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On Multivariate Fractional Taylor's and Cauchy' Mean Value Theorem

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Abstract. In this paper, a generalized multivariate fractional Taylor's and Cauchy's mean value theorem of the kind

$$f(x,y)=\sum_{j=0}^{n}\frac{D^{j\alpha}f(x_0,y_0)}{\Gamma(j\alpha+1)}+R_n^{\alpha}(\xi,\eta),\qquad \frac{f(x,y)-\sum\limits_{j=0}^{n}\frac{D^{j\alpha}f(x_0,y_0)}{\Gamma(j\alpha+1)}}{g(x,y)-\sum\limits_{j=0}^{n}\frac{D^{j\alpha}g(x_0,y_0)}{\Gamma(j\alpha+1)}}=\frac{R_n^{\alpha}(\xi,\eta)}{T_n^{\alpha}(\xi,\eta)},$$

where $0<\alpha\leq 1$, is established. Such expression is precisely the classical Taylor's and Cauchy's mean value theorem in the particular case $\alpha=1$. In addition, detailed expressions for $R_n^{\alpha}(\xi,\eta)$ and $T_n^{\alpha}(\xi,\eta)$ involving the sequential Caputo fractional derivative are also given.

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Key words: Sequential Caputo fractional derivative, generalized Taylor's mean value theorem, generalized Taylor's formula, generalized Cauchy' mean value theorem, generalized Cauchy's formula

1 Introduction

The ordinary Taylor's formula has been generalized by many authors. Riemann [1] had already written a formal version of the generalized Taylor's series:

$$f(x+h)=\sum_{m=-\infty}^{\infty}\frac{h^{m+r}}{\Gamma(m+r+1)}(D_a^{-(m+r)}f)(x), \tag{1.1}$$

where $D_a^{-(m+r)}$ is the Riemann-Liouville fractional integral of order $m+r$.

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Afterwards, Watanable [2] obtained the following relation:

$$f(x)=\sum_{k=-m}^{n-1}\frac{(x-x_0)^{\alpha+k}}{\Gamma(\alpha+k+1)}(D_a^{m+r}f)(x_0)+R_{n,m}, \tag{1.2}$$

with $m<\alpha, a\le x_0<x$, and

$$R_{n,m}=(D_{x_0}^{-(\alpha+n)}D_a^{\alpha+n}f)(x)+\frac{1}{\Gamma(-\alpha-m)}\int_a^{x_0}(x-t)^{-\alpha-m-1}(D_a^{\alpha-m-1}f)(t)dt,$$

where $D_a^{\alpha+n}$ is the Riemann-Liouville fractional derivative of order $\alpha+n$.

On the other hand, a variant of the generalized Taylor's series was given by Dzherbashyan and Nersesyan [3]. For f having all of the required continuous derivatives, they obtained

$$f(x)=\sum_{k=0}^{m-1}\frac{(D^{(\alpha_k)}f)(0)}{\Gamma(1+\alpha_k)}x^{\alpha_k}+\frac{1}{\Gamma(1+\alpha_m)}\int_0^x(x-t)^{\alpha_m-1}(D^{(\alpha_m)}f)(t)dt, \tag{1.3}$$

where $0<x,\alpha_0,\alpha_1,...,\alpha_m$ is an increasing sequence of real numbers such that $0<\alpha_k-\alpha_{k-1}\le 1, k=1,...,m$ and $D^{(\alpha_m)}f=I_0^{1-(\alpha_k-\alpha_{k-1})}D_0^{1+\alpha_{k-1}}f$.

Under certain conditions for f and $\alpha\in[0,1]$, Trujillo *et al.* [4] introduce the following generalized Taylor's mean value theorem:

$$\begin{aligned} &f(x)=\sum_{j=0}^{n}\frac{c_j(x-a)^{(j+1)\alpha-1}}{\Gamma((J+1)\alpha)}+R_n(f;\xi),\\ &R_n(f;\xi)=\frac{(D_a^{(n+1)\alpha}f)(\xi)}{\Gamma((n+1)\alpha+1)}\cdot(x-a)^{(n+1)\alpha}, \qquad a\le\xi\le x,\\ &c_j=\Gamma(a)[(x-a)^{1-\alpha}D_a^{j\alpha}f)(a+), \qquad j=0,1,...,n \end{aligned} \tag{1.4}$$

and the sequential fractional Riemann-Liouville derivative is denoted by

$$D_a^{n\alpha}=D_a^{\alpha}\cdot D_a^{\alpha}\cdot ...\cdot D_a^{\alpha}(n-times).$$

Recently, Odibat and Shawagfeh [5] obtain a new generalized Taylor's mean value theorem of this kind

$$f(x)=\sum_{j=0}^{n}\frac{(x-a)^{j\alpha}}{\Gamma(j\alpha+1)}(D_a^{j\alpha}f)(a)+\frac{(D_a^{(n+1)\alpha}f)(\xi)}{\Gamma((n+1)\alpha+1)}(x-a)^{(n+1)\alpha} \tag{1.5}$$

with $a\le\xi\le x$, where $D^{j\alpha}$ is the sequential fractional Caputo derivative.

In 2005, Pecaric *et al.* [6] deduced the Cauchy type mean value theorem for the sequence fractional Riemann-Liouville derivative from known mean value theorem of the Lagrange type.

Theorem 1.1 ([6]). *Let $\alpha \in [0,1]$, and let $f,g \in C(a,b]$ such that*

$$D_a^\alpha f, D_a^\alpha g \in C[a,b]$$

where$D_a^\alpha g(x) \neq 0$ for every $x \in [a,b]$.

Then for every $x \in [a,b]$, there is a $\xi (a \leq \xi \leq x)$ such that

$$\frac{f(x) - [(x-a)^{1-\alpha} f(x)](a+)(x-a)^{\alpha-1}}{g(x) - [(x-a)^{1-\alpha} g(x)](a+)(x-a)^{\alpha-1}} = \frac{D_a^\alpha f(\xi)}{D_a^\alpha g(\xi)}. \tag{1.6}$$

Theorem 1.2 ([6]). *Let $\alpha \in [0,1]$ and $n \in N$, and $D_a^{(n+1)\alpha} g(x) \neq 0$ for every $x \in [a,b]$. Let f be a continuous function on $(a,b]$ satisfying each of the following conditions:*

(i) $D_a^{j\alpha} f \in C(a,b]$ and $D_a^{j\alpha} f \in_a I_\alpha [a,b]$ for $j = 1,...,n$.

(ii) $D_a^{(n+1)} f$ is continuous on $[a,b]$.

(iii) If $\alpha < 1/2$, then, for each $j \in 1,...,n$ such that $(j+1)\alpha \leq 1$, $D_a^{(j+1)\alpha} f(x)$ is γ -continuous at $x = a$ for some γ $(1-(j+1)\alpha \leq \gamma \leq 1)$ or a-singular of order α.

Then for every $x \in [a,b]$, there is a $\xi (a \leq \xi \leq x)$ such that

$$\frac{R_n(f;x,a)}{R_n(g;x,a)} = \frac{D_a^{(n+1)\alpha} f(\xi)}{D_a^{(n+1)\alpha} g(\xi)}, \tag{1.7}$$

where R_n is defined by

$$\begin{aligned}
&R_n(f;x,a) = f(x) - \sum_{j=0}^{n} \frac{c_j (x-a)^{(j+1)\alpha - 1}}{\Gamma((j+1)\alpha)}, \\
&c_j = \Gamma(a)[(x-a)^{1-\alpha} D_a^{j\alpha} f)(a+), \quad j = 0,1,...,n; \\
&R_n(g;x,a) = g(x) - \sum_{j=0}^{n} \frac{c_j (x-a)^{(j+1)\alpha - 1}}{\Gamma((j+1)\alpha)}, \\
&d_j = \Gamma(a)[(x-a)^{1-\alpha} D_a^{j\alpha} g)(a+), \quad j = 0,1,...,n.
\end{aligned}$$

There are also some papers on multivariate fractional Taylor's formula. In 2006, Jumarie [7] had given the following Multivariate fractional Taylor Series

$$\begin{aligned}
f(x+h,y+l) &= E_\alpha(h^\alpha D_x^\alpha) E_\alpha(l^\alpha D_y^\alpha) f(x,y) \\
&= E_\alpha(l^\alpha D_y^\alpha) E_\alpha(h^\alpha D_x^\alpha) f(x,y) = E_\alpha[(hD_x + lD_y)^\alpha] f(x,y),
\end{aligned} \tag{1.8}$$

where $E_\alpha(x)$ denotes the Mittage-Leffler function defined by the expression [10–18]

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}.$$

But I am afraid Eq. (1.6) is incorrect, since its proof is based on the following equality

$$E_\alpha[(u+v)^\alpha]=E_\alpha(u^\alpha)E_\alpha(v^\alpha), \tag{1.9}$$

which seems incorrect unless $\alpha=1$.

In 2009, Anastassiou [8], [9, p.276] obtained an important result on multivariate fractional Taylor's formula via the Caputo fractional derivative. The fractional remainder is expressed as a composition of two Riemann-Liouville fractional integrals.

Theorem 1.3 ([8], [9]). *Let $f\in C^n(Q)$, Q compact and convex $\subset R^k$, $k\geq 2$; here $\gamma\geq 1$ such that $n=[\gamma]$. For fixed $x_0,z\in Q$, then*

$$\begin{aligned} f(z)=&\, f(x_0)+\sum_{i=1}^{k}(z_i-x_{0i})\frac{\partial f(x_0)}{\partial x_i}+\sum_{l=2}^{n-1}\frac{\{\sum_{i=1}^{k}[(z_i-x_{0k})\frac{\partial}{\partial x_i}]^l f\}(x_0)}{l!}\\ &+\frac{1}{\Gamma(\gamma)}\int_0^1(1-t)^{\gamma-1}D_0^{-(n-\gamma)}\left\{\sum_{i=1}^{k}[(z_i-x_{0k})\frac{\partial}{\partial x_i}]^n f\}(x_0+t(z-x_0))\right\}dt. \end{aligned} \tag{1.10}$$

However, there is less paper on the developments on multivariate fractional Taylor's and Cauchy type mean value theorem, the fractional reminder is expressed as the Lagrange reminder term. In this paper, we will give new kind of multivariate fractional Tayor's and Cauchy's mean value theorem via the sequence Caputo fractional derivative.

In order to establish fractional Taylor's and Cauchy' mean value theorem, the main idea seems that how to give the suitable definition of fractional integral and derivative of function with multivariate. In this paper, we will give a appropriate definition of fractional integral and derivative of function with multivariate, then derive a kind of multivariate fractional Taylor's and Cauchy's mean value theorem via the sequence Caputo fractional derivative. The organization of the work is set out as followings: Several definitions and propositions are given in Section 2; Fractional Taylor's and fractional Cauchy's mean value theorem with one variable via the sequence Caputo fractional derivative are presented in Section 3; and finally, multivariate fractional Taylor's and Cauchy's mean value theorem are obtained in Section 4.

2 Definitions and propositions

For the concept of fractional derivative we will adopt Caputo's definition which is a modification of the Riemann-Liouville definition and has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variables and their integer order which in the case in most physical processes. More detailed information on fractional calculus may be found in these books [10–18].

Definition 2.1 ([5,19]). A function $f(x)(x\geq 0)$ is said to be in the space $C_\alpha(\alpha\in R)$ if it can be written as $f(x)=x^p f_1(x)$ for $p>\alpha$ where $f_1(x)$ is continuous in $[0,\infty)$, and it is said to be in the space $C_\alpha^{(m)}$ if $f^{(m)}\in C_\alpha, m\in N$.

Definition 2.2 ([7–14]). Let $f(x)\in C_\alpha(a,\infty)$, the Riemann-Liouville integral operator of order $\alpha>0$ is defined as

$$(D_a^{-\alpha}f)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, x>a.$$

Definition 2.3 ([7–14]). Let $f(x)\in C_\alpha^{(m)}(a,\infty)$, the Caputo fractional derivative of $f(x)$ of order $\alpha>0$ is defined as

$$(D_a^{\alpha}f)(x)=(D_a^{\alpha-m}f^{(m)})(x)=\frac{1}{\Gamma(m-\alpha)}\int_a^x \frac{f^{(m)}(t)}{(x-t)^{\alpha+1-m}}dt$$

for $m-1<\alpha\leq m, m\in N, x\geq a$.

Proposition 2.1 ([7–14]). Let $D_a^{-\alpha}$ be Riemann-Liouville integral operator of order $\alpha>0$, D_a^α be Caputo fractional derivative operator of order $\alpha>0, 0<\alpha\leq 1$, $D_a^\alpha f(x)\in C(a,b)]$, then

$$[D_a^{-\alpha}D_a^{\alpha}f](x)=f(x)-f(a).$$

In order to derive fractional Taylor's formula and Cauchy formula of a function with multivariate, we will first give the following integrate definition of a function $f(x,y)$, $(x,y)\in D$, where $D\subset R^2$ is a compact and convex domain.

Definition 2.4. Let $(x_0,y_0),(x,y)\in D$, $\Delta x=x-x_0, \Delta y=y-y_0, 0\leq s\leq 1$, and $f(x,y)\in C(D)$, define

$$(D^{-1}f)(x_0+s\Delta x, y_0+s\Delta y)=\int_0^s f(x_0+t\Delta x, y_0+t\Delta y)dt, \tag{2.1}$$

when $s=1$, define

$$(D^{-1}f)(x,y)=\int_0^1 f(x_0+t\Delta x, y_0+t\Delta y)dt. \tag{2.2}$$

Proposition 2.2. Let $k\in N$, and $(x_0,y_0),(x,y)\in D$, $\Delta x=x-x_0, \Delta y=y-y_0, 0\leq s\leq 1$, then

$$(D^{-k}f)(x_0+s\Delta x, y_0+s\Delta y)=\frac{1}{(k-1)!}\int_0^s (s-t)^{k-1}f(x_0+t\Delta x, y_0+t\Delta y)dt. \tag{2.3}$$

Proof. By Definition 2.4, we have

$$\begin{aligned}
&(D^{-2}f)(x_0+s\Delta x, y_0+s\Delta y)\\
=&\int_0^s D^{-1}f(x_0+t\Delta x, y_0+t\Delta y)dt=\int_0^s dt\int_0^t f(x_0+u\Delta x, y_0+u\Delta y)du\\
=&\int_0^s du\int_u^s f(x_0+u\Delta x, y_0+u\Delta y)dt=\int_0^s (s-u)f(x_0+u\Delta x, y_0+u\Delta y)du.
\end{aligned}$$

By induction, it is not hard to prove that

$$(D^{-k}f)(x_0+s\Delta x,y_0+s\Delta y)=\frac{1}{(k-1)!}\int_0^s(s-t)^{k-1}f(x_0+t\Delta x,y_0+t\Delta y)dt. \qquad \square$$

Now we can define fractional integral of $f(x,y)$ of order γ.

Definition 2.5. Let $\gamma\in R^+$, $(x_0,y_0),(x,y)\in D$, $\Delta x=x-x_0,\Delta y=y-y_0,0\le s\le 1$, define

$$(D^{-\gamma}f)(x_0+s\Delta x,y_0+s\Delta y)=\frac{1}{\Gamma(\gamma)}\int_0^s(s-t)^{\gamma-1}f(x_0+t\Delta x,y_0+t\Delta y)dt, \tag{2.4}$$

when $s=1$, define

$$(D^{-\gamma}f)(x,y)=\frac{1}{\Gamma(\gamma)}\int_0^1(1-t)^{\gamma-1}f(x_0+t\Delta x,y_0+t\Delta y)dt. \tag{2.5}$$

For convenience, let us set

$$\varphi(t)=f(x_0+t\Delta x,y_0+t\Delta y).$$

Then we have

Proposition 2.3. Let $(x_0,y_0),(x,y)\in D$, $\Delta x=x-x_0,\Delta y=y-y_0,0\le s\le 1$, then

$$\begin{aligned}&(D^{-\gamma}f)(x_0+s\Delta x,y_0+s\Delta y)\\ =&\frac{1}{\Gamma(\gamma)}\int_0^s(s-t)^{\gamma-1}f(x_0+t\Delta x,y_0+t\Delta y)dt=(D^{-\gamma}\varphi)(s), &(2.6)\\ &(D^{-\gamma}f)(x,y)=\frac{1}{\Gamma(\gamma)}\int_0^1(1-t)^{\gamma-1}f(x_0+t\Delta x,y_0+t\Delta y)dt=(D^{-\gamma}\varphi)(1). &(2.7)\end{aligned}$$

By Proposition 2.3, it is easy to see that

Proposition 2.4. Let $\alpha,\beta\in R^+$, then

$$\begin{aligned}&D^{-\alpha}D^{-\beta}f(x_0+s\Delta x,y_0+s\Delta y)=D^{-(\alpha+\beta)}f(x_0+s\Delta x,y_0+s\Delta y),\\ &D^{-\alpha}D^{-\beta}f(x,y)=D^{-(\alpha+\beta)}f(x,y).\end{aligned}$$

Definition 2.6. If $n\in N$, define

$$D^nf(x_0+s\Delta x,y_0+s\Delta y)=(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y})^nf(x_0+s\Delta x,y_0+s\Delta y), \tag{2.8}$$

$$D^nf(x,y)=(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y})^nf(x,y). \tag{2.9}$$

Definition 2.7. Let $\mu>0$, and let n be the smallest integer exceeding μ, we define the fractional derivative of f of order μ as following

$$(D^{\mu}f)(x_0+s\Delta x,y_0+s\Delta y)=(D^{\mu-n}D^n f)(x_0+s\Delta x,y_0+s\Delta y), \tag{2.10}$$
$$(D^{\mu}f)(x,y)=(D^{\mu-n}D^n f)(x,y). \tag{2.11}$$

From Definition 2.7, it is easy to know

Proposition 2.5. Let $\varphi(s)=f(x_0+s\Delta x,y_0+s\Delta y)$, then

$$D^n f(x_0+s\Delta x,y_0+s\Delta y)=\varphi^{(n)}(s), \tag{2.12}$$
$$D^n f(x,y)=\varphi^{(n)}(1),D^n f(x_0,y_0)=\varphi^{(n)}(0). \tag{2.13}$$

By Proposition 2.4 and Proposition 2.5, we can obtain

Proposition 2.6. Let $\mu\in R^+$, then

$$(D^{\mu}f)(x_0+s\Delta x,y_0+s\Delta y)=(D^{\mu}\varphi)(s), \tag{2.14}$$
$$(D^{\mu}f)(x,y)=(D^{\mu}\varphi)(1),(D^{\mu}f)(x_0,y_0)=(D^{\mu}\varphi)(0). \tag{2.15}$$

3 Fractional Taylor's and Cauchy's mean value theorem with one variable

In this section, we will give fractional Taylor's mean value theorem and Cauchy's mean value theorem involving the sequential Caputo fractional derivative with one variable.

Let us begin with basic fractional Lagrange's mean value theorem.

Lemma 3.1 (Fractional Lagrange's mean value theorem)**.** *Suppose that $f(x)\in C[a,b]$ and $D_a^{\alpha}f(x)\in C[a,b]$, for $0\le\alpha\le 1$, then we have*

$$f(b)-f(a)=\frac{1}{\Gamma(\alpha+1)}D_a^{\alpha}f(\xi)(b-a)^{\alpha} \tag{3.1}$$

with $a\le\xi\le b$.

Proof. In view of Proposition 2.1, we have

$$f(b)-f(a)=[D_a^{-\alpha}D_a^{\alpha}f](b)=\frac{1}{\Gamma(\alpha)}\int_a^b(b-\tau)^{\alpha-1}[D_a^{\alpha}f](\tau)d\tau$$
$$=[D_a^{\alpha}f](\xi)\frac{1}{\Gamma(\alpha)}\int_a^b(b-\tau)^{\alpha-1}d\tau=[D_a^{\alpha}f](\xi)\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}.$$

From Lemma 3.1, It is easy to obtain fractional Rolle's mean value theorem which is useful in the next. □

Lemma 3.2 (Fractional Rolle's mean value theorem). *Suppose that $f(x)\in C[a,b]$, $D_a^\alpha f(x)\in C[a,b]$, for $0\le\alpha\le 1$, and $f(a)=f(b)$, then there exists $\xi\in(a,b)$, such that*

$$D_a^\alpha f(\xi)=0. \tag{3.1}$$

Now we can derive fractional Cauchy's mean value theorem with one variable.

Theorem 3.1 (Fractional Cauchy's mean value theorem). *Suppose that $f(x),g(x)\in C[a,b]$ and $D_a^\alpha f(x),D_a^\alpha g(x)\in C[a,b]$, where $D_a^\alpha g(x)\neq 0$ for $0\le\alpha\le 1$.*

Then we have

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{D_a^\alpha f(\xi)}{D_a^\alpha g(\xi)} \tag{3.2}$$

with $a\le\xi\le b$.

Proof. Set

$$F(x)=[f(b)-f(a)][g(x)-g(a)]-[f(x)-f(a)][g(b)-g(a)],$$

then $F(a)=F(b)=0$, in view of fractional Rolle's mean value Lemma 3.2, so that there exists $\xi\in(a,b)$, such that

$$D_a^\alpha F(\xi)=0.$$

Therefore, we have

$$[f(b)-f(a)]D_a^\alpha g(\xi)-D_a^\alpha f(\xi)[g(b)-g(a)]=0,$$

Theorem 3.1 is completed. □

Theorem 3.2 (Fractional Taylor's mean value theorem). *Suppose that $D_a^{k\alpha}f(x)\in C[a,b]$ for $k=0,1,\dots,m+1$, where $0<\alpha\le 1$, then we have*

$$f(b)=\sum_{k=0}^{m}\frac{(D_a^{k\alpha}f)(a)}{\Gamma(k\alpha+1)}(b-a)^{k\alpha}+\frac{(D_a^{(m+1)\alpha}f)(\xi)}{\Gamma((m+1)\alpha+1)}(b-a)^{(m+1)\alpha} \tag{3.3}$$

with $a\le\xi\le b$, where $D_a^{k\alpha}f$ is sequential Caputo fractional derivative.

Theorem 3.2 have also been established in [5] (see (1.5) in Section 1), here we give another kind of method by the use of fractional Cauchy's mean value Theorem 3.1.

Proof. By the use of fractional Cauchy's mean Theorem 3.1, we can obtain

$$\begin{aligned}&\frac{f(b)-\sum\limits_{k=0}^{m}\frac{(D_a^{k\alpha}f)(a)}{\Gamma(k\alpha+1)}(b-a)^{k\alpha}}{\frac{(b-a)^{(m+1)\alpha}}{\Gamma((m+1)\alpha+1)}}=\frac{(D_a^{\alpha}f)(\xi_1)-\sum\limits_{k=1}^{m}\frac{(D_a^{k\alpha}f)(a)}{\Gamma((k-1)\alpha+1)}(\xi_1-a)^{(k-1)\alpha}}{\frac{(\xi_1-a)^{m\alpha}}{\Gamma(m\alpha+1)}}\\
&=\frac{(D_a^{2\alpha}f)(\xi_2)-\sum\limits_{k=2}^{m}\frac{(D_a^{k\alpha}f)(a)}{\Gamma((k-2)\alpha+1)}(\xi_2-a)^{(k-2)\alpha}}{\frac{(\xi_2-a)^{(m-1)\alpha}}{\Gamma((m-1)\alpha+1)}}\\
&=\cdots=\frac{(D_a^{m\alpha}f)(\xi_m)-(D_a^{m\alpha}f)(a)}{\frac{(\xi_m-a)^{\alpha}}{\Gamma(\alpha+1)}}=(D_a^{(m+1)\alpha}f)(\xi_{m+1}),\end{aligned}$$

where $a\leqslant\xi_k\leqslant b,k=1,\cdots,m+1$.

So that we have

$$f(b)=\sum_{k=0}^{m}\frac{(D_a^{k\alpha}f)(a)}{\Gamma(k\alpha+1)}(b-a)^{k\alpha}+\frac{(D_a^{(m+1)\alpha}f)(\xi_{m+1})}{\Gamma((m+1)\alpha+1)}(b-a)^{(m+1)\alpha}.$$

The proof of Theorem 3.2 is completed. □

Theorem 3.3 (Fractional Cauchy's mean value theorem)**.** *Suppose that $D_a^{k\alpha}f(x),D_a^{k\alpha}g(x)\in C[a,b]$ for $k=0,1,...,m+1$, where $D_a^{k\alpha}g(x)\neq 0,0\leq\alpha\leq 1$.*

Then we have

$$\frac{f(b)-\sum\limits_{k=0}^{m}\frac{(D_a^{k\alpha}f)(a)}{\Gamma(k\alpha+1)}(b-a)^{k\alpha}}{g(b)-\sum\limits_{k=0}^{m}\frac{(D_a^{k\alpha}g)(a)}{\Gamma(k\alpha+1)}(b-a)^{k\alpha}}=\frac{(D_a^{(m+1)\alpha}f)(\xi)}{(D_a^{(m+1)\alpha}g)(\xi)}. \tag{3.4}$$

Proof. By the use of fractional Cauchy's mean Theorem 3.1, we have

$$\frac{f(b)-\sum\limits_{k=0}^{m}\frac{(D_a^{k\alpha}f)(a)}{\Gamma(k\alpha+1)}(b-a)^{k\alpha}}{g(b)-\sum\limits_{k=0}^{m}\frac{(D_a^{k\alpha}g)(a)}{\Gamma(k\alpha+1)}(b-a)^{k\alpha}}=\frac{(D_a^{\alpha}f)(\xi_1)-\sum\limits_{k=1}^{m}\frac{(D_a^{k\alpha}f)(a)}{\Gamma((k-1)\alpha+1)}(\xi_1-a)^{(k-1)\alpha}}{(D_a^{\alpha}g)(\xi_1)-\sum\limits_{k=1}^{m}\frac{(D_a^{k\alpha}g)(a)}{\Gamma((k-1)\alpha+1)}(\xi_1-a)^{(k-1)\alpha}},$$

and

$$\begin{aligned}&\frac{(D_a^{\alpha}f)(\xi_1)-\sum\limits_{k=1}^{m}\frac{(D_a^{k\alpha}f)(a)}{\Gamma((k-1)\alpha+1)}(\xi_1-a)^{(k-1)\alpha}}{(D_a^{\alpha}g)(\xi_1)-\sum\limits_{k=1}^{m}\frac{(D_a^{k\alpha}g)(a)}{\Gamma((k-1)\alpha+1)}(\xi_1-a)^{(k-1)\alpha}}=\frac{(D_a^{2\alpha}f)(\xi_2)-\sum\limits_{k=2}^{m}\frac{(D_a^{k\alpha}f)(a)}{\Gamma((k-2)\alpha+1)}(\xi_2-a)^{(k-2)\alpha}}{(D_a^{2\alpha}g)(\xi_2)-\sum\limits_{k=2}^{m}\frac{(D_a^{k\alpha}g)(a)}{\Gamma((k-2)\alpha+1)}(\xi_2-a)^{(k-2)\alpha}}\\ =&\cdots=\frac{(D_a^{m\alpha}f)(\xi_m)-(D_a^{m\alpha}f)(a)}{(D_a^{m\alpha}g)(\xi_m)-(D_a^{m\alpha}g)(a)}=\frac{(D_a^{(m+1)\alpha}f)(\xi_{m+1})}{(D_a^{(m+1)\alpha}g)(\xi_{m+1})},\end{aligned}$$

where $a\leqslant\xi_k\leqslant b,k=1,\cdots,m+1$. The proof of Theorem 3.3 is completed. □

Remark 3.1. (1) Theorem 3.1 and Theorem 3.3 are essentially new, which are the analogy to Theorem 1.1 and Theorem 1.2 via sequential fractional Riemann-Lionville derivative in Section 1.

(2) Set $g(x)=\frac{(x-a)^{(m+1)\alpha}}{\Gamma((m+1)\alpha+1)}$, then Theorem 3.3 reduces to Theorem 3.2.

4 Fractional multivariate Taylor's formula and Cauchy's formula

In this section, we discuss fractional Taylor's formula and Cauchy's formula with multivariate. First, We discuss multivariate fractional Taylor's formula and Cauchy's formula with the Lagrange remainder term.

Theorem 4.1 (Multivariate fractional Taylor's mean value theorem). *Let D be a compact and convex domain, $(x_0,y_0),(x,y)\in D\subset R^2$, and $D^{k\alpha}f(x,y)\in C(D),k=0,1,...,m+1$. Then*

$$f(x,y)=\sum_{k=0}^{m}\frac{D^{k\alpha}f(x_0,y_0)}{\Gamma(k\alpha+1)}+\frac{D^{(m+1)\alpha}f(\xi,\eta)}{\Gamma((m+1)\alpha+1)},\tag{4.1}$$

where $\xi=x_0+\theta(x-x_0)=x_0+\theta\Delta x,\eta=y_0+\theta\Delta y,(0<\theta<1)$ and $D^{k\alpha}f(x_0,y_0),D^{(n+1)\alpha}f(\xi,\eta)$ are defined in Section 2.

Proof. In Theorem 3.2, replacing function f by φ, and setting $a=0,b=1$, yield

$$\varphi(1)=\sum_{k=0}^{m}\frac{\varphi^{(k\alpha)}(0)}{\Gamma(k\alpha+1)}+\frac{\varphi^{((m+1)\alpha)}(\theta)}{\Gamma((m+1)\alpha+1)},\quad (0<\theta<1).\tag{4.2}$$

On the other hand, set $\varphi(t)=f(x_0+t(x-x_0),y_0+t(y-y_0))$, and by Proposition 2.6, we have

$$\varphi(1)=f(x,y),\ \ \varphi^{(k\alpha)}(0)=D^{k\alpha}f(x_0,y_0),\ \ \varphi^{((m+1)\alpha)}(\theta)=D^{(m+1)\alpha}f(\xi,\eta).\tag{4.3}$$

Substituting (4.3) into Eq. (4.2), then the proof of Theorem 4.1 is completed. □

Theorem 4.2 (Multivariate fractional Cauchy's mean value theorem). *Let D be a compact and convex domain, $(x_0,y_0),(x,y)\in D$, and $D^{k\alpha}f(x,y)$, $D^{k\alpha}f(x,y)\in C(D)$, $k=0,1,...,m+1$; $D^{(m+1)\alpha}g(x,y)\neq 0$. Then we have*

$$\frac{f(x,y)-\sum\limits_{k=0}^{m}\frac{D^{k\alpha}f(x_0,y_0)}{\Gamma(k\alpha+1)}}{g(x,y)-\sum\limits_{k=0}^{m}\frac{D^{k\alpha}g(x_0,y_0)}{\Gamma(k\alpha+1)}}=\frac{D^{(m+1)\alpha}f(\xi,\eta)}{D^{(m+1)\alpha}g(\xi,\eta)},\tag{4.4}$$

where $\xi=x_0+\theta(x-x_0)=x_0+\theta\Delta x,\eta=y_0+\theta\Delta y,(0<\theta<1)$.

Proof. In Theorem 3.3, replacing function f by φ, g by ψ and setting $a=0,b=1$, then we get

$$\frac{\varphi(1)-\sum\limits_{k=0}^{m}\frac{\varphi^{(k\alpha)}(0)}{\Gamma(k\alpha+1)}}{\psi(1)-\sum\limits_{k=0}^{m}\frac{\psi^{(k\alpha)}(0)}{\Gamma(k\alpha+1)}}=\frac{\varphi^{(m+1)\alpha}(\theta)}{\psi^{(m+1)\alpha}(\theta)}.\tag{4.5}$$

On the other hand, set $\varphi(t)=f(x_0+t(x-x_0),y_0+t(y-y_0)),\psi(t)=g(x_0+t(x-x_0),y_0+t(y-y_0))$, by Proposition 2.6, we have

$$\varphi(1)=f(x,y),\ \ \varphi^{(k\alpha)}(0)=D^{k\alpha}f(x_0,y_0),\ \ \varphi^{((n+1)\alpha)}(\theta)=D^{(n+1)\alpha}f(\xi,\eta),\tag{4.6}$$

$$\psi(1)=f(x,y),\ \ \psi^{(k\alpha)}(0)=D^{k\alpha}f(x_0,y_0),\ \ \psi^{((n+1)\alpha)}(\theta)=D^{(n+1)\alpha}f(\xi,\eta).\tag{4.7}$$

Substituting (4.6) and (4.7) into Eq. (4.5), then the proof of Theorem 4.2 is completed. □

Now, we set

$$\varphi(t)=f(x_1+t\Delta x_1,...,x_n+t\Delta x_n),$$

where $\Delta x_1=y_1-x_1,\Delta x_2=y_2-x_2,...,\Delta x_n=y_n-x_n$.

We can obtain the following proposition by a process analogous to Proposition 2.6.

Proposition 4.1. Let $(x_1,...,x_n),(y_1,...,y_n)\in D$, where $D\subset R^n$ is a compact and convex domain, then

$$(D^\mu f)(x_1+s\Delta x_1,...,x_n+s\Delta x_n)=(D^\mu\varphi)(s), \tag{4.8}$$

$$(D^\mu f)(y_1,...,y_n)=(D^\mu\varphi)(1),(D^\mu f)(x_1,...,x_n)=(D^\mu\varphi)(0). \tag{4.9}$$

By a process analogous to Theorem 4.1 and Theorem 4.2, we can obtain the following theorems.

Theorem 4.3 (Multivariate fractional Taylor's mean value theorem). *Suppose that $D_a^{k\alpha}f$ are continuous in $D\subset R^n$, for $k=0,1,...,m+1$, where $0\le\alpha\le 1$, then we have*

$$f(y_1,...,y_n)=\sum_{k=0}^{m}\frac{D^{k\alpha}f(x_1,...,x_n)}{\Gamma(k\alpha+1)}+\frac{D^{(m+1)\alpha}f(\xi_1,...,\xi_n)}{\Gamma((m+1)\alpha+1)}, \tag{4.10}$$

where $\xi_i=x_i+\theta(y_i-x_i),i=1,...,n,0<\theta<1$.

Theorem 4.4 (Multivariate fractional Cauchy's mean value theorem). *Suppose that $D_a^{k\alpha}f$, $D_a^{k\alpha}g$ are continuous in D, for $k=0,1,...,m+1$, where $D_a^{(m+1)\alpha}g\neq 0,0\le\alpha\le 1$, then we have*

$$\frac{f(y_1,...,y_n)-\sum\limits_{k=0}^{m}\frac{D^{k\alpha}f(x_1,...,x_n)}{\Gamma(k\alpha+1)}}{g(y_1,...,y_n)-\sum\limits_{k=0}^{m}\frac{D^{k\alpha}g(x_1,...,x_n)}{\Gamma(k\alpha+1)}}=\frac{D^{(m+1)\alpha}f(\xi_1,...,\xi_n)}{D^{(m+1)\alpha}g(\xi_1,...,\xi_n)}, \tag{4.11}$$

where $\xi_i=x_i+\theta(y_i-x_i),i=1,...,n$.

Next let us discuss fractional Cauchy's formula and Cauchy's formula with integral remainder term.

Lemma 4.1. *Suppose that $\varphi^{(k\alpha)}(t)\in C[0,1]$ for $k=0,1,...,m+1$, where $0<\alpha\le 1$, then we have*

$$\varphi(t)=\sum_{k=0}^{m}\frac{\varphi^{k\alpha}(0)}{\Gamma(k\alpha+1)}t^{k\alpha}+\frac{1}{\Gamma((m+1)\alpha)}\int_0^t(t-\tau)^{(m+1)\alpha-1}\varphi^{(m+1)\alpha}(\tau)d\tau, \tag{4.12}$$

when $t=1$, then

$$\varphi(1)=\sum_{k=0}^{m}\frac{\varphi^{k\alpha}(0)}{\Gamma(k\alpha+1)}+\frac{1}{\Gamma((m+1)\alpha)}\int_0^1(1-\tau)^{(m+1)\alpha-1}\varphi^{(m+1)}(\tau)d\tau. \tag{4.13}$$

Proof. By Laplace Transform, we have

$$\begin{aligned}
&L\{\frac{1}{\Gamma((m+1)\alpha)}\int_0^t (t-\tau)^{(m+1)\alpha-1}\varphi^{(m+1)}(\tau)d\tau\}(s)\\
=&L\{D^{-(m+1)\alpha}D^{(m+1)\alpha}\varphi(t)\}(s)\\
=&s^{-(m+1)\alpha}L\{D^{(m+1)\alpha}\varphi(t)\}(s)\\
=&s^{-(m+1)\alpha}[s^{(m+1)\alpha}\hat{\varphi}(s)-\sum_{k=0}^{m}s^{(m-k)\alpha}\varphi^{k\alpha}(0)]\\
=&\hat{\varphi}(s)-\sum_{k=0}^{m}\frac{\varphi^{k\alpha}(0)}{s^{k\alpha+1}}.
\end{aligned}$$

By inverse Laplace Transform, we obtain

$$\begin{aligned}
&\frac{1}{\Gamma((m+1)\alpha)}\int_0^t (t-\tau)^{(m+1)\alpha-1}\varphi^{(m+1)}(\tau)d\tau=\varphi(t)-\sum_{k=0}^{m}\frac{\varphi^{k\alpha}(0)}{\Gamma(k\alpha+1)}t^{k\alpha},\\
&\varphi(1)=\sum_{k=0}^{m}\frac{\varphi^{k\alpha}(0)}{\Gamma(k\alpha+1)}+\frac{1}{\Gamma((m+1)\alpha)}\int_0^1 (1-\tau)^{(m+1)\alpha-1}\varphi^{(m+1)}(\tau)d\tau.
\end{aligned}$$

The proof of Lemma 4.6 is completed. □

The following theorem can be obtained directly from Lemma 4.1 and Proposition 4.1.

Theorem 4.5 (Multivariate Taylor's formula with integral reminder term)**.** *Suppose that $D_a^{k\alpha}f$ are continuous in $D\subset R^n$, for $k=0,1,\dots,m+1$, where $0\le\alpha\le 1$, then we have*

$$\begin{aligned}
f(y_1,\dots,y_n)=&\sum_{k=0}^{m}\frac{D^{k\alpha}f(x_1,\dots,x_n)}{\Gamma(k\alpha+1)}\\
&+\frac{1}{\Gamma((m+1)\alpha)}\int_0^1 (1-t)^{(m+1)\alpha-1}D^{(m+1)\alpha}f(x_1+s(y_1-x_1),\dots,x_n+s(y_n-x_n))ds.
\end{aligned}\tag{4.14}$$

From Theorem 4.7, we have

Theorem 4.6 (Multivariate Cauchy's formula with integral reminder term)**.** *Let Suppose that $D_a^{k\alpha}f$, $D_a^{k\alpha}g$ are continuous in $D\subset R^n$, for $k=0,1,\dots,m+1$, where $0<\alpha\le 1$, then we have*

$$\begin{aligned}
&\frac{f(y_1,\dots,y_n)-\sum_{k=0}^{m}\frac{D^{k\alpha}f(x_1,\dots,x_n)}{\Gamma(k\alpha+1)}}{g(y_1,\dots,y_n)-\sum_{k=0}^{m}\frac{D^{k\alpha}g(x_1,\dots,x_n)}{\Gamma(k\alpha+1)}}\\
=&\frac{\int_0^1 (1-t)^{(m+1)\alpha-1}D^{(m+1)\alpha}f(x_1+s(y_1-x_1),\dots,x_n+s(y_n-x_n))ds}{\int_0^1 (1-t)^{(m+1)\alpha-1}D^{(m+1)\alpha}g(x_1+s(y_1-x_1),\dots,x_n+s(y_n-x_n))ds}.
\end{aligned}\tag{4.15}$$

Remark 4.1. Theorem 4.5 is the analogy to Theorem 1.3 via Caputo fractional derivative in Section 1.

Last, let us consider some special cases in Theorem 4.5.

(1) When $n=0,0<\alpha<1$, then we get

$$f(y_1)=\sum_{k=0}^{m}\frac{D^{k\alpha}f(x_1)}{\Gamma(k\alpha+1)}+\frac{1}{\Gamma((m+1)\alpha)}\int_0^1(1-t)^{(m+1)\alpha-1}D^{(m+1)\alpha}f(x_1+t(y_1-x_1))dt. \tag{4.16}$$

Now from Definition 2.5 we have

$$(D^{-\nu}f)(y)=\frac{1}{\Gamma(\nu)}\int_0^1(1-t)^{\nu-1}f(x_1+t(y-x_1))dt,\quad (\nu\in R^+)$$

it is easy to verify that

$$(D^{-\nu}f)(y)=(y-x_1)^{-\nu}\frac{1}{\Gamma(\nu)}\int_{x_1}^{y}(y-\tau)^{\nu-1}f(\tau)d\tau=(y-x_1)^{-\nu}[D_{x_1}^{-\nu}f](y),$$

where $[D_{x_1}^{-\nu}f](y)$ is Riemann-Liouville integral.

Similarily, from Definition 2.7 and Proposition 2.6, we can obtain

$$(D^{\mu}f)(y)=(y-x_1)^{\mu}[D_{x_1}^{\mu}f](y), \tag{4.17}$$

where $[D_{x_1}^{\mu}f](y)$ is Caputo fractional derivative.

Therefore, combining formula (4.16) with (4.17), we obtain fractional Taylor's formula with integral reminder via sequence fractional Caputo derivative:

$$f(y_1)=\sum_{k=0}^{m}\frac{(y_1-x_1)^{k\alpha}}{\Gamma(k\alpha+1)}[D_{x_1}^{k\alpha}f](x_1)+\frac{1}{\Gamma((m+1)\alpha)}\int_{x_1}^{y_1}(y_1-\tau)^{(m+1)\alpha-1}D_{x_1}^{(m+1)\alpha}f(\tau)d\tau. \tag{4.18}$$

(2) When $n=0,\alpha=1$, the fractional Taylor's formula reduced to the classical Taylor's formula

$$f(y_1)=\sum_{k=0}^{m}\frac{(y_1-x_1)^k}{k!}f^k(x_1)+\frac{1}{m!}\int_{x_1}^{y_1}(y_1-\tau)^m f^{(m+1)}(\tau)d\tau.$$

Further, let $m=0$, it reduced to the well-known Newton-Leibnitz's fundamental theorem of calculus $f(y_1)=f(x_1)+\int_{x_1}^{y_1}f'(\tau)d\tau$.

(3) When $n>1,\alpha=1$, from Definition 2.5, we have

$$\begin{aligned}f(y_1,...,y_n)=&\sum_{k=0}^{m}\frac{1}{k!}(\Delta x_1\frac{\partial}{\partial x_1}+...+\Delta x_n\frac{\partial}{\partial x_n})^k f(x_1,...,x_n)\\&+\frac{1}{m!}\int_0^1(1-t)^m D^{m+1}f(x_1+t\Delta x_1,...,x_n+t\Delta x_n)dt,\end{aligned}$$

which is the classical Taylor's formula with multivariate.

(4) Let $\alpha=1, m=0$ in Theorem 4.7, then we can get

$$f(y_1,...,y_n)=f(x_1,...,x_n)+\\ \int_0^1(\Delta x_1\frac{\partial}{\partial x_1}+...+\Delta x_n\frac{\partial}{\partial x_n})f(x_1+t\Delta x_1,...,x_n+t\Delta x_n)dt,$$

which is the famous Hadamard formula.

(5) By the use of Cauchy's mean value theorem of integrals, we have

$$\begin{aligned}&\int_0^1(1-t)^{(m+1)\alpha-1}D^{(m+1)\alpha}f(x_1+s(y_1-x_1),...,x_n+s(y_n-x_n))ds\\ =&D^{(m+1)\alpha}f(x_1+\theta(y_1-x_1),...,(x_n+s(y_n-x_n))\int_0^1(1-t)^{(m+1)\alpha-1}ds\\ =&\frac{(D_a^{(m+1)\alpha}f)(\xi_1,\cdots\xi_n)}{(m+1)\alpha},\end{aligned}$$

where $\xi_i=x_i+\theta(y_i-x_i), 0<\theta<1$. Then we can also obtain multivariate fractional Taylor's mean value Theorem 4.3 (see (4.10)) from Theorem 4.5.

5 Conclusion

We have presented multivariate fractional Taylor formulas and multivariate fractional Cauchy mean value formulas with remainders in the form of either fractional order derivatives or integrals, respectively, in the sense of sequential Caputo fractional order $0<\alpha\le 1)$ derivative. When $\alpha=1$, these formulas can be reduced to the classical Taylor formula and Cauchy mean value formula, respectively. The obtained formulas may be useful in fractional vector calculus which is an important tool for describing processes in complex media, non-local materials and distributed systems in multi-dimensional space.

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