Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **4**, No. 4, pp. 496-518

DOI: 10.4208/aamm.10-m1210
August 2012

Finite Difference/Element Method for a Two-Dimensional Modified Fractional Diffusion Equation

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Received 20 August 2011; Accepted (in revised version) 10 February 2012

Available online 10 July 2012

Abstract. We present the finite difference/element method for a two-dimensional modified fractional diffusion equation. The analysis is carried out first for the time semi-discrete scheme, and then for the full discrete scheme. The time discretization is based on the $L1$-approximation for the fractional derivative terms and the second-order backward differentiation formula for the classical first order derivative term. We use finite element method for the spatial approximation in full discrete scheme. We show that both the semi-discrete and full discrete schemes are unconditionally stable and convergent. Moreover, the optimal convergence rate is obtained. Finally, some numerical examples are tested in the case of one and two space dimensions and the numerical results confirm our theoretical analysis.

AMS subject classifications: 26A33, 65M06, 65M12, 65M60

Key words: Modified subdiffusion equation, finite difference method, finite element method, stability, convergence rate.

1 Introduction

The time fractional derivative is a useful tool for modeling anomalous subdiffusion [17, 18], e.g., the time fractional diffusion equation

$$\frac{\partial u(\mathbf{x},t)}{\partial t} = \mu \, {}_0D_t^{1-\beta} \Delta u(\mathbf{x},t) + f(\mathbf{x},t), \quad \mathbf{x} \in \Omega, \quad t \in [0,T], \tag{1.1}$$

where Δ is the usual Laplace operator and $0 < \beta < 1$, μ is a positive constant; ${}_0D_t^{1-\beta}$ denotes the Riemann-Liouville fractional derivative of order $1-\beta$

$${}_0D_t^{1-\beta} v(t) = \frac{1}{\Gamma(\beta)} \frac{\partial}{\partial t} \int_0^t \frac{v(\tau)}{(t-\tau)^{1-\beta}} d\tau.$$

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For further investigating the less anomalous subdiffusion behavior of diffusion processes, a modified time fractional diffusion equation was proposed by introducing a secondary time fractional derivative acting on the diffusion operator [11,21,22]

$$\frac{\partial u(\mathbf{x},t)}{\partial t} = \left(\mu_0 D_t{}^{1-\beta} + \nu_0 D_t{}^{1-\gamma}\right)\Delta u(\mathbf{x},t) + f(\mathbf{x},t), \quad \mathbf{x} \in \Omega, \quad t \in [0,T], \tag{1.2}$$

where $0 < \beta, \gamma \le 1$, μ and ν are positive constants. The quantity u is defined as a concentration or probability density function for the particles suspended in the liquid on a bounded domain Ω. For the particles described by (1.2), the relation between the mean square displacement of $x(t)$ of the diffusion particles and the time t is

$$\langle x^2(t)\rangle = \frac{2d\mu}{\Gamma(\beta+1)} t^{\beta} + \frac{2d\nu}{\Gamma(\gamma+1)} t^{\gamma}, \tag{1.3}$$

instead of

$$\langle x^2(t)\rangle = \frac{2d\mu}{\Gamma(\beta+1)} t^{\beta},$$

corresponding to (1.1). In (1.3), the mean square displacement of $x(t)$ is dominated by larger power for short times while for longer times it is dominated by the smaller power.

There are already some important progresses for the numerical solutions of the one-dimensional case of the time fractional diffusion equation (1.1), e.g., the finite difference method [4,6,10,23–25]; Lin and Xu discuss the spectral method [14] with the convergence rate $\mathcal{O}(\tau^{2-\beta} + \tau^{-1}N^{-m})$, and Jiang and Ma analyze the finite element method [9] and show that the optimal convergent rate $\mathcal{O}(\tau^{2-\beta} + N^{-m})$ can be obtained, where m measures the regularity of the solution in space. Liu et al. study the finite element method for the one-dimensional case of (1.2) [16] with the convergent rate $\mathcal{O}(\tau + \tau^{-1}N^{-m})$. Here we further discuss the finite element method for (1.2) by using the $L1$ approximation [5,14] to discretize the time fractional derivatives and show that the optimal convergent rate $\mathcal{O}(\tau^{1+\min\{\beta,\gamma\}} + N^{-m})$ is obtained. Instead of designing the numerical scheme straightforwardly, we first transform the Eq. (1.2) into

$$\begin{aligned}\frac{\partial u(\mathbf{x},t)}{\partial t} = &\left(\mu D_*^{1-\beta} + \nu D_*^{1-\gamma}\right)\Delta u(\mathbf{x},t) + \mu\frac{\Delta u(\mathbf{x},0)}{\Gamma(\beta)t^{1-\beta}} \\ &+ \nu\frac{\Delta u(\mathbf{x},0)}{\Gamma(\gamma)t^{1-\gamma}} + f(\mathbf{x},t), \qquad \mathbf{x} \in \Omega, \quad t \in [0,T],\end{aligned} \tag{1.4}$$

where the relation between the Caputo fractional derivative and the Riemann-Liouville fractional derivative is used, given as [19]

$${}_0D_t^{1-\vartheta} v(t) = D_*^{1-\vartheta} v(t) + \frac{v(0)}{\Gamma(\vartheta)t^{1-\vartheta}}, \quad 0 < \vartheta \le 1,$$

and the Caputo fractional derivative of order $1-\vartheta$ is defined by

$$D_*^{1-\vartheta}v(t)=\frac{1}{\Gamma(\vartheta)}\int_0^t\frac{\partial_\tau v(\tau)}{(t-\tau)^{1-\vartheta}}d\tau,\quad 0<\vartheta\le 1. \tag{1.5}$$

The initial-boundary conditions of (1.4) are given as

$$u(\mathbf{x},0)=u_0(\mathbf{x}),\quad \mathbf{x}\in\Omega;\qquad u(\mathbf{x},t)|_{\partial\Omega}=0,\quad 0<t<T. \tag{1.6}$$

The theoretical analysis is carried out first for the time discretization scheme, and then for the full discrete scheme. We show that both the semi-discrete and full discrete schemes are unconditionally stable and convergent. In the analysis of the numerical scheme, we assume that problem (1.4), (1.6) has a unique and sufficiently smooth solution.

The rest of the paper is organized as follows. In Section 2, the semi-discrete scheme is presented for the problem (1.4)-(1.6), and the stability and convergence analysis of the semi-discrete scheme are performed. The full discrete scheme is constructed by using the finite element method to discretize the spacial variables in Section 3, and the detailed stability analysis and error estimates are provided. Section 4 makes the numerical experiments to verify the theoretical results and some physical simulations are also carried out to further show the robustness of the schemes. We conclude this paper with some discussions in the last section.

2 Semi-discrete scheme and its theoretical analysis

We begin this section by introducing several necessary notations, concepts, and basic facts about the functional spaces endowed with standard norms and inner products that will be used in the subsequent discussions [1]

$$\begin{aligned}
&H^1(\Omega):=\Big\{v\in L^2(\Omega),\ D^\alpha v\in L^2(\Omega),\ |\alpha|\le 1\Big\},\\
&H_0^1(\Omega):=\Big\{v\in H^1(\Omega),\ v|_{\partial\Omega}=0\Big\},\\
&H^m(\Omega):=\Big\{v\in L^2(\Omega),\ D^\alpha v\in L^2(\Omega)\text{ for all }|\alpha|\le m\Big\},
\end{aligned}$$

where $\Omega\subset\mathbb{R}^d$, α is a d-tuple of non-negative integers α_i, the length of α is given by $|\alpha|:=\sum_{i=1}^d\alpha_i$, $D^\alpha v$ denotes the usual partial derivative $(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_d)^{\alpha_d}v$. Here we consider the case $d=2$. The standard inner products of $L^2(\Omega)$ and $H^1(\Omega)$ are defined, respectively, by

$$(u,v)=\int_\Omega uvdx,\qquad (u,v)_1=(u,v)+(\nabla u,\nabla v),$$

and the corresponding norms are defined as

$$\|v\|_0=(v,v)^{\frac{1}{2}},\qquad \|v\|_1=(v,v)_1^{\frac{1}{2}},\qquad |v|_1=(\nabla v,\nabla v)^{\frac{1}{2}}.$$

The norm $\|\cdot\|_m$ of the space $H^m(\Omega)$ is defined by

$$\|v\|_m = \Big(\sum_{0\le|\alpha|\le m} \|D^\alpha v\|_0^2\Big)^{\frac{1}{2}}.$$

In the present paper, instead of using the above standard H^1-norm, we use the following weighted H^1-norm

$$\|v\|_{w,1} = \Big[\|v\|_0^2 + \frac{\tilde{\beta}+\tilde{\gamma}}{2}|v|_1^2\Big]^{\frac{1}{2}}, \tag{2.1}$$

where

$$\tilde{\beta} = \frac{4\mu\tau^\beta}{\Gamma(1+\beta)}, \qquad \tilde{\gamma} = \frac{4\nu\tau^\gamma}{\Gamma(1+\gamma)}, \tag{2.2}$$

and τ is the time stepsize, and $0 < \beta,\gamma \le 1$. It is easy to prove that the weighted H^1-norm defined by (2.1) is equivalent to the standard H^1-norm.

Now, let's first discretize the time fractional derivatives of (1.4)

$$\begin{aligned} D_*^{1-\beta}\Delta u(\mathbf{x},t_{n+1}) =& \frac{1}{\Gamma(1+\beta)\tau^{1-\beta}}\Big[\Delta u(\mathbf{x},t_{n+1}) + \sum_{j=0}^{n-1}(a_{j+1}-a_j)\Delta u(\mathbf{x},t_{n-j}) \\ & - a_n\Delta u(\mathbf{x},0)\Big] + r_\beta^{n+1}, \end{aligned} \tag{2.3}$$

where $a_j = (j+1)^\beta - j^\beta$, $t_n = n\tau$ and

$$|r_\beta^{n+1}| \le C\tau^{1+\beta}, \tag{2.4}$$

here C is a constant depending on $\partial_t^2\Delta u$ (see, e.g., [15]). In a similar way, we have

$$\begin{aligned} D_*^{1-\gamma}\Delta u(\mathbf{x},t_{n+1}) =& \frac{1}{\Gamma(1+\gamma)\tau^{1-\gamma}}\Big[\Delta u(\mathbf{x},t_{n+1}) + \sum_{j=0}^{n-1}(b_{j+1}-b_j)\Delta u(\mathbf{x},t_{n-j}) \\ & - b_n\Delta u(\mathbf{x},0)\Big] + r_\gamma^{n+1}, \end{aligned} \tag{2.5}$$

where $b_j = (j+1)^\gamma - j^\gamma$ and

$$|r_\gamma^{n+1}| \le C\tau^{1+\gamma}, \tag{2.6}$$

C is a constant depending on $\partial_t^2\Delta u$. Furthermore, we use the second-order backward differentiation formula to discretize the first order time derivative

$$\frac{\partial u(\mathbf{x},t)}{\partial t}\Big|_{t=t_{n+1}} = \frac{3u(\mathbf{x},t_{n+1}) - 4u(\mathbf{x},t_n) + u(\mathbf{x},t_{n-1})}{2\tau} + \mathcal{O}(\tau^2), \quad n\ge 1, \tag{2.7}$$

and for the first step

$$\frac{\partial u(\mathbf{x},t)}{\partial t}\Big|_{t=t_1} = \frac{u(\mathbf{x},t_1)-u(\mathbf{x},t_0)}{\tau} + \mathcal{O}(\tau), \quad n=0. \tag{2.8}$$

For convenience, we introduce a mesh function $\{g^j\}_{j\geq 0}$, and define the fractional difference operator $\mathcal{L}_t^{\beta}$ by [15]

$$\mathcal{L}_t^{\beta} g^{n+1} = \frac{1}{\Gamma(1+\beta)}\sum_{j=0}^{n} a_j \frac{g^{n+1-j}-g^{n-j}}{\tau^{1-\beta}}, \quad n\geq 0, \tag{2.9}$$

and $\mathcal{L}_t^{\gamma}$ by

$$\mathcal{L}_t^{\gamma} g^{n+1} = \frac{1}{\Gamma(1+\gamma)}\sum_{j=0}^{n} b_j \frac{g^{n+1-j}-g^{n-j}}{\tau^{1-\gamma}}, \quad n\geq 0, \tag{2.10}$$

and the difference operator

$$\mathcal{L}_t^{1} g^{n+1} = \begin{cases} \dfrac{3g^{n+1}-4g^{n}+g^{n-1}}{2\tau}, & n\geq 1, \\ \dfrac{g^1-g^0}{\tau}, & n=0. \end{cases} \tag{2.11}$$

Combining (1.4) with (2.9)-(2.11) leads to, for $n\geq 0$,

$$\begin{aligned} \mathcal{L}_t^{1} u(\mathbf{x},t_{n+1}) =& (\mu\mathcal{L}_t^{\beta}+\nu\mathcal{L}_t^{\gamma})\Delta u(\mathbf{x},t_{n+1}) + \frac{\mu}{\Gamma(\beta)(n+1)^{1-\beta}\tau^{1-\beta}}\Delta u(\mathbf{x},0) \\ &+ \frac{\nu}{\Gamma(\gamma)(n+1)^{1-\gamma}\tau^{1-\gamma}}\Delta u(\mathbf{x},0) + f(\mathbf{x},t_{n+1}) + r^{n+1}, \end{aligned} \tag{2.12}$$

where

$$r^{n+1} = r_1^{n+1} + \mu r_{\beta}^{n+1} + \nu r_{\gamma}^{n+1} \quad \text{and} \quad r_1^{n+1} = \Big(\frac{\partial u(\mathbf{x},t)}{\partial t}\Big)\Big|_{t=t_{n+1}} - \mathcal{L}_t^1 u(\mathbf{x},t_{n+1});$$

obviously $r_1^{n+1}=\mathcal{O}(\tau^2)$ for $n\geq 1$, and $r_1^{n+1}=\mathcal{O}(\tau)$ for $n=0$. Denoting the approximation of $u(\mathbf{x},t_n)$ by u^n and omitting the truncation error, we obtain the semi-discrete scheme of (1.4), for $n\geq 0$,

$$\begin{aligned} \mathcal{L}_t^{1} u^{n+1} =& (\mu\mathcal{L}_t^{\beta}+\nu\mathcal{L}_t^{\gamma})\Delta u^{n+1} + \frac{\mu}{\Gamma(\beta)(n+1)^{1-\beta}\tau^{1-\beta}}\Delta u^0 \\ &+ \frac{\nu}{\Gamma(\gamma)(n+1)^{1-\gamma}\tau^{1-\gamma}}\Delta u^0 + f^{n+1}. \end{aligned} \tag{2.13}$$

More concretely, (2.13) is

$$\begin{aligned} \frac{3u^{n+1}-4u^n+u^{n-1}}{2\tau} =& \frac{\mu}{\Gamma(1+\beta)\tau^{1-\beta}}\Big[\Delta u^{n+1} + \sum_{j=0}^{n-1}(a_{j+1}-a_j)\Delta u^{n-j} - a_n\Delta u^0\Big] \\ &+ \frac{\nu}{\Gamma(1+\gamma)\tau^{1-\gamma}}\Big[\Delta u^{n+1} + \sum_{j=0}^{n-1}(b_{j+1}-b_j)\Delta u^{n-j} - a_n\Delta u^0\Big] \\ &+ \frac{\mu}{\Gamma(\beta)(n+1)^{1-\beta}\tau^{1-\beta}}\Delta u^0 + \frac{\nu}{\Gamma(\gamma)(n+1)^{1-\gamma}\tau^{1-\gamma}}\Delta u^0 + f^{n+1}, \quad n\geq 1, \end{aligned} \tag{2.14}$$

and for $n=0$

$$\begin{aligned}\frac{u^1-u^0}{\tau}=&\frac{\mu}{\Gamma(1+\beta)\tau^{1-\beta}}(\Delta u^1-\Delta u^0)+\frac{\nu}{\Gamma(1+\gamma)\tau^{1-\gamma}}(\Delta u^1-\Delta u^0)\\&+\frac{\mu}{\Gamma(\beta)\tau^{1-\beta}}\Delta u^0+\frac{\nu}{\Gamma(\gamma)\tau^{1-\gamma}}\Delta u^0+f^1.\end{aligned}\tag{2.15}$$

According to (1.6), we supplement (2.14) and (2.15) with the initial and boundary conditions as

$$\begin{cases}u^0(\mathbf{x})=u_0(\mathbf{x}), & x\in\Omega,\\ u^{n+1}(\mathbf{x})|_{\partial\Omega}=0, & n\geq 0.\end{cases}\tag{2.16}$$

Then the weak formulations of (2.14) and (2.15) with the boundary condition (2.16) are: for $n\geq 1$, find $u^{n+1}\in H_0^1(\Omega)$, such that for all $v\in H_0^1(\Omega)$

$$\begin{aligned}2(3u^{n+1}-4u^n+u^{n-1},v)=&\tilde{\beta}\Big[-(\nabla u^{n+1},\nabla v)+\sum_{j=0}^{n-1}(a_j-a_{j+1})(\nabla u^{n-j},\nabla v)+a_n(\nabla u^0,\nabla v)\Big]\\&+\tilde{\gamma}\Big[-(\nabla u^{n+1},\nabla v)+\sum_{j=0}^{n-1}(b_j-b_{j+1})(\nabla u^{n-j},\nabla v)+b_n(\nabla u^0,\nabla v)\Big]\\&-\tilde{\beta}_{n+1}(\nabla u^0,\nabla v)-\tilde{\gamma}_{n+1}(\nabla u^0,\nabla v)+4\tau(f^{n+1},v),\end{aligned}\tag{2.17}$$

and for $n=0$, find $u^1\in H_0^1(\Omega)$ such that for all $v\in H_0^1(\Omega)$

$$\begin{aligned}(u^1-u^0,v)=&-\frac{\tilde{\beta}}{4}(\nabla u^1-\nabla u^0,\nabla v)-\frac{\tilde{\gamma}}{4}(\nabla u^1-\nabla u^0,\nabla v)-\frac{\tilde{\beta}_1}{4}(\nabla u^0,\nabla v)\\&-\frac{\tilde{\gamma}_1}{4}(\nabla u^0,\nabla v)+\tau(f^1,v),\end{aligned}\tag{2.18}$$

where

$$\tilde{\beta}_{n+1}=\frac{4\mu\tau^{\beta}}{\Gamma(\beta)(n+1)^{1-\beta}},\qquad \tilde{\gamma}_{n+1}=\frac{4\nu\tau^{\gamma}}{\Gamma(\gamma)(n+1)^{1-\gamma}},\qquad n\geq 0.\tag{2.19}$$

Next, based on the ideas developed in [7,14,15], we will carry out the stability analysis of the semi-discrete scheme (2.17). Before this we give the following lemma which will be used later.

Lemma 2.1 (see [15]). *For the coefficients $\tilde{\beta}$, $\tilde{\beta}_{n+1}$ and $\tilde{\gamma}$, $\tilde{\gamma}_{n+1}$ defined in* (2.2) *and* (2.19), *we have*

$$\tilde{\beta}a_{n+1}\leq\tilde{\beta}_{n+1}\leq\tilde{\beta}a_n,\quad \tilde{\gamma}b_{n+1}\leq\tilde{\gamma}_{n+1}\leq\tilde{\gamma}b_n,\quad \forall n\geq 0.\tag{2.20}$$

The main stability results for the semi-discrete scheme (2.17)-(2.18) is given in the following theorem.

Theorem 2.1. *The semi-discrete scheme* (2.17)-(2.18) *is unconditionally stable in the sense that for all* $\tau > 0$*, the following inequality holds*

$$E^{n+1} \le E^1 + \frac{8C_\Omega^2 T\tau}{\tilde{\beta}_{n+1} + \tilde{\gamma}_{n+1}} \max_{1\le j\le n} \|f^{j+1}\|_0^2, \quad n \ge 1, \tag{2.21}$$

where

$$E^n = \|u^n\|_0^2 + \|2u^n - u^{n-1}\|_0^2 + \frac{\tilde{\beta}}{2}\sum_{j=0}^{n} a_j |u^{n-j}|_1^2 + \frac{\tilde{\gamma}}{2}\sum_{j=0}^{n} b_j |u^{n-j}|_1^2, \quad n \ge 1, \tag{2.22}$$

and C_Ω *given in* (2.24) *depends only on the domain* Ω*; moreover, for* (2.18)*, we have*

$$\begin{aligned}&\|u^1\|_0^2 + \frac{\tilde{\beta}}{4}\sum_{j=0}^{1} a_j|u^{1-j}|_1^2 + \frac{\tilde{\gamma}}{4}\sum_{j=0}^{1} b_j|u^{1-j}|_1^2 \\ \le &\|u^0\|_0^2 + \frac{\tilde{\beta}}{4}a_0|u^0|_1^2 + \frac{\tilde{\gamma}}{4}b_0|u^0|_1^2 + \frac{4C_\Omega^2\tau^2}{\tilde{\beta}_1 + \tilde{\gamma}_1}\|f^1\|_0^2.\end{aligned} \tag{2.23}$$

Proof. For the initial step, choosing $v = u^1$ in (2.18) leads to

$$\begin{aligned}\|u^1\|_0^2 \le \frac{1}{2}\|u^0\|_0^2 + \frac{1}{2}\|u^1\|_0^2 - \frac{\tilde{\beta}}{4}|u^1|_1^2 - \frac{\tilde{\gamma}}{4}|u^1|_1^2 + \frac{\tilde{\beta} - \tilde{\beta}_1}{8}|u^0|_1^2 + \frac{\tilde{\beta} - \tilde{\beta}_1}{8}|u^1|_1^2 \\ + \frac{\tilde{\gamma} - \tilde{\gamma}_1}{8}|u^0|_1^2 + \frac{\tilde{\gamma} - \tilde{\gamma}_1}{8}|u^1|_1^2 + \tau(f^1, u^1).\end{aligned}$$

Thus, using the Poincare inequality [20]

$$\|u^1\|_0 \le C_\Omega |u^1|_1, \tag{2.24}$$

we get

$$\begin{aligned}&\|u^1\|_0^2 + \frac{\tilde{\beta}}{4}|u^1|_1^2 + \frac{\tilde{\gamma}}{4}|u^1|_1^2 + \frac{\tilde{\beta}_1}{4}|u^0|_1^2 + \frac{\tilde{\gamma}_1}{4}|u^0|_1^2 \\ \le&\|u^0\|_0^2 + \frac{\tilde{\beta}}{4}|u^0|_1^2 + \frac{\tilde{\gamma}}{4}|u^0|_1^2 - \frac{\tilde{\beta}_1 + \tilde{\gamma}_1}{4}|u^1|_1^2 + 2\tau\|f^1\|_0\|u^1\|_0 \\ \le&\|u^0\|_0^2 + \frac{\tilde{\beta}}{4}|u^0|_1^2 + \frac{\tilde{\gamma}}{4}|u^0|_1^2 - \frac{\tilde{\beta}_1 + \tilde{\gamma}_1}{4}|u^1|_1^2 + 2C_\Omega\tau\|f^1\|_0|u^1|_1 \\ \le&\|u^0\|_0^2 + \frac{\tilde{\beta}}{4}|u^0|_1^2 + \frac{\tilde{\gamma}}{4}|u^0|_1^2 - \frac{\tilde{\beta}_1 + \tilde{\gamma}_1}{4}|u^1|_1^2 + \frac{4C_\Omega^2\tau^2}{\tilde{\beta}_1 + \tilde{\gamma}_1}\|f^1\|_0^2 + \frac{\tilde{\beta}_1 + \tilde{\gamma}_1}{4}|u^1|_1^2 \\ =&\|u^0\|_0^2 + \frac{\tilde{\beta}}{4}|u^0|_1^2 + \frac{\tilde{\gamma}}{4}|u^0|_1^2 + \frac{4C_\Omega^2\tau^2}{\tilde{\beta}_1 + \tilde{\gamma}_1}\|f^1\|_0^2,\end{aligned}$$

which reaches (2.23).

For $n \ge 1$, choosing $v = u^{n+1}$ in (2.17), and using the relation [8,15]

$$\begin{aligned}&2(3u^{n+1} - 4u^n + u^{n-1}, u^{n+1}) \\ =&\|u^{n+1}\|_0^2 - \|u^n\|_0^2 + \|2u^{n+1} - u^n\|_0^2 - \|2u^n - u^{n-1}\|_0^2 + \|u^{n+1} - 2u^n + u^{n-1}\|_0^2,\end{aligned} \tag{2.25}$$

we obtain

$$
\begin{aligned}
&\|u^{n+1}\|_0^2-\|u^n\|_0^2+\|2u^{n+1}-u^n\|_0^2-\|2u^n-u^{n-1}\|_0^2+\|u^{n+1}-2u^n+u^{n-1}\|_0^2\\
=&-\tilde{\beta}|u^{n+1}|_1^2+\tilde{\beta}\sum_{j=0}^{n-1}(a_j-a_{j+1})(\nabla u^{n-j},\nabla u^{n+1})+\tilde{\beta}a_n(\nabla u^0,\nabla u^{n+1})\\
&-\tilde{\gamma}|u^{n+1}|_1^2+\tilde{\gamma}\sum_{j=0}^{n-1}(b_j-b_{j+1})(\nabla u^{n-j},\nabla u^{n+1})+\tilde{\gamma}b_n(\nabla u^0,\nabla u^{n+1})\\
&-\tilde{\beta}_{n+1}(\nabla u^0,\nabla u^{n+1})-\tilde{\gamma}_{n+1}(\nabla u^0,\nabla u^{n+1})+4\tau(f^{n+1},u^{n+1})\\
\le&-\tilde{\beta}|u^{n+1}|_1^2+\frac{\tilde{\beta}}{2}\sum_{j=0}^{n-1}(a_j-a_{j+1})|u^{n-j}|_1^2+\frac{\tilde{\beta}}{2}\sum_{j=0}^{n-1}(a_j-a_{j+1})|u^{n+1}|_1^2\\
&-\tilde{\gamma}|u^{n+1}|_1^2+\frac{\tilde{\gamma}}{2}\sum_{j=0}^{n-1}(b_j-b_{j+1})|u^{n-j}|_1^2+\frac{\tilde{\gamma}}{2}\sum_{j=0}^{n-1}(b_j-b_{j+1})|u^{n+1}|_1^2\\
&+\frac{\tilde{\beta}a_n-\tilde{\beta}_{n+1}}{2}|u^0|_1^2+\frac{\tilde{\beta}a_n-\tilde{\beta}_{n+1}}{2}|u^{n+1}|_1^2+\frac{\tilde{\gamma}b_n-\tilde{\gamma}_{n+1}}{2}|u^0|_1^2+\frac{\tilde{\gamma}b_n-\tilde{\gamma}_{n+1}}{2}|u^{n+1}|_1^2\\
&+4\tau(f^{n+1},u^{n+1})\\
=&-\tilde{\beta}|u^{n+1}|_1^2+\frac{\tilde{\beta}}{2}\sum_{j=0}^{n-1}a_j|u^{n-j}|_1^2-\frac{\tilde{\beta}}{2}\sum_{j=1}^{n}a_j|u^{n+1-j}|_1^2+\frac{\tilde{\beta}}{2}(1-a_n)|u^{n+1}|_1^2\\
&-\tilde{\gamma}|u^{n+1}|_1^2+\frac{\tilde{\gamma}}{2}\sum_{j=0}^{n-1}b_j|u^{n-j}|_1^2-\frac{\tilde{\gamma}}{2}\sum_{j=1}^{n}b_j|u^{n+1-j}|_1^2+\frac{\tilde{\gamma}}{2}(1-b_n)|u^{n+1}|_1^2\\
&+\frac{\tilde{\beta}a_n-\tilde{\beta}_{n+1}}{2}|u^0|_1^2+\frac{\tilde{\beta}a_n-\tilde{\beta}_{n+1}}{2}|u^{n+1}|_1^2+\frac{\tilde{\gamma}b_n-\tilde{\gamma}_{n+1}}{2}|u^0|_1^2+\frac{\tilde{\gamma}b_n-\tilde{\gamma}_{n+1}}{2}|u^{n+1}|_1^2\\
&+4\tau(f^{n+1},u^{n+1})\\
\le&\frac{\tilde{\beta}}{2}\sum_{j=0}^{n}a_j|u^{n-j}|_1^2+\frac{\tilde{\gamma}}{2}\sum_{j=0}^{n}b_j|u^{n-j}|_1^2-\frac{\tilde{\beta}}{2}\sum_{j=0}^{n+1}a_j|u^{n+1-j}|_1^2-\frac{\tilde{\gamma}}{2}\sum_{j=0}^{n+1}b_j|u^{n+1-j}|_1^2\\
&-\frac{\tilde{\beta}_{n+1}}{2}|u^{n+1}|_1^2-\frac{\tilde{\gamma}_{n+1}}{2}|u^{n+1}|_1^2+4C_\Omega\tau\|f^{n+1}\|_0|u^{n+1}|_1\\
\le&\frac{\tilde{\beta}}{2}\sum_{j=0}^{n}a_j|u^{n-j}|_1^2+\frac{\tilde{\gamma}}{2}\sum_{j=0}^{n}b_j|u^{n-j}|_1^2-\frac{\tilde{\beta}}{2}\sum_{j=0}^{n+1}a_j|u^{n+1-j}|_1^2-\frac{\tilde{\gamma}}{2}\sum_{j=0}^{n+1}b_j|u^{n+1-j}|_1^2\\
&-\frac{\tilde{\beta}_{n+1}+\tilde{\gamma}_{n+1}}{2}|u^{n+1}|_1^2+\frac{8C_\Omega^2\tau^2}{\tilde{\beta}_{n+1}+\tilde{\gamma}_{n+1}}\|f^{n+1}\|_0^2+\frac{\tilde{\beta}_{n+1}+\tilde{\gamma}_{n+1}}{2}|u^{n+1}|_1^2\\
=&\frac{\tilde{\beta}}{2}\sum_{j=0}^{n}a_j|u^{n-j}|_1^2+\frac{\tilde{\gamma}}{2}\sum_{j=0}^{n}b_j|u^{n-j}|_1^2-\frac{\tilde{\beta}}{2}\sum_{j=0}^{n+1}a_j|u^{n+1-j}|_1^2-\frac{\tilde{\gamma}}{2}\sum_{j=0}^{n+1}b_j|u^{n+1-j}|_1^2\\
&+\frac{8C_\Omega^2\tau^2}{\tilde{\beta}_{n+1}+\tilde{\gamma}_{n+1}}\|f^{n+1}\|_0^2,
\end{aligned}
\tag{2.26}
$$

where Lemma 2.1 is used. Removing the last term on the left hand side of (2.26) and rearranging the inequality, we obtain

$$
\begin{aligned}
E^{n+1} \leq & E^n + \frac{8C_\Omega^2\tau^2}{\tilde{\beta}_{n+1}+\tilde{\gamma}_{n+1}}\|f^{n+1}\|_0^2 \leq E^1 + \sum_{j=1}^{n}\frac{8C_\Omega^2\tau^2}{\tilde{\beta}_{j+1}+\tilde{\gamma}_{j+1}}\|f^{j+1}\|_0^2 \\
\leq & E^1 + \frac{8C_\Omega^2 t_n\tau}{\tilde{\beta}_{n+1}+\tilde{\gamma}_{n+1}}\max_{1\leq j\leq n}\|f^{j+1}\|_0^2.
\end{aligned}
$$

Hence the proof is complete. □

Corollary 2.1. *The semi-discrete scheme* (2.17)-(2.18) *is unconditionally stable in H^1-norm, and for all $\tau > 0$, the following inequality holds*

$$
\|u^n\|_{w,1} \leq C\Big(\|u^0\|_{w,1} + T^{1-\max\{\beta,\gamma\}/2}\max_{1\leq j\leq n}\|f^j\|_0\Big), \quad n \geq 1. \tag{2.27}
$$

Proof. In view of (2.22), and using the elementary inequality $(a+b)^2 \leq 2(a^2+b^2)$, $\forall a, b \in \mathbf{R}$, we have

$$
\begin{aligned}
E^1 = & \|u^1\|_0^2 + \|2u^1 - u^0\|_0^2 + \frac{\tilde{\beta}}{2}\sum_{j=0}^{1}a_j|u^{1-j}|_1^2 + \frac{\tilde{\gamma}}{2}\sum_{j=0}^{1}b_j|u^{1-j}|_1^2 \\
\leq & 9\|u^1\|_0^2 + 2\|u^0\|_0^2 + \frac{\tilde{\beta}}{2}\sum_{j=0}^{1}a_j|u^{1-j}|_1^2 + \frac{\tilde{\gamma}}{2}\sum_{j=0}^{1}b_j|u^{1-j}|_1^2 \\
\leq & 2\|u^0\|_0^2 + 9\Big(\|u^0\|_0^2 + \frac{\tilde{\beta}}{4}a_0|u^0|_1^2 + \frac{\tilde{\gamma}}{4}b_0|u^0|_1^2 + \frac{4C_\Omega^2\tau^2}{\tilde{\beta}_1+\tilde{\gamma}_1}\|f^1\|_0^2\Big) \\
\leq & 11\Big(\|u^0\|_0^2 + \frac{\tilde{\beta}}{4}a_0|u^0|_1^2 + \frac{\tilde{\gamma}}{4}b_0|u^0|_1^2 + \frac{4C_\Omega^2\tau^2}{\tilde{\beta}_1+\tilde{\gamma}_1}\|f^1\|_0^2\Big) \\
\leq & 11\Big(\|u^0\|_{\omega,1}^2 + \frac{4C_\Omega^2\tau^2}{\tilde{\beta}_1+\tilde{\gamma}_1}\|f^1\|_0^2\Big).
\end{aligned}
$$

It follows from (2.21) that

$$
\|u^n\|_{\omega,1} \leq C\bigg(\|u^0\|_{\omega,1} + \sqrt{\frac{T\tau}{\tilde{\beta}_n+\tilde{\gamma}_n}\max_{1\leq j\leq n}\|f^j\|_0^2}\bigg), \quad n \geq 2. \tag{2.28}
$$

By elementary computations, for $n \geq 1$, we have

$$
\begin{cases}
\dfrac{\nu}{\mu}\tilde{\beta}_n \leq \tilde{\gamma}_n, & \text{if } 0<\gamma\leq\beta<1, \\
\dfrac{\mu}{\nu}\tilde{\gamma}_n \leq \tilde{\beta}_n, & \text{if } 0<\beta\leq\gamma<1.
\end{cases} \tag{2.29}
$$

Therefore

$$
\begin{cases}
\dfrac{T\tau}{\tilde{\beta}_n+\tilde{\gamma}_n} \leq \dfrac{T\tau}{(1+\nu/\mu)\tilde{\beta}_n} = \dfrac{\Gamma(\beta)n^{1-\beta}T\tau}{(1+\nu/\mu)\tau^\beta} \leq \dfrac{\Gamma(\beta)}{(1+\nu/\mu)}T^{2-\beta}, & \text{if } 0<\gamma\leq\beta<1, \\
\dfrac{T\tau}{\tilde{\beta}_n+\tilde{\gamma}_n} \leq \dfrac{T\tau}{(1+\mu/\nu)\tilde{\gamma}_n} = \dfrac{\Gamma(\gamma)n^{1-\gamma}T\tau}{(1+\mu/\nu)\tau^\gamma} \leq \dfrac{\Gamma(\gamma)}{(1+\mu/\nu)}T^{2-\gamma}, & \text{if } 0<\beta\leq\gamma<1.
\end{cases} \tag{2.30}
$$

Combining (2.28) and (2.30) leads to (2.27) for $n > 1$. For $n = 1$, by (2.23), we have

$$\|u^1\|_{w,1}^2 \leq 2\Big(\|u^0\|_{w,1}^2 + \frac{4C_\Omega^2\tau^2}{\tilde{\beta}_1+\tilde{\gamma}_1}\|f^1\|_0^2\Big),$$

and then use (2.30) again, we reach (2.27) for $n = 1$. The proof is completed. □

Next, we estimate the error of the semi-discrete scheme (2.17)-(2.18).

Theorem 2.2. *Let u be the exact solution of* (1.4)-(1.6) *and $\{u^n\}_{n\geq 0}$ be the solution of* (2.17)-(2.18) *with the initial condition given in* (2.16), *then we have*

$$\|u(\cdot,t_n) - u^n\|_{w,1} \leq CT^{1-\max\{\beta,\gamma\}/2}\tau^{1+\min\{\beta,\gamma\}}, \quad n \geq 1, \tag{2.31}$$

where C is a constant independent of T and τ.

Proof. Let $e^n = u(\cdot,t_n) - u^n$, and note that $e^0 = 0$. For $n = 0$, by (2.18), we have

$$\begin{aligned}(e^1 - e^0, v) = &-\frac{\tilde{\beta}}{4}(\nabla e^1 - \nabla e^0, \nabla v) - \frac{\tilde{\gamma}}{4}(\nabla e^1 - \nabla e^0, \nabla v) - \frac{\tilde{\beta}_1}{4}(\nabla e^0, \nabla v)\\ &- \frac{\tilde{\gamma}_1}{4}(\nabla e^0, \nabla v) + \tau(r^1, v).\end{aligned} \tag{2.32}$$

Taking $v = e^1$ in (2.32), we get

$$\begin{aligned}\|e^1\|_0^2 &= -\frac{\tilde{\beta}}{4}|e^1|_1^2 - \frac{\tilde{\gamma}}{4}|e^1|_1^2 + (\tau r^1, e^1)\\ &\leq -\frac{\tilde{\beta}}{4}|e^1|_1^2 - \frac{\tilde{\gamma}}{4}|e^1|_1^2 + \frac{1}{2}\|\tau r^1\|_0^2 + \frac{1}{2}\|e^1\|_0^2.\end{aligned} \tag{2.33}$$

Thus

$$\|e^1\|_{w,1} \leq \tau\|r^1\|_0 \leq C\tau^2. \tag{2.34}$$

Let

$$\varepsilon^n := \|e^n\|_0^2 + \|2e^n - e^{n-1}\|_0^2 + \frac{\tilde{\beta}}{2}\sum_{j=0}^{n} a_j|e^{n-j}|_1^2 + \frac{\tilde{\gamma}}{2}\sum_{j=0}^{n} b_j|e^{n-j}|_1^2, \quad n \geq 1.$$

By (2.17), we get

$$\begin{aligned}&2(3e^{n+1} - 4e^n + e^{n-1}, v)\\ =&\tilde{\beta}\Big[-(\nabla e^{n+1}, \nabla v) + \sum_{j=0}^{n-1}(a_j - a_{j+1})(\nabla e^{n-j}, \nabla v) + a_n(\nabla e^0, \nabla v)\Big]\\ &\quad + \tilde{\gamma}\Big[-(\nabla e^{n+1}, \nabla v) + \sum_{j=0}^{n-1}(b_j - b_{j+1})(\nabla e^{n-j}, \nabla v) + b_n(\nabla e^0, \nabla v)\Big]\\ &\quad - \tilde{\beta}_{n+1}(\nabla e^0, \nabla v) - \tilde{\gamma}_{n+1}(\nabla e^0, \nabla v) + 4\tau(r^{n+1}, v), \quad n \geq 1.\end{aligned} \tag{2.35}$$

Taking $v=e^{n+1}$ in (2.35), we get

$$\begin{aligned}
&\|e^{n+1}\|_0^2-\|e^n\|_0^2+\|2e^{n+1}-e^n\|_0^2-\|2e^n-e^{n-1}\|_0^2+\|e^{n+1}-2e^n+e^{n-1}\|_0^2\\
=&-\tilde{\beta}|e^{n+1}|_1^2+\tilde{\beta}\sum_{j=0}^{n-1}(a_j-a_{j+1})(\nabla e^{n-j},\nabla e^{n+1})+\tilde{\beta}a_n(\nabla e^0,\nabla e^{n+1})-\tilde{\gamma}|e^{n+1}|_1^2\\
&+\tilde{\gamma}\sum_{j=0}^{n-1}(b_j-b_{j+1})(\nabla e^{n-j},\nabla e^{n+1})+\tilde{\gamma}b_n(\nabla e^0,\nabla e^{n+1})-\tilde{\beta}_{n+1}(\nabla e^0,\nabla e^{n+1})\\
&-\tilde{\gamma}_{n+1}(\nabla e^0,\nabla e^{n+1})+4\tau(r^{n+1},e^{n+1})\\
\le&-\tilde{\beta}|e^{n+1}|_1^2+\frac{\tilde{\beta}}{2}\sum_{j=0}^{n-1}(a_j-a_{j+1})|e^{n-j}|_1^2+\frac{\tilde{\beta}}{2}\sum_{j=0}^{n-1}(a_j-a_{j+1})|e^{n+1}|_1^2\\
&-\tilde{\gamma}|e^{n+1}|_1^2+\frac{\tilde{\gamma}}{2}\sum_{j=0}^{n-1}(b_j-b_{j+1})|e^{n-j}|_1^2+\frac{\tilde{\gamma}}{2}\sum_{j=0}^{n-1}(b_j-b_{j+1})|e^{n+1}|_1^2\\
&+\frac{\tilde{\beta}a_n-\tilde{\beta}_{n+1}}{2}|e^0|_1^2+\frac{\tilde{\beta}a_n-\tilde{\beta}_{n+1}}{2}|e^{n+1}|_1^2+\frac{\tilde{\gamma}b_n-\tilde{\gamma}_{n+1}}{2}|e^0|_1^2\\
&+\frac{\tilde{\gamma}b_n-\tilde{\gamma}_{n+1}}{2}|e^{n+1}|_1^2+4\tau(r^{n+1},e^{n+1})\\
=&-\tilde{\beta}|e^{n+1}|_1^2+\frac{\tilde{\beta}}{2}\sum_{j=0}^{n-1}a_j|e^{n-j}|_1^2-\frac{\tilde{\beta}}{2}\sum_{j=1}^{n}a_j|e^{n+1-j}|_1^2+\frac{\tilde{\beta}}{2}(1-a_n)|e^{n+1}|_1^2\\
&-\tilde{\gamma}|e^{n+1}|_1^2+\frac{\tilde{\gamma}}{2}\sum_{j=0}^{n-1}b_j|e^{n-j}|_1^2-\frac{\tilde{\gamma}}{2}\sum_{j=1}^{n}b_j|e^{n+1-j}|_1^2+\frac{\tilde{\gamma}}{2}(1-b_n)|e^{n+1}|_1^2\\
&+\frac{\tilde{\beta}a_n-\tilde{\beta}_{n+1}}{2}|e^0|_1^2+\frac{\tilde{\beta}a_n-\tilde{\beta}_{n+1}}{2}|e^{n+1}|_1^2+\frac{\tilde{\gamma}b_n-\tilde{\gamma}_{n+1}}{2}|e^0|_1^2\\
&+\frac{\tilde{\gamma}b_n-\tilde{\gamma}_{n+1}}{2}|e^{n+1}|_1^2+4\tau(r^{n+1},e^{n+1})\\
=&\frac{\tilde{\beta}}{2}\sum_{j=0}^{n}a_j|e^{n-j}|_1^2+\frac{\tilde{\gamma}}{2}\sum_{j=0}^{n}b_j|e^{n-j}|_1^2-\frac{\tilde{\beta}}{2}\sum_{j=0}^{n+1}a_j|e^{n+1-j}|_1^2-\frac{\tilde{\gamma}}{2}\sum_{j=0}^{n+1}b_j|e^{n+1-j}|_1^2\\
&-\frac{\tilde{\beta}_{n+1}+\tilde{\gamma}_{n+1}}{2}|e^{n+1}|_1^2+4\tau(r^{n+1},e^{n+1}).
\end{aligned}\tag{2.36}$$

Thus, rearranging (2.36), we obtain

$$\begin{aligned}
\varepsilon^{n+1}&\le\varepsilon^n-\frac{\tilde{\beta}_{n+1}+\tilde{\gamma}_{n+1}}{2}|e^{n+1}|_1^2+4\tau(r^{n+1},e^{n+1})\\
&\le\varepsilon^n+\frac{8C_\Omega^2\tau^2}{\tilde{\beta}_{n+1}+\tilde{\gamma}_{n+1}}\|r^{n+1}\|_0^2\\
&\le\varepsilon^1+\frac{8C_\Omega^2T\tau}{\tilde{\beta}_{n+1}+\tilde{\gamma}_{n+1}}\max_{1\le j\le n}\|r^{j+1}\|_0^2.
\end{aligned}$$

By (2.34), we have

$$\varepsilon^1\le C\tau^4.$$

Then from (2.30), we obtain

$$\varepsilon^{n+1} \leq CT^{2-\max\{\beta,\gamma\}}\tau^{2+2\min\{\beta,\gamma\}}.$$

Thus, we arrive at the conclusion for $n \geq 1$. $\square$

3 Full discrete scheme and its theoretical analysis

In this section, we consider the finite element approximation for the derivative with respect to the space variable. Suppose that $\mathcal{T}_h$ is a triangulation of Ω, h_e is the element diameter, and $h = \max\{h_e\}$. Then, we associate $\mathcal{T}_h$ with the space $S_h^m \subset H_0^1(\Omega)$, which is composed of piecewise polynomials of degree at most m, $m \geq 1$, that vanish on the entire boundary $\partial\Omega$. Now, we can obtain the finite element approximations of (2.17) and (2.18): for $n \geq 1$, find $u_h^{n+1} \in S_h^m$ such that

$$\begin{aligned}
&2(3u_h^{n+1} - 4u_h^n + u_h^{n-1}, v_h)\\
=&\tilde{\beta}\Big[-(\nabla u_h^{n+1}, \nabla v_h) + \sum_{j=0}^{n-1}(a_j - a_{j+1})(\nabla u_h^{n-j}, \nabla v_h) + a_n(\nabla u_h^0, \nabla v_h)\Big]\\
&\quad + \tilde{\gamma}\Big[-(\nabla u_h^{n+1}, \nabla v_h) + \sum_{j=0}^{n-1}(b_j - b_{j+1})(\nabla u_h^{n-j}, \nabla v_h) + b_n(\nabla u_h^0, \nabla v_h)\Big]\\
&\quad - \tilde{\beta}_{n+1}(\nabla u_h^0, \nabla v_h) - \tilde{\gamma}_{n+1}(\nabla u_h^0, \nabla v_h) + 4\tau(f^{n+1}, v_h), \quad \forall v_h \in S_h^m,
\end{aligned} \tag{3.1}$$

and for $n = 0$, find $u_h^1 \in S_h^m$ such that

$$\begin{aligned}
(u_h^1 - u_h^0, v_h) = &-\frac{\tilde{\beta}}{4}(\nabla u_h^1 - \nabla u_h^0, \nabla v_h) - \frac{\tilde{\gamma}}{4}(\nabla u_h^1 - \nabla u_h^0, \nabla v_h) - \frac{\tilde{\beta}_1}{4}(\nabla u_h^0, \nabla v_h)\\
&- \frac{\tilde{\gamma}_1}{4}(\nabla u_h^0, \nabla v_h) + \tau(f^1, v_h), \quad \forall v_h \in S_h^m.
\end{aligned} \tag{3.2}$$

For the full discrete scheme, we also have the unconditional stability result as follows.

Theorem 3.1. *The full discrete scheme* (3.1)-(3.2) *is unconditionally stable, i.e., for all* $\tau > 0$, *the following inequality holds*

$$\|u_h^n\|_{w,1} \leq C\Big(\|u_h^0\|_{w,1} + T^{1-\max\{\beta,\gamma\}/2} \max_{1\leq j\leq n} \|f^j\|_0\Big). \tag{3.3}$$

The proof of Theorem 3.1 is similar to that of Theorem 2.1, so it is omitted here.

Next, we intend to estimate the error of the full discrete scheme (3.1)-(3.2). First, we introduce the standard H^1-orthogonal projection operator $P_{1,h}^m : H^{s+1}(\Omega) \to S_h^m$ as

$$(\nabla P_{1,h}^m \varphi, \nabla v_h) = (\nabla \varphi, \nabla v_h), \quad \forall \varphi \in H^{s+1}(\Omega), \quad s \geq 1, \quad \forall v_h \in S_h^m. \tag{3.4}$$

For the above H^1-orthogonal projection, we have the following result. And we would use the notation $l = \min\{m, s\}$ hereafter.

Lemma 3.1. *For the H^1-orthogonal projection operator $P_{1,h}^m$ introduced in* (3.4), *we have*

$$\|\varphi - P_{1,h}^m \varphi\|_{w,1} \leq C(h^{l+1} + \tau^{\min\{\beta,\gamma\}/2} h^l)|\varphi|_{l+1}, \tag{3.5}$$

for $\varphi \in H_0^1(\Omega) \cap H^{s+1}(\Omega)$, $s \geq 1$, where C is a positive constant independent of h, τ and φ.

By the definition of the weighted H^1-norm and the standard estimates for the standard H^1-orthogonal projection operator, it is easy to prove the estimates (3.5). Furthermore, we introduce the following lemma, which will be used in the estimate of the error of the full discrete scheme.

Lemma 3.2. *Let*

$$r_{P_{1,h}^m}^{n+1} = \begin{cases} \frac{u(\mathbf{x},t_1)-u(\mathbf{x},t_0)}{\tau} - \frac{P_{1,h}^m u(\mathbf{x},t_1) - P_{1,h}^m u(\mathbf{x},t_0)}{\tau}, & \text{for } n=0, \\ \frac{3u(\mathbf{x},t_{n+1})-4u(\mathbf{x},t_n)+u(\mathbf{x},t_{n-1})}{2\tau} - \frac{3P_{1,h}^m u(\mathbf{x},t_{n+1}) - 4P_{1,h}^m u(\mathbf{x},t_n) + P_{1,h}^m u(\mathbf{x},t_{n-1})}{2\tau}, & \text{for } n \geq 1. \end{cases} \tag{3.6}$$

Then the following estimates hold

$$\|r_{P_{1,h}^m}^{n+1}\|_0 \leq \begin{cases} C(h^{l+1}+\tau), & n=0, \\ C(h^{l+1}+\tau^2), & n \geq 1. \end{cases} \tag{3.7}$$

The proof of this lemma follows the idea in [9]. Below we just give a sketch of the proof.

Proof. For $n=0$,

$$\begin{aligned} \left|r_{P_{1,h}^m}^1\right| =& \left|\frac{u(\mathbf{x},t_1)-u(\mathbf{x},t_0)}{\tau} - \frac{P_{1,h}^m u(\mathbf{x},t_1) - P_{1,h}^m u(\mathbf{x},t_0)}{\tau}\right| \\ \leq& \left|\frac{u(\mathbf{x},t_1)-u(\mathbf{x},t_0)}{\tau} - \frac{\partial u(\mathbf{x},t_1)}{\partial t}\right| + \left|\frac{\partial u(\mathbf{x},t_1)}{\partial t} - P_{1,h}^m \frac{\partial u(\mathbf{x},t_1)}{\partial t}\right| \\ &+ \left|P_{1,h}^m \frac{\partial u(\mathbf{x},t_1)}{\partial t} - \frac{P_{1,h}^m u(\mathbf{x},t_1) - P_{1,h}^m u(\mathbf{x},t_0)}{\tau}\right| \\ \leq& C(\tau + h^{l+1}), \end{aligned} \tag{3.8}$$

and for $n \geq 1$,

$$\begin{aligned} \left|r_{P_{1,h}^m}^{n+1}\right| =& \left|\frac{3u(\mathbf{x},t_{n+1})-4u(\mathbf{x},t_n)+u(\mathbf{x},t_{n-1})}{2\tau} - \frac{3P_{1,h}^m u(\mathbf{x},t_{n+1}) - 4P_{1,h}^m u(\mathbf{x},t_n) + P_{1,h}^m u(\mathbf{x},t_{n-1})}{2\tau}\right| \\ \leq& \left|\frac{3u(\mathbf{x},t_{n+1})-4u(\mathbf{x},t_n)+u(\mathbf{x},t_{n-1})}{2\tau} - \frac{\partial u(\mathbf{x},t_{n+1})}{\partial t}\right| + \left|\frac{\partial u(\mathbf{x},t_{n+1})}{\partial t} - P_{1,h}^m \frac{\partial u(\mathbf{x},t_{n+1})}{\partial t}\right| \\ &+ \left|P_{1,h}^m \frac{\partial u(\mathbf{x},t_{n+1})}{\partial t} - \frac{3P_{1,h}^m u(\mathbf{x},t_{n+1}) - 4P_{1,h}^m u(\mathbf{x},t_n) + P_{1,h}^m u(\mathbf{x},t_{n-1})}{2\tau}\right| \\ \leq& C(\tau^2 + h^{l+1}). \end{aligned} \tag{3.9}$$

Then we reach the conclusion (3.7). ∎

With the above two lemmas, an weighted H^1-error estimate between the exact solution and the solution of the full discrete scheme (3.1)-(3.2) is derived in the following theorem.

Theorem 3.2. *Let u be the exact solution of* (1.4)-(1.6), *and* $\{u_h^n\}_{n\geq 0}$ *be the solution of the full discrete scheme* (3.1)-(3.2). *Assume that* $u(\cdot,t)\in H^{s+1}(\Omega)$, *for all* $t\in[0,T]$, $s\geq 1$, *then we have the following error estimate*

$$\begin{aligned}&\|u(\cdot,t_{n+1})-u_h^{n+1}\|_{w,1}\\ \leq& CT^{1-\max\{\beta,\gamma\}/2}\big(\tau^{1+\min\{\beta,\gamma\}}+h^{l+1}+\tau^{\min\{\beta,\gamma\}/2}h^l\big),\quad n\geq 0,\end{aligned}\tag{3.10}$$

where C is a constant independent of τ, h and T.

Proof. For convenience, we denote $U^n=u(\cdot,t_n)$. Let

$$e_h^n=U^n-u_h^n,\quad \bar{e}_h^n=U^n-P_{1,h}^mU^n,\quad \tilde{e}_h^n=P_{1,h}^mU^n-u_h^n,\quad n\geq 0.$$

It is obvious that

$$e_h^n=\bar{e}_h^n+\tilde{e}_h^n.$$

Without loss of generality, in the following we assume $\tilde{e}_h^0=0$.

For $n=0$, rearranging (3.2) leads to

$$\begin{aligned}&(u_h^1,v_h)+\frac{\tilde{\beta}+\tilde{\gamma}}{4}(\nabla u_h^1,\nabla v_h)\\ =&(u_h^0,v_h)+\frac{\tilde{\beta}+\tilde{\gamma}-\tilde{\beta}_1-\tilde{\gamma}_1}{4}(\nabla u_h^0,\nabla v_h)+\tau(f^1,v_h).\end{aligned}\tag{3.11}$$

By the definition of the H^1-orthogonal projection in (3.4) and (2.12), for the case $n=0$, we have

$$\begin{aligned}&(U^1,v_h)+\frac{\tilde{\beta}+\tilde{\gamma}}{4}(\nabla P_{1,h}^mU^1,\nabla v_h)\\ =&(U^0,v_h)+\frac{\tilde{\beta}+\tilde{\gamma}-\tilde{\beta}_1-\tilde{\gamma}_1}{4}(\nabla P_{1,h}^mU^0,\nabla v_h)+\tau(f^1,v_h)+\tau(r^1,v_h).\end{aligned}\tag{3.12}$$

Letting

$$e_{P_{1,h}^m}^{n+1}(v_h)=\begin{cases}-\big(r_{P_{1,h}^m}^1,v_h\big), & \text{for }\ n=0,\\ -\big(r_{P_{1,h}^m}^{n+1},v_h\big), & \text{for }\ n\geq 1,\end{cases}\tag{3.13}$$

and subtracting (3.12) by (3.11), we get

$$\begin{aligned}&(\tilde{e}_h^1,v_h)+\frac{\tilde{\beta}+\tilde{\gamma}}{4}(\nabla\tilde{e}_h^1,\nabla v_h)\\ =&(\tilde{e}_h^0,v_h)+\frac{\tilde{\beta}+\tilde{\gamma}-\tilde{\beta}_1-\tilde{\gamma}_1}{4}(\nabla\tilde{e}_h^0,\nabla v_h)+\tau e_{P_{1,h}^m}^1(v_h)+\tau(r^1,v_h).\end{aligned}\tag{3.14}$$

Taking $v_h=\tilde{e}_h^1$ in (3.14) and since $\tilde{e}_h^0=0$, we get

$$\|\tilde{e}_h^1\|_0^2+\frac{\tilde{\beta}+\tilde{\gamma}}{4}|\tilde{e}_h^1|_1^2=\tau e_{P_{1,h}^m}^1(\tilde{e}_h^1)+\tau(r^1,\tilde{e}_h^1).\tag{3.15}$$

Then by the Lemma 3.2, we obtain

$$\|\tilde{e}_h^1\|_{w,1} \le C\tau(h^{l+1}+\tau). \tag{3.16}$$

Therefore, using the triangle inequality $\|e_h^1\|_{w,1} \le \|\bar{e}_h^1\|_{w,1} + \|\tilde{e}_h^1\|_{w,1}$ and Lemma 3.1, the conclusion (3.10) is proved for $n=0$.

For $n \ge 1$, we first rearrange (3.1) as

$$\begin{aligned}
&2(3u_h^{n+1}-4u_h^n+u_h^{n-1},v_h)+\tilde{\beta}(\nabla u_h^{n+1},\nabla v_h)+\tilde{\gamma}(\nabla u_h^{n+1},\nabla v_h)\\
=&\tilde{\beta}\Big[\sum_{j=0}^{n-1}(a_j-a_{j+1})(\nabla u_h^{n-j},\nabla v_h)+a_n(\nabla u_h^0,\nabla v_h)\Big]\\
&+\tilde{\gamma}\Big[\sum_{j=0}^{n-1}(b_j-b_{j+1})(\nabla u_h^{n-j},\nabla v_h)+b_n(\nabla u_h^0,\nabla v_h)\Big]\\
&-\tilde{\beta}_{n+1}(\nabla u_h^0,\nabla v_h)-\tilde{\gamma}_{n+1}(\nabla u_h^0,\nabla v_h)+4\tau(f^{n+1},v_h),\quad \forall v_h\in S_h.
\end{aligned} \tag{3.17}$$

Similarly, from (2.12) we have

$$\begin{aligned}
&2(3U^{n+1}-4U^n+U^{n-1},v_h)+\tilde{\beta}(\nabla U^{n+1},\nabla v_h)+\tilde{\gamma}(\nabla U^{n+1},\nabla v_h)\\
=&\tilde{\beta}\Big[\sum_{j=0}^{n-1}(a_j-a_{j+1})(\nabla U^{n-j},\nabla v_h)+a_n(\nabla U^0,\nabla v_h)\Big]\\
&+\tilde{\gamma}\Big[\sum_{j=0}^{n-1}(b_j-b_{j+1})(\nabla U^{n-j},\nabla v_h)+b_n(\nabla U^0,\nabla v_h)\Big]\\
&-\tilde{\beta}_{n+1}(\nabla U^0,\nabla v_h)-\tilde{\gamma}_{n+1}(\nabla U^0,\nabla v_h)+4\tau(f^{n+1},v_h)+4\tau(r^{n+1},v_h),\quad \forall v_h\in S_h.
\end{aligned} \tag{3.18}$$

Subtracting (3.18) by (3.17), we obtain

$$\begin{aligned}
&2\big(3\tilde{e}_h^{n+1}-4\tilde{e}_h^n+\tilde{e}_h^{n-1},v_h\big)+\tilde{\beta}(\nabla\tilde{e}_h^{n+1},\nabla v_h)+\tilde{\gamma}(\nabla\tilde{e}_h^{n+1},\nabla v_h)\\
=&\tilde{\beta}\sum_{j=0}^{n-1}(a_j-a_{j+1})(\nabla\tilde{e}_h^{n-j},\nabla v_h)+\tilde{\gamma}\sum_{j=0}^{n-1}(b_j-b_{j+1})(\nabla\tilde{e}_h^{n-j},\nabla v_h)+(\tilde{\beta}a_n-\tilde{\beta}_{n+1})(\nabla\tilde{e}_h^0,\nabla v_h)\\
&+(\tilde{\gamma}b_n-\tilde{\gamma}_{n+1})(\nabla\tilde{e}_h^0,\nabla v_h)+4\tau e_{P_{1,h}^m}^{n+1}(v_h)+4\tau(r^{n+1},v_h),\quad \forall v_h\in S_h^m.
\end{aligned} \tag{3.19}$$

Taking $v_h=\tilde{e}_h^{n+1}$ in (3.19), we obtain

$$\begin{aligned}
&\|\tilde{e}_h^{n+1}\|_0^2+\|2\tilde{e}_h^{n+1}-\tilde{e}_h^n\|_0^2-\|\tilde{e}_h^n\|_0^2-\|2\tilde{e}_h^n-\tilde{e}_h^{n-1}\|_0^2+\|\tilde{e}_h^{n+1}-2\tilde{e}_h^n+\tilde{e}_h^{n-1}\|_0^2\\
\le&-\frac{\tilde{\beta}}{2}\sum_{j=0}^{n+1}a_j|\tilde{e}_h^{n+1-j}|_1^2+\frac{\tilde{\beta}}{2}\sum_{j=0}^{n}a_j|\tilde{e}_h^{n-j}|_1^2-\frac{\tilde{\gamma}}{2}\sum_{j=0}^{n+1}b_j|\tilde{e}_h^{n+1-j}|_1^2+\frac{\tilde{\gamma}}{2}\sum_{j=0}^{n}b_j|\tilde{e}_h^{n-j}|_1^2\\
&-\frac{\tilde{\beta}_{n+1}+\tilde{\gamma}_{n+1}}{2}|\tilde{e}_h^{n+1}|_1^2+4\tau|e_{P_{1,h}^m}^{n+1}(\tilde{e}_h^{n+1})|+4\tau|(r^{n+1},\tilde{e}_h^{n+1})|.
\end{aligned} \tag{3.20}$$

Let

$$\varepsilon^n=\|\tilde{e}_h^n\|_0^2+\|2\tilde{e}_h^n-\tilde{e}_h^{n-1}\|_0^2+\frac{\tilde{\beta}}{2}\sum_{j=0}^{n}a_j|\tilde{e}_h^{n-j}|_1^2+\frac{\tilde{\gamma}}{2}\sum_{j=0}^{n}b_j|\tilde{e}_h^{n-j}|_1^2.$$

By (3.20) and Lemma 3.2, we get

$$\begin{aligned}\varepsilon^{n+1} &\le \varepsilon^n - \frac{\tilde{\beta}_{n+1}+\tilde{\gamma}_{n+1}}{2}|\tilde{e}_h^{n+1}|_1^2 + 4\tau|e_{P_{1,h}^m}^{n+1}(\tilde{e}_h^{n+1})| + 4\tau|(r^{n+1},\tilde{e}_h^{n+1})| \\ &\le \varepsilon^1 + \frac{CT\tau}{\tilde{\beta}_{n+1}+\tilde{\gamma}_{n+1}}(h^{2l+2}+\tau^{2+2\min\{\beta,\,\gamma\}}), \quad \forall n \ge 1.\end{aligned}$$

Therefore, combining with (3.16), we obtain

$$\|\tilde{e}_h^n\|_1^2 \le CT^{2-\max\{\beta,\gamma\}}(h^{2l+2}+\tau^{2+2\min\{\beta,\gamma\}}), \quad \forall n \ge 0.$$

Consequently, by the triangle inequality and Lemma 3.1, we obtain the conclusion (3.10) for $n \ge 1$, and this completes the proof. □

4 Numerical example

In this section, we present some numerical results to confirm the performance of our numerical scheme and the convergence rate obtained in Section 3. For the finite element space S_h^m, we only consider the piecewise-linear case, i.e., $m = 1$.

4.1 One-dimensional case

Example 4.1. Consider the following one-dimensional modified fractional diffusion equation [16]:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = ({}_0D_t^{1-\beta} + {}_0D_t^{1-\gamma})\dfrac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), & x \in [0,\pi], \quad t > 0, \\ u(0,t) = u(\pi,t) = 0, & t > 0, \\ u(x,0) = 0, & x \in [0,\pi], \end{cases} \tag{4.1}$$

where the inhomogeneous term

$$f(x,t) = 2t\sin x\Big(1 + \frac{1}{\Gamma(2+\beta)}t^\beta + \frac{1}{\Gamma(2+\gamma)}t^\gamma\Big).$$

Then the exact solution of (4.1) is

$$u(x,t) = t^2\sin(x).$$

The weighted H^1-norm is measured by (2.1). TCR is the abbreviation for the theoretical convergence rate.

In Table 1, we make the space step size h sufficiently small, i.e., $h = 1/1000$. Then, the last two terms in the convergence results in (3.10) are both negligible and we can verify the convergence rate $\mathcal{O}(\tau^{1+\min\{\beta,\gamma\}})$ in the weighted H^1-norm. And, it is obvious that the numerical results coincide with theoretical analysis very well. In Table

Table 1: $h = 1/1000$, $\|u(\cdot,t_n) - u_h^n\|_{w,1}$ at $t = 1$.

τ	$\beta = 0.9, \gamma = 0.1$	rate	$\beta = 0.6, \gamma = 0.5$	rate
1/10	6.5804E-002		2.1928E-002	
1/20	2.7886E-002	1.2386	6.5181E-003	1.7503
1/40	1.2217E-002	1.1907	2.0125E-003	1.6955
1/80	5.4661E-003	1.1603	6.7074E-004	1.5852
TCR		1.1000		1.5000

Table 2: $\tau = 1/10000$, $\|u(\cdot,t_n) - u_h^n\|_{w,1}$ at $t = 1$.

h	$\beta = 0.9, \gamma = 0.1$	rate	$\beta = 0.6, \gamma = 0.5$	rate
$\pi/10$	1.0419E-001		2.1437E-002	
$\pi/20$	5.2030E-002	1.0018	1.0245E-002	1.0652
$\pi/40$	2.6007E-002	1.0004	5.0618E-003	1.0172
$\pi/80$	1.3002E-002	1.0002	2.5233E-003	1.0043
TCR		1.0000		1.0000

Table 3: $\tau = h$, $\|u(\cdot,t_n) - u_h^n\|_{w,1}$ at $t = 1$.

τ	$\beta = 0.9, \gamma = 0.1$	rate	$\beta = 0.6, \gamma = 0.5$	rate
1/100	5.9645E-003		2.2347E-003	
1/150	3.8161E-003	1.1014	1.3295E-003	1.2808
1/200	2.7836E-003	1.0966	9.2060E-004	1.2776
1/250	2.1807E-003	1.0939	6.9258E-004	1.2754
TCR		1.0500		1.2500

2, we present the weighted H^1-norm errors in the approximations to the solution u at $t = 1$ and the corresponding convergence rates. For this computation, we fix the time step size $\tau = 1/10000$ as we expect to obtain first order accuracy in space in the weighted H^1-norm. The numerical results confirm our expectation.

In order to verify the third term $\mathcal{O}(\tau^{\min\{\beta,\gamma\}/2}h^l)$ of the error estimate given in (3.10), in Table 3, we take $\tau = h$ and present the weighted H^1-norm errors in the approximations to the solution u at $t = 1$. It can be seen that the convergence rate is in well agree with the estimate.

4.2 Two-dimensional case

Example 4.2. Consider the following two-dimensional modified fractional diffusion equation:

$$\begin{cases} \dfrac{\partial u(\mathbf{x},t)}{\partial t} = ({}_0D_t^{1-\beta} + {}_0D_t^{1-\gamma})\Delta u(\mathbf{x},t) + f(\mathbf{x},t), & \mathbf{x} \in \Omega = [0,1]\times[0,1], \quad t > 0, \\ u|_{\partial\Omega} = 0, & t > 0, \\ u(\mathbf{x},0) = 0, & \mathbf{x} \in \Omega, \end{cases} \tag{4.2}$$

where

$$f(\mathbf{x},t) = 2t\sin(2\pi x_1)\sin(2\pi x_2)\Big(1 + \frac{8\pi^2}{\Gamma(2+\beta)}t^\beta + \frac{8\pi^2}{\Gamma(2+\gamma)}t^\gamma\Big).$$

Table 4: $N_{x_1} = N_{x_2} = 60$, $\|u(\cdot, t_n) - u_h^n\|_\infty$ at $t = 1$.

τ	$\beta = 0.9, \gamma = 0.1$	rate	$\beta = 0.6, \gamma = 0.5$	rate
1/4	1.0845E-001		4.4277E-002	
1/8	5.1301E-002	1.0800	1.5547E-002	1.5099
1/16	2.3594E-002	1.1206	4.8176E-003	1.6902
1/32	1.0420E-002	1.1791	1.6120E-003	1.5795
TCR		1.1000		1.5000

Table 5: $\tau = 1/10000$, $\|u(\cdot, t_n) - u_h^n\|_\infty$ at $t = 1$.

$N_{x_1} = N_{x_2}$	$\beta = 0.9, \gamma = 0.1$	rate	$\beta = 0.6, \gamma = 0.5$	rate
4	3.9245E-001		3.9218E-001	
8	9.9316E-002	1.9824	9.9200E-002	1.9831
16	2.5480E-002	1.9627	2.5461E-002	1.9621
TCR		2.0000		2.0000

Then the exact solution of (4.2) is

$$u(\mathbf{x}, t) = t^2 \sin(2\pi x_1) \sin(2\pi x_2).$$

We apply our full discrete scheme with a uniform spatial mesh with $2(N_{x_1} \times N_{x_2})$ triangles and $N_{x_1} = N_{x_2}$. Table 4 and Table 5 are devoted to verify the convergence rate for Example 4.2. However, here, we use the discrete maximum norm $\|\cdot\|_\infty$ defined as

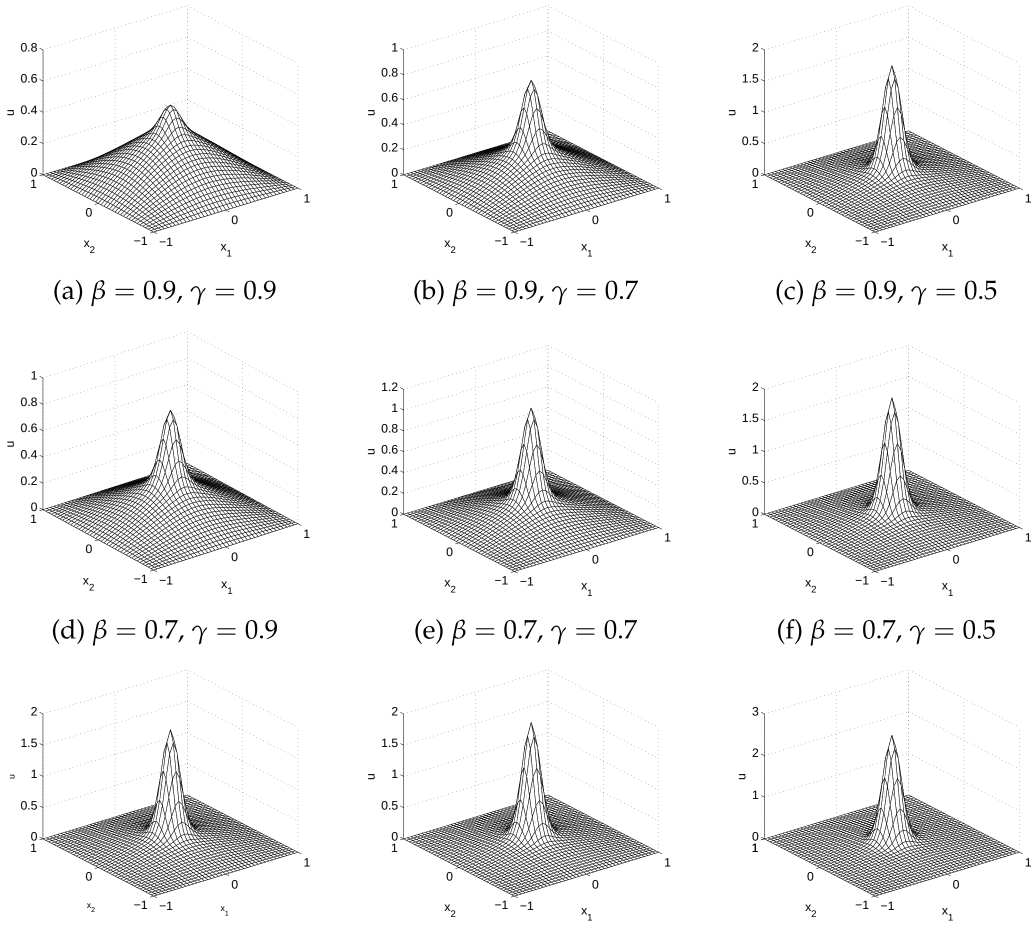
$$\|u(\cdot, t_n) - u_h^n\|_\infty = \max_{z_j} \big\{ |u(z_j, t_n) - u_h(z_j, t_n)| \big\},$$

where z_j denotes the vertexes of the triangles.

Example 4.3. Consider the following two-dimensional modified fractional diffusion equation:

$$\begin{cases} \dfrac{\partial u(\mathbf{x}, t)}{\partial t} = ({}_0D_t^{1-\beta} + {}_0D_t^{1-\gamma}) \Delta u(\mathbf{x}, t), & \mathbf{x} \in \Omega = [-1,1] \times [-1,1], \quad t > 0, \\ u|_{\partial\Omega} = 0, & t > 0, \\ u(\mathbf{x}, 0) = \dfrac{1}{2\pi\sigma^2} \exp\Big(-\dfrac{x_1^2 + x_2^2}{2\sigma^2} \Big), & \mathbf{x} \in \Omega. \end{cases} \tag{4.3}$$

In Fig. 1, we plot the surface of the solution of (4.3) at $t = 0.1$. In each row, the value of β is fixed and the value of γ is chosen to be 0.9, 07, 0.5, respectively. While for each column, the value of γ is fixed and the value of β is chosen to be 0.9, 07, 0.5, respectively. Observing the peak heights in the figures of the same row, we notice that as the decrease of the value of γ, the peak height increases. In each column, the same phenomenon is observed. This means that the solution decays slower for smaller β or γ. Observing the subfigures which are symmetrical with respect to the diagonal line $\beta = \gamma$, we find that the peck heights in these subfigures are the same, which imply that β and γ in (4.3), or (1.2), play the equal-counterpart roles.



(a) $\beta = 0.9, \gamma = 0.9$ (b) $\beta = 0.9, \gamma = 0.7$ (c) $\beta = 0.9, \gamma = 0.5$

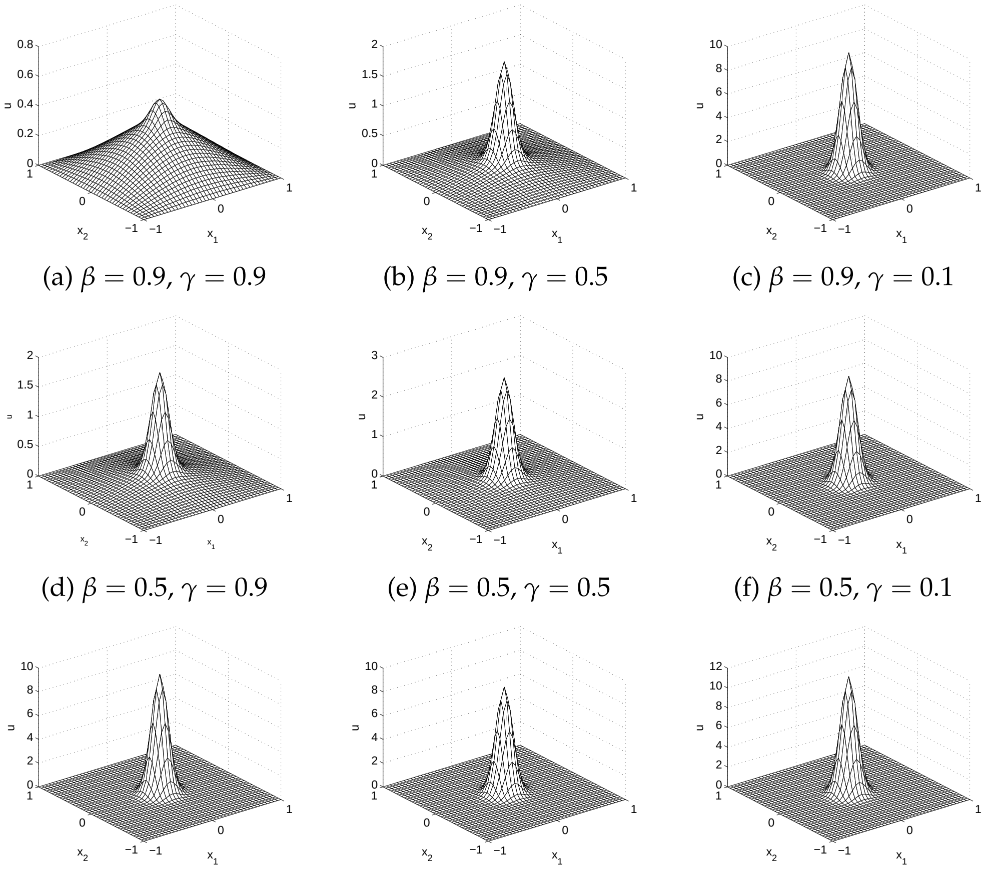
(d) $\beta = 0.7, \gamma = 0.9$ (e) $\beta = 0.7, \gamma = 0.7$ (f) $\beta = 0.7, \gamma = 0.5$

(g) $\beta = 0.5, \gamma = 0.9$ (h) $\beta = 0.5, \gamma = 0.7$ (i) $\beta = 0.5, \gamma = 0.5$

Figure 1: The surface of $u(\mathbf{x}, t)$ with $N_{x_1} = N_{x_2} = 40$, $\tau = 1/100$, $\sigma = 0.1$.

In Fig. 2, we also plot the surface of the solution of (4.3) at $t = 0.1$, but choose different values of β, γ with that in Fig. 1. This time, in each row, the value of β is fixed and the value of γ is chosen to be 0.9, 0.5, 0.1, respectively. In each column, the same thing is done. Observing the peak heights in the figures in each row, we notice that: in the first and second rows, the similar phenomenon as that in Fig. 2 is observed; However, in the third row, i.e., $\beta = 0.1$, a striking difference is that the peak height first decreases then increases as the value of γ decreases. In each column, the same phenomenon is observed.

In Fig. 3, we plot the evolution of the solution at time $t = 0.01, 0.1, 1.0$ with $\tau = 1/1000$, and Fig. 4 displays the evolution of its profiles. When $\beta = \gamma$, the Eq. (1.2) reduces to the traditional fractional diffusion equation. Observing the peak heights in Fig. 4 or those in different rows in Fig. 3, we find that, initially the solution of the modified fractional diffusion equation with $\beta = 0.9$, $\gamma = 0.7$ decays faster than the



(a) $\beta = 0.9$, $\gamma = 0.9$ (b) $\beta = 0.9$, $\gamma = 0.5$ (c) $\beta = 0.9$, $\gamma = 0.1$

(d) $\beta = 0.5$, $\gamma = 0.9$ (e) $\beta = 0.5$, $\gamma = 0.5$ (f) $\beta = 0.5$, $\gamma = 0.1$

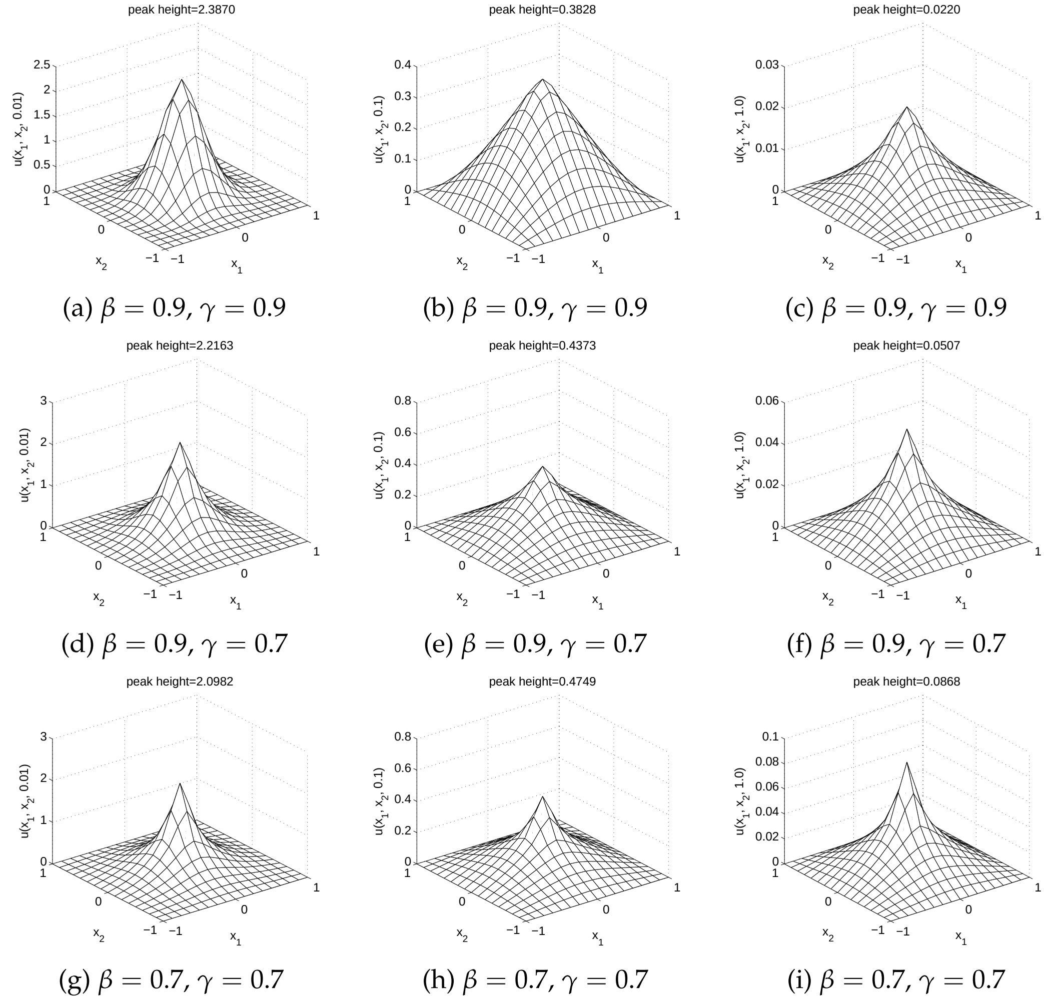
(g) $\beta = 0.1$, $\gamma = 0.9$ (h) $\beta = 0.1$, $\gamma = 0.5$ (i) $\beta = 0.1$, $\gamma = 0.1$

Figure 2: The surface of $u(\mathbf{x}, t)$ with $N_{x_1} = N_{x_2} = 40$, $\tau = 1/100$, $\sigma = 0.1$.

solution of the traditional one with $\beta = \gamma = 0.9$ while slower than the traditional one with $\beta = \gamma = 0.7$. But as time goes by, the solution of the modified fractional diffusion equation begin to decay slower than that of the traditional one with $\beta = \gamma = 0.9$ and faster than the traditional one with $\beta = \gamma = 0.7$. It means that, with different values of β and γ, the modified fractional diffusion equation displays a crossover phenomenon, see [11].

5 Concluding remarks

We have designed the finite difference/element methods for a two-dimensional modified fractional diffusion equation. The detailed and delicate error estimates and stability analysis are performed. Optimal convergent order and unconditionally stability are obtained. The extensive numerical experiments confirm our theoretical results and illustrate the robustness of the numerical algorithm and some physical observations are displayed.

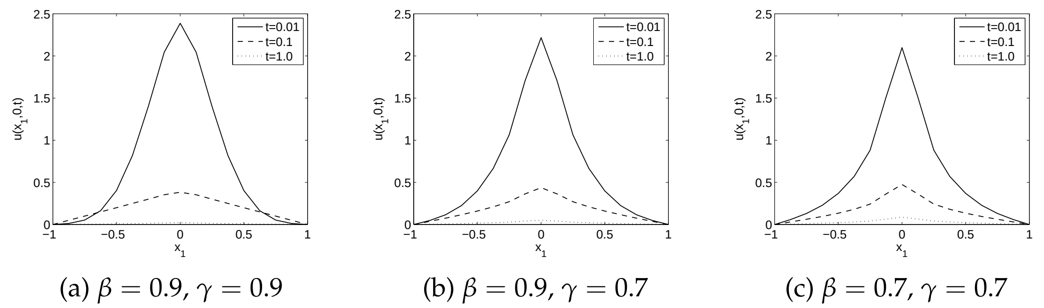


(a) $\beta = 0.9, \gamma = 0.9$ (b) $\beta = 0.9, \gamma = 0.9$ (c) $\beta = 0.9, \gamma = 0.9$

(d) $\beta = 0.9, \gamma = 0.7$ (e) $\beta = 0.9, \gamma = 0.7$ (f) $\beta = 0.9, \gamma = 0.7$

(g) $\beta = 0.7, \gamma = 0.7$ (h) $\beta = 0.7, \gamma = 0.7$ (i) $\beta = 0.7, \gamma = 0.7$

Figure 3: The surface of $u(\mathbf{x}, t)$ with $N_{x_1} = N_{x_2} = 16$, $\tau = 1/1000$, $\sigma = 0.1$.



(a) $\beta = 0.9, \gamma = 0.9$ (b) $\beta = 0.9, \gamma = 0.7$ (c) $\beta = 0.7, \gamma = 0.7$

Figure 4: The profile of $u(\mathbf{x}, t)$ with $x_2 = 0$, $N_{x_1} = N_{x_2} = 16$, $\tau = 1/1000$, $\sigma = 0.1$.

Acknowledgments

We thank the anonymous reviewers' valuable comments which greatly improved the paper. This research was partly supported by the National Basic Research Program of China, 973 Program under Grant No. 2011CB706903, the Program for New Century Excellent Talents in University under Grant No. NCET-09-0438, the National Natural Science Foundation of China under Grant No. 10801067, and the Fundamental Research Funds for the Central Universities under Grant No. lzujbky-2010-63 and No. lzujbky-2012-k26. W. H. Deng thanks Jan Hesthaven for providing the useful reference to improve the theoretical results.

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