

A Jacobi Collocation Method for the Fractional Ginzburg-Landau Differential Equation

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Abstract. In this paper, we design a collocation method to solve the fractional Ginzburg-Landau equation. A Jacobi collocation method is developed and implemented in two steps. First, we space-discretize the equation by the Jacobi-Gauss-Lobatto collocation (JGLC) method in one- and two-dimensional space. The equation is then converted to a system of ordinary differential equations (ODEs) with the time variable based on JGLC. The second step applies the Jacobi-Gauss-Radau collocation (JGRC) method for the time discretization. Finally, we give a theoretical proof of convergence of this Jacobi collocation method and some numerical results showing the proposed scheme is an effective and high-precision algorithm.

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1 Introduction

The fractional Ginzburg-Landau equation (FGLE) is known as a generalization of the classical one and has been presented to depict many kinds of nonlinear phenomena. The Ginzburg-Landau equation (GLE) has a variety of applications, e.g., in biology and chemistry. In many areas of physics, the GLE also has important applications, such as superconductivity, superfluidity, nonlinear optics, Bose-Einstein condensation and so on [1].

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At first, Ginzburg and Landau proposed the GLE to depict phase transitions in superconductors near their critical temperature. The GLE also models the dynamics of electromagnetic behavior of a superconductor in an external magnetic field [2].

In recent years, as the fractional differential equations have many applications in different fields of engineering and science, it attracted more and more scholars. A fractional Ginzburg-Landau equation is derived by Tarasov et al. [2] from the variational Euler-Lagrange equation for fractal media. Because fractals generate in a fractal media or a fractal process in nature, the FGLE has been used to depict many physical phenomena, for example, the dynamical processes in continuum with fractal dispersion or in media with a fractal mass dimension [2], the organization of a system near the phase transition point influenced by a competing nonlocal ordering [4], and a network of diffusively Hindmarsh-Rose neurons with a long-range synaptic coupling [5].

In this paper, we consider a numerical algorithm for solving the following Ginzburg-Landau equation with fractional Laplace operator ($1 < a \leq 2$):

$$u_t + (v + i\eta)(-\Delta)^{\frac{a}{2}}u + (k + i\zeta)|u|^2u - \gamma u = 0, \quad x \in \mathbb{R}, \quad t \in [0, T], \quad (1.1)$$

and the initial condition

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \quad (1.2)$$

where $i^2 = -1$, $u(x, t)$ is the unknown complex function from $\mathbb{R} \times \mathbb{R}^+$ to \mathbb{C} , $u_0(x)$ is a given smooth function, and $v > 0, k > 0, \eta, \zeta, \gamma$ are real constants. The fractional Laplace operator $(-\Delta)^{\frac{a}{2}}$ can be defined with the symbol $|\zeta|^a$ as follows:

$$-(-\Delta)^{\frac{a}{2}}u(x, t) = -\mathcal{F}^{-1}(|\zeta|^a \tilde{u}(\zeta, t)), \quad (1.3)$$

where \tilde{u} is the Fourier transform of u and \mathcal{F} denotes the Fourier transform operator. Yang et al. [6, 7] showed that the fractional derivative defined in Eq. (1.3) is equivalent to the Riesz fractional derivative, i.e.,

$$\begin{aligned} \frac{\partial^a}{\partial |x|^a} u(x, t) &= -(-\Delta)^{\frac{a}{2}} u(x, t) \\ &= -\frac{1}{2\cos(\frac{a\pi}{2})} \left[{}_{-\infty}^R D_x^a u(x, t) + {}_x^R D_{+\infty}^a u(x, t) \right]. \end{aligned} \quad (1.4)$$

The left Riemann-Liouville fractional derivative of $u(x, t)$ is defined as follows:

$${}_{-\infty}^R D_x^a u(x, t) = \begin{cases} \frac{1}{\Gamma(2-a)} \frac{\partial^2}{\partial x^2} \int_{-\infty}^x (x-s)^{1-a} u(s, t) ds, & 1 < a < 2, \\ \frac{\partial^2}{\partial x^2} u(x, t), & a = 2, \end{cases}$$

and the right Riemann-Liouville fractional derivative of $u(x, t)$ as:

$${}_x^R D_{+\infty}^a u(x, t) = \begin{cases} \frac{1}{\Gamma(2-a)} \frac{\partial^2}{\partial x^2} \int_x^{+\infty} (s-x)^{1-a} u(s, t) ds, & 1 < a < 2, \\ \frac{\partial^2}{\partial x^2} u(x, t), & a = 2. \end{cases}$$

Some theoretical analyses for the FGLE (1.1) are given in [10–13]. The global well-posedness was investigated by Pu and Guo [14]; The psi-series solution was derived and analyzed by Tarasov [13]. Guo and Huo [11] studied the inviscid limit behavior of (1.1) to the fractional Schrödinger equation. The well-posedness and asymptotic behaviors in two-dimensions are given in [10].

We know the spectral collocation method is one of the high-precision numerical approaches for solving the fractional differential equations (FDEs) [9]. We develop collocation methods for solving one-dimensional (1D) and two-dimensional (2D) FGLE. Firstly, we proposed JGLC method to discrete the FGLE in space so that the FGLE with an initial condition is converted to a system of ODEs. Secondly, we deal with the system of ODEs by a JGRC method. We will give a convergence analysis of the collocation method. Finally, several numerical examples in 1D and 2D are presented, verifying the theoretical analysis. They indicate the high precision and effectiveness of the algorithm.

The main layout of this article as follows: we mainly introduce the basic properties and recursive relations of Jacobi polynomials in the Section 2. We define the JGLC method to solve the 1D and 2D fractional Ginzburg-Landau equation in Section 3. We use the JGRC method to deal with the ordinary differential equations with initial conditions in Section 4. Next, the convergence analysis of the collocation method is investigated in Section 5. In Section 6, we give several numerical examples for testing precision of the method for the fractional Ginzburg-Landau equation in 1D and 2D. Finally, some concluding remarks are drawn in Section 7.

2 Some properties of Jacobi polynomials

In this section, we will introduce some basic properties about Jacobi polynomials that are most related to spectral collocation methods [16].

The Jacobi polynomials, denoted by $\mathcal{P}_n^{(\alpha, \beta)}(x)$, are orthogonal with the Jacobi weight function $\omega^{(\alpha, \beta)}(x) = (1-x)^\alpha (1+x)^\beta$ over $I = (-1, 1)$, namely,

$$(\mathcal{P}_i^{(\alpha, \beta)}(x), \mathcal{P}_j^{(\alpha, \beta)}(x))_{\omega^{(\alpha, \beta)}} = \int_{-1}^1 \mathcal{P}_i^{(\alpha, \beta)}(x) \mathcal{P}_j^{(\alpha, \beta)}(x) \omega^{(\alpha, \beta)}(x) dx = h_i^{(\alpha, \beta)} \delta_{ij}, \quad (2.1)$$

where $\delta_{i,j}$ is the Kronecker function and

$$h_i^{(\alpha, \beta)} = \frac{(2)^{\alpha+\beta+1} \Gamma(i+\alpha+1) \Gamma(i+\beta+1)}{(2i+\alpha+\beta+1) \Gamma(i+1) \Gamma(i+\alpha+\beta+1)}. \quad (2.2)$$

The expression of the Jacobi polynomial is as follows:

$$\mathcal{P}_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} [(1-x)^{\alpha+n} (1+x)^{\beta+n}], \quad n \in \mathbb{N}^+, \quad \alpha, \beta > -1.$$

We show that the Jacobi polynomials are the eigenfunctions of the singular Sturm-Liouville problem:

$$(x^2-1)\partial_x^2 u(x) + (\alpha-\beta - (\alpha+\beta+2)x)\partial_x u(x) = \lambda_n^{\alpha,\beta} \mathcal{P}_n^{(\alpha,\beta)}(x), \quad x \in [-1,1],$$

and the corresponding eigenvalue is:

$$\lambda_n^{\alpha,\beta} = n(n+\alpha+\beta+1).$$

The Jacobi polynomials $\mathcal{P}_i^{(\alpha,\beta)}$, ($i=0,1,\dots$) with the parameters ($\alpha > -1, \beta > -1$) satisfy the following relations:

$$\mathcal{P}_{i+1}^{(\alpha,\beta)}(x) = (a_i^{(\alpha,\beta)} x - b_i^{(\alpha,\beta)}) \mathcal{P}_i^{(\alpha,\beta)}(x) - c_i^{(\alpha,\beta)} \mathcal{P}_{i-1}^{(\alpha,\beta)}(x), \quad i \geq 1, \quad (2.3a)$$

$$\mathcal{P}_0^{(\alpha,\beta)}(x) = 1, \quad \mathcal{P}_1^{(\alpha,\beta)}(x) = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta), \quad (2.3b)$$

where

$$\begin{aligned} a_i^{(\alpha,\beta)} &= \frac{(2i+\alpha+\beta+1)(2i+\alpha+\beta+2)}{2(i+1)(i+\alpha+\beta+1)}, \\ b_i^{(\alpha,\beta)} &= \frac{(2i+\alpha+\beta+1)(\beta^2-\alpha^2)}{2(i+1)(i+\alpha+\beta+1)(2i+\alpha+\beta)}, \\ c_i^{(\alpha,\beta)} &= \frac{(2i+\alpha+\beta+2)(i+\alpha)(i+\beta)}{(i+1)(i+\alpha+\beta+1)(2i+\alpha+\beta)}. \end{aligned}$$

The value of Jacobi polynomial at $x=1$ and $x=-1$ is:

$$\begin{aligned} \mathcal{P}_i^{(\alpha,\beta)}(1) &= \frac{\Gamma(i+\alpha+1)}{i!(\alpha+1)}, \\ \mathcal{P}_i^{(\alpha,\beta)}(-1) &= (-1)^i \frac{\Gamma(i+\beta+1)}{i!(\beta+1)}. \end{aligned}$$

The Legendre polynomial is $L_n(x) = \mathcal{P}_n^{(0,0)}$ and the Chebyshev polynomial is $T_n(x) = \mathcal{P}_n^{(-\frac{1}{2},-\frac{1}{2})}$. So it makes sense to study the general Jacobi polynomials.

Now, we give the analytic form of Jacobi polynomials as follows:

$$\begin{aligned} \mathcal{P}_j^{(\alpha,\beta)}(x) &= \sum_{k=0}^j \frac{(-1)^{j-k} \Gamma(j+1+\beta) \Gamma(j+1+k+\alpha+\beta)}{\Gamma(k+\beta+1) \Gamma(j+1+\alpha+\beta) (j-k)! k! 2^k} (x+1)^k \\ &= \sum_{k=0}^j \frac{\Gamma(j+1+\alpha) \Gamma(j+1+k+\alpha+\beta)}{\Gamma(k+\alpha+1) \Gamma(j+1+\alpha+\beta) (j-k)! k! 2^k} (x-1)^k. \end{aligned} \quad (2.4)$$

In addition, the p -th derivative of polynomials reads:

$$\partial_x^{(p)} \mathcal{P}_j^{(\alpha,\beta)}(x) = \frac{\Gamma(j+\alpha+\beta+p+1)}{2^p \Gamma(j+\alpha+\beta+1)} \mathcal{P}_{j-p}^{(\alpha+p,\beta+p)}(x), \tag{2.5}$$

and we define the weighted space $L^2_{\omega^{(\alpha,\beta)}}$, the norm and the inner product as follows:

$$(u,v)_{\omega^{(\alpha,\beta)}} = \int_{-1}^1 u(x)v(x)\omega^{(\alpha,\beta)}(x)dx, \quad \|u\|_{L^2_{\omega^{(\alpha,\beta)}}} = (u,u)_{\omega^{(\alpha,\beta)}}^{\frac{1}{2}}. \tag{2.6}$$

Then, we use the Jacobi-Gauss (JG) quadrature to approximate the integrals Eq. (2.6). Let P_N be the set of polynomials whose highest degree not more than N . It follows that for all $u(x) \in P_{2N}$, there exists a unique weights $\omega_j^{(\alpha,\beta)}$, such that

$$\int_{-1}^1 \omega^{(\alpha,\beta)}(x)u(x)dx = \sum_{j=0}^N \omega_j^{(\alpha,\beta)} u(x_j), \tag{2.7}$$

where x_j and $\omega_j^{(\alpha,\beta)}$, ($j=0,1,\dots,N$) are used as usual to denote the nodes and the corresponding Christoffel numbers in the interval $[-1,1]$, respectively.

Theorem 2.1 (Jacobi-Gauss-Type Quadratures, [17]). *The Jacobi-Gauss quadrature formula (2.7) is exact for any $u(x) \in P_{2N+1}$ with the Jacobi-Gauss nodes x_j , ($j=0,1,\dots,N$) being the zeros of $\mathcal{P}_{N+1}^{(\alpha,\beta)}(x)$ and the corresponding weights $\omega_j^{(\alpha,\beta)}$, ($j=0,1,\dots,N$) given by*

$$\omega_j^{(\alpha,\beta)} = \frac{G_N^{\alpha,\beta}}{\mathcal{P}_N^{(\alpha,\beta)}(x_j)\partial_x \mathcal{P}_{N+1}^{(\alpha,\beta)}(x_j)},$$

where

$$G_N^{\alpha,\beta} = \frac{2^{\alpha+\beta} (2N+\alpha+\beta+2)\Gamma(N+\alpha+1)\Gamma(N+\beta+1)}{(N+1)!\Gamma(N+\alpha+\beta+2)}. \tag{2.8}$$

Theorem 2.2 (Jacobi-Gauss-Radau Quadratures, [17]). *The Jacobi-Gauss quadrature formula (2.7) is exact for any $u(x) \in P_{2N}$ with $x_0 = -1$ and x_j , ($j=1,\dots,N$) be the zeros of $\mathcal{P}_N^{(\alpha,\beta+1)}(x)$ and the weights are given by*

$$\omega_0^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}(\beta+1)\Gamma^2(\beta+1)N!\Gamma(N+\alpha+1)}{\Gamma(N+\beta+2)\Gamma(N+\alpha+\beta+2)}, \tag{2.9a}$$

$$\omega_j^{(\alpha,\beta)} = \frac{1}{1+x_j} \frac{G_{N-1}^{\alpha,\beta+1}}{\mathcal{P}_{N-1}^{(\alpha,\beta+1)}(x_j)\partial_x \mathcal{P}_N^{(\alpha,\beta+1)}(x_j)}, \quad 1 \leq j \leq N, \tag{2.9b}$$

where the constants $G_N^{\alpha,\beta}$ are defined in (2.8).

Theorem 2.3 (Jacobi-Gauss-Lobatto Quadratures, [17]). *The Jacobi-Gauss quadrature formula (2.7) is exact for any $u(x) \in P_{2N-1}$. Let $x_0 = -1$, $x_N = 1$, and x_j , ($j = 1, \dots, N-1$) are the zeros of $\partial_x \mathcal{P}_N^{(\alpha, \beta)}(x)$, the corresponding weights are given by:*

$$\omega_0^{(\alpha, \beta)} = \frac{2^{\alpha+\beta+1}(\beta+1)\Gamma^2(\beta+1)\Gamma(N)\Gamma(N+\alpha+1)}{\Gamma(N+\beta+1)\Gamma(N+\alpha+\beta+2)}, \quad (2.10a)$$

$$\omega_N^{(\alpha, \beta)} = \frac{2^{\alpha+\beta+1}(\alpha+1)\Gamma^2(\alpha+1)\Gamma(N)\Gamma(N+\beta+1)}{\Gamma(N+\alpha+1)\Gamma(N+\alpha+\beta+2)}, \quad (2.10b)$$

$$\omega_j^{(\alpha, \beta)} = \frac{1}{1-x_j^2} \frac{G_{N-2}^{\alpha+1, \beta+1}}{\mathcal{P}_{N-2}^{(\alpha+1, \beta+1)}(x_j) \partial_x \mathcal{P}_{N-1}^{(\alpha+1, \beta+1)}(x_j)}, \quad 1 \leq j \leq N-1, \quad (2.10c)$$

where the constants $G_N^{\alpha, \beta}$ are defined in (2.8).

Next, suppose a function $u(x)$ which is infinitely differentiable in $[-1, 1]$, then it can be written in terms as Jacobi polynomials:

$$u(x) = \sum_{j=0}^{\infty} c_j \mathcal{P}_j^{(\alpha, \beta)}(x), \quad (2.11)$$

and the coefficients c_j as follows:

$$c_j = \frac{1}{h_j^{(\alpha, \beta)}} \int_{-1}^1 u(x) \mathcal{P}_j^{(\alpha, \beta)}(x) \omega^{(\alpha, \beta)}(x) dx, \quad j = 0, 1, 2, \dots \quad (2.12)$$

In reality, we usually consider the first $N+1$ terms of the Jacobi polynomials for obtain an approximate expression of the function $u(x)$. Therefore $u(x)$ can be written as follows:

$$u(x) \approx \sum_{j=0}^N c_j \mathcal{P}_j^{(\alpha, \beta)}(x). \quad (2.13)$$

3 JGLC method for space discretization of FGLE

In this section, we propose collocation method to solving the fractional Ginzburg-Landau equations in one- and two-dimension. The proposed scheme mainly depend on the JGLC method. We discretize the FGLE in the spatial direction by using the Jacobi-Gauss-Lobatto (JGL) interpolation nodes to convert the FGLE into a system of ODEs.

3.1 Discretization in one-dimensional

In this subsection, we purpose to use a JGLC method for one-dimensional FGLE numerical solutions. In the process of JGLC method, we will chose the JGL quadrature nodes as

the interpolation points. First, we want to acquire high accuracy that these interpolation points are usually related to Gauss integration formula, see [18]. Then, the distribution of these points in $[-1,1]$ and when we choose the nodes of JGL quadrature, the initial conditions will be satisfied automatically at these interpolations points, we no need other equations to solve these initial conditions. Therefore, the JGLC method is more reasonable.

We consider the FGLE in the following form:

$$u_t + (v + i\eta)(-\Delta)^{\frac{a}{2}}u + (k + i\zeta)|u|^2u - \gamma u = 0, \tag{3.1}$$

where $1 < a < 2$, $x \in (-1,1)$, $t \in (0,T]$, $v > 0$, $k > 0$, η, ζ, γ are given real constants and the initial-boundary conditions:

$$u(x,0) = u_0(x), \quad x \in (-1,1), \tag{3.2a}$$

$$u(-1,t) = u(1,t) = 0, \quad t \in (0,T]. \tag{3.2b}$$

Due to the numerical solution $u(x,t)$ is an complex function, we decompose it into its real and imaginary parts:

$$u(x,t) = f(x,t) + ig(x,t), \quad u_0(x) = \zeta_1(x) + i\zeta_2(x), \tag{3.3}$$

where $f(x,t)$, $g(x,t)$, $\zeta_1(x)$ and $\zeta_2(x)$ are real functions.

Let us denote $f_t = f_t(x,t)$, $f = f(x,t)$, $g_t = g_t(x,t)$, and $g = g(x,t)$. We apply the Eq. (3.3) and Eq. (3.1) to get

$$f_t + ig_t + (v + i\eta)(-\Delta)^{\frac{a}{2}}(f + ig) + (k + i\zeta)(f^2 + g^2)(f + ig) - \gamma(f + ig) = 0, \tag{3.4}$$

and

$$i \left(g_t + v(-\Delta)^{\frac{a}{2}}g + \eta(-\Delta)^{\frac{a}{2}}f + k(f^2 + g^2)g + \zeta(f^2 + g^2)f - \gamma g \right) + f_t + v(-\Delta)^{\frac{a}{2}}f - \eta(-\Delta)^{\frac{a}{2}}g + k(f^2 + g^2)f - \zeta(f^2 + g^2)g - \gamma f = 0. \tag{3.5}$$

Eq. (3.5) can be written as two coupled nonlinear fractional partial differential equations:

$$f_t + v(-\Delta)^{\frac{a}{2}}f - \eta(-\Delta)^{\frac{a}{2}}g + k(f^2 + g^2)f - \zeta(f^2 + g^2)g - \gamma f = 0, \tag{3.6a}$$

$$g_t + v(-\Delta)^{\frac{a}{2}}g + \eta(-\Delta)^{\frac{a}{2}}f + k(f^2 + g^2)g + \zeta(f^2 + g^2)f - \gamma g = 0, \tag{3.6b}$$

with the initial-boundary conditions:

$$f(x,0) = \zeta_1(x), \quad g(x,0) = \zeta_2(x), \quad x \in (-1,1), \tag{3.7a}$$

$$f(-1,t) = f(1,t) = 0, \quad t \in (0,T], \tag{3.7b}$$

$$g(-1,t) = g(1,t) = 0, \quad t \in (0,T]. \tag{3.7c}$$

We write the approximate solution of $f(x, t)$ and $g(x, t)$ in the form of the Jacobi polynomials:

$$f(x, t) = \sum_{j=0}^N a_j(t) \mathcal{P}_j^{(\alpha, \beta)}(x), \quad (3.8a)$$

$$g(x, t) = \sum_{j=0}^N b_j(t) \mathcal{P}_j^{(\alpha, \beta)}(x). \quad (3.8b)$$

From Eq. (2.2) and Eq. (2.11), we obtain that

$$a_j(t) = \frac{1}{h_j^{(\alpha, \beta)}} \int_{-1}^1 f(x, t) \mathcal{P}_j^{(\alpha, \beta)}(x) \omega^{(\alpha, \beta)}(x) dx, \quad (3.9a)$$

$$b_j(t) = \frac{1}{h_j^{(\alpha, \beta)}} \int_{-1}^1 g(x, t) \mathcal{P}_j^{(\alpha, \beta)}(x) \omega^{(\alpha, \beta)}(x) dx. \quad (3.9b)$$

Let us denote

$$f_i(t) = f(x_i, t), \quad g_i(t) = g(x_i, t).$$

From the property of the Jacobi-Gauss quadrature (2.7) and Theorem 2.3, the coefficients $a_j(t)$ and $b_j(t)$ can be written as follows:

$$a_j(t) = \frac{1}{h_j^{(\alpha, \beta)}} \sum_{i=0}^N \mathcal{P}_j^{(\alpha, \beta)}(x_i) \omega_i^{(\alpha, \beta)} f_i(t), \quad (3.10a)$$

$$b_j(t) = \frac{1}{h_j^{(\alpha, \beta)}} \sum_{i=0}^N \mathcal{P}_j^{(\alpha, \beta)}(x_i) \omega_i^{(\alpha, \beta)} g_i(t). \quad (3.10b)$$

Employing Eqs. (3.9) and (3.10), the approximate solutions of Eq. (3.8b) can be written as follows:

$$f(x, t) = \sum_{i=0}^N \left(\sum_{j=0}^N \frac{1}{h_j^{(\alpha, \beta)}} \mathcal{P}_j^{(\alpha, \beta)}(x_i) \omega_i^{(\alpha, \beta)} \mathcal{P}_j^{(\alpha, \beta)}(x) \right) f_i(t), \quad (3.11a)$$

$$g(x, t) = \sum_{i=0}^N \left(\sum_{j=0}^N \frac{1}{h_j^{(\alpha, \beta)}} \mathcal{P}_j^{(\alpha, \beta)}(x_i) \omega_i^{(\alpha, \beta)} \mathcal{P}_j^{(\alpha, \beta)}(x) \right) g_i(t). \quad (3.11b)$$

Due to

$$\begin{aligned} {}_{-1}^R D_x^a (x+1)^k &= \frac{1}{\Gamma(2-a)} \frac{d^2}{dx^2} \int_{-1}^x (x-s)^{1-a} (s+1)^k ds \\ &= \frac{\Gamma(k+1)}{\Gamma(k+1-a)} (x+1)^{k-a}, \end{aligned} \quad (3.12)$$

and Eq. (2.4), the left Riemann-Liouville fractional derivative of $f(x, t)$ is as follows:

$$\begin{aligned} {}_{-1}^R D_x^\alpha f(x, t) &= \sum_{i=0}^N \left(\sum_{j=0}^N \frac{1}{h_j^{(\alpha, \beta)}} \mathcal{P}_j^{(\alpha, \beta)}(x_i) \omega_i^{(\alpha, \beta)} {}_{-1}^R D_x^\alpha \left[\mathcal{P}_j^{(\alpha, \beta)}(x) \right] \right) f_i(t) \\ &= \sum_{i=0}^N \left(\sum_{j=0}^N \frac{1}{h_j^{(\alpha, \beta)}} \mathcal{P}_j^{(\alpha, \beta)}(x_i) \omega_i^{(\alpha, \beta)} P_{L,j}^{(\alpha, \beta)}(x) \right) f_i(t) \\ &= \sum_{i=0}^N {}_L D_i(x) f_i(t), \end{aligned} \tag{3.13}$$

where

$${}_L D_i(x) = \sum_{j=0}^N \frac{1}{h_j^{(\alpha, \beta)}} \mathcal{P}_j^{(\alpha, \beta)}(x_i) \omega_i^{(\alpha, \beta)} P_{L,j}^{(\alpha, \beta)}(x), \tag{3.14a}$$

$$\begin{aligned} P_{L,j}^{(\alpha, \beta)}(x) &= \sum_{k=0}^j \frac{(-1)^{j-k} \Gamma(j+1+\beta) \Gamma(j+1+k+\alpha+\beta)}{\Gamma(k+1+\beta) \Gamma(j+1+\alpha+\beta) (j-k)! k! 2^k} {}_{-1}^R D_x^\alpha (x+1)^k \\ &= \sum_{k=0}^j \frac{(-1)^{j-k} \Gamma(j+1+\beta) \Gamma(j+1+k+\alpha+\beta) \Gamma(1+k) (x+1)^{k-a}}{\Gamma(k+1+\beta) \Gamma(j+1+\alpha+\beta) (j-k)! k! 2^k \Gamma(k+1-a)}. \end{aligned} \tag{3.14b}$$

Due to

$$\begin{aligned} {}_x^R D_1^\alpha (x-1)^k &= \frac{(-1)^k}{\Gamma(2-a)} \frac{d}{dx^2} \int_x^1 (s-x)^{1-a} (1-x)^k ds \\ &= \frac{(-1)^k \Gamma(k+1)}{\Gamma(k+1-a)} (1-x)^{k-a}. \end{aligned} \tag{3.15}$$

Then the right Riemann-Liouville fractional of $f(x, t)$ as follow:

$$\begin{aligned} {}_x^R D_1^\alpha f(x, t) &= \sum_{i=0}^N \left(\sum_{j=0}^N \frac{1}{h_j^{(\alpha, \beta)}} \mathcal{P}_j^{(\alpha, \beta)}(x_i) \omega_i^{(\alpha, \beta)} P_{R,j}^{(\alpha, \beta)}(x) \right) f_i(t) \\ &= \sum_{i=0}^N {}_R D_i(x) f_i(t), \end{aligned} \tag{3.16}$$

where

$${}_R D_i(x) = \sum_{j=0}^N \frac{1}{h_j^{(\alpha, \beta)}} \mathcal{P}_j^{(\alpha, \beta)}(x_i) \omega_i^{(\alpha, \beta)} P_{R,j}^{(\alpha, \beta)}(x), \tag{3.17a}$$

$$P_{R,j}^{(\alpha, \beta)}(x) = \sum_{k=0}^j \frac{(-1)^k \Gamma(j+1+\alpha) \Gamma(j+1+k+\alpha+\beta) \Gamma(k+1) (1-x)^{k-a}}{\Gamma(k+\alpha+1) \Gamma(j+1+\alpha+\beta) (j-k)! k! 2^k \Gamma(k+1-a)}. \tag{3.17b}$$

By Eq. (1.4), the Riesz fractional derivative of $f(x, t)$ can be expressed by Eq. (3.13) and Eq. (3.16) as follows:

$$\frac{\partial^a}{\partial|x|^a} f(x, t) = \sum_{i=0}^N \mathfrak{D}_i(x) f_i(t), \quad (3.18)$$

where

$$\mathfrak{D}_i(x) = -\frac{1}{2\cos(\frac{\pi a}{2})} [{}_L D_i(x) + {}_R D_i(x)]. \quad (3.19)$$

By the same method

$$\frac{\partial^a}{\partial|x|^a} g(x, t) = \sum_{i=0}^N \mathfrak{D}_i(x) g_i(t), \quad (3.20)$$

where

$$\mathfrak{D}_i(x) = -\frac{1}{2\cos(\frac{\pi a}{2})} [{}_L D_i(x) + {}_R D_i(x)]. \quad (3.21)$$

Employing Eqs. (3.11)-(3.21), Eq. (3.6) can be written as follows:

$$f_t - \nu \sum_{i=0}^N \mathfrak{D}_i(x) f_i(t) + \eta \sum_{i=0}^N \mathfrak{D}_i(x) g_i(t) + (f^2 + g^2)(kf - \zeta g) - \gamma f = 0, \quad (3.22a)$$

$$g_t - \nu \sum_{i=0}^N \mathfrak{D}_i(x) g_i(t) - \eta \sum_{i=0}^N \mathfrak{D}_i(x) f_i(t) + (f^2 + g^2)(kg + \zeta f) - \gamma g = 0. \quad (3.22b)$$

Let us denote

$$\dot{f}(t) = \frac{\partial}{\partial t} f(x, t), \quad \dot{g}(t) = \frac{\partial}{\partial t} g(x, t).$$

Now, we use the collocation method to convert Eq. (3.22) with its initial-boundary conditions into a system of ODEs in the time variable. From Theorem 2.3, supposing $\{x_i\}$ are the JGL interpolation nodes and applying these nodes in Eq. (3.22), Eq. (3.22) provides a system of $2(N+1)$ ODEs:

$$\begin{aligned} \dot{f}_n(t) = & \nu \sum_{i=0}^N \mathfrak{D}_i(x_n) f_i(t) - \eta \sum_{i=0}^N \mathfrak{D}_i(x_n) g_i(t) \\ & - (f_n^2(t) + g_n^2(t))(kf_n(t) - \zeta g_n(t)) + \gamma f_n(t), \end{aligned} \quad (3.23a)$$

$$\begin{aligned} \dot{g}_n(t) = & \nu \sum_{i=0}^N \mathfrak{D}_i(x_n) g_i(t) + \eta \sum_{i=0}^N \mathfrak{D}_i(x_n) f_i(t) \\ & - (f_n^2(t) + g_n^2(t))(kg_n(t) + \zeta f_n(t)) + \gamma g_n(t), \quad n=0, 1, \dots, N, \end{aligned} \quad (3.23b)$$

where $f_0(t) = f_N(t) = 0, g_0(t) = g_N(t) = 0$ and subject to the initial values:

$$f_n(0) = \zeta_1(x_n), \quad g_n(0) = \zeta_2(x_n), \quad n = 0, 1, \dots, N. \tag{3.24}$$

We will discuss a new numerical method to solve the above linear system of ODEs in Section 4.

3.2 Discretization in two-dimension

In this section, we also use the JGLC method to discretize the two-dimensional FGLE. It has the following form:

$$\begin{aligned} \frac{\partial u(x,y,t)}{\partial t} + \left((v+i\eta) \left(-\frac{\partial^a}{\partial|x|^a} - \frac{\partial^a}{\partial|y|^a} \right) \right. \\ \left. + (k+i\zeta)|u(x,y,t)|^2 - \gamma \right) u(x,y,t) = 0, \quad (x,y) \in \partial\Omega, \quad t \in (0,T], \end{aligned} \tag{3.25}$$

with the initial and boundary conditions:

$$u(x,y,0) = u_0(x,y), \quad (x,y) \in \Omega, \tag{3.26a}$$

$$u(x,y,t) = 0, \quad (x,y) \in \partial\Omega, \quad t \in (0,T], \tag{3.26b}$$

where $u(x,y,t)$ is an unknown complex function, $1 < a < 2, v > 0, k > 0, \eta, \zeta$ and γ are given real constants. $\Omega = (-1,1) \times (-1,1)$ and $\partial\Omega$ is the boundary of Ω .

We first write the unknown complex functions $u(x,y,t)$ and $u_0(x,y)$ in terms of real and imaginary parts as follows:

$$u(x,y,t) = f(x,y,t) + ig(x,y,t), \quad u_0(x,y) = \zeta_1(x,y) + i\zeta_2(x,y), \tag{3.27}$$

where $f(x,y,t), g(x,y,t), \zeta_1(x,y)$ and $\zeta_2(x,y)$ are real functions. Substituting Eq. (3.27) in Eq. (3.25) and Eq. (3.26), the fractional partial differential equations read as follows:

$$\begin{aligned} \frac{\partial}{\partial t} (f(x,y,t) + ig(x,y,t)) + (v+i\eta) \left(-\frac{\partial^a}{\partial|x|^a} - \frac{\partial^a}{\partial|y|^a} \right) (f(x,y,t) + ig(x,y,t)) \\ + ((k+i\zeta)(f^2(x,y,t) + g^2(x,y,t)) - \gamma)(f(x,y,t) + ig(x,y,t)) = 0. \end{aligned}$$

Then, the above equation can be written as:

$$\begin{aligned} \frac{\partial}{\partial t} f(x,y,t) + v \left(-\frac{\partial^a}{\partial|x|^a} - \frac{\partial^a}{\partial|y|^a} \right) f(x,y,t) - \eta \left(-\frac{\partial^a}{\partial|x|^a} - \frac{\partial^a}{\partial|y|^a} \right) g(x,y,t) \\ + (kf(x,y,t) - \zeta g(x,y,t))(f^2(x,y,t) + g^2(x,y,t)) - \gamma f(x,y,t) = 0, \\ \frac{\partial}{\partial t} g(x,y,t) + v \left(-\frac{\partial^a}{\partial|x|^a} - \frac{\partial^a}{\partial|y|^a} \right) g(x,y,t) + \eta \left(-\frac{\partial^a}{\partial|x|^a} - \frac{\partial^a}{\partial|y|^a} \right) f(x,y,t) \\ + (kg(x,y,t) + \zeta f(x,y,t))(f^2(x,y,t) + g^2(x,y,t)) - \gamma g(x,y,t) = 0, \end{aligned} \tag{3.28a}$$

with initial-boundary conditions:

$$f(x, y, 0) = \xi_1(x, y), \quad g(x, y, 0) = \xi_2(x, y), \quad (x, y) \in \Omega, \quad (3.29a)$$

$$f(-1, y, t) = f(1, y, t) = 0, \quad y \in (-1, 1), \quad t \in (0, T], \quad (3.29b)$$

$$f(x, -1, t) = f(x, 1, t) = 0, \quad x \in (-1, 1), \quad t \in (0, T], \quad (3.29c)$$

$$g(-1, y, t) = g(1, y, t) = 0, \quad y \in (-1, 1), \quad t \in (0, T], \quad (3.29d)$$

$$g(x, -1, t) = g(x, 1, t) = 0, \quad x \in (-1, 1), \quad t \in (0, T]. \quad (3.29e)$$

Next, we convert the above coupled nonlinear fractional partial differential equations with its initial conditions into a system of ODEs based on the JGLC discretization.

Let the numerical solution be a double Jacobi polynomial series:

$$f(x, y, t) = \sum_{i=0}^N \sum_{j=0}^M a_{i,j}(t) \mathcal{P}_i^{(\alpha,\beta)}(x) \mathcal{P}_j^{(\alpha,\beta)}(y), \quad (3.30a)$$

$$g(x, y, t) = \sum_{i=0}^N \sum_{j=0}^M b_{i,j}(t) \mathcal{P}_i^{(\alpha,\beta)}(x) \mathcal{P}_j^{(\alpha,\beta)}(y). \quad (3.30b)$$

Let $\{x_i: 0 \leq i \leq N\}$ and $\{y_j: 0 \leq j \leq M\}$ be the JGL interpolation nodes of Jacobi polynomials $\mathcal{P}_N^{(\alpha,\beta)}(x)$ and $\mathcal{P}_M^{(\alpha,\beta)}(y)$ respectively. From the Jacobi Gauss quadrature (2.7) and Theorem 2.3, the coefficients $a_{i,j}(t), b_{i,j}(t)$ above are determined as follows:

$$a_{i,j}(t) = \frac{1}{h_i^{(\alpha,\beta)} h_j^{(\alpha,\beta)}} \sum_{p=0}^N \sum_{q=0}^M (\mathcal{P}_j^{(\alpha,\beta)}(y_q) \omega_q^{(\alpha,\beta)} \mathcal{P}_i^{(\alpha,\beta)}(x_p) \omega_p^{(\alpha,\beta)}) f(x_p, y_q, t), \quad (3.31a)$$

$$b_{i,j}(t) = \frac{1}{h_i^{(\alpha,\beta)} h_j^{(\alpha,\beta)}} \sum_{p=0}^N \sum_{q=0}^M (\mathcal{P}_j^{(\alpha,\beta)}(y_q) \omega_q^{(\alpha,\beta)} \mathcal{P}_i^{(\alpha,\beta)}(x_p) \omega_p^{(\alpha,\beta)}) g(x_p, y_q, t). \quad (3.31b)$$

For simplicity of the formula below, denote

$$f(x_p, y_q, t) = f_{p,q}(t), \quad g(x_p, y_q, t) = g_{p,q}(t), \quad (3.32)$$

then the approximate solutions Eq. (3.30) can be written as follows:

$$f(x, y, t) = \sum_{i=0}^N \sum_{j=0}^M \sum_{p=0}^N \sum_{q=0}^M \frac{(\mathcal{P}_j^{(\alpha,\beta)}(y_q) \omega_q^{(\alpha,\beta)} \mathcal{P}_i^{(\alpha,\beta)}(x_p) \omega_p^{(\alpha,\beta)})}{h_i^{(\alpha,\beta)} h_j^{(\alpha,\beta)}} \times \mathcal{P}_i^{(\alpha,\beta)}(x) \mathcal{P}_j^{(\alpha,\beta)}(y) f_{p,q}(t), \quad (3.33a)$$

$$g(x, y, t) = \sum_{i=0}^N \sum_{j=0}^M \sum_{p=0}^N \sum_{q=0}^M \frac{(\mathcal{P}_j^{(\alpha,\beta)}(y_q) \omega_q^{(\alpha,\beta)} \mathcal{P}_i^{(\alpha,\beta)}(x_p) \omega_p^{(\alpha,\beta)})}{h_i^{(\alpha,\beta)} h_j^{(\alpha,\beta)}} \times \mathcal{P}_i^{(\alpha,\beta)}(x) \mathcal{P}_j^{(\alpha,\beta)}(y) g_{p,q}(t). \quad (3.33b)$$

Then the left and right Riemann-Liouville fractional derivative of $f(x,y,t)$ as follows:

$${}_{-1}^R D_x^a f(x,y,t) = \sum_{i=0}^N \sum_{j=0}^M a_{i,j}(t) P_{L,i}^{(\alpha,\beta)}(x) \mathcal{P}_j^{(\alpha,\beta)}(y), \tag{3.34a}$$

$${}_{-1}^R D_y^a f(x,y,t) = \sum_{i=0}^N \sum_{j=0}^M a_{i,j}(t) P_{L,j}^{(\alpha,\beta)}(y) \mathcal{P}_i^{(\alpha,\beta)}(x), \tag{3.34b}$$

$${}_x^R D_1^a f(x,y,t) = \sum_{i=0}^N \sum_{j=0}^M a_{i,j}(t) P_{R,i}^{(\alpha,\beta)}(x) \mathcal{P}_j^{(\alpha,\beta)}(y), \tag{3.34c}$$

$${}_y^R D_1^a f(x,y,t) = \sum_{i=0}^N \sum_{j=0}^M a_{i,j}(t) P_{R,j}^{(\alpha,\beta)}(y) \mathcal{P}_i^{(\alpha,\beta)}(x), \tag{3.34d}$$

where

$$P_{L,i}^{(\alpha,\beta)}(x) = \sum_{p=0}^i \frac{(-1)^{i-p} \Gamma(i+1+\beta) \Gamma(i+1+p+\alpha+\beta) \Gamma(1+p) (x+1)^{p-a}}{\Gamma(p+\beta+1) \Gamma(i+1+\alpha+\beta) (i-p)! p! 2^p \Gamma(1+p-a)}, \tag{3.35a}$$

$$P_{L,j}^{(\alpha,\beta)}(y) = \sum_{q=0}^j \frac{(-1)^{j-q} \Gamma(j+1+\beta) \Gamma(j+1+q+\alpha+\beta) \Gamma(1+q) (y+1)^{q-a}}{\Gamma(q+\beta+1) \Gamma(j+1+\alpha+\beta) (j-q)! q! 2^q \Gamma(1+q-a)}, \tag{3.35b}$$

$$P_{R,i}^{(\alpha,\beta)}(x) = \sum_{p=0}^i \frac{(-1)^p \Gamma(i+1+\alpha) \Gamma(i+1+p+\alpha+\beta) \Gamma(1+p) (1-x)^{p-a}}{\Gamma(p+\alpha+1) \Gamma(i+1+\alpha+\beta) (i-p)! p! 2^p \Gamma(1+p-a)}, \tag{3.35c}$$

$$P_{R,j}^{(\alpha,\beta)}(y) = \sum_{q=0}^j \frac{(-1)^q \Gamma(j+1+\alpha) \Gamma(j+1+q+\alpha+\beta) \Gamma(1+q) (1-y)^{q-a}}{\Gamma(q+\alpha+1) \Gamma(j+1+\alpha+\beta) (j-q)! q! 2^q \Gamma(1+q-a)}. \tag{3.35d}$$

Let

$${}_{-1}^R D_x^a f(x,y,t) = \sum_{i=0}^N \sum_{j=0}^M {}_L D_{i,j}(x,y) f_{i,j}(t), \tag{3.36a}$$

$${}_{-1}^R D_y^a f(x,y,t) = \sum_{i=0}^N \sum_{j=0}^M {}_L \bar{D}_{i,j}(x,y) f_{i,j}(t), \tag{3.36b}$$

$${}_x^R D_1^a f(x,y,t) = \sum_{i=0}^N \sum_{j=0}^M {}_R D_{i,j}(x,y) f_{i,j}(t), \tag{3.36c}$$

$${}_y^R D_1^a f(x,y,t) = \sum_{i=0}^N \sum_{j=0}^M {}_R \bar{D}_{i,j}(x,y) f_{i,j}(t), \tag{3.36d}$$

where

$${}_L D_{i,j}(t) = \sum_{p=0}^N \sum_{q=0}^M \frac{(\mathcal{P}_j^{(\alpha,\beta)}(y_q) \omega_q^{(\alpha,\beta)} \mathcal{P}_i^{(\alpha,\beta)}(x_p) \omega_p^{(\alpha,\beta)})}{h_i h_j} P_{L,i}^{(\alpha,\beta)}(x) \mathcal{P}_j^{(\alpha,\beta)}(y), \tag{3.37a}$$

$${}_L\bar{D}_{i,j}(t) = \sum_{p=0}^N \sum_{q=0}^M \frac{(\mathcal{P}_j^{(\alpha,\beta)}(y_q)\omega_q^{(\alpha,\beta)}\mathcal{P}_i^{(\alpha,\beta)}(x_p)\omega_p^{(\alpha,\beta)})}{h_i h_j} P_{L,j}^{(\alpha,\beta)}(y)\mathcal{P}_i^{(\alpha,\beta)}(x), \quad (3.37b)$$

$${}_R D_{i,j}(t) = \sum_{p=0}^N \sum_{q=0}^M \frac{(\mathcal{P}_j^{(\alpha,\beta)}(y_q)\omega_q^{(\alpha,\beta)}\mathcal{P}_i^{(\alpha,\beta)}(x_p)\omega_p^{(\alpha,\beta)})}{h_i h_j} P_{R,i}^{(\alpha,\beta)}(x)\mathcal{P}_j^{(\alpha,\beta)}(y), \quad (3.37c)$$

$${}_R\bar{D}_{i,j}(t) = \sum_{p=0}^N \sum_{q=0}^M \frac{(\mathcal{P}_j^{(\alpha,\beta)}(y_q)\omega_{h,M,q}^{(\alpha,\beta)}\mathcal{P}_i^{(\alpha,\beta)}(x_p)\omega_p^{(\alpha,\beta)})}{h_i h_j} P_{R,j}^{(\alpha,\beta)}(y)\mathcal{P}_i^{(\alpha,\beta)}(x). \quad (3.37d)$$

By the formula (1.4), the Riesz fractional derivative of $f(x,y,t)$ can be written with Eq. (3.36) as follows:

$$\frac{\partial^a}{\partial|x|^a} f(x,y,t) = \sum_{i=0}^N \sum_{j=0}^M \mathfrak{D}_{i,j}(x,y) f_{i,j}(t), \quad (3.38)$$

where

$$\mathfrak{D}_{i,j}(x,y) = -\frac{1}{2\cos(\frac{\pi a}{2})} [{}_L D_{i,j}(x,y) + {}_R D_{i,j}(x,y)]. \quad (3.39)$$

By the same method:

$$\frac{\partial^a}{\partial|y|^a} g(x,y,t) = \sum_{i=0}^N \sum_{j=0}^M \bar{\mathfrak{D}}_{i,j}(x,y) g_{i,j}(t), \quad (3.40)$$

where

$$\bar{\mathfrak{D}}_{i,j}(x,y) = -\frac{1}{2\cos(\frac{\pi a}{2})} [{}_L \bar{D}_{i,j} y(x,y) + {}_R \bar{D}_{i,j}(x,y)]. \quad (3.41)$$

Adopting Eqs. (3.38)-(3.41), Eq. (3.28) reads:

$$\begin{aligned} \dot{f}(x,y,t) = & \nu \left(\sum_{i=0}^N \sum_{j=0}^M \mathfrak{D}_{i,j}(x,y) f_{i,j}(t) + \sum_{i=0}^N \sum_{j=0}^M \bar{\mathfrak{D}}_{i,j}(x,y) f_{i,j}(t) \right) \\ & - \eta \left(\sum_{i=0}^N \sum_{j=0}^M \mathfrak{D}_{i,j}(x,y) g_{i,j}(t) + \sum_{i=0}^N \sum_{j=0}^M \bar{\mathfrak{D}}_{i,j}(x,y) g_{i,j}(t) \right) \\ & - \left(kf(x,y,t) - \zeta g(x,y,t) \right) \left(f^2(x,y,t) + g^2(x,y,t) \right) + \gamma f(x,y,t), \end{aligned} \quad (3.42a)$$

$$\begin{aligned} \dot{g}(x,y,t) = & \nu \left(\sum_{i=0}^N \sum_{j=0}^M \mathfrak{D}_{i,j}(x,y) g_{i,j}(t) + \sum_{i=0}^N \sum_{j=0}^M \bar{\mathfrak{D}}_{i,j}(x,y) g_{i,j}(t) \right) \\ & + \eta \left(\sum_{i=0}^N \sum_{j=0}^M \mathfrak{D}_{i,j}(x,y) f_{i,j}(t) + \sum_{i=0}^N \sum_{j=0}^M \bar{\mathfrak{D}}_{i,j}(x,y) f_{i,j}(t) \right) \\ & - \left(kg(x,y,t) + \zeta f(x,y,t) \right) \left(f^2(x,y,t) + g^2(x,y,t) \right) - \gamma g(x,y,t). \end{aligned} \quad (3.42b)$$

Then, we convert Eq. (3.42) into a system of ODEs with the time variable based on collocation method with the JGL interpolation nodes x_n , ($n = 0, 1, \dots, N$) of $\mathcal{P}_N^{(\alpha, \beta)}(x)$, y_m , ($m = 0, 1, \dots, M$) of $\mathcal{P}_M^{(\alpha, \beta)}(y)$. Due to the initial condition (3.29), the above equation can be written as follows:

$$\begin{aligned} \dot{f}_{n,m}(t) = & \nu \left(\sum_{i=0}^N \sum_{j=0}^M \mathfrak{D}_{i,j}(x_n, y_m) f_{i,j}(t) + \sum_{i=0}^N \sum_{j=0}^M \bar{\mathfrak{D}}_{i,j}(x_n, y_m) f_{i,j}(t) \right) \\ & - \eta \left(\sum_{i=0}^N \sum_{j=0}^M \mathfrak{D}_{i,j}(x_n, y_m) g_{i,j}(t) + \sum_{i=0}^N \sum_{j=0}^M \bar{\mathfrak{D}}_{i,j}(x_n, y_m) g_{i,j}(t) \right) \\ & - \left(k f_{n,m}(t) - \zeta g_{n,m}(t) \right) \left(f_{n,m}^2(t) + g_{n,m}^2(t) \right) + \gamma f_{n,m}(t), \end{aligned} \tag{3.43a}$$

$$\begin{aligned} \dot{g}_{n,m}(t) = & \nu \left(\sum_{i=0}^N \sum_{j=0}^M \mathfrak{D}_{i,j}(x_n, y_m) g_{i,j}(t) + \sum_{i=0}^N \sum_{j=0}^M \bar{\mathfrak{D}}_{i,j}(x_n, y_m) g_{i,j}(t) \right) \\ & + \eta \left(\sum_{i=0}^N \sum_{j=0}^M \mathfrak{D}_{i,j}(x_n, y_m) f_{i,j}(t) + \sum_{i=0}^N \sum_{j=0}^M \bar{\mathfrak{D}}_{i,j}(x_n, y_m) f_{i,j}(t) \right) \\ & - \left(k g_{n,m}(t) + \zeta f_{n,m}(t) \right) \left(f_{n,m}^2(t) + g_{n,m}^2(t) \right) - \gamma g_{n,m}(t), \end{aligned} \tag{3.43b}$$

$$n = 0, 1, \dots, N, \quad m = 0, 1, \dots, M, \tag{3.43c}$$

and subject initial conditions:

$$f_{n,m}(0) = \xi_1(x_n, y_m), \quad g_{n,m}(0) = \xi_2(x_n, y_m), \quad n = 0, 1, \dots, N, \quad m = 0, 1, \dots, M. \tag{3.44}$$

In addition, the values of $f_{0,q}(t)$, $f_{N,q}(t)$, $f_{p,0}(t)$, $f_{p,N}(t)$, $g_{0,q}(t)$, $g_{N,q}(t)$, $g_{p,0}(t)$, $g_{p,N}(t)$ are computed as follows:

$$f_{0,q}(t) = f_{N,q}(t) = 0, \quad q = 0, \dots, M, \tag{3.45a}$$

$$f_{p,0}(t) = f_{p,N}(t) = 0, \quad p = 0, \dots, N, \tag{3.45b}$$

$$g_{0,q}(t) = g_{N,q}(t) = 0, \quad q = 0, \dots, M, \tag{3.45c}$$

$$g_{p,0}(t) = g_{p,N}(t) = 0, \quad p = 0, \dots, N. \tag{3.45d}$$

The above results are a system of $2(N \times M)$ ordinary differential equations. In the next section, we proposed a new method to solve the above system of ODEs.

4 JGRC method for ODEs

In this section, we propose a new method deal with the systems of ordinary differential equation obtained from the previous section, i.e., Eq. (3.23) and Eq. (3.24), Eq. (3.43) and Eq. (3.44), namely:

$$\dot{\varphi}_r(t) = G_r(t, \varphi_1(t), \dots, \varphi_R(t)), \quad r = 1, \dots, R, \quad t \in (0, T], \tag{4.1}$$

and subject to

$$\varphi_r(0) = g_r, \quad r = 1, \dots, R, \quad (4.2)$$

where $G_r(t, \varphi_1(t), \dots, \varphi_r(t))$, $r = 1, \dots, R$ are given nonlinear functions.

In this process, we will choose the JGR quadrature nodes as interpolation nodes for discretizing above ODEs. First, Gauss quadratures can provide a high accuracy. Then, we use the nodes of the JGR quadrature so that the initial conditions satisfied automatically in the collocation method.

First, let us expand the approximate solutions of $\varphi_r(t)$ by the Jacobi polynomials in the form:

$$\varphi_r(t) = \sum_{j=0}^K a_{r,j} \mathcal{P}_{T,j}^{(\alpha,\beta)}(2t-T), \quad r = 1, \dots, R. \quad (4.3)$$

From the derivative relationship in Eq. (2.5), the first time derivative of the approximate solution is as follows:

$$\dot{\varphi}_r(t) = \sum_{j=1}^K a_{r,j} \frac{\Gamma(j+\alpha+\beta+2)}{\Gamma(j+\alpha+\beta+1)T} \mathcal{P}_{T,j-1}^{(\alpha+1,\beta+1)}(2t-T), \quad r = 1, \dots, R. \quad (4.4)$$

Then, adopting Eqs. (4.3) and (4.4), Eqs. (4.1) and (4.2) can be written as follows :

$$\begin{aligned} & \sum_{j=1}^K a_{r,j} \frac{\Gamma(j+\alpha+\beta+2)}{\Gamma(j+\alpha+\beta+1)T} \mathcal{P}_{T,j-1}^{(\alpha+1,\beta+1)}(2t-T) \\ = & G_r \left(t, \sum_{j=0}^K a_{1,j} \mathcal{P}_{T,j}^{(\alpha,\beta)}(2t-T), \dots, \sum_{j=0}^K a_{R,j} \mathcal{P}_{T,j}^{(\alpha,\beta)}(2t-T) \right), \quad r = 1, \dots, R, \quad t \in (0, T], \quad (4.5) \end{aligned}$$

subject to

$$\sum_{j=0}^K a_{r,j} \mathcal{P}_{T,j}^{(\alpha,\beta)}(-T) = g_r, \quad r = 1, \dots, R. \quad (4.6)$$

Next, we discretize the time variable at $\{t_j\}$, based on JGRC method. Let $\{t_j\}$ be the JGR interpolation nodes, where $t_0 = 0$, and t_k , ($k = 1, \dots, K$) are zeros of $\mathcal{P}_{T,K}^{(\alpha,\beta+1)}(2t-T)$. By Theorem 2.2, we get $(R \times K)$ algebraic equations:

$$\begin{aligned} & \sum_{j=1}^K a_{r,j} \frac{\Gamma(j+\alpha+\beta+2)}{\Gamma(j+\alpha+\beta+1)T} \mathcal{P}_{T,j-1}^{(\alpha+1,\beta+1)}(2t_k-T) \\ = & G_r \left(t_k, \sum_{j=0}^K a_{1,j} \mathcal{P}_{T,j}^{(\alpha,\beta)}(2t_k-T), \dots, \sum_{j=0}^K a_{R,j} \mathcal{P}_{T,j}^{(\alpha,\beta)}(2t_k-T) \right), \\ & r = 1, \dots, R, \quad k = 1, \dots, K. \quad (4.7) \end{aligned}$$

Because of the initial condition (4.6), we obtain another set of R algebraic equations. Finally, combining two algebraic systems, we obtain $(R \times (K+1))$ algebraic equations for Jacobi expansion coefficient $a_{r,j}$, $r = 1, 2, \dots, R$, $j = 0, 1, \dots, K$. We would solve the system of equations by Newton's iteration method. Therefore, we can evaluate the approximate solutions (4.3).

5 Error analysis

In this section, we estimate the error of the numerical solution obtained by using JGLC method and JGRC method in solving the fractional Ginzburg-Landua equation. We bound the error in the sense of L^∞ and L^2 .

As $u(x, t)$ is a complex function, we decompose it into its real and imaginary parts:

$$u(x, t) = f(x, t) + ig(x, t), \quad x \in [-1, 1], \quad t \in [0, T],$$

where $f(x, t)$ and $g(x, t)$ are smooth real functions.

Lemma 5.1 ([20]). *If $\alpha, \beta > -1$, then*

$$\max_{-1 \leq x \leq 1} |\mathcal{P}_n^{\alpha, \beta}| \leq Kn^2, \tag{5.1}$$

where $q = \max(\alpha, \beta, -\frac{1}{2})$ and K is a positive constant.

Let us define the discrete space by

$$\mathcal{P}_{N, M}^{\alpha, \beta} = \text{span} \{ \mathcal{P}_i^{(\alpha, \beta)}(x) \mathcal{P}_{T, k}^{(\alpha, \beta)}(t), i = 0, 1, \dots, M, k = 0, 1, \dots, N \}.$$

Let $f_{N, M}(x, t)$ and $g_{N, M}(x, t)$ be the numerical solutions approximating $f(x, t)$ and $g(x, t)$, respectively,

$$f_{N, M}(x, t), g_{N, M}(x, t) \in \mathcal{P}_{N, M}^{\alpha, \beta}.$$

Let

$$E_1(x, t) = |f(x, t) - f_{N, M}(x, t)|, \tag{5.2a}$$

$$E_2(x, t) = |g(x, t) - g_{N, M}(x, t)|, \tag{5.2b}$$

$$E_3(x, t) = |u(x, t) - u_{N, M}(x, t)|, \tag{5.2c}$$

where $u(x, t) = f(x, t) + ig(x, t)$ and $u_{N, M}(x, t) = f_{N, M}(x, t) + ig_{N, M}(x, t)$ are the exact solution and approximate solution, respectively. In particular,

$$\begin{aligned} E_3(x, t) &= |u(x, t) - u_{N, M}(x, t)| = |f(x, t) - f_{N, M}(x, t) + i(g(x, t) - g_{N, M}(x, t))| \\ &= \sqrt{E_1^2(x, t) + E_2^2(x, t)}. \end{aligned} \tag{5.3}$$

Theorem 5.1. Let $u(x, t) = f(x, t) + ig(x, t)$ be the exact solution of the original continuous problem (3.1), which is sufficiently smooth. $u_{N,M}(x, t) = f_{N,M}(x, t) + ig_{N,M}(x, t)$ is the numerical solution obtained by (4.7). The following error estimate holds,

$$\|u(x, t) - u_{N,M}(x, t)\|_{L^\infty} \leq ((C'_1 Q_1 + C'_2 Q_2 + C'_3 Q_1 Q_2)^2 + (H'_1 Q_1 + H'_2 Q_2 + H'_3 Q_1 Q_2)^2)^{\frac{1}{2}},$$

where

$$Q_1 = \frac{(M+1)^q}{(M+1)! \kappa_M^{(\alpha, \beta)}}, \quad Q_2 = \frac{(T/2)^{N+1} (N+1)^q}{(N+1)! \kappa_N^{(\alpha, \beta)}},$$

and $C'_1, C'_2, C'_3, H'_1, H'_2, H'_3$ are constants, $q = \max(\alpha, \beta, -\frac{1}{2})$ and $\kappa_M^{(\alpha, \beta)}$ is the coefficient of $\mathcal{P}_{M+1}^{(\alpha, \beta)}(x)$.

Proof. The maximum absolute error is defined by:

$$M_1 = \|E_1(x, t)\|_{L^\infty} = \max\{|E_1(x, t)| : \forall(x, t) \in [-1, 1] \times [0, T]\}, \quad (5.4a)$$

$$M_2 = \|E_2(x, t)\|_{L^\infty} = \max\{|E_2(x, t)| : \forall(x, t) \in [-1, 1] \times [0, T]\}, \quad (5.4b)$$

$$M_3 = \|E_3(x, t)\|_{L^\infty} = \max\{|E_3(x, t)| : \forall(x, t) \in [-1, 1] \times [0, T]\}, \quad (5.4c)$$

and clearly $M_3 \leq \sqrt{M_1^2 + M_2^2}$.

Because $f_{N,M}(x, t)$ is the best approximation in L^∞ -norm by the orthogonal expansion of $f(x, t)$ under the Jacobi polynomial basis, for any $v_{N,M}(x, t) \in \mathcal{P}_{N,M}^{\alpha, \beta}$, the following inequality holds:

$$\|f(x, t) - f_{N,M}(x, t)\|_{L^\infty} \leq C \|f(x, t) - v_{N,M}(x, t)\|_{L^\infty}, \quad (5.5)$$

where $v_{N,M}(x, t)$ denotes the interpolating polynomial for $f(x, t)$ at $\{x_i\}$ (the JGL interpolation points of $\mathcal{P}_M^{(\alpha, \beta)}(x)$) and $\{t_k\}$ (the JGR interpolation of $\mathcal{P}_{T,K}^{(\alpha, \beta)}(t)$).

Thus,

$$\begin{aligned} f(x, t) - v_{N,M}(x, t) &= \frac{\partial^{M+1} f(\xi, t)}{(M+1)! \partial x^{M+1}} \prod_{i=0}^M (x - x_i) + \frac{\partial^{N+1} f(x, \zeta)}{(N+1)! \partial t^{N+1}} \prod_{i=0}^N (t - t_k) \\ &\quad - \frac{\partial^{M+N+2} f(\xi', \zeta')}{(M+1)! (N+1)! \partial x^{M+1} \partial t^{N+1}} \prod_{i=0}^M (x - x_i) \prod_{i=0}^N (t - t_k), \end{aligned} \quad (5.6)$$

where $\xi, \xi' \in [-1, 1], \zeta, \zeta' \in [0, T]$,

$$\begin{aligned} &\|f(x, t) - v_{N,M}(x, t)\|_{L^\infty} \\ &\leq \max_{(x,t) \in \Omega} \left| \frac{\partial^{M+1} f(\xi, t)}{\partial x^{M+1}} \right| \frac{\|\prod_{i=0}^M (x - x_i)\|_{L^\infty}}{(M+1)!} + \max_{(x,t) \in \Omega} \left| \frac{\partial^{N+1} f(x, \zeta)}{\partial t^{N+1}} \right| \frac{\|\prod_{i=0}^N (t - t_k)\|_{L^\infty}}{(N+1)!} \\ &\quad + \max_{(x,t) \in \Omega} \left| \frac{\partial^{M+N+2} f(\xi', \zeta')}{\partial x^{M+1} \partial t^{N+1}} \right| \frac{\|\prod_{i=0}^M (x - x_i)\|_{L^\infty} \|\prod_{i=0}^N (t - t_k)\|_{L^\infty}}{(M+1)! (N+1)!}. \end{aligned} \quad (5.7)$$

For the smooth function $f(x, t)$, there exist constants C_1, C_2 and C_3 such that

$$\max_{(x,t) \in \Omega} \left| \frac{\partial^{M+1} f(\xi, t)}{\partial x^{M+1}} \right| \leq C_1, \tag{5.8a}$$

$$\max_{(x,t) \in \Omega} \left| \frac{\partial^{N+1} f(x, \zeta)}{\partial t^{N+1}} \right| \leq C_2, \tag{5.8b}$$

$$\max_{(x,t) \in \Omega} \left| \frac{\partial^{M+N+2} f(\xi', \zeta')}{\partial x^{M+1} \partial t^{N+1}} \right| \leq C_3. \tag{5.8c}$$

We estimate the factors $\|\prod_{i=0}^M (x - x_i)\|_{L^\infty}$ and $\|\prod_{k=0}^N (t - t_k)\|_{L^\infty}$.

Let $t = \frac{T}{2}(\tau + 1)$. By the affine mapping between intervals $[0, T]$ and $[-1, 1]$,

$$\begin{aligned} \max_{t \in [0, T]} \left| \prod_{k=0}^N (t - t_k) \right| &= \max_{\tau \in [-1, 1]} \left| \prod_{k=0}^N \frac{T}{2} (\tau - \tau_k) \right| = \left(\frac{T}{2}\right)^{N+1} \max_{\tau \in [-1, 1]} \left| \prod_{k=0}^N (\tau - \tau_k) \right| \\ &= \left(\frac{T}{2}\right)^{N+1} \max_{\tau \in [-1, 1]} \left| \frac{\mathcal{P}_{N+1}^{(\alpha, \beta)}(\tau)}{\kappa_N^{(\alpha, \beta)}} \right|, \end{aligned} \tag{5.9}$$

where $\kappa_N^{(\alpha, \beta)} = \frac{\Gamma(2N + \alpha + \beta + 1)}{2^N N! \Gamma(N + \alpha + \beta + 1)}$ is the coefficient of $\mathcal{P}_{N+1}^{(\alpha, \beta)}(\tau)$ and $\tau_k = \frac{2}{T}t_k - 1, t_1 = 0, t_k, (k = 1, \dots, N)$ are zeros of $\mathcal{P}_{T, N}^{(\alpha, \beta+1)}(2t - T)$. By the same way, we have

$$\max_{x \in [-1, 1]} \left| \prod_{i=0}^M (x - x_i) \right| = \max_{x \in [-1, 1]} \left| \frac{\mathcal{P}_{M+1}^{(\alpha, \beta)}(x)}{\kappa_M^{(\alpha, \beta)}} \right|, \tag{5.10}$$

where $\kappa_M^{(\alpha, \beta)} = \frac{\Gamma(2M + \alpha + \beta + 1)}{2^M M! \Gamma(M + \alpha + \beta + 1)}$ is the coefficient of $\mathcal{P}_{M+1}^{(\alpha, \beta)}(x)$ and $x_0 = -1, x_M = 1, x_i, (i = 1, \dots, M - 1)$ are the zeros of $\partial_x \mathcal{P}_M^{(\alpha, \beta)}(x)$.

By Lemma 5.1,

$$\max_{-1 \leq x \leq 1} \left| \mathcal{P}_{M+1}^{(\alpha, \beta)}(x) \right| \leq K(M+1)^q, \quad \alpha, \beta > -1, \tag{5.11}$$

where $q = \max(\alpha, \beta, -\frac{1}{2})$ and K is a positive constant. On the interval $[-1, 1]$, it reaches the maximum of $\left| \mathcal{P}_{M+1}^{(\alpha, \beta)}(x) \right|$ at $x = 1$ and $\alpha \geq \beta, \alpha \geq -\frac{1}{2}$ [21],

$$\max_{-1 \leq x \leq 1} \left| \mathcal{P}_{M+1}^{(\alpha, \beta)}(x) \right| = \mathcal{P}_{M+1}^{(\alpha, \beta)}(1) = \frac{\Gamma(M + \alpha + 2)}{(M+1)! \Gamma(\alpha + 1)} = \mathcal{O}((M+1)^\alpha).$$

Applying (5.7)-(5.11), Eq. (5.5) reads:

$$\begin{aligned} M_1 &= \|f(x,t) - f_{N,M}(x,t)\|_{L^\infty} \\ &\leq C'_1 \frac{(M+1)^q}{(M+1)! \kappa_M^{(\alpha,\beta)}} + C'_2 \frac{(T/2)^{N+1} (N+1)^q}{(N+1)! \kappa_N^{(\alpha,\beta)}} \\ &\quad + C'_3 \frac{(M+1)^q (T/2)^{N+1} (N+1)^q}{(M+1)! (N+1)! \kappa_M^{(\alpha,\beta)} \kappa_N^{(\alpha,\beta)}} \\ &= C'_1 Q_1 + C'_2 Q_2 + C'_3 Q_1 Q_2. \end{aligned}$$

By the same process, we have

$$\begin{aligned} M_2 &= \|g(x,t) - g_{N,M}(x,t)\|_{L^\infty} \\ &\leq H'_1 \frac{(M+1)^q}{(M+1)! \kappa_M^{(\alpha,\beta)}} + H'_2 \frac{(T/2)^{N+1} (N+1)^q}{(N+1)! \kappa_N^{(\alpha,\beta)}} \\ &\quad + H'_3 \frac{(M+1)^q (T/2)^{N+1} (N+1)^q}{\kappa_M^{(M+1)!(N+1)!(\alpha,\beta)} \kappa_N^{(\alpha,\beta)}} \\ &= H'_1 Q_1 + H'_2 Q_2 + H'_3 Q_1 Q_2, \end{aligned}$$

where H'_1, H'_2, H'_3 are constants and $q = \max(\alpha, \beta, -\frac{1}{2})$. So

$$\|u(x,t) - u_{N,M}(x,t)\|_{L^\infty} \leq ((C'_1 Q_1 + C'_2 Q_2 + C'_3 Q_1 Q_2)^2 + (H'_1 Q_1 + H'_2 Q_2 + H'_3 Q_1 Q_2)^2)^{\frac{1}{2}},$$

where

$$Q_1 = \frac{(M+1)^q}{(M+1)! \kappa_M^{(\alpha,\beta)}}, \quad Q_2 = \frac{(T/2)^{N+1} (N+1)^q}{(N+1)! \kappa_N^{(\alpha,\beta)}}.$$

The theorem is proved. \square

Then, we prove the above error estimate in L^2 -norm.

Theorem 5.2. Let $u(x,t) = f(x,t) + ig(x,t)$ be the exact solution of the original problem (3.1), and $f(x,t), g(x,t)$ is sufficiently smooth. $u_{N,M}(x,t) = f_{N,M}(x,t) + ig_{N,M}(x,t)$ is approximate solution of the full-discrete problem (4.7), then the following error estimate holds:

$$\|u(x,t) - u_{N,M}(x,t)\|_{L^2} \leq |(C_1^* Q_1^* + C_2^* Q_2^* + C_3^* Q_1^* Q_2^*)^2 + (H_1^* Q_1 + H_2^* Q_2 + H_3^* Q_1^* Q_2^*)^2|^{\frac{1}{2}},$$

where

$$Q_1^* = \frac{(M+1)^q}{(M+1)! \kappa_M^{(\alpha,\beta)}}, \quad Q_2^* = \frac{(T/2)^{\frac{2N-\alpha-\beta+1}{2}} (N+1)^q}{(N+1)! \kappa_N^{(\alpha,\beta)}},$$

and $C_1^*, C_2^*, C_3^*, H_1^*, H_2^*, H_3^*$ are constants, $q = \max(\alpha, \beta, -\frac{1}{2})$ and $\kappa_M^{(\alpha,\beta)}$ is the coefficient of $\mathcal{P}_{M+1}^{(\alpha,\beta)}(x)$.

Proof. Due to the best approximation in L^2 , $\forall v_{N,M}(x,t) \in \mathcal{P}_{N,M}^{\alpha,\beta}$, we obtain the following inequality

$$\|f(x,t) - v_{N,M}(x,t)\|_{L^2} \leq \|f(x,t) - v_{N,M}(x,t)\|_{L^2}, \tag{5.12}$$

where $v_{N,M}(x,t)$ denotes the interpolating polynomial for $f(x,t)$ at (x_i, t_k) , where $\{x_i\}$ is the JGL interpolation of $\mathcal{P}_M^{(\alpha,\beta)}(x)$ and $\{t_k\}$ is the JGR interpolation of $\mathcal{P}_{T,K}^{(\alpha,\beta)}(t)$.

$$\begin{aligned} f(x,t) - v_{N,M}(x,t) &= \frac{\partial^{M+1} f(\xi,t)}{(M+1)! \partial x^{M+1}} \prod_{i=0}^M (x-x_i) + \frac{\partial^{N+1} f(x,\zeta)}{(N+1)! \partial t^{N+1}} \prod_{i=0}^N (t-t_k) \\ &\quad - \frac{\partial^{M+N+2} f(\xi',\zeta')}{(M+1)!(N+1)! \partial x^{M+1} \partial t^{N+1}} \prod_{i=0}^M (x-x_i) \prod_{i=0}^N (t-t_k), \end{aligned} \tag{5.13}$$

where $\xi, \xi' \in [-1,1]$, $\zeta, \zeta' \in [0,T]$,

$$\begin{aligned} &\|f(x,t) - v_{N,M}(x,t)\|_{L^2} \\ &\leq \left\| \frac{\partial^{M+1} f(\xi,t)}{\partial x^{M+1}} \right\|_{L^2} \frac{\|\prod_{i=0}^M (x-x_i)\|_{L^2}}{(M+1)!} + \left\| \frac{\partial^{N+1} f(x,\zeta)}{\partial t^{N+1}} \right\|_{L^2} \frac{\|\prod_{i=0}^N (t-t_k)\|_{L^2}}{(N+1)!} \\ &\quad + \left\| \frac{\partial^{M+N+2} f(\xi',\zeta')}{\partial x^{M+1} \partial t^{N+1}} \right\|_{L^2} \frac{\|\prod_{i=0}^M (x-x_i)\|_{L^2} \|\prod_{i=0}^N (t-t_k)\|_{L^2}}{(M+1)!(N+1)!}. \end{aligned} \tag{5.14}$$

There exist constants C_1, C_2 and C_3 such that

$$\left\| \frac{\partial^{M+1} f(\xi,t)}{\partial x^{M+1}} \right\|_{L^2} = \left[\int_0^T \left(\frac{\partial^{M+1} f(\xi,t)}{\partial x^{M+1}} \right)^2 \omega^{(\alpha,\beta)}(t) dt \right]^{\frac{1}{2}} \leq C_1, \tag{5.15a}$$

$$\left\| \frac{\partial^{N+1} f(x,\zeta)}{\partial t^{N+1}} \right\|_{L^2} = \left[\int_{-1}^1 \left(\frac{\partial^{N+1} f(x,\zeta)}{\partial t^{N+1}} \right)^2 \omega^{(\alpha,\beta)}(x) dx \right]^{\frac{1}{2}} \leq C_2, \tag{5.15b}$$

$$\left\| \frac{\partial^{M+N+2} f(\xi',\zeta')}{\partial x^{M+1} \partial t^{N+1}} \right\|_{L^2} \leq C_3. \tag{5.15c}$$

Then

$$\begin{aligned} &\left\| \prod_{i=0}^M (x-x_i) \right\|_{L^2} \\ &= \left[\int_{-1}^1 \left(\prod_{i=0}^M (x-x_i) \right)^2 \omega^{(\alpha,\beta)}(x) dx \right]^{\frac{1}{2}} = \left[\int_{-1}^1 \left(\frac{\mathcal{P}_{M+1}^{(\alpha,\beta)}(x)}{\kappa_M^{(\alpha,\beta)}} \right)^2 \omega^{(\alpha,\beta)}(x) dx \right]^{\frac{1}{2}} \\ &\leq \frac{\sqrt{2}}{\kappa_M^{(\alpha,\beta)}} \left[\max_{x \in [-1,1]} |(\mathcal{P}_{M+1}^{(\alpha,\beta)}(x))^2 \omega^{(\alpha,\beta)}(x)| \right]^{\frac{1}{2}} \leq \frac{K_1}{\kappa_M^{(\alpha,\beta)}} \max_{x \in [-1,1]} |\mathcal{P}_{M+1}^{(\alpha,\beta)}(x)|, \end{aligned} \tag{5.16}$$

where K_1 is constant, $\kappa_M^{(\alpha,\beta)} = \frac{\Gamma(2M+\alpha+\beta+1)}{2^M M! \Gamma(M+\alpha+\beta+1)}$ is the coefficient of $\mathcal{P}_{M+1}^{(\alpha,\beta)}(x)$, $\omega^{(\alpha,\beta)}(x) = (1-x)^\alpha (1+x)^\beta$ is weight function and $x_0 = -1, x_M = 1, x_i, (i = 1, \dots, M-1)$ are the zeros of $\partial_x \mathcal{P}_M^{(\alpha,\beta)}(x)$.

Let $t = \frac{T}{2}(\tau + 1)$, then $\tau \in [-1, 1]$. With the weight function $\omega_T^{(\alpha, \beta)}(t) = (T-t)^\alpha t^\beta$ over $I = [0, T]$, we have

$$\begin{aligned}\omega_T^{(\alpha, \beta)}(t) &= (T-t)^\alpha t^\beta = \left(\frac{2}{T}\right)^{\alpha+\beta} \left(\frac{T}{2}\right)^{\alpha+\beta} \left(L - \frac{T}{2}(\tau+1)\right)^\alpha \left(\frac{T}{2}(\tau+1)\right)^\beta \\ &= \left(\frac{2}{T}\right)^{\alpha+\beta} (1-\tau)^\alpha (1+\tau)^\beta = \left(\frac{2}{T}\right)^{\alpha+\beta} \omega^{(\alpha, \beta)}(\tau), \quad \tau \in [-1, 1].\end{aligned}\quad (5.17)$$

Next,

$$\begin{aligned}\left\| \prod_{k=0}^N (t-t_k) \right\|_{L^2} &= \left[\int_0^T \left(\prod_{k=0}^N (t-t_k) \right)^2 \omega_T^{(\alpha, \beta)}(t) dt \right]^{\frac{1}{2}} \\ &= \left[\int_{-1}^1 \left(\left(\frac{T}{2}\right)^N \prod_{k=0}^N (\tau-\tau_k) \right)^2 \left(\frac{2}{T}\right)^{\alpha+\beta-1} \omega^{(\alpha, \beta)}(\tau) d\tau \right]^{\frac{1}{2}} \\ &= \left(\frac{T}{2}\right)^{\frac{2N-\alpha-\beta+1}{2}} \left[\int_{-1}^1 \left(\prod_{k=0}^N (\tau-\tau_k) \right)^2 \omega^{(\alpha, \beta)}(\tau) d\tau \right]^{\frac{1}{2}} \\ &= \left(\frac{T}{2}\right)^{\frac{2N-\alpha-\beta+1}{2}} \left[\int_{-1}^1 \left(\frac{\mathcal{P}_{N+1}^{(\alpha, \beta)}(\tau)}{\kappa_N^{(\alpha, \beta)}} \right)^2 \omega^{(\alpha, \beta)}(\tau) d\tau \right]^{\frac{1}{2}} \\ &\leq \left(\frac{T}{2}\right)^{\frac{2N-\alpha-\beta+1}{2}} \frac{\sqrt{2}}{\kappa_N^{(\alpha, \beta)}} \left[\max_{\tau \in [-1, 1]} \left| \left(\mathcal{P}_{N+1}^{(\alpha, \beta)}(\tau) \right)^2 \omega^{(\alpha, \beta)}(\tau) \right| \right]^{\frac{1}{2}} \\ &\leq \left(\frac{T}{2}\right)^{\frac{2N-\alpha-\beta+1}{2}} \frac{K_2}{\kappa_N^{(\alpha, \beta)}} \max_{\tau \in [-1, 1]} \left| \mathcal{P}_{N+1}^{(\alpha, \beta)}(\tau) \right|,\end{aligned}\quad (5.18)$$

where K_2 is constant, $\kappa_N^{(\alpha, \beta)} = \frac{\Gamma(2N+\alpha+\beta+1)}{2^{2N} N! \Gamma(N+\alpha+\beta+1)}$ is the leading coefficient of $\mathcal{P}_{N+1}^{(\alpha, \beta)}(\tau)$ and $\tau_k = \frac{2}{T}t_k - 1$, $t_1 = 0, t_k$, ($k = 1, \dots, N$) are zeros of $\mathcal{P}_{T, N}^{(\alpha, \beta+1)}(2t - T)$.

By Lemma 5.1,

$$\max_{-1 \leq x \leq 1} \left| \mathcal{P}_{M+1}^{(\alpha, \beta)}(x) \right| \leq K(M+1)^q, \quad \alpha, \beta > -1, \quad (5.19)$$

where $q = \max(\alpha, \beta, -\frac{1}{2})$ and K is a positive. Then

$$\max_{-1 \leq x \leq 1} \left| \mathcal{P}_{M+1}^{(\alpha, \beta)}(x) \right| = \mathcal{P}_{M+1}^{(\alpha, \beta)}(1) = \frac{\Gamma(M+\alpha+2)}{(M+1)! \Gamma(\alpha+1)} = \mathcal{O}((M+1)^\alpha).$$

Applying Eqs. (5.14) and (5.19), Eq. (5.12) reads

$$\begin{aligned}M_1 &= \|f(x, t) - f_{N, M}(x, t)\|_{L^2} \\ &\leq C_1^* \frac{(M+1)^q}{\kappa_M^{(\alpha, \beta)}(M+1)!} + C_2^* \frac{(T/2)^{\frac{2N-\alpha-\beta+1}{2}} (N+1)^q}{\kappa_N^{(\alpha, \beta)}(N+1)!}\end{aligned}$$

$$\begin{aligned}
 &+ C_3^* \frac{(M+1)^q (T/2)^{\frac{2N-\alpha-\beta+1}{2}} (N+1)^q}{\kappa_M^{(\alpha,\beta)} \kappa_N^{(\alpha,\beta)} (M+1)! (N+1)!} \\
 &= C_1^* Q_1^* + C_2^* Q_2^* + C_3^* Q_1^* Q_2^*.
 \end{aligned}$$

By the same process, we have

$$\begin{aligned}
 M_2 &= \|g(x,t) - g_{N,M}(x,t)\|_{L^2} \\
 &\leq H_1^* \frac{(M+1)^q}{(M+1)! \kappa_M^{(\alpha,\beta)}} + H_2^* \frac{(T/2)^{\frac{2N-\alpha-\beta+1}{2}} (N+1)^q}{(N+1)! \kappa_N^{(\alpha,\beta)}} \\
 &\quad + H_3^* \frac{(M+1)^q (T/2)^{\frac{2N-\alpha-\beta+1}{2}} (N+1)^q}{(M+1)! (N+1)! \kappa_M^{(\alpha,\beta)} \kappa_N^{(\alpha,\beta)}} \\
 &= H_1^* Q_1^* + H_2^* Q_2^* + H_3^* Q_1^* Q_2^*,
 \end{aligned}$$

where H'_1, H'_2, H'_3 are constants and $q = \max(\alpha, \beta, -\frac{1}{2})$. Finally,

$$\|u(x,t) - u_{N,M}(x,t)\|_{L^2} \leq \sqrt{M_1^2 + M_2^2}.$$

The theorem is proved. □

6 Numerical tests

In this section, we give numerical examples to illustrate the high order accuracy and efficiency of our methods.

Example 6.1 (A linear equation). The domain is $(x,t) \in (-1,1) \times (0,1)$. The $a = 3/2$ fractional order Ginzburg-Landau equation is solved:

$$u_t + (4+i)(-\Delta)^{3/4} u - u = f_1(x,t), \tag{6.1a}$$

$$u(x,0) = 0, \tag{6.1b}$$

$$u(-1,t) = u(1,t) = 0, \tag{6.1c}$$

where f_1 is defined such that the exact solution is

$$u(x,t) = (e^{4t^2} - 1)((x+1)^3(1-x)^2 + i(x+1)^2(1-x)^3). \tag{6.2}$$

In the space direction x , we use \mathcal{P}_{M+2} Jacobi-Gauss-Lobatto orthogonal polynomials on $[-1,1]$, i.e., $x_0 = -1$ and $x_{M+1} = 1$, with the weight $(x+1)^{1-a}(1-x)^{1-a}$, $a = 3/2$. In the time direction t , we use \mathcal{P}_N left Jacobi-Gauss-Radau orthogonal polynomials on $[0,1]$, i.e., $t_0 = 0$. The number of collocation points is MN , instead of $(M+2)(N+1)$, as the initial values and boundary values are given. The numerical solution and the exact solution at

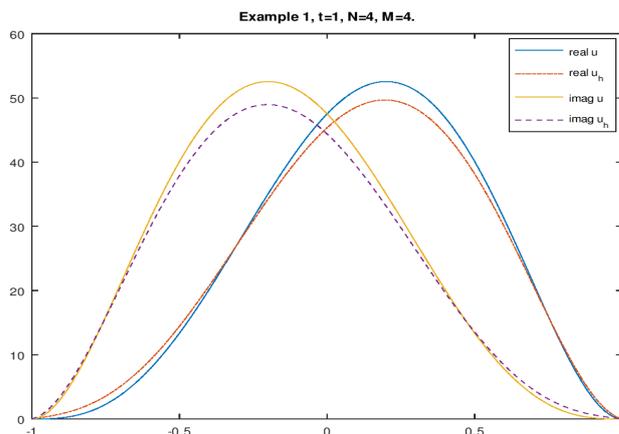


Figure 1: The numerical solution and exact solution of (6.2), $a=3/2$.

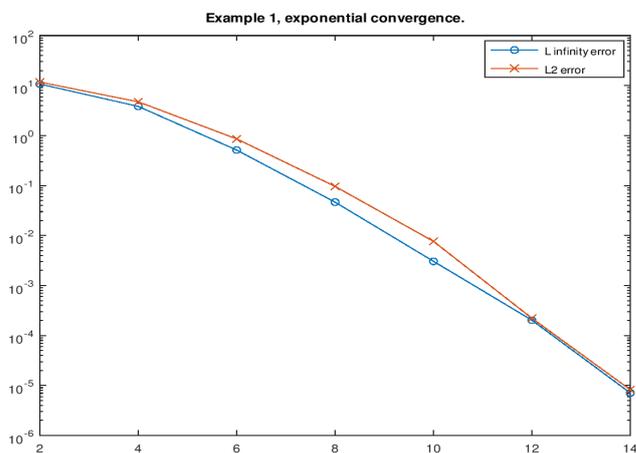


Figure 2: The (exponential) convergence of the numerical solution for (6.2), $a=3/2$.

Table 1: The L^∞ error and the L^2 error at $t=1$ for (6.1).

M, N	$\ u - u_h\ _{L^\infty}$	$\ u - u_h\ _{L^2}$
2, 2	1.0708E+001	1.1732E+001
4, 4	3.8265E-000	4.7214E-000
6, 6	5.1255E-001	8.5651E-001
8, 8	4.6525E-002	9.6116E-002
10, 10	3.0494E-003	7.6455E-003
12, 12	2.0248E-004	2.2147E-004
14, 14	7.0797E-006	8.2330E-006

time $t = 1$ are plotted in Fig. 1, for $M = 4$ and $N = 4$. We list the numerical L^∞ error and the L^2 error at time $t = 1$ in Table 1. In addition we plot the error in the semi-log scale in Fig. 2 to show the exponential convergence. Moreover, u and u_h are the exact solution and approximate solution in table and figure, respectively.

Example 6.2 (A nonlinear equation). The fractional order Ginzburg-Landau equation to be solved is

$$u_t + (6-i)(-\Delta)^{3/4}u + (2-i\frac{1}{2})|u|^2u - u = f_2(x,t), \quad (x,t) \in (-1,1) \times (0,1), \quad (6.3a)$$

$$u(x,0) = 0, \quad (6.3b)$$

$$u(-1,t) = u(1,t) = 0, \quad (6.3c)$$

where f_2 is chosen so that the exact solution is

$$u(x,t) = \sin t^2(x+1)x(1-x)(1-i). \quad (6.4)$$

Again, we use the tensor product basis of $(-\frac{1}{2}, -\frac{1}{2})$ Jacobi-Gauss-Lobatto $\mathcal{P}_{M+2}(x)$ polynomials and left Jacobi-Gauss-Radau $\mathcal{P}_N(t)$ polynomials. The numerical solution and the exact solution at time $t = 1$ are plotted in Fig. 3, for $M = 2$ and $N = 2$. We list the numerical L^∞ error and the L^2 error at time $t = 1$ in Table 2. We plot the error in the semi-log scale in Fig. 4 to see its exponential convergence. Moreover, u and u_h are the exact solution and approximate solution in table and figure, respectively.

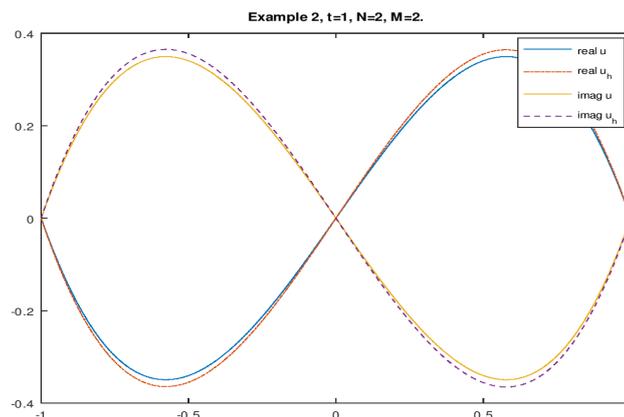


Figure 3: The numerical solution and exact solution of (6.4), $a = 3/2$.

Example 6.3 (A 2D nonlinear equation). The domain is $(x,y,t) \in (-1,1) \times (-1,1) \times (0,1)$. A 2D $a = 3/2$ fractional order Ginzburg-Landau equation is solved:

$$u_t + a_1 \left(-\frac{\partial^{3/2}u}{\partial|x|^{3/2}} - \frac{\partial^{3/2}u}{\partial|y|^{3/2}} \right) + a_2|u|^2u - a_3u = f_3, \quad (6.5a)$$

$$u(x,y,0) = 0, \quad u(\pm 1, \pm 1, t) = 0, \quad (6.5b)$$

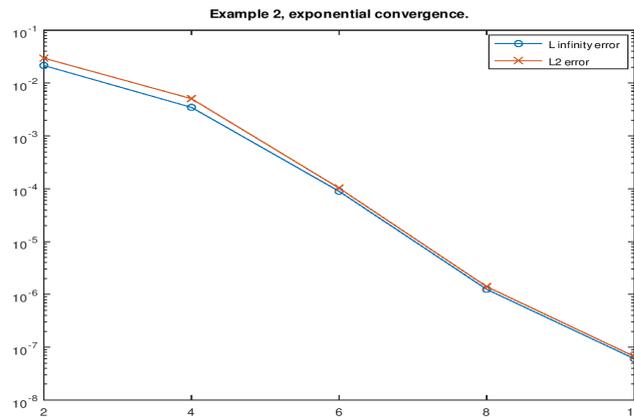


Figure 4: The (exponential) convergence of the numerical solution for (6.4).

Table 2: The L^∞ error and the L^2 error at $t=1$, for (6.3).

M, N	$\ u - u_h\ _{L^\infty}$	$\ u - u_h\ _{L^2}$
2, 2	2.1544E-002	2.9686E-002
4, 4	3.4703E-003	5.0677E-003
6, 6	8.9564E-005	1.0299E-004
8, 8	1.2479E-006	1.4046E-006
10, 10	6.0722E-008	6.7026E-008

where

$$a_1 = 6 - i, \quad a_2 = 2 - i, \quad a_3 = 8,$$

and

$$\begin{aligned}
 f_3 = & (2te^t + t^2e^t) \left((x - x^5)(y - y^3) - i(x - x^5)(y - y^3) \right) \\
 & - \frac{a_1}{2} \left(\frac{(4 - 4i)y(y+1)(y-1)}{\sqrt{x+1}\sqrt{\pi}} - \frac{(2 - 2i)(-x+i)(x+1)(x+i)(x-1)x}{\sqrt{y+1}\sqrt{\pi}} \right) \\
 & + \frac{(\frac{8}{7} - \frac{8i}{7})\sqrt{x+1}(16x^3 - 8x^2 + 6x - 5)y(y+1)(y-1)}{\sqrt{\pi}} \\
 & - \frac{(4 - 4i)\sqrt{y+1}(-x+i)(x+1)(x+i)(x-1)x(2y-1)}{\sqrt{\pi}} \\
 & - \frac{(4 - 4i)y(y+1)(y-1)}{\sqrt{1-x}\sqrt{\pi}} + \frac{(2 - 2i)(-x+i)(x+1)(x+i)(x-1)x}{\sqrt{1-y}\sqrt{\pi}} \\
 & + \frac{(\frac{8}{7} - \frac{8i}{7})\sqrt{1-x}(16x^3 + 8x^2 + 6x + 5)y(y+1)(y-1)}{\sqrt{\pi}}
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{(4-4i) \sqrt{1-y}(-x+i)(x+1)(x+i)(x-1)x(2y+1)}{\sqrt{\pi}} \sqrt{2}t^2e^t \\
 & - a_3((x-x^5)(y-y^3) - i(x-x^5)(y-y^3))t^2e^t + 2a_2(x-x^5)^2 \\
 & \times (y-y^3)^2((x-x^5)(y-y^3) - i(x-x^5)(y-y^3))t^6(e^t)^3.
 \end{aligned} \tag{6.6}$$

So the exact solution is

$$u(x,y,t) = t^2e^t(x-x^5)(y-y^3)(1-i). \tag{6.7}$$

In the space directions, x and y , we use \mathcal{P}_{M+2} Jacobi-Gauss-Lobatto orthogonal polynomials on $[-1,1]$, i.e., $x_0 = -1$ and $x_{M+1} = 1$, with the weight $(x+1)^{1-a}(1-x)^{1-a}$ (and $(y+1)^{1-a}(1-y)^{1-a}$ in y -direction), $a = 3/2$. In computing the fractional derivative in x direction, we use the left and right \mathcal{P}_{M+2} Jacobi-Gauss-Radau quadrature points. In the time direction t , we use \mathcal{P}_{N+1} left Jacobi-Gauss-Radau orthogonal polynomials on $[0,1]$, i.e., $t_0 = 0$. When computing the time-direction integrals, we use the \mathcal{P}_{N+1} Jacobi-Gauss-Radau points. The number of collocation points is M^2N , instead of $(M+2)^2(N+1)$, as the initial values and boundary values are given. The numerical solution and the error at time $t = 1$ are plotted in Fig. 5, for $M = 4$ and $N = 4$. We list the numerical L^∞ error and

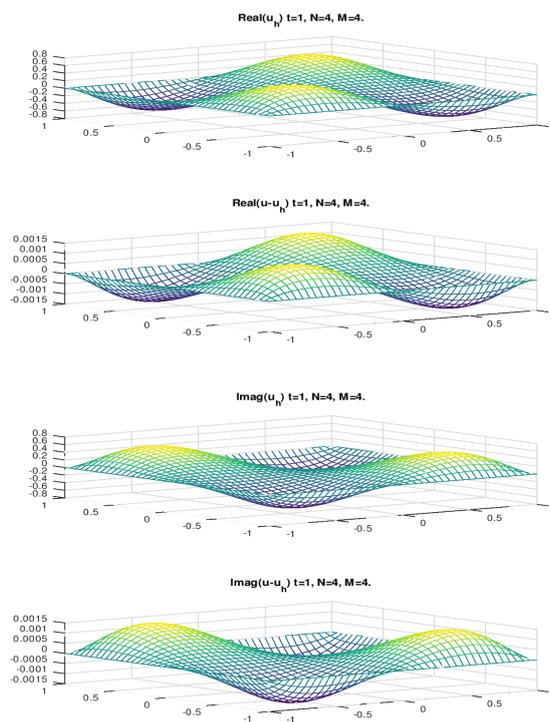


Figure 5: The numerical solution and the error for (6.7).

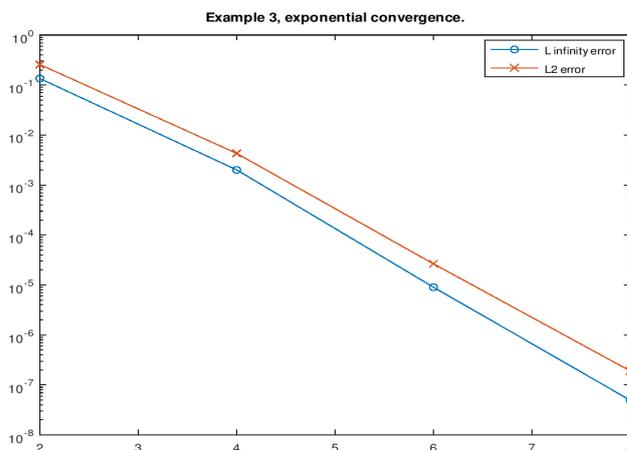


Figure 6: The (exponential) convergence of the numerical solution for (6.7).

Table 3: The L^∞ error and the L^2 error at $t=1$, for 6.7.

M, N	$\ u - u_h\ _{L^\infty}$	$\ u - u_h\ _{L^2}$
2, 2	1.3481E-001	2.5712E-001
4, 4	2.0071E-003	4.2842E-003
6, 6	9.0217E-006	2.6478E-005
8, 8	4.9333E-008	1.9056E-007
10, 10	6.4998E-008	3.0594E-007

the L^2 error at time $t=1$ in Table 3. When $N > 8$, the round-off error limits the converges of the method. We plot the error in the semi-log scale in Fig. 6 to show the exponential convergence. Moreover, u and u_h are the exact solution and approximate solution in table and figure, respectively.

According to the results of above numerical tests and convergence analysis, we know that when the solution is sufficiently smooth, the method we proposed can achieve high precision with fewer indicators (taking fewer nodes in the interval) and it converges to the exact solution with the rate of exponential power. The convergence order of the method in [27] is two orders. In addition, the method we proposed is easier to be implemented for nonlinear problems and it is suitable for long-time calculations and for large N .

7 Conclusions

In this paper, we proposed an efficient and high precision algorithm based on Jacobi-Gauss-Lobatto and Jacobi-Gauss-Radau collocation method. A collocation method is

developed and applied in two successive steps. In the first part, the JGLC method is employed for space discretization. Then we converted the equation with its initial conditions into a system of ODEs with the time variable. In the second part, we solve the system of ODEs in time from step one based on the JGRC method. Then, we obtain an algebraic system and solve it by Newton's method. The proposed scheme has many advantages. First, it is easier to be implemented for nonlinear problems. Then, it is suitable for long-time calculations and for large N (very high accuracy). In addition, we give the theoretical proof of the convergence of collocation method in both L^∞ -norm and L^2 -norm. Finally, we give specific numerical examples. The numerical results confirm the validity of scheme for solving the fractional equations and the correctness of the conclusions. This indicates that the JGLC-JGRC method has a good prospect in solving space-fractional differential equations.

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