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Homotopy Analysis Method for Solving (2+1)-dimensional Navier-Stokes Equations with Perturbation Terms

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Abstract: In this paper Homotopy Analysis Method (HAM) is implemented for obtaining approximate solutions of (2+1)-dimensional Navier-Stokes equations with perturbation terms. The initial approximations are obtained using linear systems of the Navier-Stokes equations; by the iterations formula of HAM, the first approximation solutions and the second approximation solutions are successively obtained and Homotopy Perturbation Method (HPM) is also used to solve these equations; finally, approximate solutions by HAM of (2+1)-dimensional Navier-Stokes equations without perturbation terms and with perturbation terms are compared. Because of the freedom of choice the auxiliary parameter of HAM, the results demonstrate that the rapid convergence and the high accuracy of the HAM in solving Navier-Stokes equations; due to the effects of perturbation terms, the 3rd-order approximation solutions by HAM and HPM have great fluctuation.

Key words: Navier-Stokes equation, homotopy analysis method, homotopy perturbation method, perturbation term

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1 Introduction

Most of nonlinear partial differential equations do not have a precise analytical solution, by approximate methods these nonlinear equations can be solved. So far many numerical algorithms have been developed for the approximate solutions of nonlinear partial differential

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equations, such as variational iteration method (see [1]–[5]), homotopy analysis method (see [6]–[10]), differential transform method and the homotopy analysis method (see [11]), differential quadrature method, etc. (see [12] and [13]).

Homotopy analysis method (HAM) is one of the most effective methods to find approximate solution of nonlinear partial differential equations. The HAM always provides us with a family of solution expressions in the auxiliary parameter $\hbar$, by the convergence-controller parameter $\hbar$, the region and rate of each solution might be adjusted and controlled conveniently (see [14]). The HAM has been applied to solve many types of nonlinear problems, such as the nonlinear Cauchy problem of parabolic-hyperbolic type (see [15]), differential-difference equations (see [16]), nonlinear reaction-diffusion-convection problems (see [17]), nonlocal initial boundary value problem (see [18]), fractional differential equations (see [19]–[21]), Fredholm and Volterra integral equations (see [22]). The HAM has been also used to investigate the heat conduction problems (see [23]–[29]). Convergences of the homotopy method are studied (see, for example, [30]–[36]).

The objective of this article is to implement HAM for finding new traveling wave solutions of (2+1)-dimensional Navier-Stokes equations with perturbation terms. Navier-Stokes equations are the most important equations in fluid dynamics for finding the velocity and pressure functions (see [37]–[39]). Viscosity is a characteristic of a fluid, for example, air is a kind of high viscosity fluid, water and air are fluids with low viscosity. Meanwhile, the fluid movement equation is contained Navier-Stokes equations (see [40]).

2 Navier-Stokes Equations with Perturbation Terms

The Navier-Stokes equations with perturbation terms can be written in the following basic form:

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial z} + \frac{1}{\rho}\frac{\partial P}{\partial x} - \nu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial z^2}\right) = F_1, \tag{2.1}$$

$$\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial z} + \frac{1}{\rho}\frac{\partial P}{\partial z} - \nu\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial z^2}\right) = F_2, \tag{2.2}$$

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial z} = F_3, \tag{2.3}$$

where U and V is speed component in direction to x and z, respectively, P is the pressure, ρ is the fluid density, and ν is the kinematics of fluid coefficient that is positive constant, F_i $(i = 1, 2, 3)$ are the perturbation terms. The Navier-Stokes equations are nonlinear partial differential equations and have perturbation terms, so approximate methods must be constructed.

We introduce a complex variable ξ, $\xi = x + z - ct$, where c is the speed of traveling wave.

Thus, (2.1)–(2.3) become the ordinary differential equations as follows:

$$-c\frac{\mathrm{d}U}{\mathrm{d}\xi} + U\frac{\mathrm{d}U}{\mathrm{d}\xi} + V\frac{\mathrm{d}U}{\mathrm{d}\xi} + \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}\xi} - 2\nu\frac{\mathrm{d}^2 U}{\mathrm{d}\xi^2} = F_1, \tag{2.4}$$

$$-c\frac{\mathrm{d}V}{\mathrm{d}\xi}+U\frac{\mathrm{d}V}{\mathrm{d}\xi}+V\frac{\mathrm{d}V}{\mathrm{d}\xi}+\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}\xi}-2\nu\frac{\mathrm{d}^2V}{\mathrm{d}\xi^2}=F_2, \tag{2.5}$$

$$\frac{\mathrm{d}U}{\mathrm{d}\xi}+\frac{\mathrm{d}V}{\mathrm{d}\xi}=F_3. \tag{2.6}$$

The linear system of (2.4)–(2.6) as follows:

$$\begin{aligned}&-c\frac{\mathrm{d}U}{\mathrm{d}\xi}+\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}\xi}-2\nu\frac{\mathrm{d}^2U}{\mathrm{d}\xi^2}=0,\\&-c\frac{\mathrm{d}V}{\mathrm{d}\xi}+\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}\xi}-2\nu\frac{\mathrm{d}^2V}{\mathrm{d}\xi^2}=0,\\&\frac{\mathrm{d}U}{\mathrm{d}\xi}+\frac{\mathrm{d}V}{\mathrm{d}\xi}=0.\end{aligned}$$

The solutions of the above system can be obtained

$$V=a_1+a_2\mathrm{e}^{-\lambda\xi},\quad \lambda=\frac{c}{2\nu},\quad U=a_3-a_2\mathrm{e}^{-\lambda\xi},\quad P=a_4,$$

where a_1, a_2, a_3, a_4 are constants. Obviously, for $\xi\geq 0$, $V(\xi)$ and $U(\xi)$ are asymptotically stable.

3 The Methodology

3.1 Homotopy Analysis Method (HAM)

Consider the following nonlinear differential equation:

$$N[u(\xi)]=0, \tag{3.1}$$

where N is a nonlinear operator, ξ is the independent variable and $u(\xi)$ is an unknown function. The homotopy is constructed as follows (see [10]):

$$(1-q)L[\varphi(\xi;\,q)-u_0(\xi)]=q\hbar H(\xi)N[\varphi(\xi;\,q)], \tag{3.2}$$

where $q\in[0,\,1]$ is the embedding parameter, $\hbar\neq 0$ is an auxiliary parameter, L is an auxiliary linear operator, $\varphi(\xi;\,q)$ is an unknown function, $u_0(\xi)$ is an initial guess of $u(\xi)$, $H(\xi)$ is a non-zero auxiliary function. When q varies from 0 to 1, one has

$$\varphi(\xi;\,0)=u_0(\xi),\quad \varphi(\xi;\,1)=u(\xi). \tag{3.3}$$

Expanding $\varphi(\xi;\,q)$ in Taylor series with respect to q, we have

$$\varphi(\xi;\,q)=u_0(\xi)+\sum_{m=1}^{+\infty}u_m(\xi)q^m, \tag{3.4}$$

where

$$u_m(\xi)=\frac{1}{m!}\frac{\partial^m\varphi(\xi;\,q)}{\partial q^m}\bigg|_{q=0}. \tag{3.5}$$

If the auxiliary linear operator, the initial guess and the auxiliary parameter $\hbar$ are properly chosen, when $q=1$, then the solution of (3.2) has the series form as follows:

$$u(\xi)=\varphi(\xi;\,1)=u_0(\xi)+\sum_{m=1}^{+\infty}u_m(\xi), \tag{3.6}$$

The mth-order deformation equation of (3.2) in the following

$$L[u_m(\xi)-\chi_m u_{m-1}(\xi)]=\hbar H(\xi)R_m(\boldsymbol{u}_{m-1}), \tag{3.7}$$

where

$$R_m(\boldsymbol{u}_{m-1}) = \frac{1}{(m-1)!}\,\frac{\partial^{m-1} N[\varphi(\xi;\, q)]}{\partial q^{m-1}}\bigg|_{q=0}, \tag{3.8}$$

here, $\boldsymbol{u}_{m-1} = [u_0(\xi),\, u_1(\xi),\, u_2(\xi),\, u_3(\xi),\, \cdots,\, u_{m-1}(\xi)]$ and

$$\chi_m = \begin{cases} 0, & m \le 1; \\ 1, & m > 1, \end{cases} \tag{3.9}$$

If $L = \dfrac{\mathrm{d}^2}{\mathrm{d}\xi^2}$, according to (3.7)–(3.9), then the solution of (3.1) is as follows

$$\begin{aligned} u_m(\xi) &= \chi_m u_{m-1}(\xi) + \hbar L^{-1}[R_m(\boldsymbol{u}_{m-1})] \\ &= \chi_m u_{m-1}(\xi) + \hbar \int_0^{\xi}\int_0^{\xi} R_m(\boldsymbol{u}_{m-1})\mathrm{d}\eta \mathrm{d}\eta + c_0\xi + c_1. \end{aligned} \tag{3.10}$$

For simplicity, all boundary or initial conditions are ignored, and the integral constant $c_0 = 0$, $c_1 = 0$. (3.1)–(3.10) are called the standard HAM. (3.10) shows the HAM can ensure the convergence of the series solutions by convergence-controller parameter

3.2 Homotopy Perturbation Method (HPM)

Dividing nonlinear differential equation (3.1) into linear part (L) and nonlinear part (N), one has

$$Lu + Nu - f(\xi) = 0. \tag{3.11}$$

By the homotopy technique, a homotopy of (3.11) is constructed in the form

$$H(W,\, p) = (1-p)[L(W) - L(u_0)] + p[N(W) - f(\xi)] = 0,$$

where $p \in (0,\, 1)$ is an embedding parameter, $u_2(\xi)$ is an initial approximation of (3.1).

In HPM, the embedding parameter p is used as a small parameter. Thus, the solution of (3.11) can be expressed as a power series of p in the form $W = W_0 + pW_1 + p^2W_2 + \cdots$ Set $p = 1$. Then an approximate solution of (3.1) is $u = \lim\limits_{p\to 1} W = W_0 + W_1 + W_2 + \cdots$ The combination of a small parameter (perturbation parameter) with a homotopy is called homotopy perturbation method (see [41]–[43]), essentially, the HAM logically contains the HPM (see [44]).

4 The Approximate Solutions of the Navier-Stokes Equations with Perturbation Terms

4.1 The Approximate Solutions via HAM

The Equations (2.4)–(2.6) become

$$\begin{aligned} &\frac{\mathrm{d}P}{\mathrm{d}\xi} - \rho c\frac{\mathrm{d}U}{\mathrm{d}\xi} + \rho U\frac{\mathrm{d}U}{\mathrm{d}\xi} + \rho V\frac{\mathrm{d}U}{\mathrm{d}\xi} - 2\nu\rho\frac{\mathrm{d}^2U}{\mathrm{d}\xi^2} = \rho F_1, \\ &\frac{\mathrm{d}^2V}{\mathrm{d}\xi^2} + \frac{c}{2\nu}\frac{\mathrm{d}V}{\mathrm{d}\xi} - \frac{1}{2\nu}U\frac{\mathrm{d}V}{\mathrm{d}\xi} - \frac{1}{2\nu}V\frac{\mathrm{d}V}{\mathrm{d}\xi} - \frac{1}{(2\nu\rho)}\frac{\mathrm{d}P}{\mathrm{d}\xi} = \frac{1}{2\nu}F_2, \\ &\frac{\mathrm{d}U}{\mathrm{d}\xi} + \frac{\mathrm{d}V}{\mathrm{d}\xi} = F_3. \end{aligned}$$

Take initial approximations

$$\begin{aligned} V_0 &= a_2 \mathrm{e}^{-\lambda\xi} + a_1, \\ U_0 &= -a_2 \mathrm{e}^{-\lambda\xi} + a_3, \\ P_0 &= 1. \end{aligned}$$

The auxiliary linear operators are

$$LU(\xi) = \frac{\mathrm{d}U}{\mathrm{d}\xi}, \qquad LV(\xi) = \frac{\mathrm{d}^2V}{\mathrm{d}\xi^2}, \qquad LP(\xi) = \frac{\mathrm{d}P}{\mathrm{d}\xi}.$$

For $m = 1$, $\chi_m = 0$, the iterations formulas are

$$\begin{aligned} P_1 &= \hbar \int_0^\xi \left(\frac{\mathrm{d}P_0}{\mathrm{d}\eta} - \rho c \frac{\mathrm{d}U_0}{\mathrm{d}\eta} + \rho U_0 \frac{\mathrm{d}U_0}{\mathrm{d}\eta} + \rho V_0 \frac{\mathrm{d}U_0}{\mathrm{d}\eta} - 2\nu\rho \frac{\mathrm{d}^2U_0}{\mathrm{d}\eta^2} - \rho F_1 \right) \mathrm{d}\eta, \\ V_1 &= \hbar_1 \int_0^\xi \int_0^\xi \left(\frac{\mathrm{d}^2V_0}{\mathrm{d}\eta^2} + \frac{c}{2\nu} \frac{\mathrm{d}V_0}{\mathrm{d}\eta} - \frac{1}{2\nu} U_0 \frac{\mathrm{d}V_0}{\mathrm{d}\eta} - \frac{1}{2\nu} V_0 \frac{\mathrm{d}V_0}{\mathrm{d}\eta} - \frac{1}{2\rho\nu} \frac{\mathrm{d}P_0}{\mathrm{d}\eta} - \frac{1}{2\nu} F_2 \right) \mathrm{d}\eta\mathrm{d}\eta, \\ U_1 &= \hbar \int_0^\xi \left(\frac{\mathrm{d}U_0}{\mathrm{d}\eta} + \frac{\mathrm{d}V_0}{\mathrm{d}\eta} - F_3 \right) \mathrm{d}\eta. \end{aligned}$$

For $m = 1$, $\chi_m = 1$, the iterations formulas are

$$\begin{aligned} P_2 = P_1 + \hbar \int_0^\xi \bigg(& \frac{\mathrm{d}P_1}{\mathrm{d}\eta} - \rho c \frac{\mathrm{d}U_1}{\mathrm{d}\eta} + \rho U_0 \frac{\mathrm{d}U_1}{\mathrm{d}\eta} + \rho U_1 \frac{\mathrm{d}U_0}{\mathrm{d}\eta} \\ & + \rho V_0 \frac{\mathrm{d}U_1}{\mathrm{d}\eta} + \rho V_1 \frac{\mathrm{d}U_0}{\mathrm{d}\eta} - 2\nu\rho \frac{\mathrm{d}^2U_1}{\mathrm{d}\eta^2} - \rho F_1 \bigg) \mathrm{d}\eta, \\ V_2 = V_1 + \hbar_1 \int_0^\xi \int_0^\xi \bigg(& \frac{\mathrm{d}^2V_1}{\mathrm{d}\eta^2} + \frac{c}{2\nu} \frac{\mathrm{d}V_1}{\mathrm{d}\eta} - \frac{1}{2\nu} U_0 \frac{\mathrm{d}V_1}{\mathrm{d}\eta} - \frac{1}{2\nu} U_1 \frac{\mathrm{d}V_0}{\mathrm{d}\eta} \\ & - \frac{1}{2\nu} V_1 \frac{\mathrm{d}V_0}{\mathrm{d}\eta} - \frac{1}{2\nu} V_0 \frac{\mathrm{d}V_1}{\mathrm{d}\eta} - \frac{1}{2\rho\nu} \frac{\mathrm{d}P_1}{\mathrm{d}\eta} - \frac{1}{2\nu} F_2 \bigg) \mathrm{d}\eta\mathrm{d}\eta, \\ U_2 = \hbar \int_0^\xi \bigg(& \frac{\mathrm{d}U_1}{\mathrm{d}\eta} + \frac{\mathrm{d}V_1}{\mathrm{d}\eta} - F_3 \bigg) \mathrm{d}\eta. \end{aligned}$$

For $m > 1$, $\chi_m = 1$, the iterations formulas are

$$\begin{aligned} P_m = P_{m-1} + \hbar \int_0^\xi \Bigg(& \frac{\mathrm{d}P_{m-1}}{\mathrm{d}\eta} - \rho c \frac{\mathrm{d}U_{m-1}}{\mathrm{d}\eta} + \rho \sum_{j=0}^{m-1} U_j \frac{\mathrm{d}U_{m-1-j}}{\mathrm{d}\eta} \\ & + \rho \sum_{j=0}^{m-1} V_j \frac{\mathrm{d}U_{m-1-j}}{\mathrm{d}\eta} - 2\nu\rho \frac{\mathrm{d}^2U_{m-1}}{\mathrm{d}\eta^2} - \rho F_1 \Bigg) \mathrm{d}\eta, \\ V_m = V_m + \hbar_1 \int_0^\xi \int_0^\xi \Bigg(& \frac{\mathrm{d}^2V_{m-1}}{\mathrm{d}\eta^2} + \frac{c}{2\nu} \frac{\mathrm{d}V_{m-1}}{\mathrm{d}\eta} - \frac{1}{2\nu} \sum_{j=0}^{m-1} U_j \frac{\mathrm{d}V_{m-1-j}}{\mathrm{d}\eta} \\ & - \frac{1}{2\nu} \sum_{j=0}^{m-1} V_j \frac{\mathrm{d}V_{m-1-j}}{\mathrm{d}\eta} - \frac{1}{2\nu\rho} \frac{\mathrm{d}P_{m-1}}{\mathrm{d}\eta} - \frac{1}{2\nu} F_2 \Bigg) \mathrm{d}\eta\mathrm{d}\eta, \\ U_m = U_{m-1} + \hbar \int_0^\xi \Bigg(& \frac{\mathrm{d}U_{m-1}}{\mathrm{d}\eta} + \frac{\mathrm{d}V_{m-1}}{\mathrm{d}\eta} - F_3 \Bigg) \mathrm{d}\eta. \end{aligned}$$

If the perturbation terms of the systems (2.1)–(2.3), $F_1 = \varepsilon \sin\xi$, $F_2 = 0$ and $F_3 = \varepsilon\cos\xi$, where ε is a positive small parameter, then we now successively obtain

$$\begin{aligned}
P_1 =& -\hbar(a_1a_2\rho(\mathrm{e}^{-\lambda\xi}-1) - \varepsilon\rho(\cos\xi - 1) + a_2a_3\rho(\mathrm{e}^{-\lambda\xi}-1)),\\
U_1 =& -\varepsilon\hbar\sin\xi,\\
V_1 =& \frac{a_2\hbar_1\xi(a_1+a_3)}{2\nu} - \frac{\hbar_1a_2(a_1+a_3)}{c}(1-\mathrm{e}^{-\lambda\xi}),\\
P_2 =& -\hbar(a_2(a_1+a_3)\rho(\mathrm{e}^{-\lambda\xi}-1) - \varepsilon\rho(\cos\xi-1)) - \hbar(\varepsilon - \varepsilon\cos\xi + \varepsilon\hbar\rho\\
&+ 2\varepsilon\hbar\rho\nu) - (1+2\nu)\varepsilon\hbar\rho\cos\xi + a_2(a_1+a_3)\hbar\rho(\mathrm{e}^{-\lambda\xi}-1)\\
&+ (a_1+a_3-\varepsilon)\cdot\varepsilon\hbar\rho\sin\xi + \frac{a_2^2(a_1+a_3)}{2c}\hbar\rho(\mathrm{e}^{-\lambda/2\xi}-1) + \frac{a_2^2(a_1+a_3)}{2\nu}\hbar\rho\xi\mathrm{e}^{-\lambda/\xi}\\
&- \frac{a_2\varepsilon c^2\hbar\rho}{4(c^2/4+\nu^2)}\mathrm{e}^{-\lambda/\xi}\sin\xi - \frac{a_2\varepsilon c\hbar\rho}{2(c^2/(4\nu)+\nu)}(\mathrm{e}^{-\lambda/\xi}\cos\xi - 1),\\
U_2 =& \frac{a_2(a_1+a_3)\hbar^2}{c}(\mathrm{e}^{-\lambda\xi}-1) - \varepsilon\hbar\sin\xi - 2\varepsilon\hbar^2\sin\xi + \frac{a_2(a_1+a_3)\hbar^2\xi}{2\nu},\\
V_2 =& \frac{\varepsilon\hbar_1^2\xi}{2\nu} - \frac{a_2a_3^2\hbar_2^2\xi^2}{8} - \frac{\varepsilon\hbar_1^2}{2\nu}\sin\xi - \frac{(a_1+a_3)a_2^2\hbar_1^2\lambda^{-2}}{4}(1-(\lambda\xi+1)\mathrm{e}^{-\lambda\xi})\\
&- \frac{a_1a_2(a_1+2a_3)\hbar_1^2\xi^2}{8} - \frac{a_2(a_1+a_3)\hbar_1^2\xi}{2\nu} - \frac{a_2(a_1+a_3)\hbar_1^2}{c}(\mathrm{e}^{-\lambda\xi}-1)\\
&+ \frac{a_2\hbar_1\xi(a_1+a_3)}{2\nu} - \frac{a_2\varepsilon c\hbar_1^2\xi}{4(\lambda^2+1)} + \frac{a_1a_2(a_2+a_3)\hbar_1^2\nu^2}{4c^2}(\mathrm{e}^{-\lambda/2\xi}-1)\\
&+ \frac{a_2(a_1^2+a_2^2)\hbar_1^2\nu^2}{c^2}(\mathrm{e}^{-\lambda\xi}-1) - \frac{a_2\hbar_1(a_1+a_3)}{c}(1-\mathrm{e}^{-\lambda\xi})\\
&+ \frac{a_2c\hbar_1^2\xi^2(a_1+a_3)}{8\nu^2} + a_1a_2\hbar_1^2\xi\lambda\left(a_1+\frac{a_2}{2}\right) + a_2a_3\hbar_1^2\xi\lambda\left(a_3+\frac{a_2}{2}\right)\\
&+ \frac{a_2\varepsilon c^2\hbar_1^2[2c\nu(\mathrm{e}^{-\lambda\xi}\cos\xi-1) - 4\nu^2\mathrm{e}^{-\lambda\xi}\sin\xi]}{4(c^2+4\nu^2)(\lambda^2+1)} + \frac{a_1a_2a_3\hbar_1^2\nu\xi}{c}.
\end{aligned}$$

If the perturbation terms of the systems (2.1)–(2.3) $F_1 = 0$, $F_2 = 0$ and $F_3 = 0$, namely, that is no perturbation terms, then we obtain the approximation solution as follows:

$$\begin{aligned}
P_1' =& -a_2\hbar\rho(a_1+a_3)(\mathrm{e}^{-\lambda\xi}-1),\\
V_1' =& \frac{a_2\hbar_1\xi(a_1+a_3)}{2\nu} - \frac{\hbar_1a_2(a_1+a_3)}{c}(1-\mathrm{e}^{-\lambda\xi}),\\
U_1' =& 0,\\
P_2' =& -\hbar[a_2\hbar\rho(a_1+a_3)(a_2 - a_2\mathrm{e}^{-2\lambda\xi} - 2c + 2c\mathrm{e}^{-\lambda\xi}) + \frac{a_2^2\hbar\rho\xi}{2\nu}\mathrm{e}^{-\lambda\xi}(a_1+a_3)]\\
&- a_2\hbar\rho(a_1+a_3)(\mathrm{e}^{-\lambda\xi}-1),\\
V_2' =& \hbar_1\frac{a_2\xi}{2\nu}(a_1+a_3) - \frac{a_2(a_1+a_3)}{2c}(1-\mathrm{e}^{-\lambda\xi}) + \frac{a_2(a_1+a_3)\hbar_1^2\xi}{2\nu}\\
&- \frac{a_2(a_1+a_3)\hbar^2}{c}(\mathrm{e}^{-\lambda\xi}-1) + \frac{a_2^2(a_1+a_3)\hbar_1^2}{4c^2}(\mathrm{e}^{-2\lambda\xi}-1)\\
&+ \frac{a_1a_2(a_1+a_3)\hbar_1^2}{c^2}(\mathrm{e}^{-\lambda\xi}-1) - a_2^2(a_1+a_3)(1-\mathrm{e}^{-\lambda\xi} - \lambda\xi\mathrm{e}^{-\lambda\xi})\\
&- \frac{a_2(a_1^2+a_3^2)\hbar_1^2\xi^2}{8\nu^2} + \frac{a_2^2(a_1+a_3)\hbar_1^2\xi}{4cv} + \frac{a_2(a_1^2+a_3^2)\hbar^2\xi}{2cv}
\end{aligned}$$

$$+\frac{2a_1a_2a_3\hbar_1^2}{c^2}(\mathrm{e}^{-\lambda\xi}-1)-\frac{a_1a_2a_3\hbar_1^2\xi^2}{4\nu^2}+\frac{a_2(a_1+a_3)c\hbar_1^2\xi^2}{8\nu^2}+\frac{a_1a_2a_3\hbar_1^2\xi}{c\nu},$$

$$U_2'=\frac{a_2(a_1+a_3)\hbar^2}{c}(\mathrm{e}^{-\lambda\xi}-1)+\frac{a_2\hbar^2\xi}{2\nu}(a_1+a_3).$$

By the same manner, we can obtain $U_m(\xi)$, $V_m(\xi)$, $P_m(\xi)$, $U_m'(\xi)$, $V_m'(\xi)$ and $P_m'(\xi)$ $(m=3,4,\cdots)$. Then the series solution expression by HAM can be written in the form

$$\begin{aligned}
U(\xi)&=U_0(\xi)+U_1(\xi)+U_2(\xi)+\cdots,\\
V(\xi)&=V_0(\xi)+V_1(\xi)+V_2(\xi)+\cdots,\\
P(\xi)&=P_0(\xi)+P_1(\xi)+P_2(\xi)+\cdots,\\
U'(\xi)&=U_0'(\xi)+U_1'(\xi)+U_2'(\xi)+\cdots,\\
V'(\xi)&=V_0'(\xi)+V_1'(\xi)+V_2'(\xi)+\cdots,\\
P'(\xi)&=P_0'(\xi)+P_1'(\xi)+P_2'(\xi)+\cdots,
\end{aligned}$$

where $\xi=x+z-ct$, $\lambda=\dfrac{c}{2\nu}$, $\hbar$ and $\hbar_1$ are the convergence-controller parameters that can ensure the convergence of approximate solutions. $P_1(\xi)$, $V_1(\xi)$, $U_1(\xi)$, $P_1'(\xi)$, $V_1'(\xi)$, $U_1'(\xi)$ are the first approximation solutions, $P_2(\xi)$, $V_2(\xi)$, $U_2(\xi)$, $P_2'(\xi)$, $V_2'(\xi)$, $U_2'(\xi)$ are the second approximation solutions.

4.2 The Approximate Solutions via HPM

According to HPM, the homotopies are constructed as follows:

$$(1-p)(w_\xi-P_{0\xi})+p(w_\xi-\rho cu_\xi+\rho uu_\xi+\rho vu_\xi-2\nu\rho v_{\xi\xi}-\rho\varepsilon\sin\xi)=0, \tag{4.1}$$

$$(1-p)(v_{\xi\xi}-V_{0\xi\xi})+p\Big(v_{\xi\xi}+\frac{c}{2\nu}v_\xi-\frac{1}{2\nu}uv_\xi-\frac{1}{2\nu}vv_\xi-\frac{1}{2\nu\rho}w_\xi\Big)=0, \tag{4.2}$$

$$(1-p)(u_\xi-U_{0\xi})+p(u_\xi+v_\xi-\varepsilon\cos\xi)=0, \tag{4.3}$$

where

$$\begin{aligned}
w&=w_0+pw_1+p^2w_2+\cdots,\\
u&=u_0+pu_1+p^2u_2+\cdots,\\
v&=v_0+pv_1+p^2v_2+\cdots
\end{aligned}$$

The variables P, U and V can be obtained as

$$\begin{aligned}
P&=\lim_{p\to1}w=w_0+w_1+w_2+\cdots,\\
U&=\lim_{p\to1}u=u_0+u_1+u_2+\cdots,\\
V&=\lim_{p\to1}v=v_0+v_1+v_2+\cdots
\end{aligned}$$

After expanding (4.1)–(4.3), the second power of p, one obtains

$$p^0: w_{0\xi}=P_{0\xi},\quad v_{0\xi\xi}=V_{0\xi\xi},\quad u_{0\xi}=U_{0\xi}. \tag{4.4}$$

$$\begin{aligned}
p^1:\ & w_{1\xi}-\rho cu_{0\xi}+\rho u_0u_{0\xi}+\rho v_0u_{0\xi}-2\rho\nu u_{0\xi\xi}-\rho\varepsilon\sin\xi=0,\\
& v_{1\xi\xi}+\frac{c}{2\nu}v_{0\xi}-\frac{1}{2\nu}u_0v_{0\xi}-\frac{1}{2\nu}v_0v_{0\xi}-\frac{1}{2\rho\nu}w_{0\xi}=0,\\
& u_{1\xi}+v_{0\xi}-\varepsilon\cos\xi=0.
\end{aligned} \tag{4.5}$$

$$p^2: w_{2\xi}-\rho cu_{1\xi}+\rho u_0u_{1\xi}+\rho u_1u_{0\xi}+\rho v_0u_{1\xi}+\rho v_1u_{0\xi}-2\rho\nu u_{1\xi\xi}=0,$$

$$
\begin{aligned}
& v_{2\xi\xi} + \frac{c}{2\nu} v_{1\xi} - \frac{1}{2\nu} u_0 v_{1\xi} - \frac{1}{2\nu} u_1 v_{0\xi} - \frac{1}{2\nu} v_1 v_{0\xi} - \frac{1}{2\nu} v_0 v_{1\xi} - \frac{1}{2\rho\nu} w_{1\xi} = 0, \\
& u_{2\xi} + v_{1\xi} = 0.
\end{aligned} \tag{4.6}
$$

The solutions of (4.4)–(4.6) can be obtained

$$
\begin{aligned}
w_1 &= a_1 a_2 \rho(\mathrm{e}^{-\lambda\xi} - 1) - \varepsilon\rho(\cos\xi - 1) + a_2 a_3 \rho(\mathrm{e}^{-\lambda\xi} - 1), \\
v_1 &= -\frac{a_2 \xi (a_1 + a_3 - c)}{2\nu} + \frac{a_2(a_1 + a_3 - c)}{c}(1 - \mathrm{e}^{-\lambda\xi}), \\
u_1 &= \varepsilon \sin\xi - a_2(\mathrm{e}^{-\lambda\xi} - 1), \\
w_2 &= a_2^2 \rho(\mathrm{e}^{-\lambda\xi} - 1) - a_2(a_1 + a_3)\rho - 2\varepsilon\rho\nu(1 - \cos\xi) + \rho a_2(a_1 + a_3)\mathrm{e}^{-\lambda\xi} \\
&\quad - \varepsilon\rho(a_1 + a_3 + c)\sin\xi + \frac{a_2^2\rho}{2c}(a_1 + a_3) - \frac{2\nu a_2 \varepsilon c \rho}{c^2 + 2\nu^2} \\
&\quad - \frac{a_2^2\rho}{2c}(a_1 + a_3)\mathrm{e}^{-\lambda/2\xi} - \frac{a_2^2\rho\xi}{2c}(a_1 + a_3)\mathrm{e}^{-\lambda\xi} + \frac{a_2^2 c\rho\xi}{2\nu}\mathrm{e}^{-\lambda\xi} \\
&\quad - \frac{a_2^2\rho}{2c}(a_1 + a_3)\mathrm{e}^{-\lambda/2\xi} - \frac{a_2^2\rho\xi}{2c}(a_1 + a_3)\mathrm{e}^{-\lambda\xi} + \frac{a_2^2 c\rho\xi}{2\nu}\mathrm{e}^{-\lambda\xi}, \\
v_2 &= a_2(\mathrm{e}^{-\lambda\xi} - 1) + \frac{\xi}{2\nu}(\varepsilon - a_2^2) - \frac{\varepsilon}{2\nu}\sin\xi - \frac{a_2^2}{c}(\mathrm{e}^{-\lambda\xi} - 1) \\
&\quad + \frac{a_2^2(a_1 + a_3)}{4c_2^2}(\mathrm{e}^{-\lambda/2\xi} - 1) + \frac{a_2(a_1^2 + a_3^2)}{c_2^2}(\mathrm{e}^{-\lambda\xi} - 1) - \frac{3a_2(a_1 + a_3)\xi}{2\nu} \\
&\quad + \frac{a_2 c\xi}{2\nu} - (a_1 + a_3 + c)a_2^2 c^{-2}\left[1 - \left(\frac{c\xi}{2\nu} + 1\right)\mathrm{e}^{-\lambda\xi}\right] \\
&\quad - \frac{(3(a_1 + a_3)c + 2a_1 a_3)a_2}{c^2}(\mathrm{e}^{-\lambda\xi} - 1) + \frac{a_1 a_2 \xi}{4c\nu}(a_2 + 2a_1) + \frac{a_2 a_3 \xi}{4c\nu}(2a_3 + a_2) \\
&\quad - \frac{a_2\xi^2}{8\nu^2}(a_1 + a_3 - c)^2 + \frac{a_1 a_2 a_2 \xi}{c\nu} - \frac{a_2 \varepsilon c \xi}{c^2 + 4\nu^2} + a_2 \varepsilon 2\lambda^2[(\lambda^2 + 1)^{-2} \\
&\quad - (\cos\xi + \lambda\sin\xi)]\mathrm{e}^{-\lambda\xi} - \frac{a_2 \varepsilon c[\lambda(\mathrm{e}^{-\lambda\xi}\cos\xi - 1) - \mathrm{e}^{-\lambda\xi}\sin\xi]}{c^2\lambda^2 + 2c^2 + 4\nu^2}, \\
u_2 &= \frac{a_2(a_1 + a_3 - c)}{c(\mathrm{e}^{-\lambda\xi} - 1)} + \frac{a_2(a_1 + a_3 - c)\xi}{2\nu},
\end{aligned}
$$

and so on.

Finally, the approximated solutions are given by

$$
\begin{aligned}
P &= w_0 + w_1 + w_2 + \cdots, \\
U &= u_0 + u_1 + u_2 + \cdots, \\
V &= v_0 + v_1 + v_2 + \cdots
\end{aligned}
$$

5 Comparison and Graphical Representations of the Approximation Solutions

In this section, to illustrate the efficiency of the method, comparisons are made. The reference [40] obtains the exact solutions of the systems of (2+1)-dimensional Navier-Stokes equations without perturbation terms by the expansion of the exp-function method, i.e.,

$$
\begin{aligned}
&P_{\mathrm{EXACT}}(\xi) = \frac{a_1'' s b_0 \mathrm{e}^{\xi} + a_{-1}\rho(\mathrm{i}w + \mathrm{i}a_1' s - s a_1)}{s b_0 \mathrm{e}^{\xi}},\\
&V_{\mathrm{EXACT}}(\xi) = \frac{a_1' b_0 \mathrm{e}^{\xi} + a_1' b_0 b_0' - \mathrm{i}a_{-1} - \mathrm{i}b_0' a_{-1}\mathrm{e}^{-\xi}}{b_0(\mathrm{e}^{\xi} + b_0')},\\
&U_{\mathrm{EXACT}}(\xi) = \frac{a_1 b_0 \mathrm{e}^{\xi} + a_{-1} + a_1 b_0^2 + a_{-1} b_0 \mathrm{e}^{-\xi}}{b_0(\mathrm{e}^{\xi} + b_0)},\\
&\xi = \mathrm{i}sx + sz + wt,
\end{aligned}
$$

where the subscript EXACT is abbreviation of exact solution.

When there are no perturbation terms, the 3rd-order approximation solutions by HAM are

$$
\begin{aligned}
P'_{\mathrm{HAM}}(x,\, z,\, t) &\cong P'_0(x,\, z,\, t) + P'_1(x,\, z,\, t) + P'_2(x,\, z,\, t),\\
U'_{\mathrm{HAM}}(x,\, z,\, t) &\cong U'_0(x,\, z,\, t) + U'_1(x,\, z,\, t) + U'_2(x,\, z,\, t),\\
V'_{\mathrm{HAM}}(x,\, z,\, t) &\cong V'_0(x,\, z,\, t) + V'_1(x,\, z,\, t) + V'_2(x,\, z,\, t).
\end{aligned}
$$

We now compare the approximation solutions of the (2+1)-dimensional Navier-Stokes equations without perturbation terms by HAM with the exact solutions by the expansion of the exp-function method. For some values, $a_1 = 1$, $a_2 = 2$, $a_3 = 2$, $\nu = 0.8$, $\hbar = 0.05$, $\hbar_1 = -0.02$, $c = 1$, $\rho = 1.29$, $x = 0.5$, $0 \le t \le 3$, $0 \le z \le 20$, in order to be consistent with approximation solutions by HAM, taking $s = 1$, $w = -1$; other parameters: $a_1 = -3$, $b_0 = 3$, $a_{-1} = 3$, $a_1' = 2$, $a_1'' = 2$, $b_0' = -2$, the graphical representations of $P'_{\mathrm{HAM}}(0.5,\, z,\, t)$, $V'_{\mathrm{HAM}}(0.5,\, z,\, t)$, $U'_{\mathrm{HAM}}(0.5,\, z,\, t)$, $P_{\mathrm{EXACT}}(0.5,\, z,\, t)$, $V_{\mathrm{EXACT}}(0.5,\, z,\, t)$, $U_{\mathrm{EXACT}}(0.5,\, z,\, t)$, are shown in Fig. 5.1(a)–(f), respectively.

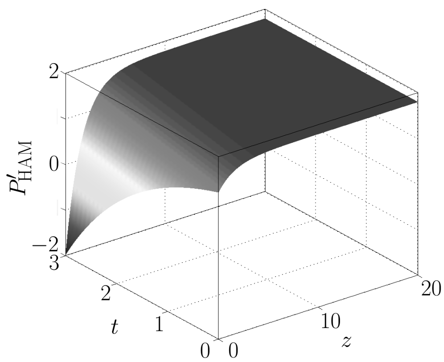
When $x = 0.5$, the 3rd-order approximation solutions by HAM of the (2+1)-dimensional Navier-Stokes equations with perturbation terms are given by

$$
\begin{aligned}
&P = P_{\mathrm{HAM}}(0.5,\, z,\, t) \cong P_0(0.5,\, z,\, t) + P_1(0.5,\, z,\, t) + P_2(0.5,\, z,\, t),\\
&U_{\mathrm{HAM}}(0.5,\, z,\, t) \cong U_0(0.5,\, z,\, t) + U_1(0.5,\, z,\, t) + U_2(0.5,\, z,\, t),\\
&V_{\mathrm{HAM}}(0.5,\, z,\, t) \cong V_0(0.5,\, z,\, t) + V_1(0.5,\, z,\, t) + V_2(0.5,\, z,\, t).
\end{aligned}
$$

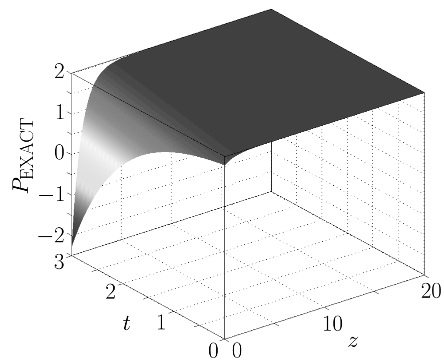
Meanwhile, the 3rd-order approximation solutions by HPM are given by

$$
\begin{aligned}
P_{\mathrm{HPM}}(0.5,\, z,\, t) &\cong w_0(0.5,\, z,\, t) + w_1(0.5,\, z,\, t) + w_2(0.5,\, z,\, t),\\
U_{\mathrm{HPM}}(0.5,\, z,\, t) &\cong u_0(0.5,\, z,\, t) + u_1(0.5,\, z,\, t) + u_2(0.5,\, z,\, t),\\
V_{\mathrm{HPM}}(0.5,\, z,\, t) &\cong v_0(0.5,\, z,\, t) + v_1(0.5,\, z,\, t) + v_2(0.5,\, z,\, t).
\end{aligned}
$$

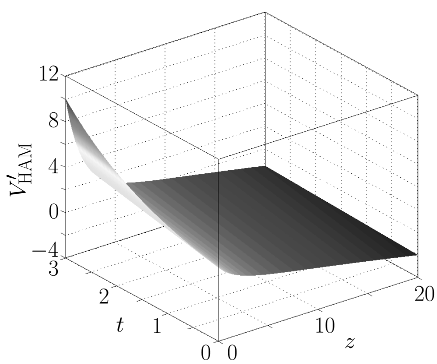
The 3rd-order approximation solutions by HAM of the (2+1)-dimensional Navier-Stokes equations with perturbation terms are compared with 3rd-order approximation solutions by HPM, the graphical representations of $P_{\mathrm{HAM}}(0.5,\, z,\, t)$, $P_{\mathrm{HPM}}(0.5,\, z,\, t)$, $V_{\mathrm{HAM}}(0.5,\, z,\, t)$, $V_{\mathrm{HPM}}(0.5,\, z,\, t)$, $U_{\mathrm{HAM}}(0.5,\, z,\, t)$, $U_{\mathrm{HPM}}(0.5,\, z,\, t)$ are shown in Fig. 5.2(a)–(f), respectively. $\varepsilon = 0.2$, the remaining parameters are the same as the above.



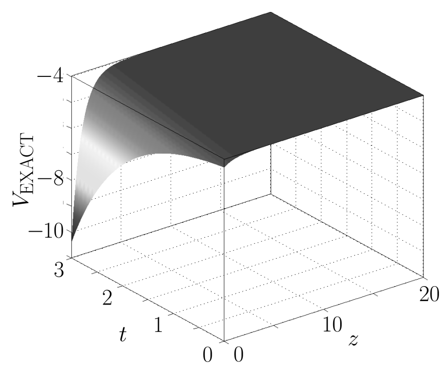
(a) $P'_{\rm HAM}(0.5,\, z,\, t)$



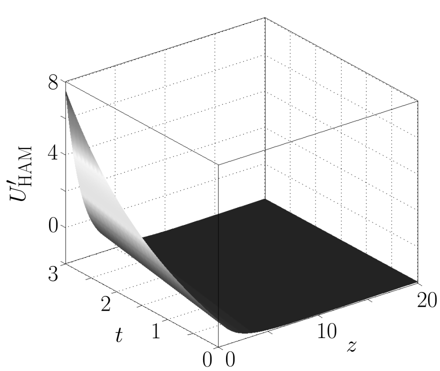
(b) $P_{\rm EXACT}(0.5,\, z,\, t)$



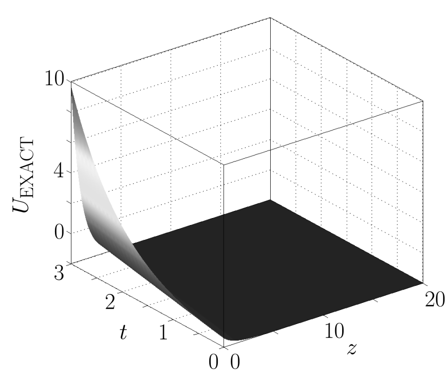
(c) $V'_{\rm HAM}(0.5,\, z,\, t)$



(d) $V_{\rm EXACT}(0.5,\, z,\, t)$

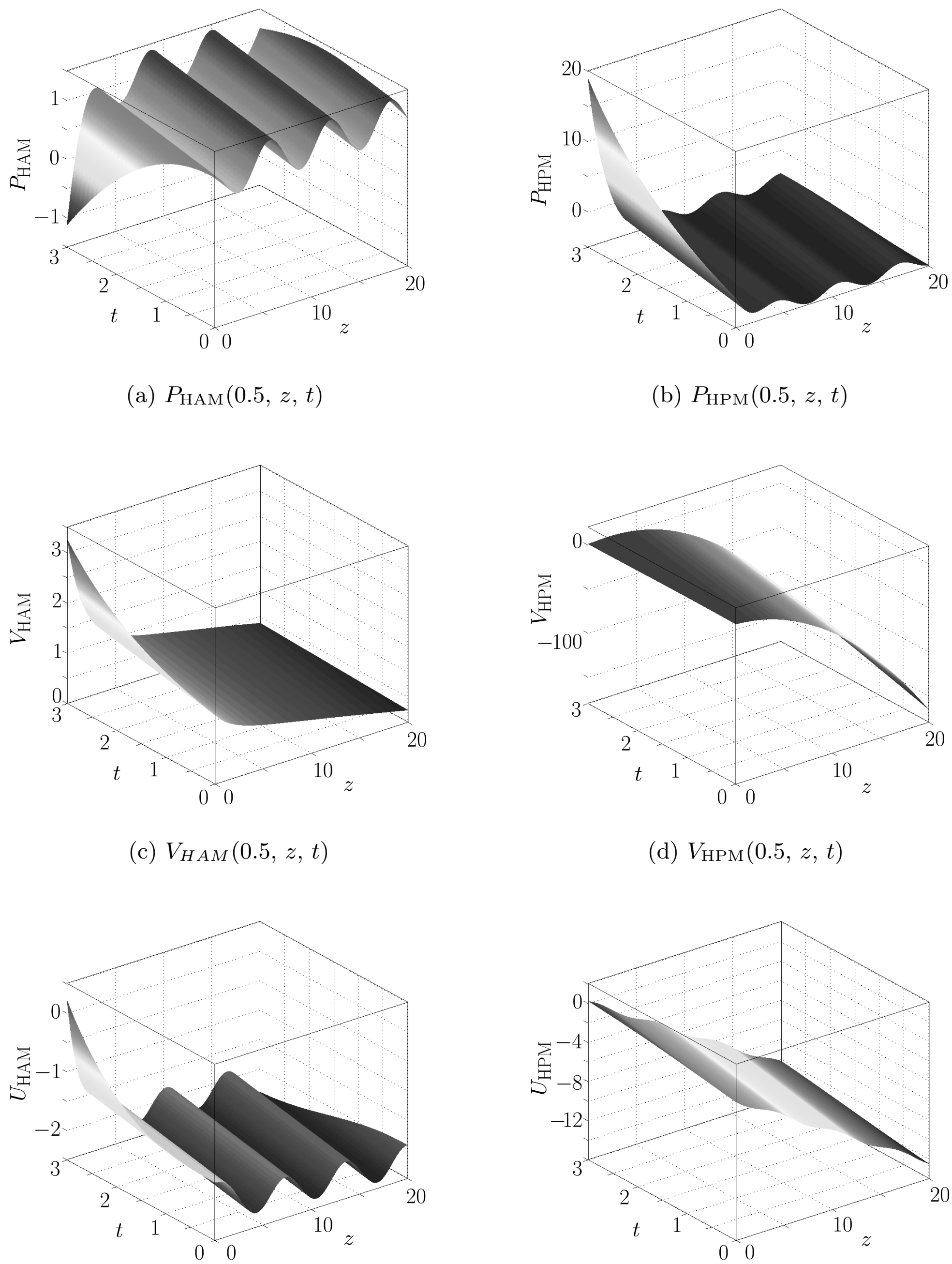


(e) $U'_{\rm HAM}(0.5,\, z,\, t)$



(f) $U_{\rm EXACT}(0.5,\, z,\, t)$

Fig. 5.1



(a) $P_{\mathrm{HAM}}(0.5,\ z,\ t)$

(b) $P_{\mathrm{HPM}}(0.5,\ z,\ t)$

(c) $V_{HAM}(0.5,\ z,\ t)$

(d) $V_{\mathrm{HPM}}(0.5,\ z,\ t)$

(e) $U_{\mathrm{HAM}}(0.5,\ z,\ t)$

(f) $U_{\mathrm{HPM}}(0.5,\ z,\ t)$

Fig. 5.2

6 Discussion

From Fig. 5.1 we observe that the solutions by HAM of $P'_{\mathrm{HAM}}(0.5,\ z,\ t)$ and $P_{\mathrm{EXACT}}(0.5,\ z,\ t)$, $U'_{\mathrm{HAM}}\ (0.5,\ z,\ t)$ and $U_{\mathrm{EXACT}}(0.5,\ z,\ t)$ are basically in good agreement, but $V'_{\mathrm{HAM}}(0.5,\ z,\ t)$ and $V_{\mathrm{EXACT}}\ (0.5,\ z,\ t)$ have some differences, because some parameters value are different. It is seen that from Fig. 5.2, due to the perturbation terms, the HAM solution and the HPM solution of have great fluctuation characteristics and the HAM solution more rapid convergence. The comparisons of Fig. 5.1 and Fig. 5.2 reveal that the perturbation terms have great effect to the (2+1)-dimensional Navier-Stokes equations, thus the perturbation effects cannot neglected in the traveling wave propagation. So HAM is very effective and convenient for solving nonlinear partial differential equations and has the great validity and potential.

7 Conclusions

Navier-Stokes equations are the most important equations in fluid dynamics for finding the velocity and pressure functions. In this study, we use the HAM for finding the 3rd-order approximate solutions of (2+1)-dimensional Navier-Stokes equations without the perturbation terms and with the perturbation terms. The 3rd-order approximate solutions of (2+1)-dimensional Navier-Stokes equations without perturbation terms by HAM are compared with the exact solutions by the expansion of the exp-function method, the 3rd-order approximate solutions of (2+1)-dimensional Navier-Stokes equations with perturbation terms by HAM are compared with the 3rd-order approximate solutions by HPM. The results show that a rapid convergence and a high accuracy of the HAM for solved (2+1)-dimensional Navier-Stokes equations and the perturbation terms have considerable effects in the traveling wave propagation.

References

[1] Sweilam N H, Khader M M. Variational iteration method for one dimensional nonlinear thermoelasticity. *Chaos, Solitons Fractals*, 2007, **32**(1): 145–149.

[2] Turkyilmazoglu M. An optimal variational iteration method. *Appl. Math. Lett.*, 2011, **24**(5): 762–765.

[3] He J H. Notes on the optimal variational iteration method. *Appl. Math. Lett.*, 2012, **25**(10): 1579–1581.

[4] He J H, Kong H Y, Hu M S, Chen Q L. Variational iteration method for Bratu-like equation arising in electrospinning. *Carbohydrate Polymers*, 2014, **105**(1): 229–230.

[5] Wu G C, Baleanu D, Deng Z G. Variational iteration method as a kernel constructive technique. *Appl. Math. Model.*, 2015, **39**(15): 4378–4384.

[6] Aslanov A. A homotopy-analysis approach for nonlinear wave-like equations with variable coefficients. *Abstr. Appl. Anal.*, 2015, Article ID 628310, 7pp.

[7] Van Gorder R A. Gaussian waves in the Fitzhugh-Nagumo equation demonstrate one role of the auxiliary function $H(x, t)$ in the homotopy analysis method. *Commun. Nonlinear Sci. Numer. Simul.*, 2012, **17**(3): 1233–1240.

[8] Khader M M, Kumar S, Abbasbandy S. New homotopy analysis transform method for solving the discontinued problems arising in nanotechnology. *Chinese Phys. B*, 2013, **22**(11): 135–139.

[9] Das D, Ray P C, Bera R K, Sarkar P. Solution of nonlinear fractional differential equation by homotopy analysis method. *Int. J. Sci. Res. Educ.*, 2015, **3**(3): 3085–3103.

[10] Liao S J. Homotopy Analysis Method in Nonlinear Differential Equations. Heidelberg: Springer and Higher Education Press, 2012.

[11] Rashidi M M, Erfani E. New analytic method for solving Burgers' and nonlinear heat transfer equations and comparison with HAM. *Comput. Phy. Commun.*, 2009, **180**(9): 1539–1544.

[12] Mittal R C, Jiwari R. A differential quadrature method for numerical solutions of Burgers'-type equations. *Internat. J. Numer. Methods Heat Fluid Flow*, 2012, **22**(6-7): 880–895.

[13] Mittal R C, Jiwari R. Differential quadrature method for numerical solution of coupled viscous Burgers' equations. *Int. J. Comput. Methods Eng. Sci. Mech.*, 2012, **13**(2): 88–92.

[14] Hassan H N, El-Tawil M A. A new technique of using homotopy analysis method for second order nonlinear differential equations. *Appl. Math. Comput.*, 2012, **219**(2): 708–728.

[15] Gupta V, Gupta S. Application of homotopy analysis method for solving nonlinear Cauchy problem. *Surv. Math. Appl.*, 2012, **7**: 105–116.

[16] Wang Q. Approximate solution for system of differential-difference equations by means of the homotopy analysis method. *Appl. Math. Comput.*, 2010, **217**(8): 4122–4128.

[17] Shidfar A, Babaei A, Molabahrami A, Alinejadmofrad M. Approximate analytical solutions of the nonlinear reaction-diffusion-convection problems. *Math. Comput. Model.*, 2011, **53**(1): 261–268.

[18] Mesloub S, Obaida S. On the application of the homotopy analysis method for a nonlocal mixed problem with Bessel operator. *Appl. Math. Comput.*, 2012, **219**(8): 3477–3485.

[19] Xu H, Cang J. Analysis of a time fractional wave-like equation with the homotopy analysis method. *Phys. Lett. A*, 2008, **372**(8): 1250–1255.

[20] Zurigat M, Momani S, Odibat Z, Alawneh A. The homotopy analysis method for handling systems of fractional differential equations. *Appl. Math. Model.*, 2010, **34**(1): 24–35.

[21] Abdulaziz O, Bataineh A, Hashim I. On convergence of homotopy analysis method and its modification for fractional modified KdV equations. *J. Appl. Math. Comput.*, 2010, **33**(1): 61–81.

[22] Vahdati S, Abbas Z. Application of homotopy analysis method to Fredholm and Volterra integral equations. *Math. Sci.*, 2010, **4**(3): 267–282.

[23] Abbasbandy S. The application of homotopy analysis method to nonlinear equations arising in heat transfer. *Phys. Lett. A*, 2006, **360**(1): 109–113.

[24] Abbasbandy S. Homotopy analysis method for heat radiation equations. *Int. Commun. Heat Mass Transfer*, 2007, **34**(3): 380–387.

[25] Domairry G, Nadim N. Assessment of homotopy analysis method and homotopy perturbation method in non-linear heat transfer equation. *Int. Commun. Heat Mass Transfer*, 2008, **35**(1): 93–102.

[26] Raftari B, Parvaneh F, Vajravelu K. Homotopy analysis of the magnetohydrodynamic flow and heat transfer of a second grade fluid in a porous channel. *Energy*, 2013, **59**: 625–632.

[27] You X C, Xu H, Pop I. Homotopy analysis of unsteady heat transfer started impulsively from rest along a symmetric wedge. *Int. Commun. Heat Mass Transfer*, 2010, **37**(1): 47–51.

[28] Hosseini K , Daneshian B, Amanifard N, Ansari R. Homotopy analysis method for a fin with temperature dependent internal heat generation and thermal conductivity. *Int. J. Nonlinear Sci.*, 2012, **14**(2): 201–210.

[29] Chauhan D, Agrawal R, Rastogi P. Magnetohydrodynamic slip flow and heat transfer in a porous medium over a stretching cylinder: homotopy analysis method. *Numer. Heat Transfer*, 2012, **62**(2): 136–157.

[30] Shidfar A, Molabahrami A. A weighted algorithm based on the homotopy analysis method: application to inverse heat conduction problems. *Commun. Nonlinear Sci. Numer. Simul.*, 2010, **15**(10): 2908–2915.

[31] Turkyilmazoglu M. A note on the homotopy analysis method. *Appl. Math. Lett.*, 2010, **23**(10): 1226–1230.

[32] Odibat Z M. A study on the convergence of homotopy analysis method. *Appl. Math. Comput.*, 2010, **217**(2): 782–789.

[33] Turkyilmazoglu M. Convergence of the homotopy analysis method. *Physics*, arXiv, 2010: 1006.4460v1.

[34] Turkyilmazoglu M. Some issues on HPM and HAM methods: a convergence scheme. *Math. Comput. Modelling*, 2011, **53**(9): 1929–1936.

[35] Russo M, Van Gorder R A. Control of error in the homotopy analysis of nonlinear Klein-Gordon initial value problems. *Appl. Math. Comput.*, 2013, **219**: 6494–6509.

[36] Abbasbandy S, Jalili M. Determination of optimal convergence-control parameter value in homotopy analysis method. *Numer. Algorithms*, 2013, **64**(4): 593–605.

[37] Ou Y B, Ren D D. Incompressible limit of global strong solutions to 3-D barotropic Navier-Stokes equations with well-prepared initial data and Navier's slip boundary conditions. *J. Math. Anal. Appl.*, 2014, **420**(2): 1316–1336.

[38] Benameur J. Long time decay to the Lei-Lin solution of 3D Navier-Stokes equations. *J. Math. Anal. Appl.*, 2015, **422**(1): 424–434.

[39] Chen W Y. A note on regularity criterion for the Navier-Stokes equations in terms of the pressure. *Appl. Math. Comput.*, 2014, **248**(1): 1–3.

[40] Ghogdi S E, Ghomanjani F, Saberi-Nadjafi J. Expansion of the Exp-function method for solving systems of two-dimensional Navier-Stokes equations. *J. Taibah Univer. Sci.*, 2015, **9**(1): 121–125.

[41] Iftikhar S S S M. Use of homotopy perturbation method for solving multi-point boundary value problems. *Res. J. Appl. Sci. Engineer. Tech.*, 2014, **7**(4): 778–785.

[42] Gupta A K, Saha R S. Comparison between homotopy perturbation method and optimal homotopy asymptotic method for the soliton solutions of Boussinesq-Burger equations. *Comput. Fluids*, 2014, **103**(258): 34–41.

[43] Fernández F M. On the homotopy perturbation method for Boussinesq-like equations. *Appl. Math. Comput.*, 2014, **230**(2): 208–210.

[44] Rashidi M M, Domairry G, Dinarvand S. Approximate solutions for the Burger and regularized long wave equations by means of the homotopy analysis method. *Commun. Nonlinear Sci. Numer. Simul.*, 2009, **14**(3): 708–717.