t}\|^2+\|\nabla\frac{\partial u_N}{\partial t}\|^2\le c\Big(\|u_N\|^2+\|\overline{\alpha_N}\|^2_{H^1(\Omega)}+\|\frac{\partial u_N}{\partial t}\|^2+\|\frac{\partial\alpha_N}{\partial t}\|^2\Big). \tag{3.38}$$

Applying the uniform Gronwall's Lemma to (3.38), using (3.11) and (3.36) we deduce that $\forall s>0$,

$$\|\frac{\partial u_N}{\partial t}(t+s)\|^2\le c(\frac{1}{s}+1)(E_{2N}(0)+1),\quad \forall t\ge 0, \tag{3.39}$$

which completes the proof of (3.37). □

**Lemma 3.2.** *There exists a constant c such that for all $r>0$*

$$\|g_N(u_N(t))\|^2\le c(\frac{1}{r}+1)(E_{2N}(0)+1),\quad \forall t\ge r>0. \tag{3.40}$$

*Proof.* Applying Gronwall's lemma to (3.9), using (3.11) we deduce that

$$\|\nabla u_N(t)\|^2+\|\frac{\partial\alpha_N}{\partial t}(t)\|^2\le E_{2N}(0)+c^{'},\quad \forall t\ge r>0. \tag{3.41}$$

By (3.37) and (3.41), we get from (3.25) the inequality (3.40). □

Using Lemma 3.2 we deduce that

$$t\|g_N(u_N(t))\|^2 \le c(T+1), \quad \forall t\in(0,T), \tag{3.42}$$

and thus

$$\begin{aligned}\Big\{\int_{E_\eta^N(t)} g_N^2(u_N)dx\Big\}^{\frac{1}{2}} &\ge |E_\eta^N(t)|^{\frac{1}{2}}.\Big\{Inf_{x\in E_\eta^N(t)}\Big(\sum_{k=0}^N \frac{(u_N(x))^{2k+1}}{2k+1}\Big)^2\Big\}^{\frac{1}{2}}\\ &\ge |E_\eta^N(t)|^{\frac{1}{2}}.Inf_{x\in E_\eta^N(t)}\sum_{k=0}^N \frac{|u_N(x)|^{2k+1}}{2k+1}\\ &\ge |E_\eta^N(t)|^{\frac{1}{2}}.\sum_{k=0}^N \frac{(1-\eta)^{2k+1}}{2k+1},\end{aligned}$$

which implies that

$$|E_\eta^N(t)|^{\frac{1}{2}} \le \frac{C}{\sqrt{t}\sum_{k=0}^N \frac{(1-\eta)^{2k+1}}{2k+1}}. \tag{3.43}$$

Thus letting $N\to+\infty$ we deduce from (3.30), (3.43) and Fatou's Lemma that

$$\begin{aligned}|E_\eta(t)| &= \int_\Omega \chi_\eta(t)dx \le \int_\Omega \liminf_{N\to+\infty}\chi_\eta^N(t)dx\\ &\le \liminf_{N\to+\infty}\int_\Omega \chi_\eta^N(t)dx \le \liminf|E_\eta^N(t)| \le \frac{4C}{t\ln^2\left(\frac{2-\eta}{\eta}\right)},\end{aligned}$$

where $|E_\eta(t)|$ and $\chi_\eta(t)$ respectively stand for the measure of the set $\{x\in\Omega,\ |u(x,t)|>1-\eta\}$ and for its characteristic function. Letting then $\eta\to 0$, it follows that for all $t\in(0,T)$

$$Meas\{x\in\Omega,\ |u(x,t)|\ge 1\}=0. \tag{3.44}$$

It follows respectively from (3.30) and (3.44) that for all $t\in(0,T)$ and almost every $x\in\Omega$

$$g_N(x,t) \longrightarrow g(x,t). \tag{3.45}$$

Then using Lions ([8], lemma 1.3, p.12) it follows from (3.26) and (3.45) that

$$g_N(u_N) \rightharpoonup g(u) \quad \text{weakly in} \quad L^2(Q), \tag{3.46}$$

so that $g^*=g(u)$.

Concerning the uniqueness, let $\left(u^{(1)},\alpha^{(1)},\frac{\partial\alpha^{(1)}}{\partial t}\right)$ and $\left(u^{(2)},\alpha^{(2)},\frac{\partial\alpha^{(2)}}{\partial t}\right)$ be two solutions with initial data $\left(u_0^{(1)},\alpha_0^{(1)},\alpha_1^{(1)}\right)$ and $\left(u_0^{(2)},\alpha_0^{(2)},\alpha_1^{(2)}\right)$, respectively. We set

$$\begin{aligned}\Big(u,\alpha,\frac{\partial\alpha}{\partial t}\Big) &= \Big(u^{(1)}-u^{(2)},\alpha^{(1)}-\alpha^{(2)},\frac{\partial\alpha^{(1)}}{\partial t}-\frac{\partial\alpha^{(2)}}{\partial t}\Big),\\ \Big(u_0,\alpha_0,\alpha_1\Big) &= \Big(u_0^{(1)}-u_0^{(2)},\alpha_0^{(1)}-\alpha_0^{(2)},\alpha_1^{(1)}-\alpha_1^{(2)}\Big).\end{aligned}$$

We then have

$$\frac{\partial u}{\partial t}-\Delta u+g(u^{(1)})-g(u^{(2)})=\frac{\partial \alpha}{\partial t}, \tag{3.47}$$

$$\frac{\partial^2 \alpha}{\partial t^2}+\frac{\partial \alpha}{\partial t}-\Delta \alpha=-\frac{\partial u}{\partial t}-u, \tag{3.48}$$

$$\frac{\partial u}{\partial \nu}=\frac{\partial \alpha}{\partial \nu}=0, \tag{3.49}$$

$$u(0)=u_0, \quad \alpha(0)=\alpha_0, \quad \frac{\partial \alpha}{\partial t}(0)=\alpha_1, \tag{3.50}$$

which is equivalent to

$$\frac{\partial u}{\partial t}-\Delta u+l(t)u=\frac{\partial \alpha}{\partial t}, \tag{3.51}$$

$$\frac{\partial^2 \alpha}{\partial t^2}+\frac{\partial \alpha}{\partial t}-\Delta \alpha=-\frac{\partial u}{\partial t}-u, \tag{3.52}$$

$$\frac{\partial u}{\partial \nu}=\frac{\partial \alpha}{\partial \nu}=0, \tag{3.53}$$

$$u(0)=u_0, \quad \alpha(0)=\alpha_0, \quad \frac{\partial \alpha}{\partial t}(0)=\alpha_1, \tag{3.54}$$

where $l(t)=\int_0^1 g'(su^{(1)}(t)+(1-s)u^{(2)}(t))ds$.

In one space dimension, by (2.41) we have for all $t\geq 0$,

$$\|u^{(i)}(t)\|_{L^\infty}\leq 1-\delta_i, \quad \delta_i=\delta_i(D(u_0^{(i)}),\|u_0^{(i)}\|_{H^2},\|\overline{\alpha_0^{(i)}}\|_{H^2},\|\alpha_1^{(i)}\|_{H^1}), \quad i=1,2.$$

We set $\delta_0=\min(\delta_1,\delta_2)$ and then deduce

$$\|su^{(1)}(t)+(1-s)u^{(2)}(t)\|_{L^\infty}\leq 1-\delta_0, \quad \forall\, 0\leq s\leq 1,$$

hence

$$\|l(t)\|_{L^\infty}\leq C(=C(\delta_0)). \tag{3.55}$$

**Remark 3.1.** In two space dimension, we have

$$\delta_i=\delta_i\Big(D(u_0^{(i)}),\|u_0^{(i)}\|_{H^3},\|\overline{\alpha_0^{(i)}}\|_{H^3},\|\alpha_1^{(i)}\|_{H^2}\Big), \quad i=1,2.$$

Multiplying (3.51) by $u+\frac{\partial u}{\partial t}$ and (3.52) by $\frac{\partial \alpha}{\partial t}$, integrating over $\Omega$ and summing the resulting equations, we obtain (note that this is where (3.55) is used)

$$\frac{dE_7}{dt}\leq cE_7, \tag{3.56}$$

where

$$E_7 = \|u\|^2 + \|\nabla u\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2 + \|\nabla \alpha\|^2. \tag{3.57}$$

This yields, owing to Gronwall's lemma,

$$E_7(t) \leq e^{ct} E_7(0). \tag{3.58}$$

Integrating then (3.52) over $\Omega$, we have, as above,

$$\langle H(t) \rangle = e^{-t} \langle H(0) \rangle, \tag{3.59}$$

where, again, $H = u + \frac{\partial \alpha}{\partial t}$, hence

$$\langle \alpha(t) \rangle^2 \leq c_T (\|u\|^2_{L^2(\Omega \times (0,T))} + E_7(0) + \|\alpha_0\|^2), \quad t \in [0,T]. \tag{3.60}$$

Noting that it follows from (3.58) that

$$\|u(t)\|^2_{H^1(\Omega)} + \|\nabla \alpha(t)\|^2 + \|\frac{\partial \alpha}{\partial t}(t)\|^2 \leq c(\|u_0\|^2_{H^1(\Omega)} + \|\alpha_0\|^2_{H^1(\Omega)} + \|\alpha_1\|^2), \tag{3.61}$$

where $c$ depends on $T$ and $\delta_0$, which yields, in particular,

$$\|u\|^2_{L^2(\Omega \times (0,T))} \leq c_T (\|u_0\|^2_{H^1} + \|\alpha_0\|^2_{H^1} + \|\alpha_1\|^2), \quad t \in [0,T], \tag{3.62}$$

we finally deduce from (3.60)-(3.62) that

$$\|u(t)\|^2_{H^1(\Omega)} + \|\alpha(t)\|^2_{H^1(\Omega)} + \|\frac{\partial \alpha}{\partial t}(t)\|^2 \leq c(\|u_0\|^2_{H^1(\Omega)} + \|\alpha_0\|^2_{H^1(\Omega)} + \|\alpha_1\|^2), \tag{3.63}$$

where $c$ depends on $T$ and $\delta_0$, hence the uniqueness, as well as the continuous dependence with respect to the initial data. $\square$

Thanks to Theorem 3.1(i), we can define the dissipative semigroup $\overline{S}(t)$ associated with problem (1.13)-(1.16) on the phase space $\overline{\Phi_0}$.

Taking $D(u_0) + \|u_0\|^2_{H^2} + \|\overline{\alpha_0}\|^2_{H^2} + \|\alpha_1\|^2_{H^1} \leq R$, $R > 0$, we obtain that $B^1_R$ is a bounded aborbing set for $\overline{S}(t)$, where

$$B^1_R = \Big\{ (u, \overline{\alpha}, \frac{\partial \alpha}{\partial t}) \in \overline{F_1},\ D(u(t)) + \|u\|^2_{H^2} + \|\overline{\alpha}\|^2_{H^2} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^1} \leq R \Big\}.$$

Indeed, by (3.2) we have

$$D(u(t)) + \|u(t)\|^2_{H^2} + \|\overline{\alpha}(t)\|^2_{H^2} + \|\frac{\partial \alpha}{\partial t}(t)\|^2_{H^1} \leq R, \quad \forall t \geq t_0. \tag{3.64}$$

Concerning the two-dimensional case, we have that

$$B^2_R = \Big\{ (u, \overline{\alpha}, \frac{\partial \alpha}{\partial t}) \in \overline{F_2},\ D(u(t)) + \|u\|^2_{H^3} + \|\overline{\alpha}\|^2_{H^3} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2} \leq R \Big\}$$

is a bounded absorbing set for $\overline{S}(t)$ in $\overline{\Psi_0}$. Indeed, we have

$$D(u(t)) + \|u(t)\|^2_{H^3} + \|\overline{\alpha}(t)\|^2_{H^3} + \|\frac{\partial \alpha}{\partial t}(t)\|^2_{H^2} \leq R, \quad \forall t \geq t_1. \tag{3.65}$$

# 4 Global attractor

We have the

**Theorem 4.1.** *(i) If $n=1$, we take the initial conditions in $B^1_R$. Then the semigroup $\overline{S}(t)$, $t\geq 0$, defined from $\overline{\Phi_0}$ to itself possesses the connected global attractor $\overline{\mathcal{A}}_1$ in $\overline{\Phi_0}$.*

*(ii) If $n=2$, the initial conditions belong to $B^2_R$. Then $\overline{S}(t)$ defined from $\overline{\Psi_0}$ to itself possesses the connected global attractor $\overline{\mathcal{A}}_2$.*

*Proof.* We use a semigroup decomposition argument (see, e.g., [6]) consisting in splitting the semigroup $\overline{S}(t)$, $t\geq 0$, into the sum of two families of operators : $\overline{S(t)}=\overline{S_1}(t)+\overline{S_2}(t)$, where operators $\overline{S_1}(t)$ go to zero as $t$ tends to infinity while operators $\overline{S_2}(t)$ are compact.

This corresponds to the following solution decomposition

$$(u,\overline{\alpha},\frac{\partial\alpha}{\partial t})=(u^d,\overline{\alpha^d},\frac{\partial\alpha^d}{\partial t})+(u^c,\overline{\alpha^c},\frac{\partial\alpha^c}{\partial t}),$$

where $(u^d,\overline{\alpha^d},\frac{\partial\alpha^d}{\partial t})$ is solution to

$$\frac{\partial u^d}{\partial t}+u^d-\Delta u^d=\frac{\partial\alpha^d}{\partial t}, \tag{4.1}$$

$$\frac{\partial^2\alpha^d}{\partial t^2}+\frac{\partial\alpha^d}{\partial t}-\Delta\alpha^d=-\frac{\partial u^d}{\partial t}-u^d, \tag{4.2}$$

$$\frac{\partial u^d}{\partial \nu}=\frac{\partial\alpha^d}{\partial \nu}=0, \tag{4.3}$$

$$u^d(0)=u_0,\quad \alpha^d(0)=\alpha_0,\quad \frac{\partial\alpha^d}{\partial t}(0)=\alpha_1, \tag{4.4}$$

and $(u^c,\overline{\alpha^c},\frac{\partial\alpha^c}{\partial t})$ solves

$$c\frac{\partial u^c}{\partial t}+u^c-\Delta u^c+f(u)=\frac{\partial\alpha^c}{\partial t}, \tag{4.5}$$

$$\frac{\partial^2\alpha^c}{\partial t^2}+\frac{\partial\alpha^c}{\partial t}-\Delta\alpha^c=-\frac{\partial u^c}{\partial t}-u^c, \tag{4.6}$$

$$\frac{\partial u^c}{\partial \nu}=\frac{\partial\alpha^c}{\partial \nu}=0, \tag{4.7}$$

$$u^c(0)=0,\quad \alpha^c(0)=0,\quad \frac{\partial\alpha^c}{\partial t}(0)=0, \tag{4.8}$$

where $f(s)=g(s)-s$ ($f$ and $g$ satisfy the same properties) and with initial data belonging to $B^1_R$. Multiplying (4.1) by $-\Delta u^d-\Delta\frac{\partial u^d}{\partial t}$, (4.2) by $-\Delta\frac{\partial\alpha^d}{\partial t}$ and summing the resulting equations, we have

$$\frac{1}{2}\frac{d}{dt}\Big(2\|\nabla u^d\|^2+\|\Delta u^d\|^2+\|\nabla\frac{\partial\alpha^d}{\partial t}\|^2+\|\Delta\alpha^d\|^2\Big)$$
$$+\|\nabla u^d\|^2+\|\Delta u^d\|^2+\|\nabla\frac{\partial u^d}{\partial t}\|^2+\|\nabla\frac{\partial\alpha^d}{\partial t}\|^2=0. \tag{4.9}$$

We set $H_1=u^d+\frac{\partial\alpha^d}{\partial t}$ and by analogy with the relation (2.8), we write

$$\frac{\partial\overline{H_1}}{\partial t}+\overline{H_1}-\Delta\overline{\alpha^d}=0. \tag{4.10}$$

Multiplying (4.10) by $-\Delta\alpha^d-\Delta\frac{\partial\alpha^d}{\partial t}$, using Hölder and Young's inequalities and

$$\|\overline{\varphi}\|^2_{H^2(\Omega)}\le c\|\Delta\varphi\|^2,\qquad \forall\varphi\in H^2(\Omega),\ c>0,$$

we get

$$\frac{d}{dt}\Big(\|\Delta\alpha^d\|^2+\|\nabla\frac{\partial\alpha^d}{\partial t}\|^2+\epsilon_3\|\nabla\alpha^d\|^2+2\epsilon_3(\nabla\frac{\partial\alpha^d}{\partial t},\nabla\alpha^d)\Big)$$
$$+c\Big(\|\nabla\frac{\partial\alpha^d}{\partial t}\|^2+\|\Delta\alpha^d\|^2\Big)\le c'\Big(\|\nabla u^d\|^2+\|\nabla\frac{\partial u^d}{\partial t}\|^2\Big),\qquad c>0, \tag{4.11}$$

where $\epsilon_3>0$ is small enough, and we have in particular

$$\|\nabla\frac{\partial\alpha^d}{\partial t}\|^2+\epsilon_3\|\nabla\alpha^d\|^2+2\epsilon_3\Big(\nabla\frac{\partial\alpha^d}{\partial t},\nabla\alpha^d\Big)\ge c\Big(\|\nabla\alpha^d\|^2+\|\nabla\frac{\partial\alpha^d}{\partial t}\|^2\Big),\quad c>0. \tag{4.12}$$

Summing (4.9) and $\epsilon_4\times$(4.11) where $\epsilon_4>0$ is small enough, we have

$$\frac{dE_8}{dt}+c(E_8+\|\nabla\frac{\partial u^d}{\partial t}\|^2)\le 0,\quad c>0, \tag{4.13}$$

where

$$E_8=2\|\nabla u^d\|^2+\|\Delta u^d\|^2+\|\nabla\frac{\partial\alpha^d}{\partial t}\|^2+\|\Delta\alpha^d\|^2$$
$$+\epsilon_4\Big(\|\Delta\alpha^d\|^2+\|\nabla\frac{\partial\alpha^d}{\partial t}\|^2+\epsilon_3\|\nabla\alpha^d\|^2+2\epsilon_3(\nabla\frac{\partial\alpha^d}{\partial t},\nabla\alpha^d)\Big). \tag{4.14}$$

Applying Gronwall's lemma to (4.13), we write

$$\|\Delta u^d\|^2+\|\overline{\alpha^d}\|^2_{H^2}+\|\nabla\frac{\partial\alpha^d}{\partial t}\|^2\le c'e^{-ct}(\|u_0\|^2_{H^2}+\|\overline{\alpha_0}\|^2_{H^2}+\|\alpha_1\|^2_{H^1}). \tag{4.15}$$

Now, we multiply (4.1) by $u^d+\frac{\partial u^d}{\partial t}$ and (4.2) by $\frac{\partial \alpha^d}{\partial t}$ and summing the resulting equations we get

$$\frac{d}{dt}\Big(2\|u^d\|^2+\|\nabla u^d\|^2+\|\nabla\alpha^d\|^2+\|\frac{\partial\alpha^d}{\partial t}\|^2\Big)$$
$$+2\Big(\|u^d\|^2+\|\nabla u^d\|^2+\|\frac{\partial u^d}{\partial t}\|^2+\|\frac{\partial\alpha^d}{\partial t}\|^2\Big)=0. \tag{4.16}$$

Considering equation (4.10) and repeating exactly the estimates that gave (2.13), we get

$$\frac{dE_9}{dt}+c(E_9+\|\frac{\partial u^d}{\partial t}\|^2)\le 0,\quad c>0, \tag{4.17}$$

where

$$E_9=2\|u^d\|^2+\|\nabla u^d\|^2+\|\nabla\alpha^d\|^2+\|\frac{\partial\alpha^d}{\partial t}\|^2$$
$$+\epsilon_6\Big(\|\frac{\partial\overline{\alpha^d}}{\partial t}\|^2+\|\nabla\alpha^d\|^2+\epsilon_5(\|\overline{\alpha^d}\|^2+2(\frac{\partial\overline{\alpha^d}}{\partial t},\overline{\alpha^d}))\Big), \tag{4.18}$$

and $\epsilon_5>0$ and $\epsilon_6>0$ are small enough so that we have in particular

$$\|\frac{\partial\overline{\alpha^d}}{\partial t}\|^2+\epsilon_5\Big(\|\overline{\alpha^d}\|^2+2(\frac{\partial\overline{\alpha^d}}{\partial t},\overline{\alpha^d})\Big)\ge c\Big(\|\overline{\alpha^d}\|^2+\|\frac{\partial\overline{\alpha^d}}{\partial t}\|^2\Big),\quad c>0. \tag{4.19}$$

Applying Gronwall's lemma to (4.17), we have

$$\|u^d\|^2_{H^1}+\|\overline{\alpha^d}\|^2_{H^1}+\|\frac{\partial\alpha^d}{\partial t}\|^2\le c'e^{-ct}(\|u_0\|^2_{H^2}+\|\overline{\alpha_0}\|^2_{H^2}+\|\alpha_1\|^2_{H^1}). \tag{4.20}$$

Combining (4.15) and (4.20), we obtain

$$\|u^d(t)\|^2_{H^2}+\|\overline{\alpha^d}(t)\|^2_{H^2}+\|\frac{\partial\alpha^d}{\partial t}(t)\|^2_{H^1}\le c'e^{-ct}(\|u_0\|^2_{H^2}+\|\overline{\alpha_0}\|^2_{H^2}+\|\alpha_1\|^2_{H^1}). \tag{4.21}$$

We can see that $\overline{S_1}(t)$ tends to zero as $t$ tends to infinity.

Now, we consider system (4.5)-(4.8).

We multiply (4.5) by $-\Delta u^c$. Integrating over $\Omega$ we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla u^c\|^2+\|\nabla u^c\|^2+\|\Delta u^c\|^2-(f(u),\Delta u^c)=\Big(\nabla\frac{\partial\alpha^c}{\partial t},\nabla u^c\Big). \tag{4.22}$$

We multiply (4.5) by $-\Delta\frac{\partial u^c}{\partial t}$. Integrating over $\Omega$ we obtain

$$\|\nabla\frac{\partial u^c}{\partial t}\|^2+\frac{1}{2}\frac{d}{dt}\|\nabla u^c\|^2+\frac{1}{2}\frac{d}{dt}\|\Delta u^c\|^2-\Big(f(u),\Delta\frac{\partial u^c}{\partial t}\Big)=\Big(\nabla\frac{\partial\alpha^c}{\partial t},\nabla\frac{\partial u^c}{\partial t}\Big). \tag{4.23}$$

We finally multiply (4.6) by $-\Delta\frac{\partial\alpha^c}{\partial t}$. Integrating over $\Omega$ we find

$$\|\nabla\frac{\partial\alpha^c}{\partial t}\|^2+\frac{1}{2}\frac{d}{dt}\|\nabla\frac{\partial\alpha^c}{\partial t}\|^2+\frac{1}{2}\frac{d}{dt}\|\Delta\alpha^c\|^2=-\Big(\nabla\frac{\partial\alpha^c}{\partial t},\nabla\frac{\partial u^c}{\partial t}\Big)-\Big(\nabla u^c,\nabla\frac{\partial\alpha^c}{\partial t}\Big). \tag{4.24}$$

Summing the resulting equations, we get

$$\begin{aligned}&\frac{d}{dt}\Big(2\|\nabla u^c\|^2+\|\Delta u^c\|^2+\|\Delta\alpha^c\|^2+\|\nabla\frac{\partial\alpha^c}{\partial t}\|^2\Big)\\&\quad+c\Big(\|\nabla u^c\|^2+\|\Delta u^c\|^2+\|\nabla\frac{\partial u^c}{\partial t}\|^2+\|\nabla\frac{\partial\alpha^c}{\partial t}\|^2\Big)\le c'\|f'(u)\nabla u\|^2,\quad c>0.\end{aligned} \tag{4.25}$$

We multiply (4.5) by $\Delta^2\frac{\partial u^c}{\partial t}$. Integrating over $\Omega$ we have

$$\frac{d}{dt}(\|\Delta u^c\|^2+\|\nabla\Delta u^c\|^2)+\|\Delta\frac{\partial u^c}{\partial t}\|^2\le\|\Delta f(u)\|^2+\|\Delta\frac{\partial\alpha^c}{\partial t}\|^2. \tag{4.26}$$

We then multiply (4.6) by $\Delta^2\frac{\partial\alpha^c}{\partial t}$ and by integrating over $\Omega$ we obtain

$$\frac{d}{dt}(\|\Delta\frac{\partial\alpha^c}{\partial t}\|^2+\|\nabla\Delta\alpha^c\|^2)+\|\Delta\frac{\partial\alpha^c}{\partial t}\|^2\le c(\|\Delta u^c\|^2+\|\Delta\frac{\partial u^c}{\partial t}\|^2). \tag{4.27}$$

Summing (4.25), (4.26) and $\epsilon_7\times$(4.27) where $\epsilon_7>0$ is small enough, we have

$$\frac{d\psi_1}{dt}\le c(\psi_1+\|\Delta f(u)\|^2+\|f'(u)\nabla u\|^2), \tag{4.28}$$

where

$$\begin{aligned}\psi_1=&2\|\Delta u^c\|^2+\|\nabla\Delta u^c\|^2+2\|\nabla u^c\|^2+\|\Delta\alpha^c\|^2\\&+\|\nabla\frac{\partial\alpha^c}{\partial t}\|^2+\epsilon_7\Big(\|\nabla\Delta\alpha^c\|^2+\|\Delta\frac{\partial\alpha^c}{\partial t}\|^2\Big).\end{aligned} \tag{4.29}$$

By (3.2), we deduce that

$$\|\Delta f(u)\|^2+\|f'(u)\nabla u\|^2\le c,\quad\forall t\ge 0, \tag{4.30}$$

where $c$ depends on $D(u_0),\|u_0\|_{H^2},\|\overline{\alpha_0}\|_{H^2}$ et $\|\alpha_1\|_{H^1}$.

By (4.30), we deduce from (4.28) the following estimate

$$\frac{d\psi_1}{dt}\le c\psi_1+c'. \tag{4.31}$$

Applying Gronwall's lemma to (4.31) (noting that $\psi_1(0)=0$) and by using (4.29) we obtain

$$\|\nabla\Delta u^c\|^2+\|\overline{\alpha^c}\|^2_{H^3}+\|\Delta\frac{\partial\alpha^c}{\partial t}\|^2\le C_{T,\|(u_0,\overline{\alpha_0},\alpha_1)\|_{F_1},B^1_R}. \tag{4.32}$$

Multiplying (4.5) by $u^c+\dfrac{\partial u^c}{\partial t}$ and (4.6) by $\dfrac{\partial \alpha^c}{\partial t}$, summing the resulting equations, we have

$$\begin{aligned}&\frac{d}{dt}\Big(2\|u^c\|^2+\|\nabla u^c\|^2+\|\nabla \alpha^c\|^2+\|\frac{\partial \alpha^c}{\partial t}\|^2\Big)\\&\quad+c\Big(\|u^c\|^2+\|\nabla u^c\|^2+\|\frac{\partial u^c}{\partial t}\|^2+\|\frac{\partial \alpha^c}{\partial t}\|^2\Big)\le c'\|f(u)\|^2.\end{aligned}\tag{4.33}$$

Summing now (4.23), (4.33) and $\epsilon_8\times$(4.24), where $\epsilon_8>0$ is small enough, we deduce that

$$\frac{d\psi_2}{dt}\le c(\psi_2+\|f(u)\|^2+\|f'(u)\nabla u\|^2),\tag{4.34}$$

where

$$\psi_2=2\|u^c\|^2+2\|\nabla u^c\|^2+\|\nabla \alpha^c\|^2+\|\frac{\partial \alpha^c}{\partial t}\|^2+\|\Delta u^c\|^2+\epsilon_8\Big(\|\Delta \alpha^c\|^2+\|\nabla\frac{\partial \alpha^c}{\partial t}\|^2\Big).\tag{4.35}$$

Applying Gronwall's lemma to (4.34), using (3.2) and (4.35) we have

$$\|u^c\|^2_{H^2}+\|\Delta \alpha^c\|^2+\|\frac{\partial \alpha^c}{\partial t}\|^2_{H^1}\le C_{T,\|u_0\|_{H^2},\|\overline{\alpha_0}\|_{H^2},\|\alpha_1\|_{H^1},B^1_R}.\tag{4.36}$$

Combining (4.32) and (4.36), we get

$$\|u^c(t)\|^2_{H^3}+\|\overline{\alpha^c}(t)\|^2_{H^3}+\|\frac{\partial \alpha^c}{\partial t}\|^2_{H^2}\le C_{T,\|u_0\|_{H^2},\|\overline{\alpha_0}\|_{H^2},\|\alpha_1\|_{H^1},B^1_R}.\tag{4.37}$$

Hence, the operator $\overline{S_2}(t)$ is asymptotically compact in the sense of the Kuratowski measure of noncompactness (see [18]), which concludes the existence part of Theorem 4.1 (i).

In order to prove part (ii) of Theorem 4.1, we now take the initial data in $B^2_R$, then multiply (4.1) by $\Delta^2 u^d+\Delta^2\dfrac{\partial u^d}{\partial t}$ and (4.2) by $\Delta^2\dfrac{\partial \alpha^d}{\partial t}$. Summing the two resulting equations, we end up with

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(2\|\Delta u^d\|^2+\|\nabla\Delta u^d\|^2+\|\Delta\frac{\partial \alpha^d}{\partial t}\|^2+\|\nabla\Delta \alpha^d\|^2\Big)\\&\quad+\|\Delta u^d\|^2+\|\nabla\Delta u^d\|^2+\|\Delta\frac{\partial u^d}{\partial t}\|^2+\|\Delta\frac{\partial \alpha^d}{\partial t}\|^2=0.\end{aligned}\tag{4.38}$$

We multiply (4.10) by $\Delta^2\dfrac{\partial \alpha^d}{\partial t}$. Integrating over $\Omega$, we have

$$\frac{d}{dt}\Big(\|\nabla\Delta \alpha^d\|^2+\|\Delta\frac{\partial \alpha^d}{\partial t}\|^2\Big)+\|\Delta\frac{\partial \alpha^d}{\partial t}\|^2\le c\Big(\|\Delta u^d\|^2+\|\Delta\frac{\partial u^d}{\partial t}\|^2\Big).\tag{4.39}$$

We multiply (4.10) by $\Delta^2\alpha^d$. Integrating over $\Omega$, and using $\|\overline{\varphi}\|^2_{H^3(\Omega)} \leq c\|\nabla\Delta\varphi\|^2$, $\forall\varphi\in H^3(\Omega), c>0$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\Delta\alpha^d\|^2+2(\Delta\frac{\partial\alpha^d}{\partial t},\Delta\alpha^d)\Big)+\|\nabla\Delta\alpha^d\|^2\\ \leq&\frac{1}{2\epsilon_9}\Big(\|\Delta u^d\|^2+\|\Delta\frac{\partial u^d}{\partial t}\|^2\Big)+c\epsilon_9\|\nabla\Delta\alpha^d\|^2+\|\Delta\frac{\partial\alpha^d}{\partial t}\|^2.\end{aligned}\tag{4.40}$$

Summing (4.39) and $\epsilon_9\times$(4.40) where $\epsilon_9>0$ is small enough, we deduce that

$$\begin{aligned}&\frac{d}{dt}\Big(\|\nabla\Delta\alpha^d\|^2+\|\Delta\frac{\partial\alpha^d}{\partial t}\|^2+\epsilon_9\|\Delta\alpha^d\|^2+2\epsilon_9(\Delta\frac{\partial\alpha^d}{\partial t},\Delta\alpha^d)\Big)\\ &\qquad+c\Big(\|\Delta\frac{\partial\alpha^d}{\partial t}\|^2+\|\nabla\Delta\alpha^d\|^2\Big)\leq c'\Big(\|\Delta u^d\|^2+\|\Delta\frac{\partial u^d}{\partial t}\|^2\Big),\end{aligned}\tag{4.41}$$

and we have, in particular,

$$\|\Delta\frac{\partial\alpha^d}{\partial t}\|^2+\epsilon_9\|\Delta\alpha^d\|^2+2\epsilon_9(\Delta\frac{\partial\alpha^d}{\partial t},\Delta\alpha^d)\geq c\Big(\|\Delta\alpha^d\|^2+\|\Delta\frac{\partial\alpha^d}{\partial t}\|^2\Big),\quad c>0.\tag{4.42}$$

Summing then (4.38) and $\epsilon_{10}\times$(4.41), where $\epsilon_{10}>0$ is small enough, we obtain

$$\frac{d\psi_3}{dt}+c\Big(\psi_3+\|\Delta\frac{\partial u^d}{\partial t}\|^2\Big)\leq 0,\quad c>0,\tag{4.43}$$

where

$$\begin{aligned}\psi_3=&2\|\Delta u^d\|^2+\|\nabla\Delta u^d\|^2+\|\Delta\frac{\partial\alpha^d}{\partial t}\|^2+\|\nabla\Delta\alpha^d\|^2\\ &+\epsilon_{10}\Big(\|\nabla\Delta\alpha^d\|^2+\|\Delta\frac{\partial\alpha^d}{\partial t}\|^2+\epsilon_9\|\Delta\alpha^d\|^2+2\epsilon_9(\Delta\frac{\partial\alpha^d}{\partial t},\Delta\alpha^d)\Big).\end{aligned}\tag{4.44}$$

Applying Gronwall's lemma to (4.43), using (4.42) and (4.44) we get

$$\|\nabla\Delta u^d\|^2+\|\overline{\alpha^d}\|^2_{H^3}+\|\Delta\frac{\partial\alpha^d}{\partial t}\|^2\leq c'e^{-ct}(\|u_0\|^2_{H^3}+\|\overline{\alpha_0}\|^2_{H^3}+\|\alpha_1\|^2_{H^2}).\tag{4.45}$$

By (4.21) and the continuous injection $F_2\subset F_1$, we have

$$\|u^d\|^2_{H^2}+\|\overline{\alpha^d}\|^2_{H^2}+\|\frac{\partial\alpha^d}{\partial t}\|^2_{H^1}\leq c'e^{-ct}(\|u_0\|^2_{H^3}+\|\overline{\alpha_0}\|^2_{H^3}+\|\alpha_1\|^2_{H^2}).\tag{4.46}$$

We then deduce from (4.45) and (4.46) the following estimate

$$\|u^d\|^2_{H^3}+\|\overline{\alpha^d}\|^2_{H^3}+\|\frac{\partial\alpha^d}{\partial t}\|^2_{H^2}\leq c'e^{-ct}(\|u_0\|^2_{H^3}+\|\overline{\alpha_0}\|^2_{H^3}+\|\alpha_1\|^2_{H^2}).\tag{4.47}$$

Concerning system (4.5)-(4.8), we multiply (4.5) by $\Delta^2 u^c$. Integrating over $\Omega$, we get

$$\frac{1}{2}\frac{d}{dt}\|\Delta u^c\|^2+\|\Delta u^c\|^2+\|\nabla\Delta u^c\|^2+(\Delta f(u),\Delta u^c)=(\Delta\frac{\partial b}{\partial t},\Delta u^c). \tag{4.48}$$

Summing (4.26), (4.27) and (4.48) we obtain

$$\begin{aligned}&\frac{d}{dt}\Big(2\|\Delta u^c\|^2+\|\nabla\Delta u^c\|^2+\|\nabla\Delta\alpha^c\|^2+\|\Delta\frac{\partial\alpha^c}{\partial t}\|^2\Big)\\&\quad+c\Big(\|\Delta u^c\|^2+\|\nabla\Delta u^c\|^2+\|\Delta\frac{\partial u^c}{\partial t}\|^2+\|\Delta\frac{\partial\alpha^c}{\partial t}\|^2\Big)\le c^{'}\|\Delta f(u)\|^2,\quad c>0.\end{aligned} \tag{4.49}$$

We multiply (4.5) by $\Delta^3\frac{\partial u^c}{\partial t}$ and we integrate over $\Omega$ to have

$$\frac{d}{dt}(\|\nabla\Delta u^c\|^2+\|\Delta^2 u^c\|^2)+\|\nabla\Delta\frac{\partial u^c}{\partial t}\|^2\le\|\nabla\Delta f(u)\|^2+\|\nabla\Delta\frac{\partial\alpha^c}{\partial t}\|^2. \tag{4.50}$$

We then multiply (4.6) by $\Delta^3\frac{\partial\alpha^c}{\partial t}$. Integrating over $\Omega$ yields

$$\frac{d}{dt}\Big(\|\nabla\Delta\frac{\partial\alpha^c}{\partial t}\|^2+\|\Delta^2\alpha^c\|^2\Big)+\|\nabla\Delta\frac{\partial\alpha^c}{\partial t}\|^2\le c\Big(\|\nabla\Delta u^c\|^2+\|\nabla\Delta\frac{\partial u^c}{\partial t}\|^2\Big). \tag{4.51}$$

Summing (4.49), (4.50) and $\epsilon_{11}\times$(4.51) where $\epsilon_{11}>0$ is small enough, we obtain

$$\frac{d\psi_4}{dt}\le c(\psi_4+\|\nabla\Delta f(u)\|^2+\|\Delta f(u)\|^2), \tag{4.52}$$

where

$$\begin{aligned}\psi_4=&2\|\nabla\Delta u^c\|^2+\|\Delta^2 u^c\|^2+2\|\Delta u^c\|^2+\|\nabla\Delta\alpha^c\|^2\\&+\|\Delta\frac{\partial\alpha^c}{\partial t}\|^2+\epsilon_{11}\Big(\|\Delta^2\alpha^c\|^2+\|\nabla\Delta\frac{\partial\alpha^c}{\partial t}\|^2\Big).\end{aligned} \tag{4.53}$$

Furthermore, we have

$$\begin{aligned}\|\Delta f(u)\|&=\|f^{''}(u)|\nabla u|^2+f^{'}(u)\Delta u\|\\&\le(\text{by}\quad(3.4))\\&\le C_{D(u_0),\|u_0\|_{H^3},\|\overline{\alpha_0}\|_{H^3},\|\alpha_1\|_{H^2}}\end{aligned} \tag{4.54}$$

$$\begin{aligned}\|\nabla\Delta f(u)\|&=\|f^{'''}(u)|\nabla u|^3+f^{''}(u)\nabla u\Delta u+f^{'}(u)\nabla\Delta u\|\\&\le(\text{by}\quad(3.4))\\&\le C_{D(u_0),\|u_0\|_{H^3},\|\overline{\alpha_0}\|_{H^3},\|\alpha_1\|_{H^2}}.\end{aligned} \tag{4.55}$$

Inserting (4.54) and (4.55) in (4.52) and applying Gronwall's lemma to the resulting estimate, we deduce by (4.53) that

$$\|\Delta^2 u^c\|^2+\|\Delta^2\alpha^c\|^2+\|\nabla\Delta\frac{\partial\alpha^c}{\partial t}\|^2\le C_{T,\|(u_0,\overline{\alpha_0},\alpha_1)\|_{F_2},B_R^2}. \tag{4.56}$$

Combining (4.37) and (4.56) we have

$$\|(u^c,\overline{\alpha^c},\frac{\partial\alpha^c}{\partial t})\|_{F_3}^2\leq C_{T,\|(u_0,\overline{\alpha_0},\alpha_1)\|_{F_2},B_R^2}, \tag{4.57}$$

which completes the proof of the theorem. □

We define for what follows the following invariant sets: in one space dimension, $\overline{X_1}=\overline{\cup_{t\geq t_0}\overline{S}(t)B_R^1}$, where $B_R^1$ is the bounded absorbing set for $\overline{S}(t)$ in $\overline{\Phi_0}$ and in two space dimensions, $\overline{X_2}=\overline{\cup_{t\geq t_1}\overline{S}(t)B_R^2}$, where $B_R^2$ is the bounded absorbing set for $\overline{S}(t)$ in $\overline{\Psi_0}$. In what follows, we will work in these two subspaces $\overline{X_1}$ and $\overline{X_2}$ which are positively invariant for $\overline{S}(t),t\geq 0$.

Now that the existence of the global attractor is proven, one natural question is to know whether this attractor has finite dimension in terms of the fractal or Hausdorff dimension. This is the aim of the final section.

## 5 Exponential attractors

The aim of this section is to prove the existence of exponential attractors for the semigroup $\overline{S}(t)$, $t\geq 0$, associated to problem (1.13)-(1.16) in one and two space dimensions using the separation property (2.41). To do so, we need the semigroup to be Lipschitz continuous and satisfy the smoothing property, but also to verify a Hölder condition in time (see [18], [19], [28–30]). This is enough to conclude on the existence of exponential attractors, but before going further, let us recall the definition of an exponential attractor which is also called inertial set.

**Definition 5.1.** A compact set $\mathcal{M}$ is called an exponential attractor for $(\{S(t)\}_{t\geq 0},\mathcal{X})$, if

(i) $\mathcal{A}\subset\mathcal{M}\subset\mathcal{X}$, where $\mathcal{A}$ is the global attractor,

(ii) $\mathcal{M}$ is positively invariant for $S(t)$, *i.e.* $S(t)\mathcal{M}\subset\mathcal{M}$ for every $t\geq 0$,

(iii) $\mathcal{M}$ has finite fractal dimension,

(iv) $\mathcal{M}$ attracts exponentially the bounded subsets of $\mathcal{X}$ in the following sense :

$$\forall B\subset\mathcal{X}\quad\text{bounded},\quad dist(S(t)B,\mathcal{M})\leq Q(\|B\|_{\mathcal{X}})\exp(-\alpha t),\quad t\geq 0,$$

where the positive constant $\alpha$ and the monotonic function $Q$ are independent of $B$, and *dist* stands for the Hausdorff semi-distance between sets in $\mathcal{X}$, defined by

$$dist(A,B)=\sup_{a\in A}\inf_{b\in B}\|a-b\|_{\mathcal{X}}.$$

We start by stating an abstract result that will be useful in what follows (see [18]).

**Theorem 5.1.** *Let $\Psi$ and $\Psi_1$ be two Banach spaces such that $\Psi_1$ is compactly embedded into $\Psi$ and $S(t): Y \longrightarrow Y$ be a semigroup acting on a closed subset $Y$ of $\Psi$. We assume that*

*(i)*

$$\forall x_1, x_2 \in Y, \quad \forall t \geq 0,\ S(t)x_1 - S(t)x_2 = S_1(t, x_1, x_2) + S_2(t, x_1, x_2),$$

*where*

$$\|S_1(t, x_1, x_2)\|_\Psi \leq d(t)\|x_1 - x_2\|_\Psi,$$

*$d$ is continuous, $t \geq 0$, $d(t) \to 0$ as $t \to +\infty$, and*

$$\|S_2(t, x_1, x_2)\|_{\Psi_1} \leq h(t)\|x_1 - x_2\|_\Psi, \quad t > 0,\ h \text{ continuous},$$

*(ii) $(t, x) \longmapsto S(t)x$ is Lipschitz/Hölder continuous on $[0, T] \times B, \forall T > 0, \forall B \subset Y$ bounded.*

*Then $S(t)$ possesses an exponential attractor $\mathcal{M}$ on $Y$.*

In order to get the existence of exponential attractors in our case, we will use Theorem 5.1. We have the following result

**Theorem 5.2.** *(i) In one space dimension, the semigroup $\overline{S}(t)$, $t \geq 0$, corresponding to equations* (1.13)-(1.16) *defined from $\overline{X_1}$ to itself satisfies a decomposition as in Theorem* 5.1.

*(ii) In two space dimension, $\overline{S}(t)$, $t \geq 0$, defined from $\overline{X_2}$ to itself also satisfies such a decomposition.*

*Proof.* Let $(u_1, \alpha_1, \frac{\partial \alpha_1}{\partial t})$ and $(u_2, \alpha_2, \frac{\partial \alpha_2}{\partial t})$ be two solutions to the problem (1.13)-(1.16) and $(u_{01}, \alpha_{01}, \alpha_{11})$ and $(u_{02}, \alpha_{02}, \alpha_{12})$ be their respective initial data. Set

$$\begin{aligned}
&(u, \alpha, \frac{\partial \alpha}{\partial t}) = \Big(u_1 - u_2, \alpha_1 - \alpha_2, \frac{\partial \alpha_1}{\partial t} - \frac{\partial \alpha_2}{\partial t}\Big),\\
&(u_0, \alpha_0, \alpha_1) = (u_{01} - u_{02}, \alpha_{01} - \alpha_{02}, \alpha_{11} - \alpha_{12}).
\end{aligned}$$

Thus $(u, \alpha, \frac{\partial \alpha}{\partial t})$ is a solution to

$$\frac{\partial u}{\partial t} - \Delta u + \kappa(t)u + u = \frac{\partial \alpha}{\partial t}, \tag{5.1}$$

$$\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t} - u, \tag{5.2}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \alpha}{\partial \nu} = 0, \tag{5.3}$$

$$u(0) = u_0, \quad \alpha(0) = \alpha_0, \quad \frac{\partial \alpha}{\partial t}(0) = \alpha_1, \tag{5.4}$$

where $\kappa(t) = \int_0^1 f'(su_1(t) + (1-s)u_2(t))ds$. Note that $\kappa(t)$ and $l(t)$ verify the same properties, hence

$$\|\kappa(t)\|_{L^\infty} \leq c \quad (= c(B_R^1)). \tag{5.5}$$

Now decompose the solution $(u,\alpha,\frac{\partial\alpha}{\partial t})$ as follows :

$$(u,\alpha,\frac{\partial\alpha}{\partial t})=(\vartheta,\eta,\frac{\partial\eta}{\partial t})+(v,\xi,\frac{\partial\xi}{\partial t}),$$

where $(\vartheta,\eta,\frac{\partial\eta}{\partial t})$ and $(v,\xi,\frac{\partial\xi}{\partial t})$ are solutions to

$$\frac{\partial\vartheta}{\partial t}-\Delta\vartheta+\vartheta=\frac{\partial\eta}{\partial t}, \tag{5.6}$$

$$\frac{\partial^2\eta}{\partial t^2}+\frac{\partial\eta}{\partial t}-\Delta\eta=-\frac{\partial\vartheta}{\partial t}-\vartheta, \tag{5.7}$$

$$\frac{\partial\vartheta}{\partial\nu}=\frac{\partial\eta}{\partial\nu}=0, \tag{5.8}$$

$$\vartheta(0)=u_0,\quad \eta(0)=\alpha_0,\quad \frac{\partial\eta}{\partial t}(0)=\alpha_1, \tag{5.9}$$

and

$$\frac{\partial v}{\partial t}-\Delta v+\kappa(t)u+v=\frac{\partial\xi}{\partial t}, \tag{5.10}$$

$$\frac{\partial^2\xi}{\partial t^2}+\frac{\partial\xi}{\partial t}-\Delta\xi=\frac{\partial v}{\partial t}-v, \tag{5.11}$$

$$\frac{\partial v}{\partial\nu}=\frac{\partial\xi}{\partial\nu}=0, \tag{5.12}$$

$$v(0)=0,\quad \xi(0)=0,\quad \frac{\partial\xi}{\partial t}(0)=0, \tag{5.13}$$

respectively. We start with the proof of $(i)$. In that case the initial conditions belong to $\overline{X_1}$. Repeating for (5.6)-(5.9) the estimates which led to (4.13) and (4.17), we then write (noting that $f\equiv 0$)

$$\frac{d\psi_5}{dt}+c(\psi_5+\|\nabla\frac{\partial\vartheta}{\partial t}\|^2)\leq 0,\quad c>0, \tag{5.14}$$

where

$$\begin{aligned}\psi_5=&2\|\nabla\vartheta\|^2+\|\Delta\vartheta\|^2+\|\nabla\frac{\partial\eta}{\partial t}\|^2+\|\Delta\eta\|^2\\&+\epsilon'\Big(\|\Delta\eta\|^2+\|\nabla\frac{\partial\eta}{\partial t}\|^2+\epsilon(\|\nabla\eta\|^2+2(\nabla\frac{\partial\eta}{\partial t},\nabla\eta))\Big),\end{aligned} \tag{5.15}$$

$$\frac{d\psi_6}{dt}+c(\psi_6+\|\frac{\partial\vartheta}{\partial t}\|^2)\leq 0,\quad c>0, \tag{5.16}$$

where

$$\begin{aligned}\psi_6=&2\|\vartheta\|^2+\|\nabla\vartheta\|^2+\|\nabla\eta\|^2+\|\frac{\partial\eta}{\partial t}\|^2\\&+\delta'\Big(\|\frac{\partial\overline{\eta}}{\partial t}\|^2+\|\nabla\eta\|^2+\delta(\|\overline{\eta}\|^2+2(\frac{\partial\overline{\eta}}{\partial t},\overline{\eta}))\Big).\end{aligned} \tag{5.17}$$

Here $\epsilon>0$ and $\delta>0$ are small enough so that we have in particular

$$\|\frac{\partial\overline{\eta}}{\partial t}\|^2+\delta\Big(\|\overline{\eta}\|^2+2(\frac{\partial\overline{\eta}}{\partial t},\overline{\eta})\Big)\geq c(\|\overline{\eta}\|^2+\|\frac{\partial\overline{\eta}}{\partial t}\|^2),\quad c>0, \tag{5.18}$$

$$\|\nabla\frac{\partial\eta}{\partial t}\|^2+\epsilon\Big(\|\nabla\eta\|^2+2(\nabla\frac{\partial\eta}{\partial t},\nabla\eta)\Big)\geq c'(\|\nabla\eta\|^2+\|\nabla\frac{\partial\eta}{\partial t}\|^2),\quad c'>0. \tag{5.19}$$

An application of Gronwall's lemma to (5.14) and (5.16) respectively yields

$$\|\Delta\vartheta\|^2+\|\overline{\eta}\|^2_{H^2}+\|\nabla\frac{\partial\eta}{\partial t}\|^2\leq c'e^{-ct}(\|u_0\|^2_{H^2}+\|\overline{\alpha_0}\|^2_{H^2}+\|\alpha_1\|^2_{H^1}), \tag{5.20}$$

$$\|\vartheta\|^2_{H^1}+\|\overline{\eta}\|^2_{H^1}+\|\frac{\partial\eta}{\partial t}\|^2\leq c'e^{-ct}(\|u_0\|^2_{H^2}+\|\overline{\alpha_0}\|^2_{H^2}+\|\alpha_1\|^2_{H^1}). \tag{5.21}$$

Combining (5.20) and (5.21), we get

$$\|(\vartheta,\overline{\eta},\frac{\partial\eta}{\partial t})\|^2_{F_1}\leq d(t)\|(u_0,\overline{\alpha_0},\alpha_1)\|^2_{F_1}. \tag{5.22}$$

Now we consider (5.10)–(5.13). We multiply (5.10) by $\Delta^2 v+\Delta^2\frac{\partial v}{\partial t}$ and (5.11) by $\Delta^2\frac{\partial\xi}{\partial t}$. Summing the resulting equations, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(2\|\Delta v\|^2+\|\nabla\Delta v\|^2+\|\nabla\Delta\xi\|^2+\|\Delta\frac{\partial\xi}{\partial t}\|^2\Big)+\|\Delta v\|^2+\|\nabla\Delta v\|^2\\&\quad+\|\Delta\frac{\partial v}{\partial t}\|^2+\|\Delta\frac{\partial\xi}{\partial t}\|^2=(\nabla(\kappa(t)u),\nabla\Delta v)-\Big(\Delta(\kappa(t)u),\Delta\frac{\partial v}{\partial t}\Big).\end{aligned} \tag{5.23}$$

Noting that

$$\Big|((\nabla(\kappa(t)u),\nabla\Delta v))\Big|\leq\frac{c}{2\epsilon}\|u\|^2_{H^2}+\frac{c\epsilon}{2}\|\nabla\Delta v\|^2, \tag{5.24}$$

due to the continuous embedding $H^2(\Omega)\subset L^\infty(\Omega)$, and by (3.2), we have

$$\|\kappa(t)\|_{H^2}\leq Q(\|(u_{01},\overline{\alpha_{01}},\alpha_{11})\|_{\overline{\Phi_0}}+\|(u_{02},\overline{\alpha_{02}},\alpha_{11})\|_{\overline{\Phi_0}})\leq c. \tag{5.25}$$

Thus,

$$\Big|((\Delta(\kappa(t)u),\Delta\frac{\partial v}{\partial t}))\Big|\leq\frac{c\epsilon}{2}\|\Delta\frac{\partial v}{\partial t}\|^2+\frac{c}{2\epsilon}\|u\|^2_{H^2}. \tag{5.26}$$

Choosing $\epsilon>0$ small enough and using (5.24) and (5.26), we deduce from (5.23) the following inequality

$$\begin{aligned}&\frac{d}{dt}\Big(2\|\Delta v\|^2+\|\nabla\Delta v\|^2+\|\nabla\Delta\xi\|^2+\|\Delta\frac{\partial\xi}{\partial t}\|^2\Big)\\&\quad+c\Big(\|\Delta v\|^2+\|\nabla\Delta v\|^2+\|\Delta\frac{\partial v}{\partial t}\|^2+\|\Delta\frac{\partial\xi}{\partial t}\|^2\Big)\leq c'\|u\|^2_{H^2},\qquad c>0.\end{aligned} \tag{5.27}$$

Integrating (5.27) over $(0,t)$, by (5.13) we have

$$\|\Delta v(t)\|^2+\|\nabla\Delta v(t)\|^2+\|\nabla\Delta\xi(t)\|^2+\|\Delta\frac{\partial\xi}{\partial t}(t)\|^2\leq c'\int_0^t\|u\|_{H^2}^2ds. \tag{5.28}$$

It only remains to estimate $\int_0^t\|u\|_{H^2}^2ds$. To do so we multiply (5.1) by $-\Delta u-\Delta\frac{\partial u}{\partial t}$ and (5.2) by $-\Delta\frac{\partial\alpha}{\partial t}$. Summing up, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(2\|\nabla u\|^2+\|\Delta u\|^2+\|\nabla\frac{\partial\alpha}{\partial t}\|^2+\|\Delta\alpha\|^2\Big)+\|\nabla u\|^2+\|\Delta u\|^2\\+\|\nabla\frac{\partial u}{\partial t}\|^2+\|\nabla\frac{\partial\alpha}{\partial t}\|^2=-(\nabla(\kappa(t)u),\nabla u)-\Big(\nabla(\kappa(t)u),\nabla\frac{\partial u}{\partial t}\Big).\end{aligned} \tag{5.29}$$

Hölder's inequality, (3.2) and (5.5) yield

$$\Big|((\nabla(\kappa(t)u),\nabla u))\Big|\leq c\|u\|_{H^1}^2. \tag{5.30}$$

Analogously, we have

$$\Big|\Big(\nabla(\kappa(t)u),\nabla\frac{\partial u}{\partial t}\Big)\Big|\leq c\|u\|_{H^1}\cdot\|\nabla\frac{\partial u}{\partial t}\|\leq\frac{c}{2\epsilon}\|u\|_{H^1}^2+\frac{c\epsilon}{2}\|\nabla\frac{\partial u}{\partial t}\|^2. \tag{5.31}$$

Choosing $\epsilon>0$ small enough and recalling (5.30) and (5.31), we obtain

$$\frac{d\psi_7(t)}{dt}+c\Big(\|\nabla u\|^2+\|\Delta u\|^2+\|\nabla\frac{\partial u}{\partial t}\|^2+\|\nabla\frac{\partial\alpha}{\partial t}\|^2\Big)\leq c'\|u\|_{H^1}^2,\quad c>0, \tag{5.32}$$

where

$$\psi_7(t)=2\|\nabla u\|^2+\|\Delta u\|^2+\|\nabla\frac{\partial\alpha}{\partial t}\|^2+\|\Delta\alpha\|^2. \tag{5.33}$$

Integrating (3.61) over $(0,t)$, we get

$$\int_0^t\|u(s)\|_{H^1}^2ds\leq c'e^{ct}(\|u_0\|_{H^2}^2+\|\overline{\alpha_0}\|_{H^2}^2+\|\alpha_1\|_{H^1}^2). \tag{5.34}$$

Integrating then (5.32) over $(0,t)$ and using (5.34) we deduce that

$$\int_0^t\|u\|_{H^2}^2ds\leq c'e^{ct}\|(u_0,\overline{\alpha_0},\alpha_1)\|_{F_1}^2, \tag{5.35}$$

hence (5.28) yields

$$\|\Delta v(t)\|^2+\|\nabla\Delta v(t)\|^2+\|\overline{\xi}(t)\|_{H^3}^2+\|\Delta\frac{\partial\xi}{\partial t}(t)\|^2\leq c'e^{ct}\|(u_0,\overline{\alpha_0},\alpha_1)\|_{F_1}^2. \tag{5.36}$$

Multiplying (5.10) by $-\Delta v-\Delta\frac{\partial v}{\partial t}$ and (5.11) by $-\Delta\frac{\partial\xi}{\partial t}$, summing up we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(2\|\nabla v\|^2+\|\Delta v\|^2+\|\Delta\xi\|^2+\|\nabla\frac{\partial\xi}{\partial t}\|^2)+\|\nabla v\|^2+\|\Delta v\|^2\\&\quad+\|\nabla\frac{\partial v}{\partial t}\|^2+\|\nabla\frac{\partial\xi}{\partial t}\|^2=-\Big(\nabla(\kappa(t)u),\nabla v\Big)-\Big(\nabla(\kappa(t)u),\nabla\frac{\partial v}{\partial t}\Big).\end{aligned}\tag{5.37}$$

Hölder's inequality and (5.5) yield

$$\frac{d\psi_8(t)}{dt}+c\Big(\|\nabla v\|^2+\|\Delta v\|^2+\|\nabla\frac{\partial v}{\partial t}\|^2+\|\nabla\frac{\partial\xi}{\partial t}\|^2\Big)\le c'\|\nabla u\|^2,\quad c>0,\tag{5.38}$$

where

$$\psi_8=2\|\nabla v\|^2+\|\Delta v\|^2+\|\Delta\xi\|^2+\|\nabla\frac{\partial\xi}{\partial t}\|^2.\tag{5.39}$$

In particular

$$\frac{d\psi_8(t)}{dt}\le c\psi_8(t)+c'\|\nabla u\|^2.\tag{5.40}$$

Applying Gronwall's lemma to (5.40) and using (5.39) we deduce that

$$\|\nabla v(t)\|^2+\|\Delta\xi(t)\|^2+\|\nabla\frac{\partial\xi}{\partial t}(t)\|^2\le c'e^{ct}\|(u_0,\overline{\alpha_0},\alpha_1)\|^2_{F_1}.\tag{5.41}$$

Finally, multiplying (5.10) by $v+\frac{\partial v}{\partial t}$ and (5.11) by $\frac{\partial\xi}{\partial t}$, and proceeding exactly as above we deduce that

$$\|v(t)\|^2_{H^1}+\|\nabla\xi(t)\|^2+\|\frac{\partial\xi}{\partial t}(t)\|^2\le c'e^{ct}\|(u_0,\overline{\alpha_0},\alpha_1)\|^2_{F_1}.\tag{5.42}$$

Combining (5.36), (5.41) and (5.42), we obtain

$$\|v(t)\|^2_{H^3}+\|\overline{\xi}(t)\|^2_{H^3}+\|\frac{\partial\xi}{\partial t}(t)\|^2_{H^2}\le h(t)\|(u_0,\overline{\alpha_0},\alpha_1)\|^2_{F_1},\tag{5.43}$$

where $h(t)=c'e^{ct}$, with $c$ and $c'$ depending on $\overline{X_1}$. We can see that $h$ is continuous.

We now turn to the two-dimensional case, and prove part $(ii)$ of Theorem 5.2. To do so we take here the initial data in $\overline{X_2}$. Repeating for (5.6)-(5.9) the estimates which led to (4.43), we then write

$$\frac{d\psi_9}{dt}+c(\psi_9+\|\Delta\frac{\partial\vartheta}{\partial t}\|^2)\le 0,\quad c>0,\tag{5.44}$$

where

$$\begin{aligned}\psi_9=&2\|\Delta\vartheta\|^2+\|\nabla\Delta\vartheta\|^2+\|\Delta\frac{\partial\eta}{\partial t}\|^2+\|\nabla\Delta\eta\|^2\\&+\epsilon'\Big(\|\nabla\Delta\eta\|^2+\|\Delta\frac{\partial\eta}{\partial t}\|^2+\epsilon(\|\Delta\eta\|^2+2(\Delta\frac{\partial\eta}{\partial t},\Delta\eta))\Big),\end{aligned}\tag{5.45}$$

and $\epsilon>0$ is small enough so that

$$\|\Delta\frac{\partial\eta}{\partial t}\|^2+\epsilon\Big(\|\Delta\eta\|^2+2(\Delta\frac{\partial\eta}{\partial t},\Delta\eta)\Big)\geq c\Big(\|\Delta\eta\|^2+\|\Delta\frac{\partial\eta}{\partial t}\|^2\Big),\quad c>0. \tag{5.46}$$

In particular

$$\frac{d\psi_9}{dt}+c\psi_9\leq 0. \tag{5.47}$$

An application of Gronwall's lemma yields

$$\|\nabla\Delta\vartheta\|^2+\|\overline{\eta}\|^2_{H^3}+\|\Delta\frac{\partial\eta}{\partial t}(t)\|^2\leq c^{'}e^{-ct}(\|u_0\|^2_{H^3}+\|\overline{\alpha_0}\|^2_{H^3}+\|\alpha_1\|^2_{H^2}). \tag{5.48}$$

Furthermore, by (5.22) and the continuous embedding $F_2\subset F_1$, we get

$$\|\vartheta\|^2_{H^2}+\|\overline{\eta}(t)\|^2_{H^2}+\|\frac{\partial\eta}{\partial t}(t)\|^2_{H^1}\leq c^{'}e^{-ct}(\|u_0\|^2_{H^3}+\|\overline{\alpha_0}\|^2_{H^3}+\|\alpha_1\|^2_{H^2}). \tag{5.49}$$

By (5.48) and (5.49) we have

$$\|\vartheta\|^2_{H^3}+\|\overline{\eta}(t)\|^2_{H^3}+\|\frac{\partial\eta}{\partial t}(t)\|^2_{H^2}\leq c^{'}e^{-ct}(\|u_0\|^2_{H^3}+\|\overline{\alpha_0}\|^2_{H^3}+\|\alpha_1\|^2_{H^2}). \tag{5.50}$$

Concerning problem (5.10)-(5.13), we multiply (5.10) by $\Delta^3 v+\Delta^3\frac{\partial v}{\partial t}$ and (5.11) by $\Delta^3\frac{\partial\xi}{\partial t}$. Summing the resulting equations, we then obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(2\|\nabla\Delta v\|^2+\|\Delta^2 v\|^2+\|\Delta^2\xi\|^2+\|\nabla\Delta\frac{\partial\xi}{\partial t}\|^2\Big)+\|\nabla\Delta v\|^2+\|\Delta^2 v\|^2\\&\qquad+\|\nabla\Delta\frac{\partial v}{\partial t}\|^2+\|\nabla\Delta\frac{\partial\xi}{\partial t}\|^2=(\Delta(\kappa(t)u),\Delta^2 v)-\Big(\nabla\Delta(\kappa(t)u),\nabla\Delta\frac{\partial v}{\partial t}\Big).\end{aligned} \tag{5.51}$$

Analogously to (5.26), we write

$$\Big|((\Delta(\kappa(t)u),\Delta^2 v))\Big|\leq\frac{c\epsilon}{2}\|\Delta^2 v\|^2+\frac{c}{2\epsilon}\|u\|^2_{H^2}. \tag{5.52}$$

By (3.4) and the continuous embedding $H^3(\Omega)\subset C(\overline{\Omega})$, we have

$$\|\kappa(t)\|_{H^3}\leq Q(\|(u_{01},\overline{\alpha_{01}},\alpha_{11})\|_{\overline{\Psi_0}}+\|(u_{02},\overline{\alpha_{02}},\alpha_{12})\|_{\overline{\Psi_0}})\leq c, \tag{5.53}$$

so that

$$\Big|\Big(\nabla\Delta(\kappa(t)u),\nabla\Delta\frac{\partial v}{\partial t}\Big)\Big|\leq\frac{c\epsilon}{2}\|\nabla\Delta\frac{\partial v}{\partial t}\|^2+\frac{c}{2\epsilon}\|u\|^2_{H^3}. \tag{5.54}$$

Choosing $\epsilon>0$ small enough and recalling (5.52) and (5.54), we deduce from (5.51) the estimate

$$\begin{aligned}&\frac{d}{dt}\Big(2\|\nabla\Delta v\|^2+\|\Delta^2 v\|^2+\|\Delta^2\xi\|^2+\|\nabla\Delta\frac{\partial\xi}{\partial t}\|^2\Big)\\&\qquad+c\Big(\|\nabla\Delta v\|^2+\|\Delta^2 v\|^2+\|\nabla\Delta\frac{\partial v}{\partial t}\|^2+\|\nabla\Delta\frac{\partial\xi}{\partial t}\|^2\Big)\leq c^{'}\|u\|^2_{H^3},\quad c>0.\end{aligned} \tag{5.55}$$

Integrating (5.55) over $(0,t)$ and by (5.13) we get

$$2\|\nabla\Delta v(t)\|^2+\|\Delta^2 v(t)\|^2+\|\Delta^2\xi(t)\|^2+\|\nabla\Delta\frac{\partial\xi}{\partial t}(t)\|^2\le c'\int_0^t\|u(s)\|_{H^3}^2ds. \tag{5.56}$$

By (5.35), we have

$$\int_0^t\|u(s)\|_{H^2}^2ds\le c'e^{ct}(\|u_0\|_{H^3}^2+\|\overline{\alpha_0}\|_{H^3}^2+\|\alpha_1\|_{H^2}^2). \tag{5.57}$$

Multiplying (5.1) by $\Delta^2u+\Delta^2\frac{\partial u}{\partial t}$ and (5.2) by $\Delta^2\frac{\partial\alpha}{\partial t}$ and summing the resulting equations, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(2\|\Delta u\|^2+\|\nabla\Delta u\|^2+\|\Delta\frac{\partial\alpha}{\partial t}\|^2+\|\nabla\Delta\alpha\|^2\Big)+\|\Delta u\|^2+\|\nabla\Delta u\|^2\\&\quad+\|\Delta\frac{\partial u}{\partial t}\|^2+\|\Delta\frac{\partial\alpha}{\partial t}\|^2=-(\kappa(t)u,\Delta^2u)-\Big(\kappa(t)u,\Delta^2\frac{\partial u}{\partial t}\Big).\end{aligned} \tag{5.58}$$

As above we have

$$\Big|((\nabla(\kappa(t)u),\nabla\Delta u))\Big|\le\frac{c\epsilon}{2}\|\nabla\Delta u\|^2+\frac{c}{2\epsilon}\|u\|_{H^1}^2, \tag{5.59}$$

$$\Big|\Big(\Delta(\kappa(t)u),\Delta\frac{\partial u}{\partial t}\Big)\Big|\le\frac{c\epsilon}{2}\|\Delta\frac{\partial u}{\partial t}\|^2+\frac{c}{2\epsilon}\|u\|_{H^2}^2. \tag{5.60}$$

Choosing $\epsilon>0$ small enough and by recalling (5.59) and (5.60), we deduce from (5.58) the estimate

$$\begin{aligned}&\frac{d}{dt}\Big(2\|\Delta u\|^2+\|\nabla\Delta u\|^2+\|\Delta\frac{\partial\alpha}{\partial t}\|^2+\|\nabla\Delta\alpha\|^2\Big)\\&\quad+c\Big(\|\Delta u\|^2+\|\nabla\Delta u\|^2+\|\Delta\frac{\partial u}{\partial t}\|^2+\|\Delta\frac{\partial\alpha}{\partial t}\|^2\Big)\le c'\|u\|_{H^2}^2,\quad c>0.\end{aligned} \tag{5.61}$$

Integrating (5.61) over $(0,t)$ and by (5.57) we have

$$\int_0^t\|\nabla\Delta u(s)\|^2ds\le c'e^{ct}(\|u_0\|_{H^3}^2+\|\overline{\alpha_0}\|_{H^3}^2+\|\alpha_1\|_{H^2}^2). \tag{5.62}$$

Combining (5.57) and (5.62), we get

$$\int_0^t\|u(s)\|_{H^3}^2ds\le c'e^{ct}(\|u_0\|_{H^3}^2+\|\overline{\alpha_0}\|_{H^3}^2+\|\alpha_1\|_{H^2}^2). \tag{5.63}$$

Inserting (5.63) in (5.56) we obtain

$$\|\nabla\Delta v(t)\|^2+\|\Delta^2 v(t)\|^2+\|\Delta^2\xi(t)\|^2+\|\nabla\Delta\frac{\partial\xi}{\partial t}(t)\|^2\le c'e^{ct}\|(u_0,\overline{\alpha_0},\alpha_1)\|_{\mathcal{F}_2}^2. \tag{5.64}$$

Noting that $\langle\overline{\xi}\rangle=0$, from (5.64) we deduce that

$$\|\nabla\Delta v(t)\|^2+\|\Delta^2 v(t)\|^2+\|\overline{\xi}(t)\|_{H^4}^2+\|\nabla\Delta\frac{\partial\xi}{\partial t}(t)\|^2\leq c^{'}e^{ct}\|(u_0,\overline{\alpha_0},\alpha_1)\|_{F_2}^2. \tag{5.65}$$

Combining (5.43) and (5.65), we obtain

$$\|v(t)\|_{H^4}^2+\|\overline{\xi}(t)\|_{H^4}^2+\|\frac{\partial\xi}{\partial t}(t)\|_{H^3}^2\leq c^{'}e^{ct}(\|u_0\|_{H^3}^2+\|\overline{\alpha_0}\|_{H^3}^2+\|\alpha_1\|_{H^2}^2), \tag{5.66}$$

which completes the proof. □

**Lemma 5.1.** *The semigroup* $\overline{S}(t)$, $t\geq 0$ *generated by the problem* (1.13)–(1.16) *is Hölder continuous on* $[0,T]\times B_R^i$, $\quad i=1,2$ *(i depending on the space dimension).*

*Proof.* We consider the one-dimensional case (the two-dimensional case can be treated similarly). The Lipschitz continuity in space is a consequence of (3.63). It just remains to prove the continuity in time (actually, a Hölder condition in time for the semigroup $\overline{S}(t),t\geq 0$). We assume that the initial data belong to $B_R^1$. For every $t_1\geq 0$ and $t_2\geq 0$, owing to the above estimates, one gets :

$$\begin{aligned}
&\|\overline{S}(t_1)(u_0,\overline{\alpha_0},\alpha_1)-\overline{S}(t_2)(u_0,\overline{\alpha_0},\alpha_1)\|_F\\
=&\|u(t_1)-u(t_2)\|_{H^1}+\|\overline{\alpha}(t_1)-\overline{\alpha}(t_2)\|_{H^1}+\Big\|\frac{\partial\alpha}{\partial t}(t_1)-\frac{\partial\alpha}{\partial t}(t_2)\Big\|\\
\leq&\Big|\int_{t_1}^{t_2}\Big\|\frac{\partial u}{\partial t}(\tau)\|_{H^1}d\tau\Big|+\Big|\int_{t_1}^{t_2}\Big\|\frac{\partial\overline{\alpha}}{\partial t}(\tau)\|_{H^1}d\tau\Big|+\Big|\int_{t_1}^{t_2}\Big\|\frac{\partial^2\alpha}{\partial t^2}(\tau)\|d\tau\Big|\\
\leq& c|t_1-t_2|^{\frac{1}{2}}+\Big(\Big|\int_{t_1}^{t_2}\Big\|\frac{\partial^2\alpha}{\partial t^2}(\tau)\|^2d\tau\Big|\Big)^{\frac{1}{2}}|t_1-t_2|^{\frac{1}{2}},
\end{aligned}$$

where $c$ depends on $T$. We multiply (1.14) by $\dfrac{\partial^2\alpha}{\partial t^2}$ to obtain

$$\frac{d}{dt}\left\{\|\frac{\partial\alpha}{\partial t}\|^2+2\left(\nabla\alpha,\nabla\frac{\partial\alpha}{\partial t}\right)\right\}+c\|\frac{\partial^2\alpha}{\partial t^2}\|^2\leq c^{'}\left(\|u\|^2+\|\frac{\partial u}{\partial t}\|^2+\|\nabla\frac{\partial\alpha}{\partial t}\|^2\right). \tag{5.67}$$

Integrating (5.67) between $t_1$ and $t_2$, we deduce from the above estimates that

$$\Big|\int_{t_1}^{t_2}\|\frac{\partial^2\alpha}{\partial t^2}(\tau)\|^2d\tau\Big|\leq c, \tag{5.68}$$

where $c$ depends on $T$ and $B_R^1$, which concludes the proof. □

We deduce from Theorem 5.2 and Lemma 5.1 the following result.

**Theorem 5.3.** *The dynamical system* $(\overline{S}(t),\overline{X_1})$ *(respectively* $(\overline{S}(t),\overline{X_2})$*) associated to* (1.13)-(1.16) *possesses, in one space dimension, an exponential attractor* $\overline{\mathcal{M}}_1$ *in* $\overline{X_1}$ *(respectively, in two space dimensions, an exponential attractor* $\overline{\mathcal{M}}_2$ *in* $\overline{X_2}$*).*

## References


[1] L.D. Landau and E.M. Lifschitz, Statistical physics (Part 1), Third edition, New York, 1980.

[2] G. Caginalp, An analysis of a phase-field model of a free boundry, Arch. Rational Mech. Anal., 92(3) (1986), 205-245.

[3] A. Miranville, Some mathematical models in phase transition, Lecture Notes, Ravello, September, 2009.

[4] G. Caginalp, The role of macroscopic anitropy in the macroscopic behavior of a phase boundry, Ann. Physics., 172(1) (1986), 136-155.

[5] A. Miranville and R. Quintanilla, Some generalizations of the Caginalp phase-field system, Appl. Anal., 88(6) (2009), 897-894.

[6] A. Miranville and R. Quintanilla, A generalization of the Caginalp phase-field system based on the Maxwell-Cattaneo law, Nonlinear Anal., 71(5-6) (2009), 2278-2298.

[7] A. Debussche and L. Dettori, On the Cahn-Hilliard equation with a logarithmic free energy, Nonlinear Anal., 24 (1995), 1491-1514.

[8] J.L. Lions, Quelques méthodes de résolution des Problèms aux limites nonlinéaires, Dunod, Paris, 1969.

[9] C. Dupaix, A singularly perturbed phase field model with a logarithmic nonlinearity: upper-semicontinuity of the attractor, Nonlinear Anal., 41(5-6) (2000), 725-744.

[10] D. Brochet, X. Chen and D. Hilhorst, Finite dimensional exponential attractor for the phase field model, Appl. Anal., 49(3-4) (1993), 197-212.

[11] C. Dupaix, D. Hilhorst and P. Laurençot, Upper-semicontinuity of the attractor for a singular perturbation of the viscous Cahn-Hilliard and Cahn-Hilliard equations, Adv. Math. Sci. Appl., 8 (1998), 115-143.

[12] J. Simon, Compact sets in the space $L^p(0,T;B)$, Ann. Mat. Pura ed Applicata IV, 146(4) (1987), 65-96.

[13] R. Temam, Infinite-dimensional dynamical systems in mechanics and physics, World Publishing Corporation, 68(5) (2000), 2135-2143.

[14] L. Cherfils and A. Miranville, On the Caginalp system with dynamics boundary conditions and singular potentials, Appl. Math., 54(2) (2009), 89-115.

[15] A. Miranville, On a phase-field model with a logarithmic nonlinearity, Appl. Math., 57(3) (2012), 215-229.

[16] M. Grasselli, A. Miranville, V. Pata and S. Zelik, Well-posedness and long time behavior of a parabolic-hyperbolic phase-field system with singular potentials, Math. Nachr., 280(13-14) (2007), 1475-1509.

[17] L. Cherfils and A. Miranville, Some results on the asymptotic behavior of the Caginalp system with singular potentials, Adv. Math. Sci. Appl., 17(1) (2007), 107-129.

[18] A. Miranville and S. Zelik, Chapter 3 attractors for dissipative partial differential equations in bounded and unbounded domains [J], Handbook of Differential Equations Evolutionary Equations, 4(08) (2008), 103-200.

[19] A. Miranville, Exponential attractors for a class of evolutionary equation by a decomposition method, C. R. Acad. Sci.-Series I-Math., 328(2) (1999), 145-150.

[20] A. Miranville and S. Zelik, Robust exponential attractors for Cahn-Hilliard type equations with singular potentials, Math. Methods Appl. Sci., 27 (2004), 545-582.

[21] A. Miranville and S. Zelik, The Cahn-Hilliard equation with singular potentials and dynamic boundry conditions, Discrete Contin. Dyn. Systems., 28 (2010), 275-310.

[22] A. Miranville and S. Zelik, Robust exponential attractors for singularly perturbed phase-

field type equations, Electronic J. Diff. Eqns., 2002 (2002), 1-28.

[23] A. Miranville and R. Quintanilla, A type III phase-field system with a logarithmic potential, Appl. Math. Lett., 24(6) (2011), 1003-1008.

[24] A. Miranville and R. Quintanilla, A phase-field model based on a three-phase-lag heat conduction, Appl. Math. Optim., 63(1) (2011), 133-150.

[25] A. Miranville and R. Quintanilla, On a phase-field system based on the Cattaneo law, Nonlinear Anal.: Theory, Methods and Applications, 75(4) (2012), 2552-2565.

[26] D. Brochet and D. Hilhorst, Universal attractor and inertial sets for the phase-field model, Appl. Math. Lett., 4(6) (1991), 59-63.

[27] D. Brochet, D. Hilhorst and A. Novick-Cohen, Finite-dimensional exponential attractor for a model for order-desorder and phase separation, Appl. Math. Lett., 7(3) (1992), 83-87.

[28] A. Babin and B. Nicolaenko, Exponential attractors for reaction-diffusion systems in an unbounded domain, J. Dynam. Diff. Equ., 7(5) (1995), 567-590.

[29] M. Conti, S. Gatti and A. Miranville, Asymptotic behavior of the Caginalp phase-field system with coupled dynamic boundary conditions, Discrete Contin. Dyn. Sys. Ser. S, 5(3) (2012), 485-505.

[30] P. Fabrie and C. Galusinski, Exponential attractors for partially dissipative reaction system, Asymptotic Anal., 12(4) (1996), 329-354.