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Solvability for a Coupled System of Fractional p-Laplacian Differential Equations at Resonance

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Abstract: In this paper, by using the coincidence degree theory, the existence of solutions for a coupled system of fractional p-Laplacian differential equations at resonance is studied. The result obtained in this paper extends some known results. An example is given to illustrate our result.

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1 Introduction

In this paper, by using the coincidence degree theory, we discuss the existence of solutions to a coupled system of fractional p-Laplacian differential equations at resonance:

$$\begin{cases} D_{0^+}^{\beta}\phi_{p_1}(D_{0^+}^{\alpha}u(t)) = f_1(t,\ u(t),\ v(t),\ D_{0^+}^{\alpha}u(t),\ D_{0^+}^{\alpha}v(t)), \qquad 0<t<1,\\ D_{0^+}^{\beta}\phi_{p_2}(D_{0^+}^{\alpha}v(t)) = f_2(t,\ u(t),\ v(t),\ D_{0^+}^{\alpha}u(t),\ D_{0^+}^{\alpha}v(t)), \qquad 0<t<1,\\ u(0)=D_{0^+}^{\alpha}u(0)=0, \qquad u(1)=\sum\limits_{i=1}^{n_1}A_iu(\epsilon_i),\\ D_{0^+}^{\gamma}\phi_{p_1}(D_{0^+}^{\alpha}u(t))|_{t=1} = \sum\limits_{i=1}^{n}a_iD_{0^+}^{\gamma}\phi_{p_1}(D_{0^+}^{\alpha}u(t))|_{t=\xi_i},\\ v(0)=D_{0^+}^{\alpha}v(0)=0, \qquad v(1)=\sum\limits_{i=1}^{m_1}B_iv(\sigma_i),\\ D_{0^+}^{\delta}\phi_{p_2}(D_{0^+}^{\alpha}v(t))|_{t=1} = \sum\limits_{i=1}^{m}b_iD_{0^+}^{\delta}\phi_{p_2}(D_{0^+}^{\alpha}v(t))|_{t=\eta_i}, \end{cases} \tag{1.1}$$

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where $1 < \alpha, \beta \leq 2$, and $3 < \alpha + \beta \leq 4$; $0 < \gamma,\ \delta \leq \beta - 1$; $\phi_{p_i}(x) = |x|^{p_i - 2}x$, $p_i > 1$, $\phi_{q_i} = \phi_{p_i}^{-1}$, $\dfrac{1}{p_i} + \dfrac{1}{q_i} = 1$, $i = 1, 2$; $0 < \epsilon_1 < \epsilon_2 < \cdots < \epsilon_{n_1} < 1$, $0 < \sigma_1 < \sigma_2 < \cdots < \sigma_{m_1} < 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_m < 1$; $A_r,\ a_j,\ B_k,\ b_l \in \mathbf{R}$, $r = 1, 2, \cdots, n_1$, $j = 1, 2, \cdots, n$, $k = 1, 2, \cdots, m_1$, $l = 1, 2, \cdots, m$. D^α, D^β, D^γ and D^δ are the standard Riemann-Liouville fractional derivatives.

In this paper, we always suppose that the following conditions hold.

(H_1) $\sum\limits_{i=1}^{n_1} A_i \epsilon_i^{\alpha-1} = 1$, $\sum\limits_{i=1}^{m_1} B_i \sigma_i^{\alpha-1} = 1$, $\sum\limits_{i=1}^{n} a_i \xi_i^{\beta-\gamma-1} = 1$, $\sum\limits_{i=1}^{m} b_i \eta_i^{\beta-\delta-1} = 1$, $\sum\limits_{i=1}^{n_1} A_i \epsilon_i^{\alpha} \neq 1$, $\sum\limits_{i=1}^{m_1} B_i \sigma_i^{\alpha} \neq 1$, $\sum\limits_{i=1}^{n} a_i \xi_i^{\beta-\gamma} \neq 1$, $\sum\limits_{i=1}^{m} b_i \eta_i^{\beta-\delta} \neq 1$.

(H_2) f_i: $[0, 1] \times \mathbf{R}^4 \to \mathbf{R}$ satisfied Carathéodory conditions, $i = 1, 2$, that is,

(i) $f(\,\cdot\,;\ x_1, x_2, x_3, x_4)$: $[0, 1] \to \mathbf{R}$ is measurable for all $(x_1, x_2, x_3, x_4) \in \mathbf{R}^4$;

(ii) $f(t;\ \cdot\,,\ \cdot\,,\ \cdot\,,\ \cdot\,)$: $\mathbf{R}^4 \to \mathbf{R}$ is continuous for a.e. $t \in [0, 1]$;

(iii) for each compact set $\mathcal{K} \subset \mathbf{R}^4$ there is a function $\varphi_{\mathcal{K}} \in L^\infty[0, 1]$ such that

$$|f(t, x_1, x_2, x_3, x_4)| \leq \varphi_{\mathcal{K}}(t)$$

for a.e. $t \in [0, 1]$ and all $(x_1, x_2, x_3, x_4) \in \mathcal{K}$.

The existence of solutions for boundary value problem of integer order differential equations at resonance has been studied by many authors (see [1]–[10] and references cited therein). Since the extensive applicability of fractional differential equations (see [11] and [12]), recently, more and more authors pay their close attention to the boundary value problems of fractional differential equations (see [13]–[20]). In papers [13] and [14], the existence of solutions to coupled system of fractional differential equations at nonresonance has been given. In papers [15] and [16], the solvability of fractional differential equations at resonance has been investigated.

Paper [16] investigates the following coupled system of fractional differential equations at resonance:

$$\begin{cases} D_{0+}^{\alpha} u(t) = f_1(t,\ u(t),\ v(t)), & 0 < t < 1, \\ D_{0+}^{\beta} v(t) = f_2(t,\ u(t),\ v(t)), & 0 < t < 1, \\ u(0) = 0, \quad D_{0+}^{\gamma} u(t)|_{t=1} = \sum\limits_{i=1}^{n} a_i D_{0+}^{\gamma} u(t)|_{t=\xi_i}, & \\ v(0) = 0, \quad D_{0+}^{\delta} v(t)|_{t=1} = \sum\limits_{i=1}^{m} b_i D_{0+}^{\delta} v(t)|_{t=\eta_i}, & \end{cases} \tag{1.2}$$

where $1 < \alpha, \beta \leq 2$, $0 < \gamma \leq \alpha - 1$, $\delta \leq \beta - 1$; $0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_m < 1$; $\sum\limits_{i=1}^{n} a_i \xi_i^{\beta-\gamma-1} = 1$, $\sum\limits_{i=1}^{m} b_i \eta_i^{\beta-\delta-1} = 1$. By using the coincidence degree theory due to Mawhin and constructing suitable operators, the existence of solutions for (1.2) is obtained.

In the past few decades, in order to meet the demands of research, the p-Laplacian equation is introduced in some BVP, such as [17] and [18].

The turbulent flow in a porous medium is a fundamental mechanics problem. For studying this type of problems, Leibenson[17] introduced the p-Laplacian equation as follows

$$(\phi_p(x'(t)))' = f(t,\ x(t),\ x(t)),$$

where $\phi_p(s) = |s|^{p-2}s,\ p > 1$. Obviously, ϕ_p is invertible and its inverse operator is ϕ_q, where $q > 1$ is a constant such that $\frac{1}{p} + \frac{1}{q} = 1$.

Paper [18] investigated the existence of solutions for the BVP of fractional p-Laplacian equation with the following form

$$\begin{cases} D^{\beta}_{0^+}(\phi_p(D^{\alpha}_{0^+}x(t))) = f(t,\ x(t),\ D^{\alpha}_{0^+}x(t)), & t \in [0,1], \\ D^{\alpha}_{0^+}x(0) = D^{\alpha}_{0^+}x(1) = 0, \end{cases}$$

where $0 < \alpha, \beta \le 1$ with $1 < \alpha+\beta \le 2$, and $p > 1$, $\phi_p(s) = |s|^{p-2}s$ is a p-Laplacian operator, D^{α} is a Caputo fractional derivative. By using the coincidence degree theory, a new result on the existence of solutions for the above fractional boundary value problem is obtained.

Inspired by above works, our work presented in this paper has the following new features. On the one hand, the method used in this paper is the coincidence degree theory and the system has p-Laplacian, which bring about many argument difficulties. On the other hand, our study is on fractional p-Laplacian differential system with multipoint boundary conditions. To the best of our knowledge, there are relatively few results on boundary value problems for fractional p-Laplacian equations at resonance. We fill this gap in the literature. Hence we improve and generalize the results of previous papers to some degree, and so it is interesting and important to study the existence of solutions for system (1.1).

This paper is organized as follows. In Section 2, we present the transformation of the system (1.1), some results of fractional calculus theory and some lemmas, which are used in the next two sections. In Section 3, basing on the coincidence degree theory of Mawhin[19], we get the existence of solutions for system (1.1). In Section 4, one example is given to illustrate our result. Our result is different from those of bibliographies listed above.

2 Preliminaries

For abbreviation, we write $D^{\gamma}_{0^+}u(\xi) = D^{\gamma}_{0^+}u(t)|_{t=\xi}$.

In order to use the coincidence degree theory to study the existence of solutions for BVP (1.1), let $w_1(t) = \phi_{p_1}(D^{\alpha}_{0^+}u(t))$, $w_2(t) = \phi_{p_2}(D^{\alpha}_{0^+}v(t))$. Then we can rewrite (1.1) in the following form:

$$\begin{cases} D^{\alpha}_{0^+}u(t) = \phi_{q_1}(w_1(t)), \quad 0 < t < 1, \\ D^{\alpha}_{0^+}v(t) = \phi_{q_2}(w_2(t)), \quad 0 < t < 1, \\ D^{\beta}_{0^+}w_1(t) = f_1(t,\ u(t),\ v(t),\ \phi_{q_1}(w_1(t)),\ \phi_{q_2}(w_2(t))), \quad 0 < t < 1, \\ D^{\beta}_{0^+}w_2(t) = f_2(t,\ u(t),\ v(t),\ \phi_{q_1}(w_1(t)),\ \phi_{q_2}(w_2(t))), \quad 0 < t < 1, \\ u(0) = w_1(0) = 0, \quad u(1) = \sum\limits_{i=1}^{n_1} A_i u(\epsilon_i), \quad D^{\gamma}_{0^+}w_1(t)|_{t=1} = \sum\limits_{i=1}^{n} a_i D^{\gamma}_{0^+}w_1(t)|_{t=\xi_i}, \\ v(0) = w_2(0) = 0, \quad v(1) = \sum\limits_{i=1}^{m_1} B_i v(\sigma_i), \quad D^{\delta}_{0^+}w_2(t)|_{t=1} = \sum\limits_{i=1}^{m} b_i D^{\delta}_{0^+}w_2(t)|_{t=\eta_i}. \end{cases} \tag{2.1}$$

Clearly, if $\boldsymbol{x}(t) = (u(t),\ v(t),\ w_1(t),\ w_2(t))^{\mathrm{T}}$ is a solution of (2.1), then $(u(t),\ v(t))^{\mathrm{T}}$ must be a solution of (1.1). So the problem of finding a solution for (1.1) is converted to find a solution for (2.1).

Next we present here the necessary definitions and Lemmas from fractional calculus theory. These definitions and Lemmas can be found in the recent literatures [11] and [12].

Definition 2.1[11] *The fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbf{R}$ is defined by*

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s$$

provided that the right-hand side exists.

Definition 2.2[11] *The Riemann-Liouville fractional order derivative of order $\alpha \in (n-1,\ n]$ of a function $f : (0, \infty) \to \mathbf{R}$ is defined by*

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t (t-s)^{n-\alpha-1} f(s)\mathrm{d}s$$

provided that the right-hand side exists.

Lemma 2.1[12] *Assume that $f \in L[0,\ 1]$, $q > p \geq 0$. Then*

$$D_{0+}^{p} I_{0+}^{q} f(t) = I_{0+}^{q-p} f(t).$$

Lemma 2.2[12] *Assume that $\alpha > 0$, $\lambda > -1$. Then*

$$D_{0+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(n+\lambda-\alpha+1)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} (t^{n+\lambda-\alpha}),$$

where $n = [\alpha] + 1$.

Lemma 2.3[12] *Let $\alpha \in (n-1,\ n]$, $u \in C(0,\ 1) \cap L^1(0,\ 1)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in \mathbf{R}$, $i = 1, 2, \cdots, n$.

Now, we briefly recall some notations and an abstract existence result, which can be found in [19].

Let X, Y be real Banach spaces, $L : \mathrm{dom}L \subset X \to Y$ be a Fredholm operator with index zero, and $P : X \to X$, $Q : Y \to Y$ be projectors such that $\mathrm{Im}P = \ker L$, $\mathrm{Im}L = \ker Q$. then

$$X = \ker L \oplus \ker P, \qquad Y = \mathrm{Im}L \oplus \mathrm{Im}Q,$$

and

$$L|_{\mathrm{dom}L \cap \ker P} : \mathrm{dom}L \cap \ker P \to \mathrm{Im}L$$

is invertible. We denote the inverse by K_P.

If $\varOmega$ is an open bounded subset of X such that $\mathrm{dom}L \cap \bar{\varOmega} \neq \emptyset$, then the map $N : X \to Y$ will be called L-compact on $\bar{\varOmega}$ if $QN(\bar{\varOmega})$ is bounded and $K_P(I-Q)N : \bar{\varOmega} \to X$ is compact.

Lemma 2.4[19] *Let $L:\mathrm{dom}L\subset X\to Y$ be a Fredholm operator with index zero and $N:X\to Y$ be L-compact on $\bar{\Omega}$. Assume that the following conditions are satisfied:*

(1) *$Lx\neq\lambda Nx$ for every $(x,\ \lambda)\in[(\mathrm{dom}L\backslash\ker L)\cap\partial\Omega]\times(0,\ 1)$;*

(2) *$Nx\notin\mathrm{Im}L$ for every $x\in\ker L\cap\partial\Omega$;*

(3) *$\deg(JQN|_{\ker L},\ \Omega\cap\ker L,\ 0)\neq 0$, where $J:\mathrm{Im}Q\to\ker L$ is a isomorphism, $Q:Y\to Y$ is a projection such that $\mathrm{Im}L=\ker Q$.*

Then the equation $Lx=Nx$ has at least one solution in $\mathrm{dom}L\cap\bar{\Omega}$.

Let

$$C^{\alpha-1}[0,\ 1]=\{x\mid x,\ D_{0^+}^{\alpha-1}x\in C[0,\ 1]\}$$

with norm $\|x\|_\alpha=\max\{\|x\|_\infty,\ \|D_{0^+}^{\alpha-1}x\|_\infty\}$,

$$C^{\beta-1}[0,\ 1]=\{x\mid x,\ D_{0^+}^{\beta-1}x\in C[0,\ 1]\}$$

with norm $\|x\|_\beta=\max\{\|x\|_\infty,\ \|D_{0^+}^{\beta-1}x\|_\infty\}$, where $\|x\|_\infty=\max\limits_{t\in[0,1]}|x(t)|$.

Set

$$X=\{\boldsymbol{x}=(u(\cdot),\ v(\cdot),\ w_1(\cdot),\ w_2(\cdot))^{\mathrm{T}}\mid u,v\in C^{\alpha-1}[0,\ 1],\ w_1,w_2\in C^{\beta-1}[0,\ 1]\}$$

with norm

$$\|\boldsymbol{x}\|_X=max\{\|u\|_\alpha,\ \|v\|_\alpha,\ \|w_1\|_\beta,\ \|w_2\|_\beta\},$$

and

$$Y=\{\boldsymbol{y}=(y_1(\cdot),\ y_2(\cdot),\ y_3(\cdot),\ y_4(\cdot))^{\mathrm{T}}\in L([0,\ 1],\ \mathbf{R}^4)\}$$

with norm

$$\|\boldsymbol{y}\|_Y=\max\{\|y_1\|_1,\ \|y_2\|_1,\ \|y_3\|_1,\ \|y_4\|_1\},$$

where $\|y_i\|_1=\displaystyle\int_0^1|y_i(x)|\mathrm{d}x$, $i=1,2,3,4$. By means of the linear functional analysis theory, we can prove that X, Y are Banach spaces.

Define the operator $L:\mathrm{dom}L\subset X\to Y$ by

$$L\boldsymbol{x}=\begin{pmatrix}D_{0^+}^{\alpha}u\\ D_{0^+}^{\alpha}v\\ D_{0^+}^{\beta}w_1\\ D_{0^+}^{\beta}w_2\end{pmatrix},\tag{2.2}$$

where

$$\begin{aligned}\mathrm{dom}L=\Big\{&\boldsymbol{x}=(u(\cdot),\ v(\cdot),\ w_1(\cdot),\ w_2(\cdot))^{\mathrm{T}}\in X\mid(D_{0^+}^{\alpha}u,\ D_{0^+}^{\alpha}v,\ D_{0^+}^{\beta}w_1,\ D_{0^+}^{\beta}w_2))^{\mathrm{T}}\in Y,\\ &u(0)=v(0)=w_1(0)=w_2(0)=0,\ u(1)=\sum_{i=1}^{n_1}A_iu(\epsilon_i),\ v(1)=\sum_{i=1}^{m_1}B_iv(\sigma_i),\\ &D_{0^+}^{\gamma}w_1(t)|_{t=1}=\sum_{i=1}^{n}a_iD_{0^+}^{\gamma}w_1(t)|_{t=\xi_i},\ D_{0^+}^{\delta}w_2(t)|_{t=1}=\sum_{i=1}^{m}b_iD_{0^+}^{\delta}w_2(t)|_{t=\eta_i}\Big\}.\end{aligned}$$

Let $N:X\to Y$ be the operator

$$N\boldsymbol{x}(t)=\begin{pmatrix} N_1\boldsymbol{x}(t)\\ N_2\boldsymbol{x}(t)\\ N_3\boldsymbol{x}(t)\\ N_4\boldsymbol{x}(t)\end{pmatrix}=\begin{pmatrix} \phi_{q_1}(w_1(t))\\ \phi_{q_2}(w_2(t))\\ f_1(t,\ u(t),\ v(t),\ \phi_{q_1}(w_1(t)),\ \phi_{q_2}(w_2(t)))\\ f_2(t,\ u(t),\ v(t),\ \phi_{q_1}(w_1(t)),\ \phi_{q_2}(w_2(t)))\end{pmatrix}. \tag{2.3}$$

Then BVP (2.1) is equivalent to the operator equation

$$L\boldsymbol{x}=N\boldsymbol{x}.$$

3 Main Results

Define operators $T_1,\ T_2,\ T_3,\ T_4 : L[0,\ 1]\to\mathbf{R}$ as follows:

$$T_1y=\int_0^1(1-s)^{\alpha-1}\boldsymbol{y}(s)\mathrm{d}s-\sum_{i=1}^{n_1}A_i\int_0^{\epsilon_i}(\epsilon_i-s)^{\alpha-1}\boldsymbol{y}(s)\mathrm{d}s,$$

$$T_2y=\int_0^1(1-s)^{\alpha-1}\boldsymbol{y}(s)\mathrm{d}s-\sum_{i=1}^{m_1}B_i\int_0^{\sigma_i}(\sigma_i-s)^{\alpha-1}\boldsymbol{y}(s)\mathrm{d}s,$$

$$T_3y=\int_0^1(1-s)^{\beta-\gamma-1}\boldsymbol{y}(s)\mathrm{d}s-\sum_{i=1}^{n}a_i\int_0^{\xi_i}(\xi_i-s)^{\beta-\gamma-1}\boldsymbol{y}(s)\mathrm{d}s,$$

$$T_4y=\int_0^1(1-s)^{\beta-\delta-1}\boldsymbol{y}(s)\mathrm{d}s-\sum_{i=1}^{m}b_i\int_0^{\eta_i}(\eta_i-s)^{\beta-\delta-1}\boldsymbol{y}(s)\mathrm{d}s.$$

In order to obtain our main results, we first present the following lemmas.

Lemma 3.1 *Suppose that* (H_1) *holds, and let* L *be defined by* (2.2). *Then*

$$\ker L=\Big\{(u,v,w_1,w_2)^{\mathrm{T}}=(c_{11}t^{\alpha-1},\ c_{12}t^{\alpha-1},\ c_{21}t^{\beta-1},\ c_{22}t^{\beta-1})^{\mathrm{T}},\quad c_{11},\ c_{12},\ c_{21},\ c_{22}\in\mathbf{R},\ t\in[0,\ 1]\Big\}, \tag{3.1}$$

$$\mathrm{Im}L=\{(y_1,\ y_2,\ y_3,\ y_4)^{\mathrm{T}}\in Y\mid T_1y_1=T_2y_2=T_3y_3=T_4y_4=0\}. \tag{3.2}$$

Proof. Since $(u,v,w_1,w_2)^{\mathrm{T}}\in\ker L$, we get

$$(D_{0+}^{\alpha}u(t),\ D_{0+}^{\alpha}v(t),\ D_{0+}^{\beta}w_1(t),\ D_{0+}^{\beta}w_2(t))^{\mathrm{T}}=(0,0,0,0)^{\mathrm{T}}.$$

By Lemma 2.3, $D_{0+}^{\alpha}u(t)=0$ has solution

$$u(t)=c_{11}t^{\alpha-1}+e_{11}t^{\alpha-2},\qquad c_{11},e_{11}\in\mathbf{R}.$$

Combining with the boundary value condition $u(0)=0$, we get $e_{11}=0$. So

$$u(t)=c_{11}t^{\alpha-1}.$$

Likewisely,

$$v(t)=c_{12}t^{\alpha-1}.$$

Similarly, by Lemma 2.3, $D_{0+}^{\beta}w_1(t)=0$ has solution

$$w_1(t)=c_{21}t^{\beta-1}+e_{21}t^{\beta-2},\qquad c_{21},e_{21}\in\mathbf{R}.$$

Together with the boundary value condition $w_1(0)=0$, we get $e_{21}=0$. So

$$w_1(t)=c_{21}t^{\beta-1}.$$

Likewisely,

$$w_2(t) = c_{22}t^{\beta-1}.$$

One has that (3.1) holds.

If $\boldsymbol{y} = (y_1,\ y_2,\ y_3,\ y_4)^{\mathrm{T}} \in \mathrm{Im}L$, then there exists an $\boldsymbol{x} = (u,\ v,\ w_1,\ w_2)^{\mathrm{T}} \in \mathrm{dom}L$ such that $L\boldsymbol{x} = \boldsymbol{y}$. That is,

$$y_1(t) = D^{\alpha}_{0+}u(t), \quad y_2(t) = D^{\alpha}_{0+}v(t), \quad y_3(t) = D^{\alpha}_{0+}w_1(t), \quad y_4(t) = D^{\alpha}_{0+}w_2(t).$$

Basing on Lemma 2.3, we have

$$u(t) = I^{\alpha}_{0+}y_1(t) + c_1t^{\alpha-1} + e_2t^{\alpha-2}, \qquad c_1, e_1 \in \mathbf{R}.$$

From condition $u(0) = 0$, we get $e_1 = 0$. It follows from (H_1) and the boundary conditions of $u(1) = \sum\limits_{i=1}^{n_1} A_i u(\epsilon_i)$ that y_1 satisfies

$$\int_0^1 (1-s)^{\alpha-1}y_1(s)\mathrm{d}s = \sum_{i=1}^{n_1} A_i \int_0^{\epsilon_i} (\epsilon_i - s)^{\alpha-1}y_1(s)\mathrm{d}s.$$

Likewisely, y_2 satisfies

$$\int_0^1 (1-s)^{\alpha-1}y_2(s)\mathrm{d}s = \sum_{i=1}^{m_1} B_i \int_0^{\sigma_i} (\sigma_i - s)^{\alpha-1}y_2(s)\mathrm{d}s.$$

Similarly, by Lemma 2.3, we have

$$w_1(t) = I^{\beta}_{0+}y_3(t) + c_2t^{\beta-1} + e_2t^{\beta-2}, \qquad c_2, e_2 \in \mathbf{R}.$$

From condition $w_1(0) = 0$, we get $e_2 = 0$. It follows from (H_1) and the boundary conditions of $D^{\gamma}_{0+}w_1(1) = \sum\limits_{i=1}^{n} a_i D^{\gamma}_{0+}w_1(\xi)$ that y_3 satisfies

$$\int_0^1 (1-s)^{\beta-\gamma-1}y_3(s)\mathrm{d}s = \sum_{i=1}^{n} a_i \int_0^{\xi_i} (\xi_i - s)^{\beta-\gamma-1}y_3(s)\mathrm{d}s.$$

Likewisely, y_4 satisfies

$$\int_0^1 (1-s)^{\beta-\delta-1}y_4(s)\mathrm{d}s = \sum_{i=1}^{m} b_i \int_0^{\eta_i} (\eta_i - s)^{\beta-\delta-1}y_4(s)\mathrm{d}s.$$

So,

$$T_1y_1 = T_2y_2 = T_3y_3 = T_4y_4 = 0.$$

That is,

$$\mathrm{Im}L \subseteq \{(\boldsymbol{x},\ \boldsymbol{y}) \in \{(y_1,\ y_2,\ y_3,\ y_4)^{\mathrm{T}} \in Y \mid T_1y_1 = T_2y_2 = T_3y_3 = T_4y_4 = 0\}.$$

On the other hand, let $\boldsymbol{y} = (y_1,\ y_2,\ y_3,\ y_4)^{\mathrm{T}} \in Y$ satisfy $T_1y_1 = T_2y_2 = T_3y_3 = T_4y_4 = 0$, and $\boldsymbol{x} = (u, v, w_1, w_1)^{\mathrm{T}}$. Take

$$u(t) = I^{\alpha}_{0+}y_1, \quad v(t) = I^{\alpha}_{0+}y_2, \quad w_1(t) = I^{\beta}_{0+}y_3(t), \quad w_2(t) = I^{\beta}_{0+}y_4(t).$$

It follows from Lemma 2.1 that

$$L\boldsymbol{x} = \boldsymbol{y}.$$

Obviously, $(u,\ v,\ w_1,\ w_2)^{\mathrm{T}} \in X$, $(D^{\alpha}_{0+}u,\ D^{\alpha}_{0+}v,\ D^{\alpha}_{0+}w_1,\ D^{\alpha}_{0+}w_2)^{\mathrm{T}} \in Y$, and

$$u(0) = v(0) = w_1(0) = w_2(0) = 0.$$

By $T_1y_1 = T_2y_2 = T_3y_3 = T_4y_4 = 0$, we get that $u,\ v,\ w_1,\ w_2$ satisfy

$$u(1) = \sum_{i=1}^{n_1} A_i u(\epsilon_i), \qquad v(1) = \sum_{i=1}^{m_1} B_i v(\sigma_i),$$

$$D_{0^+}^{\gamma} w_1(1) = \sum_{i=1}^{n} a_i D_{0^+}^{\gamma} w_1(\xi_i), \qquad D_{0^+}^{\delta} w_2(1) = \sum_{i=1}^{m} b_i D_{0^+}^{\delta} \phi_{p_2} w_2(\eta_i),$$

respectively. So, $(u,\ v,\ w_1,\ w_2)^{\mathrm{T}} \in \mathrm{dom}L$, we get $(y_1,\ y_2,\ y_3,\ y_4)^{\mathrm{T}} \in \mathrm{Im}L$. That is,

$$\big\{(\boldsymbol{x},\ \boldsymbol{y}) \in \{(y_1,\ y_2,\ y_3,\ y_4)^{\mathrm{T}} \in Y \mid T_1y_1 = T_2y_2 = T_3y_3 = T_4y_4 = 0\big\} \subseteq \mathrm{Im}L.$$

The proof of Lemma 3.1 is completed.

Lemma 3.2 *Let L be defined by (2.2). If (H_1) holds, then L is a Fredholm operator of index zero, and the linear continuous projector operators $P : X \to X$ and $Q : Y \to Y$ can be defined as*

$$P\begin{pmatrix} u(t) \\ v(t) \\ w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} \dfrac{D_{0^+}^{\alpha-1}u(0)}{\Gamma(\alpha)} t^{\alpha-1} \\ \dfrac{D_{0^+}^{\alpha-1}v(0)}{\Gamma(\alpha)} t^{\alpha-1} \\ \dfrac{D_{0^+}^{\beta-1}w_1(0)}{\Gamma(\beta)} t^{\beta-1} \\ \dfrac{D_{0^+}^{\beta-1}w_2(0)}{\Gamma(\beta)} t^{\beta-1} \end{pmatrix},$$

$$Q\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{pmatrix} = \begin{pmatrix} Q_1y_1(t) \\ Q_2y_2(t) \\ Q_3y_3(t) \\ Q_4y_4(t) \end{pmatrix} = \begin{pmatrix} \dfrac{\alpha}{1 - \sum\limits_{i=1}^{n_1} A_i \epsilon_i^{\alpha}} T_1y_1(t) \\ \dfrac{\alpha}{1 - \sum\limits_{i=1}^{m_1} B_i \sigma_i^{\alpha}} T_2y_2(t) \\ \dfrac{\beta-\gamma}{1 - \sum\limits_{i=1}^{n} a_i \xi_i^{\beta-\gamma}} T_3y_3(t) \\ \dfrac{\beta-\delta}{1 - \sum\limits_{i=1}^{m} b_i \eta_i^{\beta-\delta}} T_4y_4(t) \end{pmatrix}$$

for $t \in [0,\ 1]$, and the operator $K_P : \mathrm{Im}L \to \mathrm{dom}L \cap \ker P$ can be written as

$$K_P\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} K_{P_1}y_1 \\ K_{P_2}y_2 \\ K_{P_3}y_3 \\ K_{P_4}y_4 \end{pmatrix} = \begin{pmatrix} I_{0^+}^{\alpha}y_1 \\ I_{0^+}^{\alpha}y_2 \\ I_{0^+}^{\beta}y_3 \\ I_{0^+}^{\beta}y_4 \end{pmatrix},$$

where K_P is the inverse of $L|_{\mathrm{dom}L\cap\ker P}$.

Proof. We divide the proof into two steps.

Step 1. We prove that L is a Fredholm operator of index zero.

(I) Since Lemma 3.1, we know

$$\ker L = \big\{(u,\ v,\ w_1,\ w_2)^{\mathrm{T}} = (c_{11}t^{\alpha-1},\ c_{12}t^{\alpha-1},\ c_{21}t^{\beta-1},\ c_{22}t^{\beta-1})^{\mathrm{T}},\ c_{11},\ c_{12},\ c_{21},\ c_{22} \in \mathbf{R},\ t \in [0,\ 1]\big\}.$$

By $u(t) = c_{11}t^{\alpha-1}$ and Lemma 2.2, we get

$$D_{0^+}^{\alpha-1}u(0) = c_{11}\Gamma(\alpha).$$

So

$$c_{11} = \frac{D_{0^+}^{\alpha-1}u(0)}{\Gamma(\alpha)}.$$

Likewisely,

$$c_{12} = \frac{D_{0^+}^{\alpha-1}v(0)}{\Gamma(\alpha)}, \qquad c_{21} = \frac{D_{0^+}^{\beta-1}w_1(0)}{\Gamma(\beta)}, \qquad c_{22} = \frac{D_{0^+}^{\beta-1}w_2(0)}{\Gamma(\beta)}.$$

So $\mathrm{Im}P = \ker L$.

We show that $P^2(u,\ v,\ w_1,\ w_2)^{\mathrm{T}} = P(u,\ v,\ w_1,\ w_2)^{\mathrm{T}}$ in the follows. In fact, by Lemma 2.2, we get

$$\begin{aligned}
P^2\begin{pmatrix} u(t)\\ v(t)\\ w_1(t)\\ w_2(t)\end{pmatrix} &= P\left(P\begin{pmatrix} u(t)\\ v(t)\\ w_1(t)\\ w_2(t)\end{pmatrix}\right) = P\begin{pmatrix} c_{11}t^{\alpha-1}\\ c_{12}t^{\alpha-1}\\ c_{21}t^{\beta-1}\\ c_{22}t^{\beta-1}\end{pmatrix}\\
&= \begin{pmatrix} \dfrac{D_{0^+}^{\alpha-1}(c_{11}t^{\alpha-1})|_{t=0}}{\Gamma(\alpha)}t^{\alpha-1}\\ \dfrac{D_{0^+}^{\alpha-1}(c_{12}t^{\alpha-1})|_{t=0}}{\Gamma(\alpha)}t^{\alpha-1}\\ \dfrac{D_{0^+}^{\beta-1}(c_{21}t^{\beta-1})|_{t=0}}{\Gamma(\beta)}t^{\beta-1}\\ \dfrac{D_{0^+}^{\beta-1}(c_{22}t^{\beta-1})|_{t=0}}{\Gamma(\beta)}t^{\beta-1}\end{pmatrix} = \begin{pmatrix} \dfrac{c_{11}\Gamma(\alpha)}{\Gamma(\alpha)}t^{\alpha-1}\\ \dfrac{c_{12}\Gamma(\alpha)}{\Gamma(\alpha)}t^{\alpha-1}\\ \dfrac{c_{21}\Gamma(\beta)}{\Gamma(\beta)}t^{\beta-1}\\ \dfrac{c_{22}\Gamma(\beta)}{\Gamma(\beta)}t^{\beta-1}\end{pmatrix} = \begin{pmatrix} c_{11}t^{\alpha-1}\\ c_{12}t^{\alpha-1}\\ c_{21}t^{\beta-1}\\ c_{22}t^{\beta-1}\end{pmatrix}\\
&= P\begin{pmatrix} u(t)\\ v(t)\\ w_1(t)\\ w_2(t)\end{pmatrix}, \qquad t \in [0,\ 1].
\end{aligned}$$

Then P is the linear continuous projector operator. So, we have $X = \ker L \oplus \ker P$.

(II) For $\boldsymbol{y} = (y_1,\ y_2,\ y_3,\ y_4)^{\mathrm{T}} \in Y$, we prove $Q^2\boldsymbol{y} = Q\boldsymbol{y}$, that is, $Q_i^2 y_i = Q_i y_i$, $i = 1, 2, 3, 4$. In fact,

$$\begin{aligned}
Q_1^2 y_1(t) &= Q_1(Q_1 y_1(t))\\
&= \frac{\alpha}{1 - \sum\limits_{i=1}^{n_1} A_i\epsilon_i^{\alpha}} T_1(Q_1 y_1(t))\\
&= \frac{\alpha}{1 - \sum\limits_{i=1}^{n_1} A_i\epsilon_i^{\alpha}} \left(\int_0^1 (1-s)^{\alpha-1}Q_1 y_1(t)\mathrm{d}s - \sum_{i=1}^{n_1} A_i \int_0^{\epsilon_i} (\epsilon_i - s)^{\alpha-1}Q_1 y_1(t)\mathrm{d}s\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha}{1-\sum\limits_{i=1}^{n_1} A_i\epsilon_i^{\alpha}}\left(\int_0^1 (1-s)^{\alpha-1}\mathrm{d}s - \sum_{i=1}^{n_1} A_i \int_0^{\epsilon_i} (\epsilon_i - s)^{\alpha-1}\mathrm{d}s\right) Q_1 y_1(t)\\
&= \frac{\alpha}{1-\sum\limits_{i=1}^{n_1} A_i\epsilon_i^{\alpha}}\left(\frac{1}{\alpha} - \frac{\sum\limits_{i=1}^{n_1} A_i\epsilon_i^{\alpha}}{\alpha}\right) Q_1 y_1(t)\\
&= Q_1 y_1(t).
\end{aligned}
$$

$$
\begin{aligned}
Q_3^2 y_3(t) &= Q_3(Q_3 y_3(t))\\
&= \frac{\beta-\gamma}{1-\sum\limits_{i=1}^{n} a_i\xi_i^{\beta-\gamma}} T_3(Q_3 y_3(t))\\
&= \frac{\beta-\gamma}{1-\sum\limits_{i=1}^{n} a_i\xi_i^{\beta-\gamma}}\left(\int_0^1 (1-s)^{\beta-\gamma-1} Q_3 y_3(t)\mathrm{d}s - \sum_{i=1}^{n} a_i \int_0^{\xi_i} (\xi_i - s)^{\beta-\gamma-1} Q_3 y_3(t)\mathrm{d}s\right)\\
&= \frac{\beta-\gamma}{1-\sum\limits_{i=1}^{n} a_i\xi_i^{\beta-\gamma}}\left(\int_0^1 (1-s)^{\beta-\gamma-1}\mathrm{d}s - \sum_{i=1}^{n} a_i \int_0^{\xi_i} (\xi_i - s)^{\beta-\gamma-1}\mathrm{d}s\right) Q_3 y_3(t)\\
&= \frac{\beta-\gamma}{1-\sum\limits_{i=1}^{n} a_i\xi_i^{\beta-\gamma}}\left(\frac{1}{\beta-\gamma} - \frac{\sum\limits_{i=1}^{n} a_i\xi_i^{\beta-\gamma}}{\beta-\gamma}\right) Q_3 y_3(t)\\
&= Q_3 y_3(t).
\end{aligned}
$$

Likewisely,

$$
Q_2^2 y_2(t) = Q_2 y_2(t), \qquad Q_4^2 y_4(t) = Q_4 y_4(t).
$$

So

$$
Q^2\boldsymbol{y} = Q\boldsymbol{y}.
$$

From the definition of Q and (3.2), we can easily get that

$$
\ker Q = \mathrm{Im} L.
$$

So, we have

$$
Y = \mathrm{Im} L \oplus \mathrm{Im} Q.
$$

Thus

$$
\dim \ker L = \dim \mathrm{Im} Q = \mathrm{codim}\, \mathrm{Im} L = 4.
$$

This means that L is a Fredholm operator of index zero.

Step 2. We prove that the inverse of $L|_{\mathrm{dom}L\cap\ker P}$ is K_P.

For $\boldsymbol{y} = (y_1,\ y_2,\ y_3,\ y_4)^{\mathrm{T}} \in \mathrm{Im}L$, $\boldsymbol{z} = (z_1,\ z_2,\ z_3,\ z_4)^{\mathrm{T}}$, let $\boldsymbol{z} = K_P\boldsymbol{y}$, that is, $\boldsymbol{z}$ satisfy $z_i = K_{P_i} y_i$, $i = 1, 2, 3, 4$, and $\boldsymbol{z} \in \mathrm{dom}L \cap \ker P$. Since $LK_P\boldsymbol{y} = \boldsymbol{y}$, we get $L\boldsymbol{z} = \boldsymbol{y}$. By (2.2), we know

$$
\left(D_{0^+}^{\alpha} z_1(t),\ D_{0^+}^{\alpha} z_2(t),\ D_{0^+}^{\beta} z_3(t),\ D_{0^+}^{\beta} z_4(t)\right)^{\mathrm{T}} = (y_1(t),\ y_2(t),\ y_3(t),\ y_4(t))^{\mathrm{T}}.
$$

By Lemma 2.3, we have

$$\begin{cases} z_j(t) = I^{\alpha}_{0+}y_j(t) + c_{1j}t^{\alpha-1} + e_{1j}t^{\alpha-2}, & c_{1j}, e_{1j} \in \mathbf{R},\ j = 1, 2, \\ z_k(t) = I^{\beta}_{0+}y_k(t) + c_{1k}t^{\beta-1} + e_{1k}t^{\beta-2}, & c_{1k}, e_{1k} \in \mathbf{R},\ k = 3, 4. \end{cases} \tag{3.3}$$

By $\boldsymbol{z} \in \mathrm{dom}L$, we know $z_i(0) = 0$, $i = 1, 2, 3, 4$. So,

$$e_{1j} = e_{1k} = 0, \qquad j = 1, 2;\ k = 3, 4. \tag{3.4}$$

By $\boldsymbol{z} \in \ker P$, we know

$$\frac{D^{\alpha-1}_{0+}z_j(0)}{\Gamma(\alpha)}t^{\alpha-1} = 0, \quad \frac{D^{\beta-1}_{0+}z_k(0)}{\Gamma(\beta)}t^{\beta-1} = 0, \qquad j = 1, 2;\ k = 3, 4,\ t \in [0,\ 1].$$

It follows from (3.3)–(3.4) and Lemma 2.2 that

$$D^{\alpha-1}_{0+}I^{\alpha}_{0+}y_j(0) + c_{1j}\Gamma(\alpha) = 0, \quad D^{\beta-1}_{0+}I^{\beta}_{0+}y_k(0) + c_{1k}\Gamma(\beta) = 0, \qquad j = 1, 2;\ k = 3, 4.$$

We get

$$c_{1j} = -\frac{1}{\Gamma(\alpha)}I^{1}_{0+}y_j(0) = 0, \quad c_{1k} = -\frac{1}{\Gamma(\beta)}I^{1}_{0+}y_k(0) = 0, \qquad j = 1, 2;\ k = 3, 4. \tag{3.5}$$

It follows from (3.3)–(3.5) that

$$(z_1(t),\ z_2(t),\ z_3(t),\ z_4(t))^{\mathrm{T}} = \big(I^{\alpha}_{0+}y_1(t),\ I^{\alpha}_{0+}y_2(t),\ I^{\beta}_{0+}y_3(t),\ I^{\beta}_{0+}y_4(t)\big)^{\mathrm{T}}.$$

That is,

$$K_P\boldsymbol{y} = (I^{\alpha}_{0+}y_1,\ I^{\alpha}_{0+}y_2,\ I^{\beta}_{0+}y_3,\ I^{\beta}_{0+}y_4)^{\mathrm{T}}.$$

The proof of Lemma 3.2 is completed.

Lemma 3.3 *Suppose that* (H$_1$) *and* (H$_2$) *hold. If $\varOmega \subset X$ is an open bounded subset and* $\mathrm{dom}L \cap \bar{\varOmega} \neq \emptyset$, *then N is L-compact on $\bar{\varOmega}$.*

Proof. By the condition (H$_2$), the continuity of ϕ_{q_1}, ϕ_{q_2} and the definition of Q, we can know that $QN(\bar{\varOmega})$ is bounded. Now we show that $K_P(I - Q)N : \bar{\varOmega} \to X$ is compact. For this, we prove firstly: (i) $K_P(I - Q)N(\bar{\varOmega})$ is uniformly bounded; (ii) $K_P(I - Q)N(\bar{\varOmega})$, $D^{\alpha-1}_{0+}K_P(I_0 - Q_j)N_j(\bar{\varOmega})$ and $D^{\beta-1}_{0+}K_P(I_0 - Q_k)N_k(\bar{\varOmega})$ are equicontinuous on $[0, 1]$, where $I_0 : L[0,\ 1] \to L[0,\ 1]$ is a identity mapping, $j = 1, 2$, $k = 3, 4$.

(i) The condition (H$_2$) and the continuity of ϕ_{q_1}, ϕ_{q_2} mean that there exist constant $M_i > 0$ such that

$$|(I_0 - Q_i)N_i\boldsymbol{x}| \le M_i, \qquad t \in [0,\ 1],\ \boldsymbol{x} \in \bar{\varOmega},\ i = 1, 2, 3, 4.$$

For $\boldsymbol{x} \in \bar{\varOmega}$, $t \in [0,\ 1]$, we have

$$\begin{aligned} &K_P(I - Q)N\boldsymbol{x}(t) \\ &= \big(I^{\alpha}_{0+}(I_0 - Q_1)N_1\boldsymbol{x}(t),\ I^{\alpha}_{0+}(I_0 - Q_2)N_2\boldsymbol{x}(t),\ I^{\beta}_{0+}(I_0 - Q_3)N_3\boldsymbol{x}(t),\ I^{\beta}_{0+}(I_0 - Q_4)N_4\boldsymbol{x}(t)\big). \end{aligned}$$

And we can know

$$\begin{aligned} |I^{\alpha}_{0+}(I_0 - Q_j)N_j\boldsymbol{x}(t)| &= \frac{1}{\Gamma(\alpha)}\bigg|\int_0^t (t-s)^{\alpha-1}(I_0 - Q_j)N_j\boldsymbol{x}(s)\mathrm{d}s\bigg| \\ &\le \frac{M_j}{\Gamma(\alpha)}\bigg|\int_0^1 (1-s)^{\alpha-1}\mathrm{d}s\bigg| \\ &= \frac{M_j}{\Gamma(\alpha+1)}, \qquad j = 1, 2, \end{aligned} \tag{3.6}$$

$$|D_{0^+}^{\alpha-1} I_{0^+}^{\alpha}(I_0 - Q_j)N_j \boldsymbol{x}(t)| \le \int_0^t |(I_0 - Q_j)N_j \boldsymbol{x}(s)|\mathrm{d}s \le M_j, \qquad j = 1, 2, \tag{3.7}$$

$$|I_{0^+}^{\beta}(I_0 - Q_j)N_k \boldsymbol{x}(t)| \le \frac{M_k}{\Gamma(\beta+1)}, \quad |D_{0^+}^{\beta-1} I_{0^+}^{\beta}(I_0 - Q_k)N_k \boldsymbol{x}(t)| \le M_k, \qquad k = 3, 4. \tag{3.8}$$

From (3.6)–(3.8), we get

$$\|K_P(I - Q)N\boldsymbol{x}\|_X \le M,$$

where $M = \max\left\{M_1,\ M_2,\ M_3,\ M_4,\ \dfrac{M_1}{\Gamma(\alpha+1)},\ \dfrac{M_2}{\Gamma(\alpha+1)},\ \dfrac{M_3}{\Gamma(\beta+1)},\ \dfrac{M_4}{\Gamma(\beta+1)}\right\}$. That is, $K_P(I - Q)N(\bar{\Omega})$ is uniformly bounded.

(ii) For $0 \le t_1 < t_2 \le 1$, $\boldsymbol{x} \in \bar{\Omega}$, we have

$$\begin{aligned}
&K_P(I - Q)N\boldsymbol{x}(t_2) - K_P(I - Q)N\boldsymbol{x}(t_1))\\
&= \big(I_{0^+}^{\alpha}(I_0 - Q_1)N_1\boldsymbol{x}(t_2) - I_{0^+}^{\alpha}(I_0 - Q_1)N_1\boldsymbol{x}(t_1),\\
&\quad I_{0^+}^{\alpha}(I_0 - Q_2)N_2\boldsymbol{x}(t_2) - I_{0^+}^{\alpha}(I_0 - Q_2)N_2\boldsymbol{x}(t_1),\\
&\quad I_{0^+}^{\alpha}(I_0 - Q_3)N_3\boldsymbol{x}(t_2) - I_{0^+}^{\alpha}(I_0 - Q_3)N_3\boldsymbol{x}(t_1),\\
&\quad I_{0^+}^{\alpha}(I_0 - Q_4)N_4\boldsymbol{x}(t_2) - I_{0^+}^{\alpha}(I_0 - Q_4)N_4\boldsymbol{x}(t_1)\big),
\end{aligned}$$

$$\begin{aligned}
&|I_{0^+}^{\alpha}(I_0 - Q_j)N_j\boldsymbol{x}(t_2) - I_{0^+}^{\alpha}(I_0 - Q_j)N_j\boldsymbol{x}(t_1)|\\
&= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2}(t_2 - s)^{\alpha-1}(I_0 - Q_j)N_j\boldsymbol{x}(s)\mathrm{d}s - \int_0^{t_1}(t_1 - s)^{\alpha-1}(I_0 - Q_j)N_j\boldsymbol{x}(s)\mathrm{d}s\right|\\
&\le \frac{M_j}{\Gamma(\alpha+1)}|t_2^{\alpha} - t_1^{\alpha} + 2(t_2 - t_1)^{\alpha}|, \qquad j = 1, 2.
\end{aligned}$$

By Lemma 2.1, we get

$$\begin{aligned}
&|D_{0^+}^{\alpha-1}I_{0^+}^{\alpha}(I_0 - Q_j)N_j\boldsymbol{x}(t_2) - D_{0^+}^{\alpha-1}I_{0^+}^{\alpha}(I_0 - Q_j)N_j\boldsymbol{x}(t_1)|\\
&= \left|\int_0^{t_2}(I_0 - Q_j)N_j\boldsymbol{x}(s)\mathrm{d}s - \int_0^{t_1}(I_0 - Q_j)N_j\boldsymbol{x}(s)\mathrm{d}s\right|\\
&\le M_j(t_2 - t_1), \qquad j = 1, 2.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\big|I_{0^+}^{\beta}(I_0 - Q_k)N_k\boldsymbol{x}(t_2) - I_{0^+}^{\beta}(I_0 - Q_k)N_k\boldsymbol{x}(t_1)\big|\\
&\le \frac{M_k}{\Gamma(\beta+1)}\big|t_2^{\beta} - t_1^{\beta} + 2(t_2 - t_1)^{\beta}\big|,\\
&\big|D_{0^+}^{\beta-1}I_{0^+}^{\beta}(I_0 - Q_k)N_k\boldsymbol{x}(t_2) - D_{0^+}^{\beta-1}I_{0^+}^{\beta}(I_0 - Q_k)N_k\boldsymbol{x}(t_1)\big|\\
&\le M_k(t_2 - t_1)
\end{aligned}$$

for $k = 3, 4$.

Since t^{α}, t^{β} are uniformly continuous on $[0,\ 1]$, we can obtain that $K_P(I - Q)N(\bar{\Omega})$, $D_{0^+}^{\alpha-1}K_P(I_0 - Q_j)N_j(\bar{\Omega})$ $(j = 1, 2)$ and $D_{0^+}^{\beta-1}K_P(I_0 - Q_k)N_k(\bar{\Omega})$ $(k = 3, 4)$ are equicontinuous on $[0,\ 1]$.

Applying the Arzelà-Ascoli theorem, we get $K_P(I - Q)N : \bar{\Omega} \to X$ is compact. So, N is L-compact on $\bar{\Omega}$. The proof of Lemma 3.3 is completed.

To obtain our main results, we need the following conditions.

(H$_3$) There exist functions $\zeta_i,\ \psi_i,\ \varphi_i,\ h_i,\ g_i \in L[0,\ 1]$, $i=1,2$, such that

$$\begin{aligned}&|f_i(t,\ x_1,\ x_2,\ x_3,x_4)|\\ &\le \zeta_i(t)+\psi_i(t)|x_1|^{p_1-1}+\varphi_i(t)|x_2|^{p_2-1}+h_i(t)|x_3|^{p_1-1}+g_i(t)|x_4|^{p_2-1}\end{aligned}$$

for $t\in[0,\ 1]$, $(x_1,x_2,x_3,x_4)\in\mathbf{R}^4$, where $\psi_i,\ \varphi_i,\ h_i,\ g_i$, $i=1,2$, satisfying

$$l := \frac{1+\xi_n^{\beta-1}}{\xi_n^{\beta-1}}\left(\frac{2}{\Gamma(\alpha+1)}\right)^{p_1-1}\|\psi_1\|_1+\Gamma(\beta+1)\|h_1\|_1<\Gamma(\beta+1),$$

$$k := \frac{1+\eta_m^{\beta-1}}{\eta_m^{\beta-1}}\left(\frac{2}{\Gamma(\alpha+1)}\right)^{p_2-1}\|\varphi_2\|_1+\Gamma(\beta+1)\|g_2\|_1<\Gamma(\beta+1),$$

$$0\le \frac{(1+\xi_n^{\beta-1})(1+\eta_m^{\beta-1})\|\varphi_1\|_1\|\psi_2\|_1}{(\Gamma(\beta+1)-l)(\Gamma(\beta+1)-k)\xi_n^{\beta-1}\eta_m^{\beta-1}}\left(\frac{2}{\Gamma(\alpha+1)}\right)^{p_1+p_2-2}<1, \tag{3.9a}$$

$$0\le \frac{\Gamma^2(\beta)(1+\beta\xi_n^{\beta-1})(1+\beta\eta_m^{\beta-1})\|\varphi_1\|_1\|\psi_2\|_1}{(\Gamma(\beta+1)-l)(\Gamma(\beta+1)-k)\xi_n^{\beta-1}\eta_m^{\beta-1}}\left(\frac{2}{\Gamma(\alpha+1)}\right)^{p_1+p_2-2}<1. \tag{3.9b}$$

(H$_4$) For $\boldsymbol{x}\in \mathrm{dom}L$, there exist constants $R_i>0$, $i=1,2,3,4$, such that if at least one of the inequations

(1) $|u(t)|>R_1$, $t\in[\epsilon_{n_1},\ 1]$;

(2) $|w_1(t)|>R_3$, $t\in[\xi_n,\ 1]$;

(3) $|v(t)|>R_2$, $t\in[\sigma_{m_1},\ 1]$;

(4) $|w_2(t)|>R_4$, $t\in[\eta_m,\ 1]$

holds, then at least one of the following inequations holds:

$$T_1N_1\boldsymbol{x}(t)\ne 0,\quad T_3N_3\boldsymbol{x}(t)\ne 0,\quad T_2N_2\boldsymbol{x}(t)\ne 0,\quad T_4N_4\boldsymbol{x}(t)\ne 0.$$

(H$_5$) For $\boldsymbol{x}=(c_1t^{\alpha-1},\ c_2t^{\alpha-1},\ c_3t^{\beta-1},\ c_4t^{\beta-1})^{\mathrm{T}}\in\ker L$, there exist constants $e_i>0$, $i=1,2,3,4$, such that either

(1) $c_iT_iN_i\boldsymbol{x}>0$ if $|c_i|>e_i$, $i=1,2,3,4$,

or

(2) $c_iT_iN_i\boldsymbol{x}<0$, if $|c_i|>e_i$, $i=1,2,3,4$

holds.

Lemma 3.4 *Suppose that* (H$_1$)–(H$_4$) *hold. Then the set*

$$\varOmega_1=\{\boldsymbol{x}\in\mathrm{dom}L\backslash\ker L\mid L\boldsymbol{x}=\lambda N\boldsymbol{x},\ \lambda\in(0,\ 1)\}$$

is bounded in X.

Proof. Take

$$\boldsymbol{x}=(u,\ v,\ w_1,\ w_2)^{\mathrm{T}}\in\varOmega_1.$$

By $L\boldsymbol{x}=\lambda N\boldsymbol{x}$, Lemma 2.3 and $(u(0),\ v(0),\ w_1(0),\ w_2(0))^{\mathrm{T}}=(0,\ 0,\ 0,\ 0)^{\mathrm{T}}$, we have

$$\begin{pmatrix}u(t)\\ v(t)\\ w_1(t)\\ w_2(t)\end{pmatrix}=\begin{pmatrix}\lambda I_{0^+}^{\alpha}\phi_{q_1}(w_1(t))+c_{11}t^{\alpha-1}\\ \lambda I_{0^+}^{\alpha}\phi_{q_2}(w_2(t))+c_{12}t^{\alpha-1}\\ \lambda I_{0^+}^{\beta}f_1(t,u(t),v(t),\phi_{q_1}(w_1(t)),\phi_{q_2}(w_2(t)))+c_{21}t^{\beta-1}\\ \lambda I_{0^+}^{\beta}f_2(t,u(t),v(t),\phi_{q_1}(w_1(t)),\phi_{q_2}(w_2(t)))+c_{22}t^{\beta-1}\end{pmatrix}, \tag{3.10}$$
$$c_{11},c_{12},c_{21},c_{22}\in\mathbf{R},\ t\in[0,\ 1].$$

By $N\boldsymbol{x} \in \mathrm{Im}L$, we get

$$T_i N_i \boldsymbol{x} = 0, \qquad i = 1, 2, 3, 4.$$

These, together with (H_4), mean that there exist constants $t_{11} \in [\epsilon_{n_1},\ 1]$, $t_1 \in [\xi_n,\ 1]$, $t_{22} \in [\sigma_{m_1},\ 1]$, $t_2 \in [\eta_m,\ 1]$ such that

$$|u(t_{11})| \le R_1, \quad |w_1(t_1)| \le R_3, \quad |v(t_{22})| \le R_2, \quad |w_2(t_2)| \le R_4.$$

By (3.10), we have

$$|c_{21}|t_1^{\beta-1} \le R_3 + \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} |f_1(s, u(s), v(s), \phi_{q_1}(w_1(s)), \phi_{q_2}(w_2(s)))| \mathrm{d}s, \tag{3.11}$$

$$|c_{22}|t_2^{\beta-1} \le R_4 + \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} |f_2(s, u(s), v(s), \phi_{q_1}(w_1(s)), \phi_{q_2}(w_2(s)))| \mathrm{d}s. \tag{3.12}$$

By Lemma 2.3, we have

$$\begin{aligned} |u(t)| &= |u(t_{11}) + I_{t_{11}^+}^{\alpha} D_{0^+}^{\alpha} u(t)| \\ &\le |u(t_{11})| + \frac{1}{\Gamma(\alpha)} \int_{t_{11}}^{t} (t-s)^{\alpha-1} |D_{0^+}^{\alpha} u(s)| \mathrm{d}s \\ &\le R_1 + \frac{\|D_{0^+}^{\alpha} u\|_\infty}{\Gamma(\alpha+1)}, \end{aligned}$$

that is,

$$\|u\|_\infty \le R_1 + \frac{\|D_{0^+}^{\alpha} u\|_\infty}{\Gamma(\alpha+1)}. \tag{3.13}$$

Similarly, we have

$$\|v\|_\infty \le R_2 + \frac{\|D_{0^+}^{\alpha} v\|_\infty}{\Gamma(\alpha+1)}. \tag{3.14}$$

By (3.10)–(3.12), we know

$$\begin{aligned} |w_1(t)| &\le |\lambda I_{0^+}^{\beta} f_1(t, u(t), v(t), \phi_{q_1}(w_1(t)), \phi_{q_2}(w_2(t)))| + |c_{21}|t^{\beta-1} \\ &\le \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} |f_1(s, u(s), v(s), \phi_{q_1}(w_1(s)), \phi_{q_2}(w_2(s)))| \mathrm{d}s \\ &\quad + \left(\frac{t}{t_1}\right)^{\beta-1} |c_{21}|t_1^{\beta-1} \\ &\le \frac{R_3}{\xi_n^{\beta-1}} + \frac{1}{\Gamma(\beta)}\left(1 + \frac{1}{\xi_n^{\beta-1}}\right) \int_0^1 (1-s)^{\beta-1} \big[\zeta_1(s) + \psi_1(s)|u(s)|^{p_1-1} \\ &\quad + \varphi_1(s)|v(s)|^{p_2-1} + h_1(s)|\phi_{q_1}(w_1(s))|^{p_1-1} + g_1(s)|\phi_{q_2}(w_2(s))|^{p_2-1}\big] \mathrm{d}s \\ &\le \frac{R_3}{\xi_n^{\beta-1}} + \frac{1+\xi_n^{\beta-1}}{\Gamma(\beta+1)\xi_n^{\beta-1}} \big(\|\zeta_1\|_1 + \|\psi_1\|_1 \|u\|_\infty^{p_1-1} + \|\varphi_1\|_1 \|v\|_\infty^{p_2-1} \\ &\qquad + \|h_1\|_1 \|w_1\|_\infty + \|g_1\|_1 \|w_2\|_\infty\big), \end{aligned}$$

which together with

$$|\phi_p(x+y)| \le 2^{p-1}(x^{p-1} + y^{p-1}), \qquad x, y > 0$$

(see [20]) and (3.13)–(3.14), we get

$$\|w_1\|_\infty \le \frac{R_3}{\xi_n^{\beta-1}} + \frac{1+\xi_n^{\beta-1}}{\Gamma(\beta+1)\xi_n^{\beta-1}} \Bigg[\|\zeta_1\|_1 + \|\psi_1\|_1 \left(R_1 + \frac{\|\phi_{q_1}(w_1)\|_\infty}{\Gamma(\alpha+1)}\right)^{p_1-1}$$

$$
\begin{aligned}
&+\|\varphi_1\|_1\left(R_2+\frac{\|\phi_{q_2}(w_2)\|_\infty}{\Gamma(\alpha+1)}\right)^{p_2-1}+\|h_1\|_1\|w_1\|_\infty+\|g_1\|_1\|w_2\|_\infty\Bigg]\\
\le{}& \frac{R_3}{\xi_n^{\beta-1}}+\frac{1+\xi_n^{\beta-1}}{\Gamma(\beta+1)\xi_n^{\beta-1}}\Bigg[\|\zeta_1\|_1+2^{p_1-1}\|\psi_1\|_1\left(R_1^{p_1-1}+\left(\frac{\|\phi_{q_1}(w_1)\|_\infty}{\Gamma(\alpha+1)}\right)^{p_1-1}\right)\\
&+2^{p_2-1}\|\varphi_1\|_1\left(R_2^{p_2-1}+\left(\frac{\|\phi_{q_2}(w_2)\|_\infty}{\Gamma(\alpha+1)}\right)^{p_2-1}\right)+\|h_1\|_1\|w_1\|_\infty+\|g_1\|_1\|w_2\|_\infty\Bigg].
\end{aligned}
$$

So

$$
\begin{aligned}
\|w_1\|_\infty\le{}&\frac{\Gamma(\beta+1)}{\Gamma(\beta+1)-l}\Bigg\{\frac{R_3}{\xi_n^{\beta-1}}+\frac{1+\xi_n^{\beta-1}}{\Gamma(\beta+1)\xi_n^{\beta-1}}\Bigg[\|\zeta_1\|_1+2^{p_1-1}\|\psi_1\|_1R_1^{p_1-1}\\
&+2^{p_2-1}\|\varphi_1\|_1R_2^{p_2-1}+\left(\|g_1\|_1+\left(\frac{2}{\Gamma(\alpha+1)}\right)^{p_2-1}\|\varphi_1\|_1\right)\|w_2\|_\infty\Bigg]\Bigg\}.
\end{aligned}
$$

Likewisely,

$$
\begin{aligned}
\|w_2\|_\infty\le{}&\frac{\Gamma(\beta+1)}{\Gamma(\beta+1)-k}\Bigg\{\frac{R_4}{\eta_m^{\beta-1}}+\frac{1+\eta_m^{\beta-1}}{\Gamma(\beta+1)\eta_m^{\beta-1}}\Bigg[\|\zeta_2\|_1+2^{p_1-1}\|\psi_2\|_1R_1^{p_1-1}\\
&+2^{p_2-1}\|\varphi_2\|_1R_2^{p_2-1}+\left(\|h_2\|_1+\left(\frac{2}{\Gamma(\alpha+1)}\right)^{p_1-1}\|\psi_2\|_1\right)\|w_1\|_\infty\Bigg]\Bigg\}.
\end{aligned}
$$

In view of (3.9a), we can see that there exist constants $\bar{M}_1$, $\bar{M}_2>0$ such that

$$
\|w_1\|_\infty\le\bar{M}_1,\qquad \|w_2\|_\infty\le\bar{M}_2. \tag{3.15}
$$

So

$$
\|D_{0^+}^{\alpha}u\|_\infty=\|\phi_{q_1}(w_1)\|_\infty\le\phi_{q_1}(\bar{M}_1),\qquad \|D_{0^+}^{\alpha}v\|_\infty=\|\phi_{q_2}(w_2)\|_\infty\le\phi_{q_2}(\bar{M}_2).
$$

Combing (3.13) with (3.14), we get

$$
\|u\|_\infty\le R_1+\frac{\phi_{q_1}(\bar{M}_1)}{\Gamma(\alpha+1)},\qquad \|v\|_\infty\le R_2+\frac{\phi_{q_2}(\bar{M}_2)}{\Gamma(\alpha+1)}. \tag{3.16}
$$

On the other hand, by (3.10), we have

$$
|c_{11}|t_{11}^{\alpha-1}\le R_1+\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}|\phi_{q_1}(w_1(s))|\mathrm{d}s.
$$

So

$$
\begin{aligned}
|D_{0^+}^{\alpha-1}u(t)|={}&|\lambda\int_0^1\phi_{q_1}(w_1(s))\mathrm{d}s+c_{11}\Gamma(\alpha)|\\
\le{}&\int_0^1|\phi_{q_1}(w_1(s))|\mathrm{d}s+\left(\frac{1}{t_{11}}\right)^{\alpha-1}|c_{11}|t_{11}^{\alpha-1}\Gamma(\alpha)\\
\le{}&\int_0^1|\phi_{q_1}(w_1(s))|\mathrm{d}s+\frac{\Gamma(\alpha)}{\epsilon_{n_1}^{\alpha-1}}R_1+\frac{1}{\epsilon_{n_1}^{\alpha-1}}\int_0^1(1-s)^{\alpha-1}|\phi_{q_1}(w_1(s))|\mathrm{d}s\\
\le{}&\left(1+\frac{1}{\alpha\epsilon_{n_1}^{\alpha-1}}\right)\|\phi_{q_1}(w_1)\|_\infty+\frac{\Gamma(\alpha)}{\epsilon_{n_1}^{\alpha-1}}R_1\\
\le{}&\left(1+\frac{1}{\alpha\epsilon_{n_1}^{\alpha-1}}\right)\phi_{q_1}(\bar{M}_1)+\frac{\Gamma(\alpha)}{\epsilon_{n_1}^{\alpha-1}}R_1.
\end{aligned}
$$

Likewisely,

$$
|D_{0^+}^{\alpha-1}v(t)|\le\left(1+\frac{1}{\alpha\sigma_{m_1}^{\alpha-1}}\right)\phi_{q_2}(\bar{M}_2)+\frac{\Gamma(\alpha)}{\epsilon_{m_1}^{\sigma-1}}R_2.
$$

That is,

$$
\|D_{0^+}^{\alpha-1}u\|_\infty\le\bar{M}_3,\qquad \|D_{0^+}^{\alpha-1}v\|_\infty\le\bar{M}_4, \tag{3.17}
$$

where $\bar{M}_3=\left(1+\frac{1}{\alpha\epsilon_{n_1}^{\alpha-1}}\right)\phi_{q_1}(\bar{M}_1)+\frac{\Gamma(\alpha)}{\epsilon_{n_1}^{\alpha-1}}R_1,\ \bar{M}_4=\left(1+\frac{1}{\alpha\sigma_{m_1}^{\alpha-1}}\right)\phi_{q_2}(\bar{M}_2)+\frac{\Gamma(\alpha)}{\epsilon_{m_1}^{\sigma-1}}R_2.$

Since

$$D_{0^+}^{\beta-1}w_1(t)=\lambda\int_0^1 f_1(s,u(s),v(s),\phi_{q_1}(w_1(s)),\phi_{q_2}(w_2(s)))\mathrm{d}s+c_{21}\Gamma(\beta),$$

likewisely (3.15) and (3.17) obtained, and the condition (3.9b), we can know there exist constants $\bar{M}_5,\ \bar{M}_6>0$ such that

$$\|D_{0^+}^{\beta-1}w_1\|_\infty\le\bar{M}_5,\qquad\|D_{0^+}^{\beta-1}w_2\|_\infty\le\bar{M}_6.\tag{3.18}$$

By (3.15)–(3.18), we have

$$\|(u,v,w_1,w_2)^{\mathrm{T}}\|_X=\max\{\|u\|_\alpha,\ \|v\|_\alpha,\ \|w_1\|_\beta,\ \|w_2\|_\beta\}\le r_1,$$

where

$$r_1=\max\left\{\bar{M}_1,\ \bar{M}_2,\ \bar{M}_3,\ \bar{M}_4,\ \bar{M}_5,\ \bar{M}_6,\ R_1+\frac{\phi_{q_1}(\bar{M}_1)}{\Gamma(\alpha+1)},\ R_2+\frac{\phi_{q_2}(\bar{M}_2)}{\Gamma(\alpha+1)}\right\}.$$

Therefore, Ω_1 is bounded. The proof of Lemma 3.4 is completed.

Lemma 3.5 *Suppose that* (H$_1$), (H$_2$) *and* (H$_5$) *hold. Then the set*

$$\Omega_2=\{\boldsymbol{x}\in\ker L\mid N\boldsymbol{x}\in\mathrm{Im}L\}$$

is bounded in X.

Proof. For $\boldsymbol{x}=(u,\ v,\ w_1,\ w_2)^{\mathrm{T}}\in\Omega_2$, we have

$$\boldsymbol{x}=(c_1t^{\alpha-1},\ c_2t^{\alpha-1},\ c_3t^{\beta-1},\ c_4t^{\beta-1})^{\mathrm{T}},\qquad c_i\in\mathbf{R},\ t\in[0,\ 1],\ i=1,2,3,4.$$

By $N\boldsymbol{x}\in\mathrm{Im}L$, we know

$$T_iN_i\boldsymbol{x}=0,\qquad i=1,2,3,4.$$

By (H$_5$), we know there exist constants $e_i>0$ such that

$$|c_i|\le e_i,\qquad i=1,2,3,4.$$

So

$$|u(t)|=|c_1t^{\alpha-1}|\le|c_1|\le e_1,$$

that is,

$$\|u\|_\infty\le e_1.$$

Likewisely,

$$\|v\|_\infty\le e_2,\quad\|w_1\|_\infty\le e_3,\quad\|w_2\|_\infty\le e_4.$$

By Lemma 2.2, we can get

$$|D_{0^+}^{\alpha-1}u(t)|=|c_1\Gamma(\alpha)|\le e_1\Gamma(\alpha),$$

that is,

$$\|D_{0^+}^{\alpha-1}u\|_\infty\le e_1\Gamma(\alpha).$$

Likewisely,

$$\|D_{0^+}^{\alpha-1}v\|_\infty\le e_2\Gamma(\alpha),\quad\|D_{0^+}^{\beta-1}w_1\|_\infty\le e_3\Gamma(\beta),\quad\|D_{0^+}^{\beta-1}w_2\|_\infty\le e_4\Gamma(\beta).$$

Thus

$$\|(u,\ v,\ w_1,\ w_2)^{\mathrm{T}}\|_X=\max\{\|u\|_\alpha,\ \|v\|_\alpha,\ \|w_1\|_\beta,\ \|w_2\|_\beta\}\le r_2,$$

where

$$r_2=\max\{e_1,\ e_2,\ e_3,\ e_4,\ e_1\Gamma(\alpha),\ e_2\Gamma(\alpha),\ e_3\Gamma(\beta),\ e_4\Gamma(\beta)\}.$$

Therefore, Ω_2 is bounded. The proof of Lemma 3.5 is completed.

Lemma 3.6 *Suppose that* (H_1), (H_2) *and* (H_5) *hold. Then the set*

$$\Omega_3 = \{\boldsymbol{x} \in \ker L \mid \lambda \boldsymbol{x} + (1-\lambda)\theta JQN\boldsymbol{x} = 0,\ \lambda \in [0,\ 1]\}$$

is bounded in X, *where* $J : \mathrm{Im}Q \to \ker L$ *is a isomorphism given by*

$$J\begin{pmatrix} \dfrac{\alpha}{1-\sum\limits_{i=1}^{n_1} A_i\epsilon_i^{\alpha}}e_1 \\ \dfrac{\alpha}{1-\sum\limits_{i=1}^{m_1} B_i\sigma_i^{\alpha}}e_2 \\ \dfrac{\beta-\gamma}{1-\sum\limits_{i=1}^{n} a_i\xi_i^{\beta-\gamma}}e_3 \\ \dfrac{\beta-\delta}{1-\sum\limits_{i=1}^{m} b_i\eta_i^{\beta-\delta}}e_4 \end{pmatrix} = \begin{pmatrix} e_1t^{\alpha-1} \\ e_2t^{\alpha-1} \\ e_3t^{\beta-1} \\ e_4t^{\beta-1} \end{pmatrix}, \qquad t \in [0,\ 1],\ e_i \in \mathbf{R},\ i = 1,2,3,4.$$

$$\theta = \begin{cases} 1, & \text{if } (\mathrm{H}_5)(1) \text{ holds;} \\ -1, & \text{if } (\mathrm{H}_5)(2) \text{ holds.} \end{cases}$$

Proof. For $\boldsymbol{x} = (u,\ v,\ w_1,\ w_2)^{\mathrm{T}} \in \ker L$, $(u,\ v,\ w_1,\ w_2)^{\mathrm{T}} = (c_1t^{\alpha-1},\ c_2t^{\alpha-1},\ c_3t^{\beta-1},\ c_4t^{\beta-1})^{\mathrm{T}}$, $c_i \in \mathbf{R}$, $t \in [0,\ 1]$, $i = 1,2,3,4$. There exists $\lambda \in [0,\ 1]$ such that

$$\lambda \boldsymbol{x} = -(1-\lambda)\theta JQN\boldsymbol{x},$$

that is,

$$\lambda\begin{pmatrix} c_1t^{\alpha-1} \\ c_2t^{\alpha-1} \\ c_3t^{\beta-1} \\ c_4t^{\beta-1} \end{pmatrix} = -(1-\lambda)\theta\begin{pmatrix} T_1N_1\boldsymbol{x}t^{\alpha-1} \\ T_2N_2\boldsymbol{x}t^{\alpha-1} \\ T_3N_3\boldsymbol{x}t^{\beta-1} \\ T_4N_4\boldsymbol{x}t^{\beta-1} \end{pmatrix}.$$

We get

$$\lambda c_i = -(1-\lambda)\theta T_iN_i\boldsymbol{x}, \qquad i = 1,2,3,4.$$

If $\lambda = 0$, by (H_5), we get

$$|c_i| \le e_i, \qquad i = 1,2,3,4.$$

If $\lambda = 1$, we get

$$c_i = 0, \qquad i = 1,2,3,4.$$

For $\lambda \in (0,1)$, one has

$$|c_1| > e_1, \qquad |c_2| > e_2, \qquad |c_3| > e_3, \qquad |c_4| > e_4. \tag{3.19}$$

If at least one of the inequalities in (3.19) holds, we have that at least one of the following inequations holds:

$$\begin{aligned} \lambda c_1^2 &= -(1-\lambda)\theta c_1T_1N_1\boldsymbol{x} < 0, \\ \lambda c_2^2 &= -(1-\lambda)\theta c_2T_2N_2\boldsymbol{x} < 0, \\ \lambda c_3^2 &= -(1-\lambda)\theta c_3T_3N_3\boldsymbol{x} < 0, \\ \lambda c_4^2 &= -(1-\lambda)\theta c_4T_4N_4\boldsymbol{x} < 0, \end{aligned}$$

this is a contradiction. So, for $\lambda \in [0,1]$, we get

$$|c_i| \le e_i, \qquad i = 1,2,3,4.$$

Similar to the proof of Lemma 3.5, we can get

$$\|(u,\ v,\ w_1,\ w_2)^{\mathrm{T}}\| \le r_2.$$

Therefore, we obtain Ω_3 is bounded. The proof of Lemma 3.6 is completed.

Theorem 3.1 *Suppose that* (H_1)–(H_5) *hold. Then the problem* (1.1) *has at least one solution in* X.

Proof. Set

$$\Omega = \{\boldsymbol{x} \in X \mid \|\boldsymbol{x}\|_X < r_1 + r_2 + 1\}.$$

Obviously, Ω is a bounded open subset of X and $\Omega_1 \cup \Omega_2 \cup \Omega_3 \subset \Omega$. It follows from Lemmas 3.2 and 3.3 that L (defined by (2.2)) is a Fredholm operator of index zero and N (defined by (2.3)) is L-compact on $\bar{\Omega}$. By Lemmas 3.4 and 3.5, we get that the following two conditions are satisfied:

(1) $L\boldsymbol{x} \ne \lambda N\boldsymbol{x}$ for every $(\boldsymbol{x},\ \lambda) \in [(\mathrm{dom}L\backslash\ker L) \cap \partial\Omega] \times (0,\ 1)$;

(2) $N\boldsymbol{x} \notin \mathrm{Im}L$ for every $\boldsymbol{x} \in \ker L \cap \partial\Omega$.

Next, we need only to prove

(3) $\deg(JQN|_{\ker L},\ \Omega \cap \ker L,\ 0) \ne 0$.

Take

$$H(\boldsymbol{x},\ \lambda) = \lambda\boldsymbol{x} + \theta(1-\lambda)JQN\boldsymbol{x}, \qquad \boldsymbol{x} \in (\mathrm{dom}L\backslash\ker L) \cap \partial\Omega,\ \lambda \in (0,\ 1).$$

According to Lemma 3.6, we know

$$H(\boldsymbol{x},\ \lambda) \ne 0, \qquad x \in \partial\Omega \cap \ker L.$$

By the homotopy of degree, we have

$$\begin{aligned} &\deg(JQN|_{\ker L},\ \Omega \cap \ker L,\ 0) \\ &= \deg(\theta H(\,\cdot\,,\ 0),\ \Omega \cap \ker L,\ 0) \\ &= \deg(\theta H(\,\cdot\,,\ 1),\ \Omega \cap \ker L,\ 0) \\ &= \deg(\theta I,\ \Omega \cap \ker L,\ 0) \\ &\ne 0. \end{aligned}$$

By Lemma 2.4, we can get that $L\boldsymbol{x} = N\boldsymbol{x}$ has at least one solution on $\mathrm{dom}L \cap \bar{\Omega}$. That is, (2.1) has at least one solution in X. Then we know (1.1) has at least one solution in X. The proof of Theorem 3.1 is completed.

4 Example

Let us consider the following coupled system of fractional p-Laplacian differential equations at resonance

$$\begin{cases} D_{0^+}^{\frac{3}{2}}\phi_3(D_{0^+}^{\frac{5}{4}}u(t)) = f_1(t,\ u(t),\ v(t),\ D_{0^+}^{\frac{1}{2}}u(t),\ D_{0^+}^{\frac{1}{2}}v(t)), \qquad 0<t<1, \\ D_{0^+}^{\frac{3}{2}}\phi_2(D_{0^+}^{\frac{5}{4}}v(t)) = f_2(t,\ u(t),\ v(t),\ D_{0^+}^{\frac{1}{2}}u(t),\ D_{0^+}^{\frac{1}{2}}v(t)), \qquad 0<t<1, \\ u(0) = D_{0^+}^{\alpha}u(0) = 0, \quad u(1) = \dfrac{2}{3}u\Big(\dfrac{1}{16}\Big) + 2u\Big(\dfrac{1}{81}\Big), \\ D_{0^+}^{\frac{1}{2}}\phi_3(D_{0^+}^{\frac{5}{4}}u(1)) = D_{0^+}^{\frac{1}{2}}\phi_3\Big(D_{0^+}^{\frac{5}{4}}u\Big(\dfrac{1}{4}\Big)\Big), \\ v(0) = D_{0^+}^{\alpha}v(0) = 0, \quad v(1) = \dfrac{\sqrt{2}}{3}v\Big(\dfrac{1}{4}\Big) + \dfrac{\sqrt{3}}{3}v\Big(\dfrac{1}{9}\Big) + \dfrac{\sqrt{5}}{3}v\Big(\dfrac{1}{25}\Big), \\ D_{0^+}^{\frac{1}{4}}\phi_2(D_{0^+}^{\frac{5}{4}}v(1)) = \sqrt{3}D_{0^+}^{\frac{1}{4}}\phi_2\Big(D_{0^+}^{\frac{5}{4}}u\Big(\dfrac{1}{9}\Big)\Big), \end{cases} \tag{4.1}$$

where

$$f_1(t,\ x_1,\ x_2,\ x_3,\ x_4) = t^3\cos(x_1x_2) + \frac{1}{32}e^{-(1-t)}x_1^2 + \frac{t}{64}\sin x_2 + \frac{t}{6}x_3^2 + \frac{t}{3}x_4,$$

$$f_2(t,\ x_1,\ x_2,\ x_3, x_4) = \sqrt{t}\sin(x_1x_2) + \frac{1}{16}\cos t\sin(x_1^2) + \frac{1}{32}e^{-(1-t)}x_2 + \frac{t}{4}x_3^2 + \frac{t}{6}x_4.$$

Corresponding to BVP (1.1), we have that $m = n = 1$, $m_1 = 3$, $n_1 = 2$, $\alpha = \frac{5}{4}$, $\beta = \frac{3}{2}$, $\gamma = \frac{1}{2}$, $\delta = \frac{1}{4}$, $\epsilon_1 = \frac{1}{16}$, $\epsilon_2 = \frac{1}{81}$, $\sigma_1 = \frac{1}{4}$, $\sigma_1 = \frac{1}{9}$, $\sigma_1 = \frac{1}{25}$, $A_1 = \frac{2}{3}$, $A_2 = 2$, $B_1 = \frac{\sqrt{2}}{3}$, $B_2 = \frac{\sqrt{3}}{3}$, $B_3 = \frac{\sqrt{5}}{3}$, $\xi_1 = \frac{1}{4}$, $\eta_1 = \frac{1}{9}$, $a_1 = 1$, $b_1 = \sqrt{3}$. Take

$$\zeta_1 = t^3, \quad \psi_1 = \frac{1}{32}e^{-(1-t)}, \quad \varphi_1 = \frac{t}{64}, \qquad h_1 = \frac{t}{6}, \quad g_1 = \frac{t}{3},$$

$$\zeta_2 = \sqrt{t}, \quad \psi_2 = \frac{1}{16}\cos t, \quad \varphi_2 = \frac{1}{32}e^{-(1-t)}, \quad h_2 = \frac{t}{4}, \quad g_2 = \frac{t}{6}.$$

Then

$$\begin{aligned} l &= \frac{1+\xi_n^{\beta-1}}{\xi_n^{\beta-1}}\Big(\frac{2}{\Gamma(\alpha+1)}\Big)^{p_1-1}\|\psi_1\|_1 + \Gamma(\beta+1)\|h_1\|_1 \\ &< 3\times\Big(\frac{2}{1.133}\Big)^2\times\frac{1}{32} + 1.330\times\frac{1}{6} \\ &\approx 0.434 < 1.329 \approx \Gamma\Big(\frac{5}{2}\Big) \\ &= \Gamma(\beta+1), \\ k &= \frac{1+\eta_m^{\beta-1}}{\eta_m^{\beta-1}}\Big(\frac{2}{\Gamma(\alpha+1)}\Big)^{p_2-1}\|\varphi_2\|_1 + \Gamma(\beta+1)\|g_2\|_1 \\ &< 4\times\Big(\frac{2}{1.133}\Big)\times\frac{1}{32} + 1.330\times\frac{1}{6} \approx 0.409 < 1.329 \\ &\approx \Gamma\Big(\frac{5}{2}\Big) = \Gamma(\beta+1), \end{aligned}$$

and

$$\frac{(1+\xi_n^{\beta-1})(1+\eta_m^{\beta-1})\|\varphi_1\|_1\|\psi_2\|_1}{(\Gamma(\beta+1)-l)(\Gamma(\beta+1)-k)\xi_n^{\beta-1}\eta_m^{\beta-1}}\Big(\frac{2}{\Gamma(\alpha+1)}\Big)^{p_1+p_2-2} \approx 0.484 < 1,$$

$$\frac{\Gamma^2(\beta)(1+\beta\xi_n^{\beta-1})(1+\beta\eta_m^{\beta-1})\|\varphi_1\|_1\|\psi_2\|_1}{(\Gamma(\beta+1)-l)(\Gamma(\beta+1)-k)\xi_n^{\beta-1}\eta_m^{\beta-1}}\Big(\frac{2}{\Gamma(\alpha+1)}\Big)^{p_1+p_2-2}\approx 0.012<1.$$

By simple calculation, we can get that (H_1)–(H_4) and $(\mathrm{H}_5)(1)$ hold. By Theorem 3.1, we obtain that the problem (4.1) has at least one solution.

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