Communications
in
Mathematical
Research
33(1)(2017), 19–25

Bonnesen-style Isoperimetric Inequalities of an n-simplex

Wang Wen[1,2], Chen Ya-ping[3] and Yang Shi-guo[1]

(*1. School of Mathematics and Statistics, Hefei Normal University, Hefei, 230601*)

(*2. School of Mathematical Science, University of Science and Technology of China, Hefei, 230026*)

(*3. International Branch of Nan Men Primary School at Shangcheng, Hefei, 231600*)

Communicated by Lei Feng-chun

Abstract: In this paper, by the theory of geometric inequalities, some new Bonnesen-style isoperimetric inequalities of n-dimensional simplex are proved. In several cases, these inequalities imply characterizations of regular simplex.

Key words: simplex, isopermetric deficit, Bonnesen-style isoperimetric inequality

2010 MR subject classification: 51K05, 52A38, 52A40

Document code: A

Article ID: 1674-5647(2017)01-0019-07

DOI: 10.13447/j.1674-5647.2017.01.03

1 Introduction

Let Ω_n be an n-simplex in the n-dimensional Euclidean space E^n with vertices A_1, A_2, $\cdots$, A_{n+1}. Denote by a_{ij} $(i, j = 1, 2, \cdots, n+1)$ the edge lengths of Ω_n (sometimes, we can set $a_1, a_2, \cdots, a_{\frac{1}{2}n(n+1)}$ in some order). If all edge lengths are equal, the simplex is said to be regular. Let F_i denote the $(n-1)$-dimensional volume of the facet $f_i = \{A_1, \cdots, A_{i-1}, A_{i+1}, \cdots, A_{n+1}\}$ opposite to the vertex P_i $(i = 1, 2, \cdots, n+1)$. Setting $F = \sum\limits_{i=1}^{n+1} F_i$, hence F is the surface area of Ω_n.

As a well known result, for a simple closed curve $\mathcal{C}$ (in the Euclidian plane) of length L enclosing a domain of area A, then

$$L^2 - 4\pi A \geq 0, \tag{1.1}$$

with equality holds if and only if the curve is a Euclidean circle. The quantity $L^2 - 4\pi A$ is said to be the isoperimetric deficit of $\mathcal{C}$ (see [1]–[3]).

Received date: April 29, 2015.

Foundation item: The Doctoral Programs Foundation (20113401110009) of Education Ministry of China, Universities Natural Science Foundation (KJ2016A310) of Anhui Province.

E-mail address: wenwang1985@163.com (Wang W).

As an extension, Bonnesen proved the following inequality (see [1]):

$$L^2 - 4\pi A \geq \pi^2 (R - r)^2, \tag{1.2}$$

where R is the circumradius and r is the inradius of the curve $\mathcal{C}$. Note that if the right hand side of (1.2) equals zero, then $R = r$. This means that $\mathcal{C}$ is a circle and $L^2 - 4\pi A = 0$.

More generally, inequalities of the form

$$L^2 - 4\pi A \geq K \tag{1.3}$$

are called Bonnesen-style isoperimetric inequalities if equality is only attained for the Euclidean circle (see [1]). See references [4]–[9] for more details.

When the simple closed curve $\mathcal{C}$ is a triangle (in the Euclidean plane) of area S and with side lengths a_1, a_2, a_3, the following inequality is known:

$$P^2 \geq 3\sqrt{3} S, \tag{1.4}$$

where $P = \dfrac{1}{2}(a_1 + a_2 + a_3)$. Equality holds if and only if this triangle is regular.

Inequality (1.4) may be deemed isoperimetric inequality for triangles.

Veljan-Korchmaros inequality (see [10]) concerning the volume and the edge lengths of Ω_n states as follows:

$$\prod_{1 \leq i < j \leq n+1} a_{ij}^{\frac{2}{n+1}} \geq \left(\frac{2^n n!^2}{n+1} \right)^{\frac{1}{2}} V \tag{1.5}$$

with equality holds if and only if Ω_n is regular.

By utilize the arithmetic-geometric mean inequality to (1.5), we have

$$L^{2(n+1)} \geq \frac{n^{2(n+1)} (n+1)^{\frac{(n+1)(2n-1)}{n}}}{2^{n+1}} (n! \cdot V)^{\frac{2(n+1)}{n}} \tag{1.6}$$

with equality holds if and only if Ω_n is regular.

The inequality (1.6) may be deemed isoperimetric inequality of an n-simplex. The deficit value between the right-hand side and left-hand side of inequality (1.6) can be considered to be the isopermetric deficit for Ω_n:

$$\Delta_1 = L^{2(n+1)} - \frac{n^{2(n+1)} (n+1)^{\frac{(n+1)(2n-1)}{n}}}{2^{n+1}} (n! \cdot V)^{\frac{2(n+1)}{n}}. \tag{1.7}$$

In addition, the volume V and the facet areas of the simplex Ω_n satisfy the following inequality:

$$(V)^{\frac{2}{n}} \leq [(n-1)!]^{\frac{2}{n-1}} \frac{(n+1)^{\frac{1}{n}}}{n^{\frac{1}{n-1}}} \left(\prod_{i=1}^{n+1} F_i \right)^{2(n^2-1)} \tag{1.8}$$

with equality holds if and only if Ω_n is regular (see [11]).

By applying the arithmetic-geometric mean inequality to (1.8), we have

$$F^{2(n^2-1)} \geq \left[\frac{n \cdot (n+1)^{\frac{1}{n}}}{(n-1)!^2} (n! \cdot V)^{\frac{2(n-1)}{n}} \right]^{n^2-1} \tag{1.9}$$

with equality holds if and only if Ω_n is regular.

The inequality (1.9) may be also called isoperimetric inequality for an n-simplex. The deficit value between the right-hand side and left-hand side of inequality (1.9) can be regarded as the other isopermetric deficit for the n-simplex Ω_n:

$$\Delta_2 = F^{2(n^2-1)} - \left[\frac{n \cdot (n+1)^{\frac{1}{n}}}{(n-1)!^2} (n! \cdot V)^{\frac{2(n-1)}{n}} \right]^{n^2-1}.$$

2 Main Results

Our main results are stated as follows.

Theorem 2.1 *Let Ω_n be an n-simplex. Then*

$$\Delta_1 \geq \frac{n^{2n}(n+1)^{2n+1}}{3 \times 2^{n+2}}(n! \cdot V)^2 \left(R - \frac{\sqrt{n}}{(n+1)^{\frac{n+1}{2n}}}(n! \cdot V)^{\frac{1}{n}}\right)^2 \tag{2.1}$$

with equality holds if and only if Ω_n is regular.

Theorem 2.2 *Let Ω_n be an n-simplex. Then*

$$\Delta_2 \geq \frac{n^{3(n^2-1)}(n+1)^{n^2+n-1}}{3 \times n!^{2n}} V^{2(n^2-n-1)} \left(R - \frac{\sqrt{n}}{(n+1)^{\frac{n+1}{2n}}}(n! \cdot V)^{\frac{1}{n}}\right)^2 \tag{2.2}$$

with equality holds if and only if Ω_n is regular.

Corollary 2.1 *Suppose that ABC is a triangle of area S with the side lengths a_1, a_2, a_3. Then*

$$P^6 - 2^3 3^{\frac{9}{2}} S^3 \geq 324 S^2 \left(R - \frac{2}{\sqrt[4]{27}}\sqrt{S}\right)^2 \tag{2.3}$$

with equality holds if and only if the triangle is regular, where $P = \frac{1}{2}(a_1 + a_2 + a_3)$.

Corollary 2.2 *For a tetrahedron $ABCD$, we have*

$$L^8 - 2^{12} 3^{\frac{32}{3}} V^{\frac{8}{3}} \geq 3^7 \times 2^{11} \left(R - \frac{2}{\sqrt[6]{243}}\sqrt[3]{V}\right)^2, \tag{2.4}$$

$$F^{16} - \frac{3^{16}}{2^{\frac{8}{3}}} V^{\frac{32}{3}} \geq 3^{17} \times 2^{20} \left(R - \frac{2}{\sqrt[6]{243}}\sqrt[3]{V}\right)^2, \tag{2.5}$$

and the equalities are attained if and only if the tetrahedron is regular, where F is the surface area of $ABCD$.

3 The Proofs of Theorems

To prove the above theorems, we need some lemmas.

Lemma 3.1[11] *For an n-simplex Ω_n, we have*

$$\sum_{1 \leq i < j \leq n+1} a_{ij}^2 \leq (n+1)^2 R^2, \tag{3.1}$$

$$\left(\prod_{i=1}^{\frac{1}{2}n(n+1)} a_i\right)^{\frac{4}{n}} \geq \frac{2^{n+1} n!^2}{n} V^2 \cdot R^2, \tag{3.2}$$

$$\left(\prod_{i=1}^{n+1} F_i\right)^{n-1} \geq \frac{n^{\frac{3n^2-4}{2}}}{(n+1)^{\frac{(n+1)(n-2)}{2}} n!^n} V^{n^2-n-1} \cdot R, \tag{3.3}$$

and the equalities are attained if and only if Ω_n is regular.

Lemma 3.2[12] *Let Ω_n be an n-simplex. Then*

$$R^2 \geq \frac{n}{(n+1)^{\frac{n+1}{n}}}(n!\cdot V)^{\frac{2}{n}} + \frac{1}{2(n+1)^2}\sum_{i=1}^{\frac{1}{2}n(n+1)}\left(a_i - \sqrt{\frac{2(n+1)}{n}}R\right)^2, \tag{3.4}$$

and the equality is attained if and only if Ω_n is regular.

Lemma 3.3 *Let Ω_n be an n-simplex. Then*

$$R^2 \geq \frac{n}{(n+1)^{\frac{n+1}{n}}}(n!\cdot V)^{\frac{2}{n}} + \frac{1}{(n+1)^2}\sum_{i=1}^{\frac{1}{2}n(n+1)}\left(a_i - \frac{\sqrt{2}}{(n+1)^{\frac{1}{2n}}}(n!\cdot V)^{\frac{1}{n}}\right)^2, \tag{3.5}$$

and the equality is attained if and only if Ω_n is regular.

Proof. By suitable calculation, we get

$$\begin{aligned}
&\sum_{i=1}^{\frac{1}{2}n(n+1)}\left(a_i - \frac{\sqrt{2}}{(n+1)^{\frac{1}{2n}}}(n!\cdot V)^{\frac{1}{n}}\right)^2 \\
= &\sum_{i=1}^{\frac{1}{2}n(n+1)} a_i^2 + \frac{2}{(n+1)^{\frac{1}{n}}}(n!\cdot V)^{\frac{2}{n}}\cdot\frac{1}{2}n(n+1) - \frac{2\sqrt{2}}{(n+1)^{\frac{1}{2n}}}(n!\cdot V)^{\frac{1}{n}}\sum_{i=1}^{\frac{1}{2}n(n+1)} a_i.
\end{aligned} \tag{3.6}$$

By (3.6), we have

$$\begin{aligned}
\sum_{i=1}^{\frac{1}{2}n(n+1)} a_i^2 &= \frac{2\sqrt{2}}{(n+1)^{\frac{1}{2n}}}(n!\cdot V)^{\frac{1}{n}}\sum_{i=1}^{\frac{1}{2}n(n+1)} a_i - \frac{2}{(n+1)^{\frac{1}{n}}}(n!\cdot V)^{\frac{2}{n}}\cdot\frac{1}{2}n(n+1) \\
&\quad + \sum_{i=1}^{\frac{1}{2}n(n+1)}\left(a_i - \frac{\sqrt{2}}{(n+1)^{\frac{1}{2n}}}(n!\cdot V)^{\frac{1}{n}}\right)^2 \\
&\geq n(n+1)^{\frac{n-1}{n}}(n!\cdot V)^{\frac{2}{n}} + \sum_{i=1}^{\frac{1}{2}n(n+1)}\left(a_i - \frac{\sqrt{2}}{(n+1)^{\frac{1}{2n}}}(n!\cdot V)^{\frac{1}{n}}\right)^2.
\end{aligned} \tag{3.7}$$

From (3.1) and (3.7), we get (3.5).

Lemma 3.4 *Let X, Y, Z be any real numbers. Then*

$$(X-Y)^2 \leq 2[(X-Z)^2 + (Y-Z)^2]. \tag{3.8}$$

Proof. By using the absolute value inequality and the arithmetic-geometric means inequality, we get

$$\begin{aligned}
(X-Y)^2 &= |X-Y|^2 \\
&\leq (|X-Z| + |Y-Z|)^2 \\
&= |X-Z|^2 + |Y-Z|^2 + 2|x-Z|\cdot|Y-Z| \\
&\leq 2[|X-Z|^2 + |Y-Z|^2] \\
&= 2[(X-Z)^2 + (Y-Z)^2].
\end{aligned}$$

The Proof of Theorem 2.1 By using the arithmetic-geometric means inequality, (3.2) and (3.4), we find that

$$\begin{aligned}
L^{2(n+1)} &= \left(\sum_{i=1}^{\frac{n(n+1)}{2}} a_i\right)^{2(n+1)} \\
&\geq \left(\frac{n(n+1)}{2}\right)^{2(n+1)} \left(\prod_{i=1}^{\frac{n(n+1)}{2}} a_i\right)^{\frac{4}{n}} \\
&\geq \left(\frac{n(n+1)}{2}\right)^{2(n+1)} \frac{2^{n+1}n!^2}{n} V^2 \cdot R^2 \\
&\geq \left(\frac{n(n+1)}{2}\right)^{2(n+1)} \frac{2^{n+1}n!^2}{n} V^2 \\
&\quad \cdot \left\{ \frac{(n!)^{\frac{2}{n}} n}{(n+1)^{\frac{n+1}{n}}} V^{\frac{2}{n}} + \frac{1}{2(n+1)^2} \sum_{i=1}^{\frac{n(n+1)}{2}} \left(a_i - \sqrt{\frac{2(n+1)}{n}} R\right)^2 \right\} \\
&= \frac{n^{2(n+1)}(n+1)^{\frac{(n+1)(2n-1)}{n}}}{2^{n+1}} (n! \cdot V)^{\frac{2(n+1)}{n}} \\
&\quad + \frac{n^{2n+1}(n+1)^{2n}}{2^{n+2}} (n! \cdot V)^2 \sum_{i=1}^{\frac{n(n+1)}{2}} \left(a_i - \sqrt{\frac{2(n+1)}{n}} R\right)^2. \qquad (3.9)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
L^{2(n+1)} &= \left(\sum_{i=1}^{\frac{n(n+1)}{2}} a_i\right)^{2(n+1)} \\
&\geq \left(\frac{n(n+1)}{2}\right)^{2(n+1)} \left(\prod_{i=1}^{\frac{n(n+1)}{2}} a_i\right)^{\frac{4}{n}} \\
&\geq \left(\frac{n(n+1)}{2}\right)^{2(n+1)} \frac{2^{n+1}n!^2}{n} V^2 \cdot R^2 \\
&\geq \left(\frac{n(n+1)}{2}\right)^{2(n+1)} \frac{2^{n+1}n!^2}{n} V^2 \\
&\quad \cdot \left\{ \frac{(n!)^{\frac{2}{n}} n}{(n+1)^{\frac{n+1}{n}}} V^{\frac{2}{n}} + \frac{1}{(n+1)^2} \sum_{i=1}^{\frac{1}{2}n(n+1)} \left(a_i - \frac{\sqrt{2}}{(n+1)^{\frac{1}{2n}}} (n! \cdot V)^{\frac{1}{n}}\right)^2 \right\} \\
&= \frac{n^{2(n+1)}(n+1)^{\frac{(n+1)(2n-1)}{n}}}{2^{n+1}} (n! \cdot V)^{\frac{2(n+1)}{n}} \\
&\quad + \frac{n^{2n+1}(n+1)^{2n}}{2^{n+1}} (n! \cdot V)^2 \sum_{i=1}^{\frac{1}{2}n(n+1)} \left(a_i - \frac{\sqrt{2}}{(n+1)^{\frac{1}{2n}}} (n! \cdot V)^{\frac{1}{n}}\right)^2. \qquad (3.10)
\end{aligned}$$

From (3.9) and (3.10), furthermore, applying (3.8), we obtain

$$
\begin{aligned}
3\Delta_1 \geq\ & \frac{n^{2n+1}(n+1)^{2n}}{2^{n+1}}(n!\cdot V)^2 \\
& \cdot \sum_{i=1}^{\frac{n(n+1)}{2}} \left[\left(a_i - \sqrt{\frac{2(n+1)}{n}}R\right)^2 + \left(a_i - \frac{\sqrt{2}}{(n+1)^{\frac{1}{2n}}}(n!\cdot V)^{\frac{1}{n}}\right)^2\right] \\
\geq\ & \frac{n^{2n+1}(n+1)^{2n}}{2^{n+1}}(n!\cdot V)^2 \sum_{i=1}^{\frac{n(n+1)}{2}} \frac{1}{2}\left(\sqrt{\frac{2(n+1)}{n}}R - \frac{\sqrt{2}}{(n+1)^{\frac{1}{2n}}}(n!\cdot V)^{\frac{1}{n}}\right)^2 \\
=\ & \frac{n^{2n}(n+1)^{2n+1}}{2^{n+2}}(n!\cdot V)^2\left(R - \frac{\sqrt{n}}{(n+1)^{\frac{n+1}{2n}}}(n!\cdot V)^{\frac{1}{n}}\right)^2.
\end{aligned}
$$

Thus equality (2.1) is valid. From Lemmas 3.1–3.4, it is easy to see that equality holds in (2.1) if and only if Ω_n is regular.

The Proof of Theorem 2.2 Similar to the proof of Theorem 2.1, by the arithmetic-geometric mean inequality, the inequalities (3.3), (3.4) and (3.5), it follows that

$$
\begin{aligned}
F^{2(n^2-1)} =\ & \left(\sum_{i=1}^{n+1} F_i\right)^{2(n^2-1)} \\
\geq\ & (n+1)^{2(n^2-1)}\left(\prod_{i=1}^{n+1} F_i\right)^{2(n-1)} \\
\geq\ & (n+1)^{2(n^2-1)}\left[\frac{n^{\frac{3n^2-4}{2}}}{(n+1)^{\frac{(n+1)(n-2)}{2}}n!^n}\right]^2 V^{2(n^2-n-1)}\cdot R^2 \\
\geq\ & \left[\frac{n\cdot(n+1)^{\frac{1}{n}}}{(n-1)!^2}(n!\cdot V)^{\frac{2(n-1)}{n}}\right]^{n^2-1} + \left[\frac{n^{\frac{3n^2-4}{2}}(n+1)^{\frac{n(n+1)}{2}}}{n!^n}\right]^2 V^{2(n^2-n-1)} \\
& \times \frac{1}{2(n+1)^2}\sum_{i=1}^{\frac{n(n+1)}{2}}\left(a_i - \sqrt{\frac{2(n+1)}{n}}R\right)^2. \qquad (3.11)
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
F^{2(n^2-1)} =\ & \left(\sum_{i=1}^{n+1} F_i\right)^{2(n^2-1)} \\
\geq\ & (n+1)^{2(n^2-1)}\left(\prod_{i=1}^{n+1} F_i\right)^{2(n-1)} \\
\geq\ & (n+1)^{2(n^2-1)}\left[\frac{n^{\frac{3n^2-4}{2}}}{(n+1)^{\frac{(n+1)(n-2)}{2}}n!^n}\right]^2 V^{2(n^2-n-1)}\cdot R^2 \\
\geq\ & \left[\frac{n\cdot(n+1)^{\frac{1}{n}}}{(n-1)!^2}(n!\cdot V)^{\frac{2(n-1)}{n}}\right]^{n^2-1} + \left[\frac{n^{\frac{3n^2-4}{2}}(n+1)^{\frac{n(n+1)}{2}}}{n!^n}\right]^2 V^{2(n^2-n-1)} \\
& \times \frac{1}{(n+1)^2}\sum_{i=1}^{\frac{1}{2}n(n+1)}\left(a_i - \frac{\sqrt{2}}{(n+1)^{\frac{1}{2n}}}(n!\cdot V)^{\frac{1}{n}}\right)^2. \qquad (3.12)
\end{aligned}
$$

From (3.11) and (3.12), furthermore, applying (3.8), we obtain

$$
\begin{aligned}
3\Delta_2 \geq & \left[\frac{n^{\frac{3n^2-4}{2}}(n+1)^{\frac{n(n+1)}{2}}}{n!^n(n+1)}\right]^2 V^{2(n^2-n-1)} \\
& \cdot \sum_{i=1}^{\frac{n(n+1)}{2}} \left[\left(a_i - \sqrt{\frac{2(n+1)}{n}}R\right)^2 + \left(a_i - \frac{\sqrt{2}}{(n+1)^{\frac{1}{2n}}}(n!\cdot V)^{\frac{1}{n}}\right)^2\right] \\
\geq & \left[\frac{n^{\frac{3n^2-4}{2}}(n+1)^{\frac{n(n+1)}{2}}}{n!^n(n+1)}\right]^2 V^{2(n^2-n-1)} \sum_{i=1}^{\frac{n(n+1)}{2}} \frac{1}{2}\left(\sqrt{\frac{2(n+1)}{n}}R - \frac{\sqrt{2}}{(n+1)^{\frac{1}{2n}}}(n!\cdot V)^{\frac{1}{n}}\right)^2 \\
= & \frac{n^{3(n^2-1)}(n+1)^{n^2+n-1}}{n!^{2n}} V^{2(n^2-n-1)}\left(R - \frac{\sqrt{n}}{(n+1)^{\frac{n+1}{2n}}}(n!\cdot V)^{\frac{1}{n}}\right)^2.
\end{aligned}
$$

Thus equality (2.2) is true. From Lemmas 3.1–3.4, it is easy to see that equality holds in (2.2) if and only if Ω_n is regular.

References

[1] Osserman R. Bonnesen-style isoperimetric inequalities. *Amer. Math. Monthly*, 1979, **86**: 1–29.

[2] Bokowski J, Heil E. Integral representation of quermassintegrals and Bonnesen-style inequalities. *Arch. Math.* (*Basel*), 1986, **47**(1): 79–89.

[3] Bonnesen T. Les Problèms des Isopérimètres et des Isépiphanes. Paris: Gauthier-Villars, 1929.

[4] Bonnesen T, Fenchel W. Theorie der konvexen Körper (German). Berichtigter Reprint. Berlin-New York: Springer-Verlag, 1974.

[5] Zhou J Z, Xia Y W, Zeng C N. Some new Bonnesen-style inequalities. *J. Korean Math. Soc.*, 2011, **48**(2): 421–430.

[6] Zhang G Y, Zhou J Z. Containment Measures in Integral Geometry. Integral Geometry and Convexity. Hackensack, NJ,: World Sci. Publ., 2006, 153–168.

[7] Martini H, Mustafaev Z. Extensions of a Bonnesen-style inequality to Minkowski spaces. *Math. Inequal. Appl.*, 2008, **11**: 739–748.

[8] Cianchi A, Pratelli A. On the isoperimetric deficit in Gauss space. *Amer. J. Math.*, 2011, **133**(1): 131–186.

[9] Figalli A, Maggi F, Pratelli A. A mass transportation approach to quantitative isoperimetric inequalities. *Invent. Math.*, 2010, **182**(1): 167–211.

[10] Zun S. Geometric Inequalities in China (in Chinese). Nanjing: Jiangsu Education Press, 1996.

[11] Mitrinović D S, Pečarić J E, Volenec V. Recent Advances in Geometric Inequalities. Mathematics and its Applications (East European Series), vol. 28. Dordrecht: Kluwer Academic Publishers Group, 1989.

[12] Wang W, Yang S G. On Bonnesen-style isoperimetric inequalities for n-simplices. *Math. Inequal. Appl.*, 2015, **18**(1): 133–144.