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Reversible Properties of Monoid Crossed Products

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Abstract: We study the reversible properties of monoid crossed products. The new class of strongly CM-reversible rings is introduced and characterized. This class of rings is a generalization of those of strongly reversible rings, skew strongly reversible rings and strongly M-reversible rings. Some well-known results on this subject are generalized and extended.

Key words: monoid crossed product, strongly reversible ring, strongly CM-reversible ring

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1 Introduction

Throughout, unless otherwise indicated, R denotes an associative ring with identity and M is a monoid. In [1], Cohn introduced the notion of a reversible ring. A ring R is said to be reversible if $ab = 0$ implies $ba = 0$ for all $a, b \in R$. Anderson and Camillo[2] used the term of ZC_2 for what is called reversible. It was proved in [3] that polynomial rings over reversible rings need not be reversible. A ring R is called reduced if it has no non-zero nilpotent elements (see [4]), i.e., $a^2 = 0$ implies $a = 0$ for all $a \in R$. Recall from [5] that a ring R is strongly reversible if polynomials $f(x)$, $g(x) \in R[x]$ with $f(x)g(x) = 0$ implies $g(x)f(x) = 0$. It is clear that all reduced rings are strongly reversible, but the inverse is not true. Rage and Chhawchharia[6] introduced the concept of an Armendariz ring. A

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ring R is an Armendariz ring, whenever polynomials $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$ are in $R[x]$ and if $f(x)g(x) = 0$, then $a_ib_j = 0$ for all i, j. In the following, we denote by $R[M]$ the monoid ring constructed from ring R and the monoid M, and e always stands for the identity of M. According to [7], a ring R is called an M-Armendariz if $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n$, $\beta = b_1h_1 + b_2h_2 + \cdots + b_mg_m \in R[M]$ satisfy $\alpha\beta = 0$, then $a_ib_j = 0$ for all i, j. A ring R is strongly M-reversible if $\alpha\beta = 0$ implies $\beta\alpha = 0$ for all $\alpha, \beta \in R[M]$ (see [8]). Recall from [9] that a ring R is skew strongly M-reversible whenever $\alpha\beta = 0$ implies $\beta\alpha = 0$, where $\alpha, \beta \in R * M$.

A monoid M is a u.p.-monoid (unique product monoid) if for any two nonempty finite subsets $A, B \in M$ there exists an element $g \in M$ uniquely in the form of ab with $a \in A$ and $b \in B$. If there exists a monoid homomorphism $\omega : M \to \mathrm{Aut}(R)$, we denote by $\omega_g(r)$ the image of r under $\omega(g)$ with $g \in M$ and $r \in R$. We can form a skew monoid ring $R * M$ (see [10]) (induced by the monoid homomorphism ω) by taking its elements to be finite formal combinations $\sum\limits_{i=1}^{n} a_ig_i$ with the multiplication induced by $(ag)(bh) = (a\omega_g(b))(gh)$. The map $\omega : M \to \mathrm{Aut}(R)$ defined by $\omega_g(r) = r$ for each $g \in M$ and $r \in R$ is called the trivial monoid homomorphism. More generally, if R is a ring and M is a monoid, then the crossed product $R\sharp M$ over R consists of all finite sums $R\sharp M = \{\sum r_gg \mid r_g \in R,\ g \in M\}$ with addition defined componentwise and multiplication defined by the distributive law and two rules that are called the twisting and the action explained below. Specifically, we have the twisting operation $gh = f(g, h)gh$ for every $g, h \in M$, where $f : M \times M \to U = U(R)$. For every $r \in R$ and $g \in M$, we have $gr = \omega_g(r)g$ with $\omega : M \to \mathrm{Aut}(R)$. If $R\sharp M$ is the crossed product over R, then the twisted function f and the weak action ω of M on R must satisfy

$$\begin{aligned}
&\omega_g(\omega_h(r)) = f(g, h)\omega_{gh}(r)f(g, h)^{-1},\\
&\omega_g(f(h, k))f(g, hk) = f(g, h)f(gh, k),\\
&f(e, g) = f(g, e) = 1
\end{aligned}$$

for all g, h, $k \in M$.

Monoid crossed products are a quite general ring construction. Let $R\sharp M$ be a monoid crossed product with twisting f and action ω. If the twisting f is trivial, i.e., $f(x, y) = 1$ for all x, $y \in M$, then $R\sharp M$ is the skew monoid ring $R * M$. If the action ω is trivial, i.e., $\omega_g = i_R$ with i_R the identity map over R, then $R\sharp M$ is the twisted monoid ring $R^\tau[M]$. If both the twisting f and the action ω are trivial, then $R\sharp M$ is a monoid ring, denoted by $R[M]$. Motivated by the results of [3], [5], [8] and [9], in this paper we introduce and study the concept of strongly CM-reversible rings, which is a generalization of strongly reversible rings, strongly M-reversible rings and skew strongly M-reversible rings. The main idea is to study the reversible condition defined for the monoid ring crossed product $R\sharp M$. It is shown that if R is an M-rigid ring, then R is strongly CM-reversible. Moreover, if R is a right Ore ring with classical right quotient ring Q, then we show that R is strongly CM-reversible if and only if Q is strongly CM-reversible. Suppose that R/I is strongly CM-reversible for some ω-invariant ideal I of R. If I is an M-rigid ring, it is proved that R is strongly

CM-reversible. Some well-known results on this subject are generalized and extended.

2 Main Results

In this section, we introduce the notion of strongly CM-reversible rings and investigate its properties. Some characterizations of this class of rings are given.

We start with the following definition.

Definition 2.1 *Let R be a ring, M be a monoid with a twisting $f\colon M\times M\to U(R)$ and an action $\omega\colon M\to \mathrm{Aut}(R)$. We call that the ring R is a strongly CM-reversible ring if $\alpha\beta=0$ implies $\beta\alpha=0$ for all $\alpha,\beta\in R\sharp M$.*

Remark 2.1 Let R be a strongly CM-reversible ring. Then we have the following facts:

(1) If R is an arbitrary ring and $M=\{e\}$, then the trivial monoid homomorphism $\omega\colon M\to\mathrm{Aut}(R)$ is the only monoid homomorphism and the twisting f is trivial. Clearly, R is strongly CM-reversible if and only if R is strongly M-reversible.

(2) Let $M=(\mathbf{N},+)$. If the monoid homomorphism $\omega\colon M\to\mathrm{Aut}(R)$ and the twisting f are trivial, then it is clear that a ring R is strongly CM-reversible if and only if R is strongly M-reversible if and only if R is strongly reversible.

(3) If the twisting f is trivial, then the class of strongly CM-reversible rings is precisely the class of skew strongly M-reversible rings.

(4) If R is a strongly CM-reversible ring with a trivial twisting f, then every M-invariant subring S (i.e., $\omega_g(S)\subseteq S$ for all $g\in M$) is also strongly CM-reversible.

The next proposition gives the relationship between the strongly CM-reversible property of a ring R and that of its subrings induced by a central idempotent.

Proposition 2.1 *Let R be a ring, M be a monoid with a twisting $f\colon M\times M\to U(R)$ and an action $\omega\colon M\to\mathrm{Aut}(R)$. If a is a central idempotent of R such that $\omega_g(a)=a$ for each $g\in M$, then the following statements are equivalent:*

(1) *R is a strongly CM-reversible ring;*

(2) *aR and $(1-a)R$ are strongly CM-reversible rings.*

Proof. (1) $\Rightarrow$ (2). It is straightforward.

(2) $\Rightarrow$ (1). Let aR and $(1-a)R$ be strongly CM-reversible rings. Suppose that $\alpha=a_1g_1+\cdots+a_ng_n$, $\beta=b_1h_1+\cdots+b_mh_m\in R\sharp M$ such that $\alpha\beta=0$. Let

$$\alpha_1=\sum_{i=1}^{n}aa_ig_i,\quad \beta_1=\sum_{j=1}^{m}ab_jh_j,\quad \alpha_2=\sum_{i=1}^{n}(1-a)a_ig_i,\quad \beta_2=\sum_{j=1}^{m}(1-a)b_jh_j.$$

It is easy to see that α_1, $\beta_1\in(aR)\sharp M$ and α_2, $\beta_2\in((1-a)R)\sharp M$. Since a is a central idempotent of R such that $\omega_g(a)=a$ for each $g\in M$, we have

$$\begin{aligned}\alpha_1\beta_1&=aa_1\omega_{g_1}(ab_1)f(g_1,\,h_1)g_1h_1+\cdots+aa_n\omega_{g_n}(ab_m)f(g_n,\,h_m)g_nh_m\\&=aa_1\omega_{g_1}(a)\omega_{g_1}(b_1)f(g_1,\,h_1)g_1h_1+\cdots+aa_n\omega_{g_n}(a)\omega_{g_n}(b_m)f(g_n,\,h_m)g_nh_m\end{aligned}$$

$$
\begin{aligned}
&= aa_1a\omega_{g_1}(b_1)f(g_1,\, h_1)g_1h_1 + \cdots + aa_na\omega_{g_n}(b_m)f(g_n,\, h_m)g_nh_m\\
&= a^2a_1\omega_{g_1}(b_1)f(g_1,\, h_1)g_1h_1 + \cdots + a^2a_n\omega_{g_n}(b_m)f(g_n,\, h_m)g_nh_m\\
&= aa_1\omega_{g_1}(b_1)f(g_1,\, h_1)g_1h_1 + \cdots + aa_n\omega_{g_n}(b_m)f(g_n,\, h_m)g_nh_m\\
&= a(a_1\omega_{g_1}(b_1)f(g_1,\, h_1)g_1h_1 + \cdots + a_n\omega_{g_n}(b_m)f(g_n,\, h_m)g_nh_m)\\
&= a\alpha\beta\\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
\alpha_2\beta_2 &= (1-a)a_1\omega_{g_1}((1-a)b_1)f(g_1,\, h_1)g_1h_1 + \cdots\\
&\quad + (1-a)a_n\omega_{g_n}((1-a)b_m)f(g_1,\, h_1)g_nh_m\\
&= (1-a)a_1\omega_{g_1}(b_1)f(g_1,\, h_1)g_1h_1 + \cdots + (1-a)a_n\omega_{g_n}(b_m)f(g_n,\, h_m)g_nh_m\\
&= (1-a)(a_1\omega_{g_1}(b_1)f(g_1,\, h_1)g_1h_1 + \cdots + a_n\omega_{g_n}(b_m)f(g_n,\, h_m)g_nh_m)\\
&= (1-a)\alpha\beta\\
&= 0.
\end{aligned}
$$

Because aR and $(1-a)R$ are strongly CM-reversible subrings of R, we conclude that

$$\beta_1\alpha_1 = 0, \qquad \beta_2\alpha_2 = 0.$$

Therefore, we have

$$\beta\alpha = \beta_1\alpha_1 + \beta_2\alpha_2 = a\beta\alpha + (1-a)\beta\alpha = 0.$$

This implies that R is strongly CM-reversible. The proof is completed.

According to Krempa[11], an endomorphism α of a ring R is rigid if $a\alpha(a) = 0$ implies that $a = 0$ for $a \in R$. A ring R is α-rigid if there exists a rigid endomorphism α of R. A ring R is α-compatible if for every $a, b \in R$, $ab = 0$ if and only if $a\alpha(b) = 0$. By Lemma 2.2 of [12], a ring R is α-rigid if and only if R is α-compatible and reduced.

For a ring R and a monoid M with $\omega\colon M \to \mathrm{End}(R)$ a monoid homomorphism, we say that R is M-compatible (resp., M-rigid) if ω_g is compatible (resp., rigid) for any $g \in M$.

Corollary 2.1 *Let R be an M-compatible ring and M be a monoid with a twisting $f\colon M\times M \to U(R)$ and an action $\omega\colon M \to \mathrm{Aut}(R)$. If a is a central idempotent of R, then R is strongly CM-reversible if and only if aR and $(1-a)R$ are both strongly CM-reversible.*

Proof. If R is M-compatible, then $\omega_g(a) = a$ for each idempotent $a \in R$ and $g \in M$ by Lemma 2.11 of [13], and the result follows from Proposition 2.1. This completes the proof.

According to [14], a ring R is said to be CM-Armendariz if $\alpha = a_1g_1 + \cdots + a_ng_n$, $\beta = b_1h_1 + \cdots + b_mh_m \in R\sharp M$ such that $\alpha\beta = 0$, then $a_i\omega_{g_i}(b_j) = 0$ for all i, j.

Lemma 2.1 *Let R be a ring and M be a u.p.-monoid with a twisting $f\colon M\times M \to U(R)$ and an action $\omega\colon M \to \mathrm{Aut}(R)$. If R is M-rigid, then $R\sharp M$ is reduced.*

Proof. Suppose that $\alpha = a_1g_1 + \cdots + a_ng_n \in R\sharp M$ such that $\alpha^2 = 0$. Then R is a CM-Armendariz ring by Proposition 2.2 of [14], and thus $a_i\omega_{g_i}(a_j) = 0$ for all i, j. Since every M-rigid ring is M-compatible and reduced, we have $a_i = 0$ for all $1 \le i \le n$. It follows that $\alpha = 0$. This implies that $R\sharp M$ is reduced.

Corollary 2.2 *Let M be a u.p.-monoid and R be a reduced ring. Then $R[M]$ is reduced.*

Proposition 2.2 *Let M be a u.p.-monoid with a twisting $f\colon M\times M\to U(R)$ and an action $\omega\colon M\to \mathrm{Aut}(R)$. If R is an M-rigid ring, then R is strongly CM-reversible.*

Proof. Let $\alpha=\sum\limits_{i=1}^{n}a_ig_i$, $\beta=\sum\limits_{j=1}^{m}b_jh_j\in R\sharp M$ with $\alpha\beta=0$, where $a_i, b_j\in R$ and $g_i, h_j\in M$ for each i, j. Then we have

$$(\beta\alpha)^2=(\beta\alpha)(\beta\alpha)=\beta(\alpha\beta)\alpha=0.$$

Since R is M-rigid, we have $\beta\alpha=0$ by Lemma 2.1. Hence R is strongly CM-reversible.

Lemma 2.2 *Direct products of strongly CM-reversible rings are strongly CM-reversible.*

Proposition 2.3 *Let R be a ring, M be a commutative cancellative monoid with a twisting $f\colon M\times M\to U(R)$ and $\omega\colon M\to \mathrm{End}(R)$ a monoid homomorphism. Suppose that N is an ideal of M such that $\omega_g(r)=1_R$ for every $g\in N$ and $r\in R$. If R is strongly CN-reversible, then R is strongly CM-reversible.*

Proof. Let $\alpha=\sum\limits_{i=1}^{n}a_ig_i$, $\beta=\sum\limits_{j=1}^{m}b_jh_j\in R\sharp M$ such that $\alpha\beta=0$. Since M is a cancellative monoid, we have $gg_i\neq gg_j$ and $h_ig\neq h_jg$ whenever $i\neq j$. If we take $g\in N$, then gg_1, gg_2, $\cdots$, gg_n, h_1g, h_1g, $\cdots$, $h_mg\in N$.

Let $\alpha_1=\sum\limits_{i=1}^{n}a_igg_i$, $\beta_1=\sum\limits_{j=1}^{m}b_jh_jg$. It is clear that α_1, $\beta_1\in R\sharp N$, and thus we have

$$\begin{aligned}
\alpha_1\beta_1&=\left(\sum_{i=1}^{n}a_igg_i\right)\left(\sum_{j=1}^{m}b_jh_jg\right)\\
&=\sum_{i=1}^{n}\sum_{j=1}^{m}(a_igg_i)(b_jh_jg)\\
&=a_1\omega_{gg_1}(b_1)f(gg_1,\,h_1g)gg_1h_1g+\cdots+a_n\omega_{gg_n}(b_m)f(gg_n,\,h_mg)gg_nh_mg\\
&=a_1\omega_{g_1}(b_1)f(gg_1,\,h_1g)gg_1h_1g+\cdots+a_n\omega_{g_n}(b_m)f(gg_n,\,h_mg)gg_nh_mg\\
&=0.
\end{aligned}$$

Since R is strongly CN-reversible, we have

$$\begin{aligned}
\beta_1\alpha_1&=\left(\sum_{j=1}^{m}b_jh_jg\right)\left(\sum_{i=1}^{n}a_igg_i\right)\\
&=\sum_{j=1}^{m}\sum_{i=1}^{n}(b_jh_jg)(a_igg_i)\\
&=b_1\omega_{h_1g}(a_1)f(h_1g,\,gg_1)h_1ggg_1+\cdots+b_m\omega_{h_mg}(a_n)f(h_mg,\,gg_n)h_mggg_n\\
&=b_1\omega_{h_1}(a_1)f(h_1g,\,gg_1)h_1ggg_1+\cdots+b_m\omega_{h_m}(a_n)f(h_mg,\,gg_n)h_mggg_n\\
&=0.
\end{aligned}$$

This implies that

$$b_j\omega_{h_j}(a_i)f(h_jg,\,gg_i)h_jggg_i=0$$

for each i, j. Therefore, we have $b_j\omega_{h_j}(a_i) = 0$ for all i, j, and thus

$$\beta\alpha = b_1\omega_{h_1}(a_1)f(h_1, g_1)h_1g_1 + \cdots + b_m\omega_{h_m}(a_n)f(h_m, g_n)h_mg_n = 0.$$

This proves that R is strongly CM-reversible.

Example 2.1 Let R be a ring with unity and $M = \{e, g, g^2, \cdots, g^{n-1}\}$ a cyclic group of order n. Let

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in R \right\}.$$

For every $e \neq g \in M$, we define $\omega\colon M \to \mathrm{Aut}(S)$ by

$$\omega_g\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}.$$

If the twisting f is trivial (i.e., $f(x, y) = 1$ for all $x, y \in M$), then S is not strongly CM-reversible. In fact, let

$$\alpha = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} e + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} g, \qquad \beta = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} e + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} g$$

be elements in $S\sharp M$. It is easy to see that $\alpha\beta = 0$. But

$$\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \omega_g\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \neq 0.$$

This implies that $\beta\alpha \neq 0$. Therefore, S is not strongly CM-reversible.

Lemma 2.3 *Let M be a monoid and N be a submonoid of M. If R is a strongly CM-reversible ring, then R is strongly CN-reversible.*

Lemma 2.4[7] *If M and N are u.p.-monoids, then so is $M \times N$.*

Let $T(G)$ be the set of elements of finite order in an Abelian group G. Then $T(G)$ is fully invariant subgroup of G. G is said to be torsion-free if $T(G) = \{e\}$.

Theorem 2.1 *Let G be a finitely generated Abelian group. Then the following conditions on G are equivalent*:

(1) *G is torsion-free*;

(2) *There exists a ring R with $|R| \geq 2$ such that R is strongly CG-reversible.*

Proof. (2) $\Rightarrow$ (1). If $g \in T(G)$ and $g \neq e$, then $N = \langle g \rangle$ is cyclic group of finite order. If a ring $R \neq \{0\}$ is strongly CG-reversible, then R is strongly CN-reversible by Lemma 2.3. Since $N = \langle g \rangle$ is a submonoid of G, by Example 2.1, R is not CN-reversible, a contradiction. Therefore, every ring $R \neq \{0\}$ is not strongly CG-reversible.

(1) $\Rightarrow$ (2). Let G be a finitely generated Abelian group with $T(G) = \{e\}$. Then $G = \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ is a finite direct product of group $\mathbb{Z}$. By Lemma 2.4, G is a u.p.-monoid. If R is a commutative M-rigid ring, then R is strongly CG-reversible by Proposition 2.2.

Let $\triangle$ be a multiplicative monoid consisting of central regular elements of R. Then it is easy to see that $\triangle^{-1}R = \{u^{-1}a \mid u \in \triangle,\ a \in R\}$ is a ring. Let M be a monoid with

$\omega\colon M \to \mathrm{Aut}(R)$ a monoid homomorphism. If $\omega_g(\triangle) \subseteq \triangle$ for every $g \in M$, then ω can be extended to $\bar{\omega}\colon M \to \mathrm{Aut}(\triangle^{-1}R)$ defined by $\bar{\omega}_g(u^{-1}a) = \omega_g(u)^{-1}\omega_g(a)$. Note that if $f\colon M \times M \to U(R)$ is a twisted function, then f is also a twisted function from $M \times M$ to $\triangle^{-1}R$ since $U(R) \subseteq U(\triangle^{-1}R)$.

Proposition 2.4 *Let M be a cancellative monoid with a twisting $f\colon M \times M \to U(R)$ and an action $\omega\colon M \to \mathrm{Aut}(R)$. Then R is strongly CM-reversible if and only if $\triangle^{-1}R$ is strongly CM-reversible.*

Proof. It suffices to show the necessity. Assume that R is strongly CM-reversible. Let

$$\alpha = \sum_{i=1}^{m} u_i^{-1}a_ig_i, \qquad \beta = \sum_{j=1}^{n} v_j^{-1}b_jh_j \in (\triangle^{-1}R)\sharp M$$

such that $\alpha\beta = 0$. Since $\triangle$ is a multiplicative monoid consisting of central regular elements of R, we have

$$\begin{aligned}\alpha\beta &= \left(\sum_{i=1}^{m} u_i^{-1}a_ig_i\right)\left(\sum_{j=1}^{n} v_j^{-1}b_jh_j\right)\\ &= \sum_{i,j} u_i^{-1}a_i\omega_{g_i}(v_j^{-1}b_j)f(g_i,\, h_j)g_ih_j\\ &= \sum_{i,j} a_i\omega_{g_i}(b_j)(u_i\omega_{g_i}(v_j))^{-1}f(g_i,\, h_j)g_ih_j\\ &= 0.\end{aligned}$$

Let

$$\tilde{\alpha} = \sum_{i=1}^{m} a_ig_i, \qquad \tilde{\beta} = \sum_{j=1}^{n} b_jh_j.$$

Then $\tilde{\alpha},\, \tilde{\beta} \in R\sharp M$ and

$$\tilde{\alpha}\tilde{\beta} = \sum_{i,j} a_i\omega_{g_i}(b_j)f(g_i,\, h_j)g_ih_j = 0.$$

Since R is strongly CM-reversible, we have

$$\tilde{\beta}\tilde{\alpha} = \sum_{i,j} b_j\omega_{h_j}(a_i)f(h_j,\, g_i)h_jg_i = 0.$$

This implies that

$$\beta\alpha = \sum_{i,j} v_j^{-1}b_j\omega_{h_j}(a_i)\omega_{h_j}(u_i)^{-1}(f(h_j,\, g_i)h_jg_i = 0,$$

since u_i, v_j are central regular elements of R.

Let I be an ideal of R and $\omega\colon M \to \mathrm{Aut}(R)$ a monoid homomorphism. An ideal I of R is said to be ω-invariant in the case $\omega_g(I) \subseteq I$ for every $g \in M$. Note that $\bar{\omega}\colon M \to \mathrm{Aut}(R/I)$ defined by $\bar{\omega}_g(r + I) = \omega_g(r) + I$ is a monoid homomorphism. Moreover, it is easy to see that the twisting $f\colon M \times M \to U(R)$ induces a twisting $\bar{f}\colon M \times M \to U(R/I)$ given by $\bar{f}(x,\, y) = f(x,\, y) + I$.

For all $\alpha = \sum\limits_{i=1}^{n} a_ig_i$ in $R\sharp M$, we denote $\bar{\alpha} = \sum\limits_{i=1}^{n} \bar{a}_ig_i$ in $(R/I)\sharp M \cong (R\sharp M)/(I\sharp M)$, where

$\bar{a}_i = a_i + I$ for $1 \le i \le n$. And the map $\mu\colon R\sharp M \to (R/I)\sharp M$ defined by $\mu(\alpha) = \bar{\alpha}$ is a ring epimorphism.

For a ring S and $n \ge 2$, let

$$R = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a,\, a_{ij} \in S \right\}.$$

Let M be a monoid and $\omega\colon M \to \mathrm{Aut}(S)$ a monoid homomorphism. For every $g \in M$, the map ω can be extended to a monoid homomorphism $\bar{\omega}$ from M to $\mathrm{Aut}(R)$ defined by $\bar{\omega}_g((a_{ij})) = (\omega_g(a_{ij}))$.

The following example shows that there exists a ring R such that R/I is strongly CM-reversible for every non-zero strongly CM-reversible proper ideal I (as a ring without identity), but R is not strongly CM-reversible.

Example 2.2 Let S be a division ring, and

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a,\, b,\, c,\, d \in S \right\}.$$

It is clear that R is not strongly CM-reversible since it is not a reversible ring. Let M be a monoid with $|M| \ge 2$. Take a non-zero proper ideal

$$I = \begin{pmatrix} 0 & 0 & S \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that I is a strongly CM-reversible ideal of R. Next we show that R/I is a strongly CM-reversible ring. To this end, if

$$\alpha = \sum_{i=1}^{n} \begin{pmatrix} a_i & b_i & 0 \\ 0 & a_i & c_i \\ 0 & 0 & a_i \end{pmatrix} g_i, \qquad \beta = \sum_{j=1}^{m} \begin{pmatrix} u_j & v_j & 0 \\ 0 & u_j & w_j \\ 0 & 0 & u_j \end{pmatrix} h_j$$

are elements in $(R/I)\sharp M$ such that $\alpha\beta = 0$, then we have

$$\begin{pmatrix} \sum\limits_{i=1}^{n} a_i g_i & \sum\limits_{i=1}^{n} b_i g_i & 0 \\ 0 & \sum\limits_{i=1}^{n} a_i g_i & \sum\limits_{i=1}^{n} c_i g_i \\ 0 & 0 & \sum\limits_{i=1}^{n} a_i g_i \end{pmatrix} \begin{pmatrix} \sum\limits_{j=1}^{m} u_j h_j & \sum\limits_{j=1}^{m} v_j h_j & 0 \\ 0 & \sum\limits_{j=1}^{m} u_j h_j & \sum\limits_{j=1}^{m} w_j h_j \\ 0 & 0 & \sum\limits_{j=1}^{m} u_j h_j \end{pmatrix} = 0.$$

Therefore, we have

$$\left(\sum_{i=1}^{n} a_i g_i\right)\left(\sum_{j=1}^{m} u_j h_j\right) = \sum_{i,j} a_i \omega_{g_i}(u_j) f(g_i,\, h_j) g_i h_j = 0.$$

This implies that $a_i\omega_{g_i}(u_j) = 0$, and thus $a_iu_j = 0$, since S is an M-rigid ring. Because S is a division ring, we have $\sum\limits_{i=1}^{n} a_ig_i = 0$ or $\sum\limits_{j=1}^{m} u_jh_j = 0$. In any case, it can be easily checked that $\beta\alpha = 0$, as desired.

However, we have an affirmative answer as the following proposition.

Proposition 2.5 *Let R be a ring and M be a monoid with a twisting $f\colon M \times M \to U(R)$ and an action $\omega\colon M \to \mathrm{Aut}(R)$. Suppose that R/I is strongly CM-reversible for some ω-invariant ideal I of R. If I is M-rigid, then R is strongly CM-reversible.*

Proof. Suppose that $\alpha = \sum\limits_{i=1}^{n} a_ig_i$, $\beta = \sum\limits_{j=1}^{m} b_jh_j \in R\sharp M$ with $\alpha\beta = 0$. Then we have

$$\bar{\alpha} = \sum_{i=1}^{n} \bar{a}_ig_i,\ \bar{\beta} = \sum_{j=1}^{m} \bar{b}_jh_j \in (R/I)\sharp M,$$

where $\bar{a}_i = a_i + I$, $\bar{b}_j = b_j + I$. On the other hand, since

$$\alpha\beta = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i\omega_{g_i}(b_j)f(g_i, h_j)g_ih_j = 0,$$

we have

$$\begin{aligned}
\bar{0} &= \bar{\alpha}\bar{\beta} \\
&= (\bar{a}_1g_1 + \cdots + \bar{a}_ng_n)(\bar{b}_1h_1 + \cdots + \bar{b}_mh_m) \\
&= (a_1 + I)\bar{\omega}_{g_1}(b_1 + I)\bar{f}(g_1, h_1)g_1h_1 + \cdots + (a_n + I)\bar{\omega}_{g_n}(b_m + I)\bar{f}(g_n, h_m)g_nh_m \\
&= (a_1\omega_{g_1}(b_1)f(g_1, h_1) + I)g_1h_1 + \cdots + (a_n\omega_{g_n}(b_m)f(g_n, h_m) + I)g_nh_m
\end{aligned}$$

in $(R/I)\sharp M$. Therefore, we have

$$\bar{\beta}\bar{\alpha} = \left(\sum_{j=1}^{m} \bar{b}_jh_j\right)\left(\sum_{i=1}^{n} \bar{a}_ig_i\right) = \bar{0},$$

since R/I is strongly CM-reversible, and thus $\beta\alpha \in I\sharp M$. Since I is an M-rigid ring, $I\sharp M$ is reduced by Lemma 2.5. It follows that

$$(\beta\alpha)^2 = (\beta\alpha)(\beta\alpha) = \beta(\alpha\beta)\alpha = 0,$$

which implies that $\beta\alpha = 0$. This shows that R is a strongly CM-reversible ring.

A ring R is called right Ore, if for any $a, b \in R$ with b regular, there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. Note that R is right Ore if and only if the classical right quotient ring Q of R exists. Note that if there exists the classical right quotient ring Q of R and M is a monoid with $\omega\colon M \to \mathrm{End}(R)$ a monoid homomorphism, then the induced map $\bar{\omega}\colon M \to \mathrm{End}(Q)$ defined by $\bar{\omega}_g(ab^{-1}) = \omega_g(a)\cdot\omega_g(b)^{-1}$ extends ω and is also a monoid homomorphism with $ab^{-1} \in Q$, where $a, b \in R$, $g \in M$ and b is regular.

It was shown in Theorem 2.6 of [3] that a ring R is reversible if and only if its classical right quotient ring is reversible. Moreover, the authors of [15] also proved that a ring R is strongly right α-reversible if and only if its classical right quotient ring is strongly right α-reversible. More generally, we have the following theorem.

Theorem 2.2 *Let M be a monoid with a twisting $f\colon M\times M\to U(R)$ and an action $\omega\colon M\to \mathrm{Aut}(R)$. If R is a right Ore ring with a classical right quotient ring Q, then R is strongly CM-reversible if and only if Q is strongly CM-reversible.*

Proof. Assume that $\alpha=\sum\limits_{i=1}^{n}a_ig_i$, $\beta=\sum\limits_{j=1}^{m}b_jh_j\in Q\sharp M$ such that $\alpha\beta=0$, where $a_i,b_j\in Q$ and $g_i,h_j\in M$ for all i, j. Since ω_{g_i}, $\omega_{h_j}\in \mathrm{Aut}(R)$ and R is a right Ore ring with classical right quotient ring Q, by Proposition 2.1.16 of [16], we can assume that

$$a_i=p_i\omega_{g_i}(u^{-1}),\qquad b_j=q_j\omega_{h_j}(v^{-1})$$

with p_i, $q_j\in R$ for all i, j and regular elements $u,v\in R$. Also by Proposition 2.1.16 of [16], there exists a $c_j\in R$ and a regular element $s\in R$ such that $u^{-1}q_j=c_js^{-1}$ for each j. Let $\alpha_1=\sum\limits_{i=1}^{n}p_ig_i$, $\beta_1=\sum\limits_{j=1}^{m}q_jh_j$ and $\beta_2=\sum\limits_{j=1}^{m}c_jh_j$. Then we have

$$\begin{aligned}0&=\alpha\beta\\&=\sum_{i=1}^{n}\sum_{j=1}^{m}p_i\omega_{g_i}(u^{-1})\omega_{g_i}(q_j\omega_{h_j}(v^{-1}))f(g_i,\,h_j)g_ih_j\\&=\sum_{i=1}^{n}\sum_{j=1}^{m}p_i\omega_{g_i}(u^{-1}q_j\omega_{h_j}(v^{-1}))f(g_i,\,h_j)g_ih_j\\&=\sum_{i=1}^{n}p_ig_i\sum_{j=1}^{m}(u^{-1}q_j\omega_{h_j}(v^{-1}))h_j\\&=\sum_{i=1}^{n}p_ig_i\sum_{j=1}^{m}(c_js^{-1}\omega_{h_j}(v^{-1}))h_j\\&=\alpha_1\beta_2\omega_{h_j}^{-1}(s^{-1}\omega_{h_j}(v^{-1}))\end{aligned}$$

since ω_{h_j} is an automorphism of R for each h_j. Therefore, we have $\alpha_1\beta_2=0$, and hence $\alpha_1\beta_1=0$ in $R\sharp M$.

Moreover, there exists a $d_i\in R$ and a regular element $t\in R$ such that $v^{-1}p_i=d_it^{-1}$ for each i again by Proposition 2.1.16 of [16]. Let $\alpha_2=\sum\limits_{i=1}^{n}d_ig_i\in R\sharp M$. Then we have

$$0=\alpha_1t\beta_1=\sum_{i=1}^{n}(p_it)g_i\sum_{j=1}^{m}q_jh_j=\sum_{i=1}^{n}(vd_i)g_i\sum_{j=1}^{m}q_jh_j=v\alpha_2\beta_1.$$

It follows that $\alpha_2\beta_1=0$ in $R*M$, and thus $\beta_1\alpha_2=0$ since R is a strongly CM-reversible ring. Therefore, we have

$$\begin{aligned}0&=\beta\alpha\\&=\sum_{j=1}^{m}\sum_{i=1}^{n}q_j\omega_{h_j}(v^{-1})\omega_{h_j}(p_i\omega_{g_i}(u^{-1}))f(h_j,\,g_i)h_jg_i\\&=\sum_{j=1}^{m}\sum_{i=1}^{n}q_j\omega_{h_j}(v^{-1}p_i\omega_{g_i}(u^{-1}))f(h_j,\,g_i)h_jg_i\\&=\sum_{j=1}^{m}\sum_{i=1}^{n}q_j\omega_{h_j}(d_it^{-1}\omega_{g_i}(u^{-1}))f(h_j,\,g_i)\cdot h_jg_i\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{m} q_j h_j \sum_{i=1}^{n} d_i t^{-1} \omega_{g_i}(u^{-1}) g_i \\
&= \beta_1 \alpha_2 \omega_{g_i}^{-1}(t^{-1} \omega_{g_i}(u^{-1})).
\end{aligned}
$$

By the definition of a strongly CM-reversible ring, Q is a strongly CM-reversible ring and we are done.

Proposition 2.6 *Let M be a monoid with a twisting $f\colon M \times M \to U(R)$ and an action $\omega\colon M \to \mathrm{Aut}(R)$. If R is an M-rigid CM-Armendariz ring, then R is strongly CM-reversible.*

Proof. Let $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n$, $\beta = b_1h_1 + b_2h_2 + \cdots + b_mh_m \in R\sharp M$ such that $\alpha\beta = 0$. Since R is CM-Armendariz, we get $a_i\omega_{g_i}(b_j) = 0$ for all i, j. This implies that $a_ib_j = 0$ for all i, j since R is M-compatible. Because R is a reversible ring, $b_ja_i = 0$ for all i, j. Then $b_j\omega_{h_j}(a_i) = 0$ for all i, j, and hence

$$
\beta\alpha = \sum_{i,j} b_j \omega_{h_j}(a_i) f(h_m, g_n) h_m g_n = 0.
$$

This implies that R is strongly CM-reversible.

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