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Collocation Methods for A Class of Volterra Integral Functional Equations with Multiple Proportional Delays

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Abstract. In this paper, we apply the collocation methods to a class of Volterra integral functional equations with multiple proportional delays (VIFEMPDs). We shall present the existence, uniqueness and regularity properties of analytic solutions for this type of equations, and then analyze the convergence orders of the collocation solutions and give corresponding error estimates. The numerical results verify our theoretical analysis.

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Key words: Volterra integral functional equation, multiple proportional delays, collocation method.

1 Introduction

The Volterra integral functional equations with proportional delays (VIFEPDs) provide a powerful model of phenomena when processes are modeled evolving in time, where the rate of change of the process is not only determined by its present state but also by a certain past state. VIFEPDs play an important role in explaining many different phenomena in biology, economy, control theory, electrodynamics, demography, viscoelastic materials and insurance. Numerical methods based on finite difference methods, discontinuous Galerkin methods and spectral methods etc., have also been developed for various VIFEPDs and we refer to [2–5, 8, 9, 11–13, 17], and references therein for details about the rich literature.

In this paper, we shall study the collocation method for Volterra integral functional equations (VIFE) with multiple delay (or: lag) functions $\theta_k = \theta_k(t)$, $k = 1, 2, \cdots, p$ of the form

$$u(t) = \sum_{k=1}^{p} a_k(t)u(\theta_k(t)) + f(t) + (\mathcal{V}u)(t) + \sum_{k=1}^{p} (\mathcal{V}_{\theta_k}u)(t), \quad t \in I := [0, T], \tag{1.1}$$

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where p is some positive integer. The Volterra integral operators $\mathcal{V}$ and $\mathcal{V}_{\theta_k}$ $(k = 1,2,\cdots,p)$ are defined by

$$(\mathcal{V}u)(t) := \int_0^t K_0(t,s)u(s)ds, \qquad (\mathcal{V}_{\theta_k}u)(t) := \int_0^{\theta_k(t)} K_k(t,s)u(s)ds,$$

where a_k, f, K_0 and K_k are given smooth functions. The delay functions $\theta_k(t)$, $k = 1,2,\cdots,p$ are assumed to have the following properties:

(P1) $\theta_k(0) = 0$, and θ_k is strictly increasing on I;

(P2) $\theta_k(t) \leq \bar{q}_k t$ on I for some $\bar{q}_k \in (0,1)$;

(P3) $\theta_k \in C^{\nu_k}(I)$ for some integer $\nu_k \geq 0$.

An important special case is the linear vanishing delay or proportional delay, i.e.,

$$\theta_k(t) = q_k t = t - (1-q_k)t := t - \tau_k(t) \quad \text{with } 0 < q_k < 1,$$

which are known as the pantograph delay functions (see [1,7,14,16]). In rest of this paper, we shall concern on the corresponding VIFEMPDs given by

$$u(t) = \sum_{k=1}^{p} a_k(t)u(q_k t) + f(t) + (\mathcal{V}u)(t) + \sum_{k=1}^{p}(\mathcal{V}_{q_k}u)(t), \qquad t \in I, \tag{1.2}$$

where

$$(\mathcal{V}_{q_k}u)(t) := \int_0^{q_k t} K_k(t,s)u(s)ds, \quad k = 1,2,\cdots,p,$$

as the multi-pantograph Volterra integral functional equations.

The collocation method for the Volterra integral equation with proportion delay (VIEPD) of the form

$$u(t) = f(t) + \int_0^t K_0(t,s)u(s)ds + \int_0^{qt} K_1(t,s)u(s)ds, \tag{1.3}$$

with $t \in [0,T]$ is discussed in [6], and recently Hermann and his collaborators also study the collocation method for functional equation

$$u(t) = b(t)u(qt) + f(t), \tag{1.4}$$

where b and f are given functions (see [10]). To the best of our knowledge, there is few work on collocation method for VIFEMPDs of form (1.2). In order to gain some insight approaches for VIFE of first and second kinds, we present a study of piecewise polynomial collocation solutions for (1.2).

There are two main challenges for these VIFEMPDs:

◇ the situations for the multiple proportional delays $(\mathcal{V}_{q_k}u)(t)$ in (1.2) are more complicated than single proportional delay;

- ⋄ the q-difference terms $u(q_k t)$ in the right hand side of (1.2) make the corresponding numerical schemes more difficult to be solved than schemes for VIFE.

For the former one, we shall present an algorithm to enumerate all possible cases for multiple proportional delays, and we give the numerical schemes for the particular cases of $p = 2$ and 3; For the latter issue, we shall show the q-difference terms $u(q_k t)$ always make the numerical schemes lose superconvergence properties both in theoretically and numerically.

The rest of this paper is organized as follows: in Section 2, the existence, uniqueness and regularity of the analytic solution to (1.2) is proposed. Section 3 is devoted to construct the collocation schemes. The conditions for the uniqueness and the attainable global convergence order of collocation scheme are presented in Section 4, and finally in Section 5, we give some numerical experiments to verify our theoretic results.

2 Existence, uniqueness and regularity

For the simplification, we introduce the linear operator $\mathcal{K} : L^\infty(I) \to L^\infty(I)$, $I = [0, T]$, by setting

$$(\mathcal{K}\varphi)(t) = \sum_{k=1}^{p} a_k(t)\varphi(q_k t) + (\mathcal{V}\varphi)(t) + \sum_{k=1}^{p} (\mathcal{V}_{q_k}\varphi)(t), \qquad t \in I. \tag{2.1}$$

Then the Eq. (1.2) can be rewritten as

$$(\mathcal{I} - \mathcal{K})u = f, \tag{2.2}$$

where $\mathcal{I}$ denotes the identity operator.

We begin with a result on the existence and uniqueness of the analytic solution of (1.2).

Theorem 2.1. *Assume that the functions a_k, f and K_k in* (1.2) *satisfy*

(i) $a_k, f \in C(I)$, $K_0 \in C(D)$ *and* $K_k \in C(D_{q_k})$, $k = 1, \cdots, p$, *where*

$$D = \{(t,s) | 0 \le s \le t \le T\}, \qquad D_{q_k} = \{(t,s) | 0 \le s \le q_k t\};$$

(ii) $\sum_{k=1}^{p} \|a_k\|_\infty < 1$, *where* $\|v\| := \max_{t \in I} v(t)$.

Then there exists a unique solution $u \in C(I)$ *of* (1.2).

Proof. We shall prove this result by Banach fixed point theorem and mathematical induction. Since the given kernels functions K_k $(k = 0, 1, \cdots, p)$ are continuous on their closed domains respectively, there exist positive constants M_k $(k = 0, 1, \cdots, p)$ such that $|K_k(t,s)| \le M_k$. From the condition (ii), we can choose δ such that

$$0 < \delta < \frac{1 - \sum_{k=1}^{p} \|a_k\|_\infty}{M_0 + \sum_{k=1}^{p} q_k M_k}.$$

Denote $S = \{u : u \in C[0,\delta]\}$. Using the inequalities

$$\begin{aligned}\|\mathcal{K}u\|_\infty =& \Big\| \sum_{k=1}^{p} a_k(t)u(q_kt) + (\mathcal{V}u)(t) + \sum_{k=1}^{p} (\mathcal{V}_{q_k}u)(t) \Big\|_\infty \\ \le& \Big(\sum_{k=1}^{p} \|a_k\|_\infty \Big) \|u\|_\infty + \delta\Big(M_0 + \sum_{k=1}^{p} q_k M_k\Big)\|u\|_\infty < \|u\|_\infty,\end{aligned}$$

we know that $\mathcal{K} : S \to S$ is a contraction map. Hence the operator $\mathcal{I} - \mathcal{K}$ has a bounded inverse, which implies (1.2) has a unique continuous solution on $[0,\delta]$.

Assume (1.2) has a unique continuous solution on $[0,k\delta]$ for some positive integer k, we want to prove that it is also true on $[k\delta,(k+1)\delta]$, that is mathematical induction with index k. For $t \in [k\delta,(k+1)\delta]$, we have

$$\begin{aligned} u(t) &= \sum_{k=1}^{p} a_k(t)u(q_kt) + f(t) + \int_0^t K_0(t,s)u(s)ds + \sum_{k=1}^{p} \int_0^{q_kt} K_k(t,s)u(s)ds \\ &= \sum_{k=1}^{p} a_k(t)u(q_kt) + \widetilde{f}(t) + \int_{k\delta}^t K_0(t,s)u(s)ds + \sum_{k=1}^{p} \int_{q_kk\delta}^{q_kt} K_k(t,s)u(s)ds, \end{aligned} \tag{2.3}$$

with

$$\widetilde{f}(t) = f(t) + \int_0^{k\delta} K_0(t,s)u(s)ds + \sum_{k=1}^{p} \int_0^{q_kk\delta} K_k(t,s)u(s)ds.$$

Using the same argument on interval $[0,\delta]$, it follows that the Eq. (2.3) has a unique continuous solution on $[k\delta,(k+1)\delta]$, therefore $u \in [0,(k+1)\delta]$ is continuous. This completes the proof. □

Before the discussion of the regularity for VIFEMPDs of form (1.2), we first give a regularity result about the corresponding multiple delays functional equation.

Lemma 2.1. *Consider the multiple delays functional equation*

$$u(t) = \sum_{k=1}^{p} a_k(t)u(q_kt) + f(t), \qquad t \in I, \tag{2.4}$$

if a_k, $f \in C^\nu(I)$ for some integer $\nu \ge 1$ and $\sum_{k=1}^{p} \|a_k\|_\infty < 1$, then the solution satisfies $u \in C^\nu(I)$.

Proof. The proof is similar to the proof for the functional equation with proportional delay in [10]. Here for the sake of reader's convenience, we give a detail proof for functional equation with multiple delays. By Theorem 2.1 with $K_0 = K_1 = \cdots = K_p = 0$, we know that $u \in C(I)$. For a_k,$f \in C^1(I)$, differentiate both sides of the Eq. (2.4) formally leading to

$$u'(t) = \sum_{k=1}^{p} q_k a_k(t) u'(q_kt) + \widetilde{f}(t),$$

with

$$\widetilde{f}(t) = \sum_{i=1}^{p} a_i'(t)u(q_k t) + f'(t).$$

Since the existence of a differentiable solution of the Eq. (2.4) is still unknown, we consider the equation

$$\widetilde{u}(t) = \sum_{k=1}^{p} q_k a_k(t)\widetilde{u}(q_k t) + \widetilde{f}(t). \tag{2.5}$$

Noting a_k, $f \in C^1(I)$ and the fact that $u \in C(I)$, we know $q_k a_k$, $\widetilde{f} \in C(I)$. It now follows from Theorem 2.1 that there exists a unique solution $\widetilde{u} \in C(I)$ for Eq. (2.5).

Next, we will prove the unique solution $\widetilde{u}(t)$ equals to $u'(t)$, that is to show

$$\lim_{h\to 0}\Big|\widetilde{u}(t) - \frac{u(t+h)-u(t)}{h}\Big| = 0. \tag{2.6}$$

For any t, $t+h \in I$ with $h \neq 0$, we have

$$\begin{aligned}
&\widetilde{u}(t) - \frac{u(t+h)-u(t)}{h}\\
=&\sum_{k=1}^{p} q_k a_k(t)\Big[\widetilde{u}(q_k t) - \frac{u(q_k(t+h)) - u(q_k t)}{q_k h}\Big]\\
&\quad + \sum_{k=1}^{p}\Big[a_k'(t)u(q_k t) - \frac{a_k(t+h)-a_k(t)}{h}u(q_k(t+h))\Big] + f'(t) - \frac{f(t+h)-f(t)}{h}.
\end{aligned}$$

Let

$$\begin{aligned}
\omega[f,v,s] &:= \sup_{t\in I,\ 0<|s|<h}\Big|f(t) - \frac{v(t+s)-v(t)}{s}\Big|,\\
A_1 &:= \sum_{k=1}^{p} q_k a_k(t)\Big[\widetilde{u}(q_k t) - \frac{u(q_k(t+h)) - u(q_k t)}{q_k h}\Big],\\
A_2 &:= \sum_{k=1}^{p}\Big[a_k'(t)u(q_k t) - \frac{a_k(t+h)-a_k(t)}{h}u(q_k(t+h))\Big],\\
A_3 &:= f'(t) - \frac{f(t+h)-f(t)}{h}.
\end{aligned}$$

Then we have

$$\Big|\widetilde{u}(t) - \frac{u(t+h)-u(t)}{h}\Big| \leq |A_1| + |A_2| + |A_3|. \tag{2.7}$$

Noting that

$$\begin{aligned}
|A_1| &\leq \sum_{k=1}^{p}|q_k a_k(t)|\Big|\widetilde{u}(q_k t) - \frac{u(q_k(t+h)) - u(q_k t)}{q_k h}\Big|\\
&\leq \sum_{k=1}^{p}|q_k a_k(t)|\omega[\widetilde{u},u,h] \leq \Big(\sum_{k=1}^{p} q_k\|a_k\|_\infty\Big)\omega[\widetilde{u},u,h],
\end{aligned}$$

and taking sup-norm on both sides of (2.7), we obtain

$$\Big(1-\sum_{k=1}^{p} q_k\|a_k\|_\infty\Big)\omega[\widetilde{u},u,h] \le |A_2|+|A_3|.$$

From the assumptions a_k, $f\in C^1(I)$ and $u\in C(I)$, we know that

$$\lim_{h\to 0}|A_i|=0,\quad i=2,3.$$

Hence the relation (2.6) holds, which implies $\widetilde{u}(t)=u'(t)$ for any $t\in I$. Therefore, the solution of (2.4) satisfies $u\in C^1(I)$.

Furthermore, assume that $u\in C^\mu(I)$ holds with

$$0\le \mu\le \nu-1,$$

since a_k, $f\in C^\nu(I)\subset C^{\mu+1}(I)$, we can obtain that $u\in C^{\mu+1}(I)$. By the mathematical induction, we reach the conclusion. □

Our regularity result about the solution of (1.2) is given by following theorem.

Theorem 2.2. *Assume that the functions a_k, f and K_k in* (1.2) *satisfy*

(i) a_k, $f\in C^\nu(I)$, $K_0\in C^\nu(D)$ *and* $K_k\in C^\nu(D_{q_k})$, $k=1,\cdots,p$, *for some integer* $v\ge 1$;

(ii) $\sum_{k=1}^{p}\|a_k\|_\infty<1$.

Then the solution of (1.2) *satisfies* $u\in C^\nu(I)$.

Proof. By Theorem 2.1, the solution u is continuous. Since the given functions are smooth, differentiating both sides of the Eq. (1.2) formally and replace $u'(t)$ by $\widetilde{u}(t)$ yields

$$\widetilde{u}(t)=\sum_{k=1}^{p} q_k a_k(t)\widetilde{u}(q_k t)+\widetilde{f}(t), \tag{2.8}$$

where

$$\begin{aligned}\widetilde{f}(t)=&\sum_{k=1}^{p} a_k'(t)u(q_kt)+f'(t)+K_0(t,t)u(t)+\int_0^t\frac{\partial K_0}{\partial t}(t,s)u(s)ds\\ &+\sum_{k=1}^{p} q_kK_k(t,q_kt)u(q_kt)+\sum_{k=1}^{p}\int_0^{q_kt}\frac{\partial K_k}{\partial t}(t,s)u(s)ds.\end{aligned}$$

Since a_k, $f\in C^1(I)$, $K_0\in C^1(D)$ and $K_k\in C^1(D_{q_k})$, $k=1,2,\cdots,p$ and $u\in C(I)$, it follows that q_ka_k, $\widetilde{f}\in C(I)$. Thus Lemma 2.1 implies that Eq. (2.8) has a unique solution $u\in C^1(I)$.

Furthermore, assume that $u\in C^\mu(I)$ holds with $0\le\mu\le\nu-1$, since a_k, $f\in C^{\mu+1}(I)$, $K_0\in C^{\mu+1}(D)$ and $K_k\in C^{\mu+1}(D_{q_k})$, $k=1,2,\cdots,p$, we can obtain that $u\in C^{\mu+1}(I)$ by Lemma 2.1. By the mathematical induction, we reach the conclusion of the theorem. □

3 Collocation methods

In this section, it will propose an algorithm to enumerate the all possible cases for multiple proportional delays in (1.2). In particularly for $p = 2$ and $p = 3$, we give the numerical scheme for each case.

For the simplification, we first introduce some notations. Let

$$I_h = \{t_n = nh,\ \ n = 0, \cdots, N\} \quad \text{with}\ \ t_N = Nh = T,$$

be a given uniform mesh on I and set $e_n := (t_n, t_{n+1}]$, $n = 0, \cdots, N-1$. We shall be concerned with the collocation solution u_h lying in the piecewise polynomial space

$$S_{m-1}^{(-1)}(I_h) := \{v : v|_{e_n} \in \pi_{m-1},\ \ 0 \le n \le N-1\}, \tag{3.1}$$

where π_{m-1} $(m \ge 1)$ denotes the set of polynomials of degree not exceeding $m-1$. The dimension of the space $S_{m-1}^{(-1)}(I_h)$ equals Nm. Hence, we are natural to choose the set of collocation points to be

$$X_h := \{t_{n,i} = t_n + c_i h : 0 < c_1 < \cdots < c_m \le 1,\ \ n = 0, \cdots, N-1\},$$

as its cardinality is Nm. Here, $\{c_i\}_{i=1}^m$ is a given set of collocation parameters in $(0, 1]$. Hence, We are looking for $u_h \in S_{m-1}^{(-1)}(I_h)$ satisfying the collocation equation

$$u_h(t) = \sum_{k=1}^{p} a_k(t) u_h(q_k t) + f(t) + (\mathcal{V} u_h)(t) + \sum_{k=1}^{p} (\mathcal{V}_{q_k} u_h)(t), \qquad t \in X_h. \tag{3.2}$$

Setting

$$U_{n,j} = u_h(t_n + c_j h), \qquad j = 1, \cdots, m,$$

we can express u_h^n (the restriction of u_h on interval e_n) by interpolation

$$u_h|_{e_n} = u_h^n(t) = u_h(t_n + sh) = \sum_{j=1}^{m} L_j(s) U_{n,j}, \qquad 0 < s \le 1, \tag{3.3}$$

with Lagrange interpolation polynomials

$$L_j(s) = \prod_{k=1, k\neq j}^{m} \frac{s - c_k}{c_j - c_k}, \quad 0 < s \le 1, \quad j = 1, \cdots, m. \tag{3.4}$$

Therefore, the global collocation solution u_h on I is given by

$$u_h(t) = \sum_{n=0}^{N-1} \chi_n(t) u_h^n(t),$$

where $\chi_n(t)$ is the characteristic function on e_n.

3.1 Properties of the images $q_k t$ in the vanishing delays

The main difficulty in the numerical analysis of VIFEMPDs on uniform meshes is the overlap of the images $q_k t$ $(k = 1, \cdots, p)$ of the collocation points of the vanishing delays.

To be more precise, for a uniform mesh I_h and $t_{n,i} := t_n + c_i h \in X_h$, we first discuss some properties of $\{qt_{n,i}\}_{i=1}^m$ for each fixed n, that is the single proportional delay case for (1.2). Note that

$$qt_{n,i} = q(t_n + c_i h) = q(nh + c_i h) = q(n + c_i)h.$$

Let

$$qt_{n,i} := (q_{n,i} + \gamma_{ni})h = q_{n,i}h + \gamma_{ni}h = t_{q_{n,i}} + \gamma_{ni}h \in (t_{q_{n,i}}, t_{q_{n,i}+1}],$$

with

$$q_{n,i} = \lfloor q(n + c_i) \rfloor, \quad \gamma_{ni} = q(n + c_i) - q_{n,i} \in (0,1), \quad q_n = \min_{1 \le i \le m} \{q_{n,i}\}.$$

Here for any $x \in \mathcal{R}$, $\lfloor x \rfloor$ is the greatest integer not exceeding x, and similarly, $\lceil x \rceil$ denotes the smallest integer exceeding x.

Remark 3.1. Noting that $qt_{n,m} - qt_{n,1} = q(c_m - c_1) < h$, then for each fixed n, $\{qt_{n,i}\}_{i=1}^m$ at most belongs to two subintervals of I_h.

Denoting

$$q^I = \left\lceil \frac{q}{1-q} c_1 \right\rceil, \qquad q^{II} = \left\lceil \frac{q}{1-q} c_m \right\rceil, \tag{3.5}$$

the following lemma characterize the "overlap" of the images qt of the collocation points.

Lemma 3.1. $(cf.$ [6]$)$ *Let $q \in (0,1)$ and $0 < c_1 < \cdots < c_m \le 1$ be given, and assume that I_h is a uniform mesh with mesh diameter $h = T/N$. Then*

(i) For $0 \le n < q^I$, we have $\{qt_{n,i}\}_{i=1}^m \subset (t_n, t_{n+1})$;

(ii) For $q^I \le n < q^{II}$, there exists $\nu_n \in \{1, \cdots, m-1\}$ so that

$$qt_{n,i} \in \begin{cases} (t_{n-1}, t_n], & 1 \le i \le \nu_n, \\ (t_n, t_{n+1}], & \nu_n < i \le m; \end{cases}$$

(iii) For $q^{II} \le n \le N-1$, then $qt_{n,i} \le t_n$ $(i = 1, \cdots, m)$, therefore we have two cases:

(iiia) $\{qt_{n,i}\}_{i=1}^m$ belong to one interval, which means $q_{n,i} = q_n$ and $\{qt_{n,i}\}_{i=1}^m \subset (t_{q_n}, t_{q_n+1}]$.

(iiib) $\{qt_{n,i}\}_{i=1}^{m}$ *belong to two intervals, then there exists* $\nu_n \in \{1,\cdots,m-1\}$*, such that*

$$qt_{n,i} \in \begin{cases} (t_{q_n}, t_{q_n+1}], & 1 \le i \le \nu_n, \\ (t_{q_n+1}, t_{q_n+2}], & \nu_n < i \le m. \end{cases}$$

For $q^{II} \le n \le N-1$*, we can consider the case (iiia) as a special state of case (iiib) with* $\nu_n = m$*.*

Next, we shall study the images $q_k t$ $(k = 1,2,\cdots,p)$ of the collocation points of the multiple proportional vanishing delays (i.e., $\{q_k t_{n,i}\}_{i=1}^{m}$). Without loss of generality, we assume $q_1 \ge q_2 \cdots \ge q_p$, otherwise, one can rearrange $\{q_k\}_{k=1}^{p}$ and rename them such that the inequalities still hold. Similar to the single proportional vanishing delays, we introduce following notations

$$q_k^I = \Big\lceil \frac{q_k}{1-q_k} c_1 \Big\rceil, \qquad q_k^{II} = \Big\lceil \frac{q_k}{1-q_k} c_m \Big\rceil, \qquad q_{n,i}^{(k)} = \lfloor q_k(n+c_i) \rfloor, \tag{3.6a}$$

$$q_n^{(k)} = \min_{1 \le i \le m} \{q_{n,i}^{(k)}\}, \quad \gamma_{ni}^{(k)} = q_k(n+c_i) - q_{n,i}^{(k)}, \quad k = 1,2,\cdots,p. \tag{3.6b}$$

From the definitions (3.6) and $c_1 \le c_m$, it is obvious that

$$q_k^{II} \ge q_k^I, \quad k = 1,2,\cdots,p; \quad q_1^{II} \ge q_2^{II} \cdots \ge q_p^{II}, \quad q_1^I \ge q_2^I \cdots \ge q_p^I. \tag{3.7}$$

In order to give the numerical schemes, it is necessary to study the images $q_k t$ $(k = 1,2,\cdots,p)$ of the collocation points, which is equivalent to enumerate all possible arrangements of $\{q_k^I, q_k^{II}\}_{k=1}^{p}$ under constrains (3.7). Before solve this problem, we first introduce some concepts and a lemma about Catalan number.

Definition 3.1. *A Dyck word is a string consisting of p X's and p Y's such that no initial segment of the string has more Y's than X's.*

Definition 3.2. *Given a* $p \times p$ *square cells, a monotonic path is one which starts in the lower-left corner, finishes in the upper-right corner, and consists entirely of edges pointing rightwards or upwards.*

Lemma 3.2. (*see* [15]) *The number of monotonic paths which do not pass above the diagonal (as showed in Fig. 1) is* C_p *(Catalan number), which is give by*

$$C_p = C_{2p}^{p} - C_{2p}^{p-1} = \frac{1}{p+1} C_{2p}^{p} = \frac{1}{p} C_{2p}^{p-1}, \quad p \ge 1,$$

where $C_n^k = n!/k!(n-k)!$ *is the combination number.*

The following theorem gives the number of rangements of $\{q_k^I, q_k^{II}\}_{k=1}^{p}$ with constrains (3.7).

Theorem 3.1. *The following three problems are equivalent.*

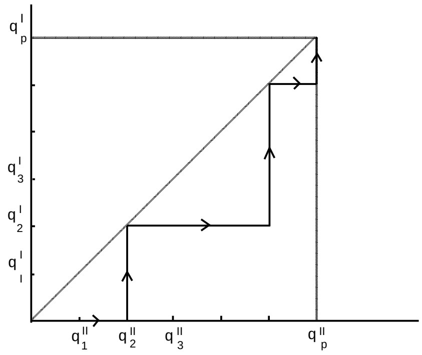

Figure 1: Diagram of monotonic paths which do not pass above the diagonal.

(P1) The number of rangements of $\{q_k^I, q_k^{II}\}_{k=1}^p$ *with constrains* (3.7)*;*

(P2) The number of Dyck words of length $2p$*;*

(P3) Given a $p \times p$ *square cells, the number of monotonic paths which do not pass above the diagonal.*

Furthermore, the number problem (P1) is

$$C_p = C_{2p}^p - C_{2p}^{p-1},$$

where C_p *is the Catalan number.*

Proof. For a given choose p from $2p$ position, it can denotes ordered series $q_1^{II} \geq q_2^{II} \cdots \geq q_p^{II}$ or p X's without order; the rest p positions denote ordered series $q_1^I \geq q_2^I \cdots \geq q_p^I$ or p Y's without order. Setting p X's represent q_k^{II} and p Y's represent q_k^I, the constrains $q_k^{II} \geq q_k^I$, $k = 1, 2, \cdots, p$ equivalent to no initial segment of the string has more Y's than X's. Hence, problem (P1) is equivalent to problem (P2).

Let X stand for "move right" and Y stand for "move up", then no initial segment of the string has more Y's than X's equivalent to paths which do not pass above the diagonal. Therefore, problem (P2) is equivalent to problem (P3). Using these equivalences and Lemma 3.2, we have the conclusion. □

Theorem 3.1 tells us the number of rangements of $\{q_k^I, q_k^{II}\}_{k=1}^p$ with constrains (3.7) is equal to the number of monotonic paths which do not pass above the diagonal, and shows the number is Catalan number C_p. But we still need an algorithm to enumerate all C_p cases, and following algorithm or the proof of Lemma 3.2 solve this problem.

Algorithm 3.1. *For* $l = p, p-1, \cdots, 1$*, enumerate the cases that the number of elements of* $\{q_p^{II} \geq q_k^I,\ k = 1, 2 \cdots, p\}$ *is* l*.*

Using above algorithm and mathematical induction, we can get the formulation for Catalan number in Lemma 3.2. It is easy to see that the number under unit squares the monotonic paths which do not pass above the diagonal increasing as l decreasing. Next, we use $p = 2$ and $p = 3$ as illustrations for this algorithm.

Remark 3.2. When $p=2$, $C_2 = C^2_{2\times 2} - C^1_{2\times 2} = 2$, the two monotonic paths are shown in Fig. 2.

Then corresponding rangements are

Case 2.1: $q^{II}_1 \ge q^{II}_2 \ge q^I_1 \ge q^I_2$;

Case 2.2: $q^{II}_1 \ge q^I_1 > q^{II}_2 \ge q^I_2$.

When $p=3$, $C_3 = C^3_{2\times 3} - C^2_{2\times 3} = 5$, the five monotonic paths are shown in Fig. 3. Then corresponding rangements are

Case 3.1: $q^{II}_1 \ge q^{II}_2 \ge q^{II}_3 \ge q^I_1 \ge q^I_2 \ge q^I_3$;

Case 3.2: $q^{II}_1 \ge q^{II}_2 \ge q^I_1 > q^{II}_3 \ge q^I_2 \ge q^I_3$;

Case 3.3: $q^{II}_1 \ge q^I_1 > q^{II}_2 \ge q^{II}_3 \ge q^I_2 \ge q^I_3$;

Case 3.4: $q^{II}_1 \ge q^{II}_2 \ge q^I_1 \ge q^I_2 > q^{II}_3 \ge q^I_3$;

Case 3.5: $q^{II}_1 \ge q^I_1 > q^{II}_2 \ge q^I_2 > q^{II}_3 \ge q^I_3$.

Although Theorem 3.1 shows that the number of all possible rangements of $\{q^I_k, q^{II}_k\}^p_{k=1}$ under constrains (3.7) is the Catalan number $C_n = C^n_{2n} - C^{n-1}_{2n}$, the collocation schemes for each rangement are totally different. Once the $q'_k s$ in (1.2) are given, we should compute q^I_k and q^{II}_k, and specify the corresponding case by using Theorem 3.1 and Algorithm 3.1, and then adopt concrete collocation scheme for this case finally.

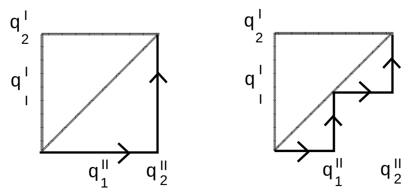

Figure 2: Diagram of monotonic paths which do not pass above the diagonal when $p=2$.

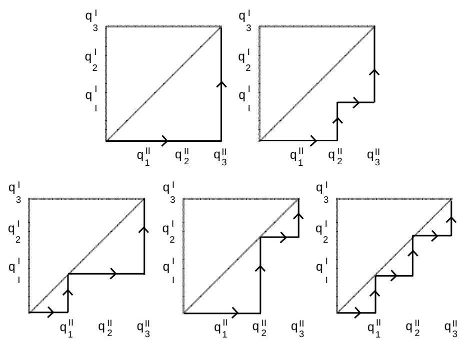

Figure 3: Diagram of monotonic paths which do not pass above the diagonal when $p=3$.

3.2 Collocation schemes for VIFEMPDs with Cases 2.1 and 3.1

In this subsection, we shall give the numerical schemes for VIFEMPDs with $p = 2$ (Case 2.1) and $p = 3$ (Case 3.1) corresponding to Remark 3.2 as illustrations.

Using notations (3.6), for each collocation point $t_{n,i} \in X_h$, the collocation equation (3.2) becomes

$$u_h(t_{n,i}) = \sum_{k=1}^{p} \Big[a_k(t_{n,i}) \sum_{j=1}^{m} L_j(\gamma_{n,i}^{(k)}) U_{q_n^{(k)},j}\Big] + f(t_{n,i}) + (\mathcal{V}u_h)(t_{n,i}) + \sum_{k=1}^{p} (\mathcal{V}_{q_k} u_h)(t_{n,i}), \tag{3.8}$$

where

$$\begin{aligned}(\mathcal{V}u_h)(t_{n,i}) =& h \sum_{l=0}^{n-1} \sum_{j=1}^{m} \Big(\int_0^1 K_0(t_{n,i}, t_l + sh) L_j(s) ds\Big) U_{l,j} \\ &+ h \sum_{j=1}^{m} \Big(\int_0^{c_i} K_0(t_{n,i}, t_n + sh) L_j(s) ds\Big) U_{n,j},\end{aligned}$$

and for $k = 1, \cdots, p$,

$$\begin{aligned}(\mathcal{V}_{q_k} u_h)(t_{n,i}) =& h \sum_{l=0}^{q_{n,i}^{(k)}-1} \sum_{j=1}^{m} \Big(\int_0^1 K_k(t_{n,i}, t_l + sh) L_j(s) ds\Big) U_{l,j} \\ &+ h \sum_{j=1}^{m} \Big(\int_0^{\gamma_{n,i}^{(k)}} K_k(t_{n,i}, t_{q_{n,i}^{(k)}} + sh) L_j(s) ds\Big) U_{q_{n,i}^{(k)},j}.\end{aligned}$$

For simplification, we introduce some notations ($k = 1, 2, 3$; $n, l = 0, 1, \cdots, N-1$; $i, j = 1, \cdots, m$)

$$\begin{aligned}
&B_n^{(k)} = \big(a_k(t_{n,i}) L_j(\gamma_{n,i}^{(k)})\big)_{ij}, && F_n = \big(f(t_n + c_1 h), \cdots, f(t_n + c_m h)\big)^T, \\
&M_n = \Big(\int_0^{c_i} K_0(t_{n,i}, t_n + sh) L_j(s) ds\Big)_{ij}, && M_{n,l} = \Big(\int_0^1 K_0(t_{n,i}, t_l + sh) L_j(s) ds\Big)_{ij}, \\
&N_{n,l}^{(k)} = \Big(\int_0^1 K_k(t_{n,i}, t_l + sh) L_j(s) ds\Big)_{ij}, && R_{n,l}^{(k)} = \Big(\int_0^{\gamma_{n,i}^{(k)}} K_k(t_{n,i}, t_l + sh) L_j(s) ds\Big)_{ij}, \\
&U_n = (U_{n,1}, \cdots, U_{n,m})^T, && T_{\nu_n^{(k)}} = \mathrm{diag}(\underbrace{1, \cdots, 1}_{\nu_n^{(k)}}, 0, \cdots, 0).
\end{aligned}$$

3.2.1 Collocation schemes for Case 2.1: $q_1^{II} \geq q_2^{II} \geq q_1^{I} \geq q_2^{I}$

We have fives phases with respective to interval $(t_n, t_{n+1}]$ for this case.

Phase I (q_1 and q_2 complete overlap)

$$0 \leq n < q_2^I = \Big\lceil \frac{q_2}{1 - q_2} c_1 \Big\rceil.$$

We know that $\{q_k t_{n,i}\}_{i=1}^m \subset (t_n, t_{n+1}), k = 1,2$ only belong to one interval with $q_{n,i}^{(k)} = n$. Then the corresponding matrix form for Eq. (3.8) is given by

$$\Big[I_m - \sum_{k=1}^{2}(B_n^{(k)} + hR_{n,n}^{(k)}) - hM_n\Big]U_n = F_n + h\sum_{l=0}^{n-1}\Big(M_{n,l} + \sum_{k=1}^{2} N_{n,l}^{(k)}\Big)U_l. \tag{3.9}$$

Phase II (q_1 complete overlap and q_2 partial overlap)

$$q_2^I \le n < q_1^I = \Big\lceil \frac{q_1}{1-q_1}c_1 \Big\rceil.$$

Here, $q_{n,i}^{(1)} = n$ for all values of $i = 1,\cdots,m$ and for given integer n, let $\nu_n^{(2)}$ with $1 \le \nu_n^{(2)} < m$ such that

$$\begin{aligned} &t_{n-1} < q_2 t_{n,i} \le t_n, && i = 1,\cdots,\nu_n^{(2)},\\ &t_n < q_2 t_{n,i} < t_{n+1}, && i = \nu_n^{(2)}+1,\cdots,m. \end{aligned}$$

Then the collocation equation (3.8) can be led to the linear algebraic system

$$\begin{aligned} &\big[I_m - B_n^{(1)} - hR_{n,n}^{(1)} - (I_m - T_{\nu_n^{(2)}})(B_n^{(2)} + hR_{n,n}^{(2)}) - hM_n\big]U_n\\ =&T_{\nu_n^{(2)}}(B_n^{(2)} + hR_{n,n-1}^{(2)})U_{n-1} + h\sum_{l=0}^{n-1}(M_{n,l} + N_{n,l}^{(1)})U_l + F_n\\ &+ h\sum_{l=0}^{n-2} N_{n,l}^{(2)}U_l + (I_m - T_{\nu_n^{(2)}})hN_{n,n-1}^{(2)}U_{n-1}. \end{aligned} \tag{3.10}$$

Phase III (q_1 and q_2 partial overlap)

$$q_1^I \le n < q_2^{II} = \Big\lceil \frac{q_2}{1-q_2}c_m \Big\rceil.$$

For given n and $k = 1,2$, we know that

$$\{q_k t_{n,i}\}_{i=1}^m \subset (t_{n-1}, t_{n+1}),$$

and there exist $\nu_n^{(k)} \in \{1,\cdots,m-1\}$, such that

$$\begin{cases} t_{n-1} < q_1 t_{n,i} \le t_n, & i = 1,\cdots,\nu_n^{(1)},\\ t_n < q_1 t_{n,i} < t_{n+1}, & i = \nu_n^{(1)}+1,\cdots,m, \end{cases} \qquad \begin{cases} t_{n-1} < q_2 t_{n,i} \le t_n, & i = 1,\cdots,\nu_n^{(2)},\\ t_n < q_2 t_{n,i} < t_{n+1}, & i = \nu_n^{(2)}+1,\cdots,m. \end{cases}$$

The corresponding linear algebraic system yields

$$\begin{aligned} &\Big[I_m - \sum_{k=1}^{2}(I_m - T_{\nu_n^{(k)}})(B_n^{(k)} + hR_{n,n}^{(k)}) - hM_n\Big]U_n\\ =&\sum_{k=1}^{2} T_{\nu_n^k}(B_n^{(k)} + hR_{n,n-1}^{(k)})U_{n-1} + F_n + h\sum_{l=0}^{n-1} M_{n,l}U_l\\ &+ h\sum_{l=0}^{n-2}\Big(\sum_{k=1}^{2} N_{n,l}^{(k)}\Big)U_l + \sum_{k=1}^{2}(I_m - T_{\nu_n^{(k)}})hN_{n,n-1}^{(k)}U_{n-1}. \end{aligned} \tag{3.11}$$

Phase IV (q_1 partial overlap and q_2 non-overlap)

$$q_2^{II} \le n < q_1^{II} = \Big\lceil \frac{q_1}{1-q_1} c_m \Big\rceil.$$

According to Lemma 3.1, we know that $\{q_1 t_{n,i}\}_{i=1}^m \subset (t_{n-1}, t_{n+1})$ and $q_2 t_{n,i} \le t_n$ $(i = 1, \cdots, m)$, which means, for given n, there exist two integers $\nu_n^{(1)} \in \{1, \cdots, m-1\}$ and $\nu_n^{(2)} \in \{1, \cdots, m\}$ such that

$$\begin{aligned} &q_{n,i}^{(1)} = n-1 (i = 1, \cdots, \nu_n^{(1)}), && q_{n,i}^{(1)} = n (i = \nu_n^{(1)}+1, \cdots, m), \\ &q_{n,i}^{(2)} = q_n^{(2)} (i = 1, \cdots, \nu_n^{(2)}), && q_{n,i}^{(2)} = q_n^{(2)} + 1 (i = \nu_n^{(2)}+1, \cdots, m). \end{aligned}$$

The linear algebraic system for this phase satisfies

$$\begin{aligned} &\big[I_m - (I_m - T_{\nu_n^{(1)}})(B_n^{(1)} + hR_{n,n}^{(1)}) - hM_n\big] U_n \\ =&T_{\nu_n^{(1)}}(B_n^{(1)} + hR_{n,n-1}^{(1)})U_{n-1} + T_{\nu_n^{(2)}}(B_n^{(2)} + hR^{(2)}_{n,q_n^{(2)}})U_{q_n^{(2)}} + F_n \\ &+ h\sum_{l=0}^{n-1} M_{n,l} U_l + h\sum_{l=0}^{n-2} N_{n,l}^{(1)} U_l + h\sum_{l=0}^{q_n^{(2)}-1} N_{n,l}^{(2)} U_l + (I_m - T_{\nu_n^{(1)}})hN_{n,n-1}^{(1)} U_{n-1} \\ &+ \big(I_m - T_{\nu_n^{(2)}}\big)\big(B_n^{(2)} U_{q_n^{(2)}+1} + hN^{(2)}_{n,q_n^{(2)}} U_{q_n^{(2)}} + hR^{(2)}_{n,q_n^{(2)}+1} U_{q_n^{(2)}+1}\big). \end{aligned} \tag{3.12}$$

Phase V (q_1 and q_2 non-overlap)

$$q_1^{II} \le n \le N-1.$$

Given n, there exists integers $\nu_n^{(k)} (k = 1, 2) \in \{1, \cdots, m\}$, such that

$$\begin{cases} t_{q_n^{(1)}} < q_1 t_{n,i} \le t_{q_n^{(1)}+1}, & i = 1, \cdots, \nu_n^{(1)}, \\ t_{q_n^{(1)}+1} < q_1 t_{n,i} < t_{q_n^{(1)}+2}, & i = \nu_n^{(1)}+1, \cdots, m, \end{cases}$$
$$\begin{cases} t_{q_n^{(2)}} < q_2 t_{n,i} \le t_{q_n^{(2)}+1}, & i = 1, \cdots, \nu_n^{(2)}, \\ t_{q_n^{(2)}+1} < q_2 t_{n,i} < t_{q_n^{(2)}+2}, & i = \nu_n^{(2)}+1, \cdots, m. \end{cases}$$

The system of linear equations describing the last phase is given by

$$\begin{aligned} (I_m - hM_n)U_n = &\sum_{k=1}^{2} T_{\nu_n^{(k)}}(B_n^{(k)} + hR^{(k)}_{n,q_n^k})U_{q_n^{(k)}} + F_n + h\sum_{l=0}^{n-1} M_{n,l} U_l + \sum_{k=1}^{2} h \sum_{l=0}^{q_n^{(k)}-1} N_{n,l}^{(k)} U_l \\ &+ \sum_{k=1}^{2} (I_m - T_{\nu_n^{(k)}})\big(B_n^{(k)} U_{q_n^{(k)}+1} + hN^{(k)}_{n,q_n^{(k)}} U_{q_n^{(k)}} + hR^{(k)}_{n,q_n^{(k)}+1} U_{q_n^{(k)}+1}\big). \end{aligned} \tag{3.13}$$

3.2.2 Collocation schemes for Case 3.1: $q_1^{II} \geq q_2^{II} \geq q_3^{II} \geq q_1^I \geq q_2^I \geq q_3^I$

We have seven phases with respective to interval $(t_n, t_{n+1}]$ for this case.

Phase I (q_1, q_2 and q_3 complete overlap)

$$0 \leq n < q_3^I = \left\lceil \frac{q_3}{1-q_3} c_1 \right\rceil.$$

We know that $\{q_k t_{n,i}\}_{i=1}^m \subset (t_n, t_{n+1}), k = 1,2,3$ only belong to one interval with $q_{n,i}^{(k)} = n$. Then the corresponding matrix form for Eq. (3.8) is given by

$$\Big[I_m - \sum_{k=1}^{3} (B_n^{(k)} + hR_{n,n}^{(k)}) - hM_n\Big] U_n = F_n + h\sum_{l=0}^{n-1} \Big(M_{n,l} + \sum_{k=1}^{3} N_{n,l}^{(k)}\Big) U_l. \tag{3.14}$$

Phase II (q_1 and q_2 complete overlap, q_3 partial overlap)

$$q_3^I \leq n < q_2^I = \left\lceil \frac{q_2}{1-q_2} c_1 \right\rceil.$$

Here, $q_{n,i}^{(k)} = n, k = 1,2$ for all values of $i = 1, \cdots, m$. For given integer n, let $\nu_n^{(3)}$ with $1 \leq \nu_n^{(3)} < m$ be such that

$$\begin{aligned} &t_{n-1} < q_3 t_{n,i} \leq t_n, && i = 1, \cdots, \nu_n^{(3)}, \\ &t_n < q_3 t_{n,i} < t_{n+1}, && i = \nu_n^{(3)} + 1, \cdots, m. \end{aligned}$$

Then the collocation equation (3.8) leads to

$$\begin{aligned} &\Big[I_m - \sum_{k=1}^{2} (B_n^{(k)} + hR_{n,n}^{(k)}) - (I_m - T_{\nu_n^{(3)}})(B_n^{(3)} + hR_{n,n}^{(3)}) - hM_n\Big] U_n \\ =& T_{\nu_n^{(3)}} (B_n^{(3)} + hR_{n,n-1}^{(3)}) U_{n-1} + h\sum_{l=0}^{n-1} \Big(M_{n,l} + \sum_{k=1}^{2} N_{n,l}^{(k)}\Big) U_l + F_n \\ &+ h\sum_{l=0}^{n-2} N_{n,l}^{(3)} U_l + (I_m - T_{\nu_n^{(3)}}) hN_{n,n-1}^{(3)} U_{n-1}. \end{aligned} \tag{3.15}$$

Phase III (q_1 complete overlap, q_2 and q_3 partial overlap)

$$q_2^I \leq n < q_1^I = \left\lceil \frac{q_1}{1-q_1} c_1 \right\rceil.$$

According to Lemma 3.1, we know that $q_{n,i}^{(1)} = n$ for all values of $i = 1, \cdots, m$ and $\{q_k t_{n,i}\}_{i=1}^m \subset (t_{n-1}, t_{n+1})$, $k = 2,3$, which means for given n, there exist two integers $\nu_n^{(2)} \in \{1, \cdots, m-1\}$ and $\nu_n^{(3)} \in \{1, \cdots, m-1\}$ such that

$$\begin{aligned} &q_{n,i}^{(2)} = n-1 (i = 1, \cdots, \nu_n^{(2)}), && q_{n,i}^{(2)} = n (i = \nu_n^{(2)} + 1, \cdots, m), \\ &q_{n,i}^{(3)} = n-1 (i = 1, \cdots, \nu_n^{(3)}), && q_{n,i}^{(3)} = n (i = \nu_n^{(3)} + 1, \cdots, m). \end{aligned}$$

The linear algebraic system corresponding to this phase can be written as

$$\begin{aligned}&\Big[I_m-(B_n^{(1)}+hR_{n,n}^{(1)})-\sum_{k=2}^{3}(I_m-T_{\nu_n^{(k)}})(B_n^{(k)}+hR_{n,n}^{(k)})-hM_n\Big]U_n\\ =&\sum_{k=2}^{3}T_{\nu_n^{(k)}}(B_n^{(k)}+hR_{n,n-1}^{(k)})U_{n-1}+h\sum_{l=0}^{n-1}(M_{n,l}+N_{n,l}^{(1)})U_l\\ &+F_n+h\sum_{k=2}^{3}\Big[\sum_{l=0}^{n-2}N_{n,l}^{(k)}U_l+(I_m-T_{\nu_n^{(k)}})N_{n,n-1}^{(k)}U_{n-1}\Big]. \qquad (3.16)\end{aligned}$$

Phase IV (q_1, q_2 and q_3 partial overlap)

$$q_1^I\le n<q_3^{II}=\Big\lceil\frac{q_3}{1-q_3}c_m\Big\rceil.$$

Using Lemma 3.1, we have $\{q_k t_{n,i}\}_{i=1}^m\subset(t_{n-1},t_{n+1})$, $k=1,2,3$, which means for given n, there exist three integers $\nu_n^{(k)}\in\{1,\cdots,m-1\}$, $k=1,2,3$ such that

$$q_{n,i}^{(k)}=n-1(i=1,\cdots,\nu_n^{(k)}),\quad q_{n,i}^{(k)}=n(i=\nu_n^{(k)}+1,\cdots,m),\quad k=1,2,3.$$

The equation is given by

$$\begin{aligned}&\Big[I_m-\sum_{k=1}^{3}(I_m-T_{\nu_n^{(k)}})(B_n^{(k)}+hR_{n,n}^{(k)})-hM_n\Big]U_n\\ =&\sum_{k=1}^{3}T_{\nu_n^{(k)}}(B_n^{(k)}+hR_{n,n-1}^{(k)})U_{n-1}+F_n+h\sum_{l=0}^{n-1}M_{n,l}U_l\\ &+h\sum_{k=1}^{3}\Big[\sum_{l=0}^{n-2}N_{n,l}^{(k)}U_l+(I_m-T_{\nu_n^{(k)}})N_{n,n-1}^{(k)}U_{n-1}\Big]. \qquad (3.17)\end{aligned}$$

Phase V (q_1 and q_2 partial overlap, q_3 non-overlap)

$$q_3^{II}\le n<q_2^{II}=\Big\lceil\frac{q_2}{1-q_2}c_m\Big\rceil. \qquad (3.18)$$

By Lemma 3.1, we obtain $\{q_k t_{n,i}\}_{i=1}^m\subset(t_{n-1},t_{n+1})$, $k=1,2$ and $q_3t_{n,i}\le t_n$ for $i=1,\cdots,m$, which means for given n, there exist three integers $\nu_n^{(k)}\in\{1,\cdots,m-1\}$, $k=1,2$ and $\nu_n^{(3)}\in\{1,\cdots,m\}$ so that

$$\begin{aligned}&q_{n,i}^{(k)}=n-1(i=1,\cdots,\nu_n^{(k)}),\qquad q_{n,i}^{(k)}=n(i=\nu_n^{(k)}+1,\cdots,m),\qquad k=1,2,\\ &q_{n,i}^{(3)}=q_n^{(3)}(i=1,\cdots,\nu_n^{(3)}),\qquad q_{n,i}^{(3)}=q_n^{(3)}+1(i=\nu_n^{(3)}+1,\cdots,m).\end{aligned}$$

The collocation scheme for this phase can be written as

$$\begin{aligned}
&\Big[I_m - \sum_{k=1}^{2}(I_m - T_{\nu_n^{(k)}})(B_n^{(k)} + hR_{n,n}^{(k)}) - hM_n\Big]U_n \\
=&\sum_{k=1}^{2} T_{\nu_n^{(k)}}(B_n^{(k)} + hR_{n,n-1}^{(k)})U_{n-1} + h\sum_{k=1}^{2}\Big[\sum_{l=0}^{n-2} N_{n,l}^{(k)}U_l + (I_m - T_{\nu_n^{(k)}})N_{n,n-1}^{(k)}U_{n-1}\Big] \\
&+ T_{\nu_n^{(3)}}(B_n^{(3)} + hR_{n,q_n^{(3)}}^{(3)})U_{q_n^{(3)}} + F_n + h\sum_{l=0}^{n-1} M_{n,l}U_l + h\sum_{l=0}^{q_n^{(3)}-1} N_{n,l}^{(3)}U_l \\
&+ (I_m - T_{\nu_n^{(3)}})\big(B_n^{(3)}U_{q_n^{(3)}+1} + hN_{n,q_n^{(3)}}^{(3)}U_{q_n^{(3)}} + hR_{n,q_n^{(3)}+1}^{(3)}U_{q_n^{(3)}+1}\big).
\end{aligned} \tag{3.19}$$

Phase VI (q_1 partial overlap, q_2 and q_3 non-overlap)

$$q_2^{II} \le n < q_1^{II} = \Big\lceil \frac{q_1}{1-q_1} c_m \Big\rceil.$$

From Lemma 3.1, we see $\{q_1 t_{n,i}\}_{i=1}^m \subset (t_{n-1}, t_{n+1})$ and $q_k t_{n,i} \le t_n$, $k = 2,3$ for $i = 1,\cdots,m$, which means for given n, there exist three integers $\nu_n^{(1)} \in \{1,\cdots,m-1\}$ and $\nu_n^{(k)} \in \{1,\cdots,m\}$, $k = 2,3$ such that

$$\begin{aligned}
&q_{n,i}^{(1)} = n-1(i = 1,\cdots,\nu_n^{(1)}), \quad && q_{n,i}^{(1)} = n(i = \nu_n^{(1)}+1,\cdots,m), \\
&q_{n,i}^{(k)} = q_n^{(k)}(i = 1,\cdots,\nu_n^{(k)}), \quad && q_{n,i}^{(k)} = q_n^{(k)}+1(i = \nu_n^{(k)}+1,\cdots,m), \quad k = 2,3.
\end{aligned}$$

The linear algebraic system corresponding to this phase given by

$$\begin{aligned}
&\big[I_m - (I_m - T_{\nu_n^{(1)}})(B_n^{(1)} + hR_{n,n}^{(1)}) - hM_n\big]U_n \\
=&T_{\nu_n^{(1)}}(B_n^{(1)} + hR_{n,n-1}^{(1)})U_{n-1} + h\sum_{l=0}^{n-2} N_{n,l}^{(1)}U_l + (I_m - T_{\nu_n^{(1)}})hN_{n,n-1}^{(1)}U_{n-1} \\
&+ \sum_{k=2}^{3} T_{\nu_n^{(k)}}(B_n^{(k)} + hR_{n,q_n^{(k)}}^{(k)})U_{q_n^{(k)}} + F_n + h\sum_{l=0}^{n-1} M_{n,l}U_l + \sum_{k=2}^{3} h\sum_{l=0}^{q_n^{(k)}-1} N_{n,l}^{(k)}U_l \\
&+ \sum_{k=2}^{3}(I_m - T_{\nu_n^{(k)}})\big(B_n^{(k)}U_{q_n^{(k)}+1} + hN_{n,q_n^{(k)}}^{(k)}U_{q_n^{(k)}} + hR_{n,q_n^{(k)}+1}^{(k)}U_{q_n^{(k)}+1}\big).
\end{aligned} \tag{3.20}$$

Phase VII (q_1, q_2 and q_3 non-overlap)

$$q_1^{II} \le n \le N-1.$$

Here, there is no longer any overlap of the images $q_k t_{n,i}$, $k = 1,2,3$ with interval $(t_n, t_{n+1}]$. For each value of n, there exist integers $\nu_n^{(k)} \in \{1,\cdots,m\}$, $k = 1,2,3$, such that

$$q_{n,i}^{(k)} = q_n^{(k)}(i = 1,\cdots,\nu_n^{(k)}), \quad q_{n,i}^{(k)} = q_n^{(k)}+1(i = \nu_n^{(k)}+1,\cdots,m), \quad k = 1,2,3.$$

The collocation scheme for the last phase is given by:

$$
\begin{aligned}
&(I_m - hM_n)U_n \\
=&\sum_{k=1}^{3} T_{\nu_n^{(k)}}(B_n^{(k)} + hR^{(k)}_{n,q_n^{(k)}})U_{q_n^{(k)}} + F_n + h\sum_{l=0}^{n-1} M_{n,l}U_l + h\sum_{k=1}^{3}\sum_{l=0}^{q_n^{(k)}-1} N_{n,l}^{(k)}U_l \\
&+\sum_{k=1}^{3}(I_m - T_{\nu_n^{(k)}})\big(B_n^{(k)}U_{q_n^{(k)}+1} + hN^{(k)}_{n,q_n^{(k)}}U_{q_n^{(k)}} + hR^{(k)}_{n,q_n^{(k)}+1}U_{q_n^{(k)}+1}\big).
\end{aligned}
\tag{3.21}
$$

4 Theoretic results for the collocation solution on uniform mesh I_h

In this section, we shall present the existence and uniqueness of the collocation solution. Using traditional technique (see [6]), we give a convergence result about collocation method for VIFEMPDs (1.2) with $p = 2$, and then using projection operators, we propose a theorem about convergence of VIFEMPDs for general p. The last part of this section is devoted to some comments on superconvergence of collocation method for VIFEMPDs.

4.1 The existence and uniqueness of the collocation solution

In order to study existence and uniqueness, we firstly introduce a lemma as follows

Lemma 4.1. *(see [10]) Consider multiple delays functional equation (2.4), if $a_k, f \in C^v(I)$ for some integer $v \geq 1$ and*

$$\sum_{k=1}^{p} \|a_k\|_\infty < 1,$$

then there exists an $\bar{h} > 0$ (depend only on q_k), for any uniform mesh I_h with $h < \bar{h}$, the Eq. (3.8) with $\mathcal{V} = \mathcal{V}_{q_k} = 0$ defines a unique collocation solution $u_h \in S_{m-1}^{(-1)}$ for all $q_k \in (0,1)$.

Using above lemma, the existence of a unique solution for (3.8) is given by following theorem.

Theorem 4.1. *Assume that the functions a_k, f and K_k in (1.2) satisfy*

(i) $a_k, f \in C(I)$, $K_0 \in C(D)$ and $K_k \in C(D_{q_k})$, $k = 1, 2, \cdots, p$;

(ii) $\sum_{k=1}^{p} \|a_k\|_\infty < 1$.

Then there exists a constant $\bar{h} > 0$ (depend only on q_k), for any uniform mesh I_h with $h < \bar{h}$, the Eq. (3.8) defines a unique collocation solution $u_h \in S_{m-1}^{(-1)}$ for all $q_k \in (0,1)$.

Proof. We only discuss when $p=2$ with Case 2.1 here, the proof for the other case is similar. For the Phase I, the collocation solution of (3.8) can be rewritten as (3.9), and according to Lemma 4.1, we know that there exists a constant $\bar{h}_1>0$, such that for any $h<\bar{h}_1$, the matrix $I_m-\sum_{k=1}^{p}B_n^{(k)}$ is nonsingular. Then we obtain

$$\Big\|\Big[I_m-\sum_{k=1}^{p}B_n^{(k)}-h\Big(M_n+\sum_{k=1}^{p}R_{n,n}^{(k)}\Big)\Big]-\Big(I_m-\sum_{k=1}^{p}B_n^{(k)}\Big)\Big\|\le h\Big\|M_n+\sum_{k=1}^{p}R_{n,n}^{(k)}\Big\|,$$

which means $I_m-\sum_{k=1}^{p}B_n^{(k)}-h(M_n+\sum_{k=1}^{p}R_{n,n}^{(k)})$ is nonsingular for h small enough. Therefore the linear algebraic systems (3.9) has a unique solution. The statement that the linear algebraic system (3.10) in Phase II, (3.11) in Phase III, (3.12) in Phase IV and (3.13) in Phase V has a unique solution, can be carried out in a similar way. $\square$

4.2 The convergence results for collocation solution

We now analyze the attainable global convergence order of the collocation solution $u_h\in S_{m-1}^{(-1)}(I_h)$ for the VIFEMPDs (1.2). First, we give the analysis of convergence by the traditional method for (1.2) with $p=2$.

Theorem 4.2. *Let $u_h\in S_{m-1}^{(-1)}(I_h)$ be the collocation solution with $p=2$ defined in Section* 3.2. *Assume that the functions a_k, f and K_k in* (1.2) *satisfy*

(i) $a_k,f\in C^m(I)$, $K_0\in C^m(D)$ *and* $K_k\in C^m(D_{q_k})$, $k=1,2$;

(ii) $\sum_{k=1}^{2}\|a_k\|_\infty<1$.

Then there exists a constant $\bar{h}>0$, for any uniform mesh I_h with $h<\bar{h}$, the following estimate holds

$$\|u-u_h\|_\infty\le Ch^m.$$

Here, the constant C is independent on h.

Proof. By Theorem 2.2, we have $u\in C^m(I)$. Using Peano's Theorem for interpolation to y on e_n, we obtain

$$u(t_n+sh)=\sum_{j=1}^{m}L_j(s)Y_{n,j}+h^mR_{m,n}(s),\qquad s\in(0,1],\tag{4.1}$$

where L_j $(j=1,2,\cdots,m)$ are the Lagrange interpolation basis defined in (3.4) and $Y_{n,j}=u(t_{n,j})$. Here, the Peano remainder term and Peano kernel are given by

$$\begin{aligned}R_{m,n}(s)&=\int_0^1\widetilde{K}_m(s,z)u^{(m)}(t_n+zh)dz,\\ \widetilde{K}_m(s,z)&=\frac{1}{(m-1)!}\Big\{(s-z)_+^{m-1}-\sum_{k=1}^{m}L_k(s)(c_k-z)_+^{m-1}\Big\}.\end{aligned}$$

Let $\mathcal{E}_{n,j} = Y_{n,j} - U_{n,j}$, using formulations (3.3) and (4.1), the collocation error $e_h = y - u_h$ can be written as

$$e_h(t_n + sh) = \sum_{j=1}^{m} L_j(s)\mathcal{E}_{n,j} + h^m R_{m,n}(s), \quad s \in (0,1]. \tag{4.2}$$

Subtracting (3.8) from (1.2) with $p = 2$, we obtain the error at collocation points satisfy following equation

$$e_h(t_{n,i}) = \sum_{k=1}^{2} a_k(t_{n,i})e_h(q_k t_{n,i}) + \int_0^t K_0(t,s)e_h(s)ds + \sum_{k=1}^{2}\int_0^{q_k t} K_k(t,s)e_h(s)ds, \tag{4.3}$$

with $e_h(t_{n,i}) = \mathcal{E}_{n,i}$. Using (4.2) and (4.3), we get

$$\mathcal{E}_{n,i} = \sum_{k=1}^{2} r_1^{(k)} + r_2 + \sum_{k=1}^{2} r_3^{(k)}, \tag{4.4}$$

where

$$\begin{aligned}
r_1^{(k)} &= a_k(t_{n,i})\Big[\sum_{j=1}^{m} L_j(\gamma_{n,i}^{(k)})\mathcal{E}_{q_{n,i}^{(k)},j} + h^m R_{m,q_{n,i}^{(k)}}(\gamma_{n,i}^{(k)})\Big], \quad k=1,2, \\
r_2 &= h\sum_{l=0}^{n-1}\int_0^1 K_0(t_{n,i}, t_l + sh)\Big[\sum_{j=1}^{m} L_j(s)\mathcal{E}_{l,j} + h^m R_{m,l}(s)\Big]ds \\
&\quad + h\int_0^{c_i} K_0(t_{n,i}, t_n + sh)\Big[\sum_{j=1}^{m} L_j(s)\mathcal{E}_{n,j} + h^m R_{m,n}(s)\Big]ds, \\
r_3^{(k)} &= h\sum_{l=0}^{q_{n,i}^{(k)}-1}\int_0^1 K_k(t_{n,i}, t_l + sh)\Big[\sum_{j=1}^{m} L_j(s)\mathcal{E}_{l,j} + h^m R_{m,l}(s)\Big]ds \\
&\quad + h\int_0^{\gamma_{n,i}^{(k)}} K_k\big(t_{n,i}, t_{q_{n,i}^{(k)}} + sh\big)\Big[\sum_{j=1}^{m} L_j(s)\mathcal{E}_{q_{n,i}^{(k)},j} + h^m R_{m,q_{n,i}^{(k)}}(s)\Big]ds, \quad k=1,2.
\end{aligned}$$

Next, without loss of generality, we shall discuss the convergence of the Case 2.1. For the simplifications, we introduce following notations

$$\begin{aligned}
&\bar{a}_k = \|a_k\|_\infty, \qquad M_m = \|u^{(m)}\|_\infty, && \Phi_{n,l}^{(k)} = \big(a_k(t_{n,i})R_{m,l}(\gamma_{n,i}^{(k)})\big)_i, \\
&\rho_n = \Big(\int_0^{c_i} K_0(t_{n,i}, t_n + sh)R_{m,n}(s)ds\Big)_i, && \rho_{n,l} = \Big(\int_0^1 K_0(t_{n,i}, t_l + sh)R_{m,l}(s)ds\Big)_i, \\
&\rho_{n,l}^{(k)} = \Big(\int_0^1 K_k(t_{n,i}, t_l + sh)R_{m,l}(s)ds\Big)_i, && \widehat{\rho}_{n,l}^{(k)} = \Big(\int_0^{\gamma_{n,i}^{(k)}} K_k(t_{n,i}, t_l + sh)R_{m,l}(s)ds\Big)_i, \\
&\bar{L} = \max_j \|L_j\|_\infty, \quad k_m = \max_{s\in[0,1]}\int_0^1 |\widetilde{K}_m(s,z)|dz, && \bar{K} = \max_{k=0,1,2}\Big\{\max_{t\in[0,1]}\int_0^t |K_k(t,s)|ds\Big\}.
\end{aligned}$$

For Phase I, we know $q_{n,i}^{(k)} = n$ $(k=1,2)$ for $i=1,\cdots,m$, using above notations, the corresponding matrix form for Eq. (4.4) is given by

$$\begin{aligned}(I_m - \mathcal{B}_n^I)\mathcal{E}_n =& h^m \sum_{k=1}^{2} \Phi_{n,n}^{(k)} + h\sum_{l=0}^{n-1}\Big(M_{n,l} + \sum_{k=1}^{2} N_{n,l}^{(k)}\Big)\mathcal{E}_l \\ &+ h^{m+1}\Big[\sum_{l=0}^{n-1}\Big(\rho_{n,l} + \sum_{k=1}^{2}\rho_{n,l}^{(k)}\Big) + \rho_n + \sum_{k=1}^{2}\widehat{\rho}_{n,n}^{(k)}\Big], \end{aligned} \tag{4.5}$$

with

$$\mathcal{B}_n^I = B_n^{(1)} + B_n^{(2)} + h(M_n + R_{n,n}^{(1)} + R_{n,n}^{(2)}) \quad \text{and} \quad \mathcal{E}_n = (\mathcal{E}_{n,1},\cdots,\mathcal{E}_{n,m})^T.$$

According to Theorem 4.1, the above linear algebraic systems possesses a unique solution for uniform meshes I_h with $h \in (0,\bar{h})$. Thus, there exists a constant D_0 such that

$$\|(I_m - \mathcal{B}_n^I)^{-1}\|_1 \le D_0, \quad n = 0,\cdots,q_1^I - 1.$$

Using (4.5), we have

$$\begin{aligned}\|\mathcal{E}_n\|_1 \le& D_0\Big[mh^m(\bar{a}_1+\bar{a}_2)k_m M_m + 3mh\bar{K}\bar{L}\sum_{l=0}^{n-1}\|\mathcal{E}_l\|_1 \\ &+ 3mk_m M_m \bar{K} h^m \sum_{l=0}^{n-1} h + 3mk_m M_m \bar{K} h^{m+1}\Big],\end{aligned}$$

which leads to,

$$\|\mathcal{E}_n\|_1 \le \gamma_{0,1}\sum_{l=0}^{n-1} h\|\mathcal{E}_l\|_1 + \gamma_{1,1} M_m h^m, \qquad 0 \le n < q_2^I, \tag{4.6}$$

where

$$\gamma_{0,1} = 3m\bar{K}\bar{L}D_0, \qquad \gamma_{1,1} = \big(m(\bar{a}_1+\bar{a}_2)k_m + 3mk_m\bar{K}T + 3mk_m\bar{K}h\big)D_0.$$

Applying discrete Gronwall inequality to (4.6) yields

$$\|\mathcal{E}_n\|_1 \le \gamma_{1,1} M_m h^m \exp(\gamma_{0,1}T) := C_1 M_m h^m, \quad 0 \le n < q_1^I.$$

Similar arguments can be done for Phase II to Phase V. Combining these five phases, we know there exists a constant $C < \infty$ such that

$$\|\mathcal{E}_n\|_1 \le CM_m h^m, \qquad \text{for all } 0 \le n \le N-1. \tag{4.7}$$

Substituting the above estimate (4.7) into (4.2) leads to

$$|e_h(t_n + sh)| \le (C\bar{L} + k_m)M_m h^m, \quad 0 \le n \le N-1.$$

This completes the proof. □

From the proof presented above on two proportional delays, it seems that the traditional technique is not suitable for analysis about multiple proportional delays, since the situations for multiple proportional delays are too complicated. By using projection operators, we can present another proof of the convergence results on multiple proportional delays which is also hold for proportional delays (see [10]).

Theorem 4.3. *Let $u_h \in S^{(-1)}_{m-1}(I_h)$ is the collocation solution defined in* (3.8). *Assume that the functions a_k, f and K_k in* (1.2) *satisfy*

(i) $a_k, f \in C^m(I)$, $K_0 \in C^m(D)$ *and* $K_k \in C^m(D_{q_k})$, $k = 1, 2, \cdots, p$;

(ii) $\sum\limits_{k=1}^{p} \|a_k\|_\infty < 1$.

Then for all sufficiently small $h > 0$, we have

$$\|u - u_h\|_\infty \le C\big(\|(\mathcal{I} - \mathcal{P}_h)f\|_\infty + \|(\mathcal{I} - \mathcal{P}_h)\mathcal{K}u\|_\infty\big), \tag{4.8}$$

where the operator $\mathcal{K}$ is given by (2.1), *$\mathcal{P}_h$ is the Lagrange interpolate projection operator corresponding to the collocation parameters $\{c_i\}$, and the constant C is independent on h.*

Furthermore, if the exact solution $u \in W^{m,\infty}(I)$, we obtain

$$\|u - u_h\|_\infty \le Ch^m \|u\|_{m,\infty}, \tag{4.9}$$

where

$$\|v\|_{m,\infty} := \max_{0\le j\le m} \Big(\sup_{t\in I} \Big| \frac{d^j v(t)}{dt^j} \Big| \Big).$$

Proof. The operator formulations for VIFEMPDs (1.2) and its collocation equation (3.8) are given by

$$\begin{cases} u = f + \mathcal{K}u, \\ u_h = \mathcal{P}_h f + \mathcal{P}_h \mathcal{K} u_h. \end{cases} \tag{4.10}$$

Based on the solvability of the VIFEMPDs and its collocation equation, we obtain

$$\begin{cases} u = (\mathcal{I} - \mathcal{K})^{-1} f, \\ u_h = (\mathcal{I} - \mathcal{P}_h\mathcal{K})^{-1} \mathcal{P}_h f. \end{cases} \tag{4.11}$$

The error between u and u_h can be expressed in the form

$$\begin{aligned} u - u_h =& (\mathcal{I} - \mathcal{K})^{-1} f - (\mathcal{I} - \mathcal{P}_h\mathcal{K})^{-1}\mathcal{P}_h f \\ =& (\mathcal{I} - \mathcal{K})^{-1}(f - \mathcal{P}_h f) + (\mathcal{I} - \mathcal{K})^{-1}\mathcal{P}_h f - (\mathcal{I} - \mathcal{P}_h\mathcal{K})^{-1}\mathcal{P}_h f \\ =& (\mathcal{I} - \mathcal{K})^{-1}(f - \mathcal{P}_h f) + (\mathcal{I} - \mathcal{P}_h\mathcal{K})^{-1}(\mathcal{K} - \mathcal{P}_h\mathcal{K})(\mathcal{I} - \mathcal{K})^{-1}\mathcal{P}_h f \\ =& (\mathcal{I} - \mathcal{K})^{-1}(\mathcal{I} - \mathcal{P}_h) f + (\mathcal{I} - \mathcal{P}_h\mathcal{K})^{-1}(\mathcal{K} - \mathcal{P}_h\mathcal{K})(\mathcal{I} - \mathcal{K})^{-1}(\mathcal{P}_h f - f) \\ &+ (\mathcal{I} - \mathcal{P}_h\mathcal{K})^{-1}(\mathcal{K} - \mathcal{P}_h\mathcal{K})(\mathcal{I} - \mathcal{K})^{-1} f \\ =& (\mathcal{I} - \mathcal{K})^{-1}(\mathcal{I} - \mathcal{P}_h) f + (\mathcal{I} - \mathcal{P}_h\mathcal{K})^{-1}(\mathcal{K} - \mathcal{P}_h\mathcal{K})(\mathcal{I} - \mathcal{K})^{-1}(\mathcal{P}_h - \mathcal{I}) f \\ &+ (\mathcal{I} - \mathcal{P}_h\mathcal{K})^{-1}(\mathcal{I} - \mathcal{P}_h)\mathcal{K}u, \end{aligned} \tag{4.12}$$

which implies

$$\|u-u_h\|_\infty \leq C\big(\|(\mathcal{I}-\mathcal{P}_h)f\|_\infty + \|(\mathcal{I}-\mathcal{P}_h)\mathcal{K}u\|_\infty\big). \tag{4.13}$$

If $u \in W^{m,\infty}$, from the error estimates of the interpolation operator $\mathcal{P}_h$, we know that

$$\|(\mathcal{I}-\mathcal{P}_h)f\|_\infty \leq Ch^m\|f\|_{m,\infty} \leq Ch^m\|u\|_{m,\infty}, \tag{4.14a}$$

$$\|(\mathcal{I}-\mathcal{P}_h)\mathcal{K}u\|_\infty \leq Ch^m\|\mathcal{K}u\|_{m,\infty} \leq Ch^m\|u\|_{m,\infty}, \tag{4.14b}$$

which leads to (4.9) Thus, the proof is complete. □

4.3 Comments on superconvergence

In the rest of this section, we discuss the superconvergence of collocation method for VIFEMPDs. Define the iterated collocation solution u_h^{it} associated with u_h by

$$u_h^{it}(t) = \sum_{k=1}^{p} a_k(t)u_h(q_kt) + f(t) + (\mathcal{V}u_h)(t) + \sum_{k=1}^{p}(\mathcal{V}_{q_k}u_h)(t)ds. \tag{4.15}$$

Then the iterated error $e_h^{it} = u - u_h^{it}$ is given by

$$e_h^{it}(t) = e_h(t) - \delta_h(t) = \sum_{k=1}^{p} a_k(t)e_h(q_kt) + (\mathcal{V}e_h)(t) + \sum_{k=1}^{p}(\mathcal{V}_{q_k}e_h)(t),$$

where

$$\delta_h(t) := -u_h(t) + \sum_{k=1}^{p} a_k(t)u_h(q_kt) + f(t) + (\mathcal{V}u_h)(t) + \sum_{k=1}^{p}(\mathcal{V}_{q_k}u_h)(t).$$

We present a superconvergence result for a special case of (4.15) with $a_k = 0$ $(k = 1,\cdots,p)$.

Theorem 4.4. *Assume that the functions a_k, f and K_k in* (1.2) *satisfy*

(i) $a_k = 0, f \in C^{m+1}(I)$, $K_0 \in C^{m+1}(D)$ *and* $K_k \in C^{m+1}(D_{q_k})$, $k = 1,\dots,p$.

Let $u_h \in S_{m-1}^{(-1)}(I_h)$ *is the collocation solution defined in* (3.8) *with collocation parameters* $\{c_i\}$ *satisfying the orthogonality condition*

$$J_0 := \int_0^1 \prod_{i=1}^{m}(s-c_i)ds = 0.$$

Then the iterated collocation solution defined by (4.15) is globally superconvergent on I with

$$\|u-u_h^{it}\|_\infty \leq Ch^{m+1},$$

where the constant C is independent on h.

The proof is similar to collocation method for VIFEPDs (see [6]).

Using Theorem 4.2, it is easy to see that $\delta_h(t) = \mathcal{O}(h^m)$, and at collocation points, $\delta_h(t) = 0$. Hence, we obtain $e_h^{it}(t) = e_h(t)$, $t \in X_h$. As for classical VIES, e_h^{it} can exhibit a higher order of convergence for properly chosen collocation points $\{c_i\}$ like Theorem 4.4. But for VIFEMPDs of form (1.2), we point out that e_h^{it} can not achieve a higher convergence order since the existence of the q-difference term $\{u(q_k t)\}_{k=1}^{p}$, and we shall show this numerically in next section.

5 Numerical examples

In this section, we apply the collocation methods described in Section 3 to several VIFEMPDs examples. The first three examples are about the Eq. (1.2) with $p = 2$, and the last example is concerned with $p = 3$. Results of the numerical simulations verify our convergence analysis in Section 4. Simultaneously, we also address, by means of numerical tests in Example 5.2, the iterative collocation method e_h^{it} can not achieve a higher convergence order.

Example 5.1. Consider the VIFEMPDs (1.2) with $p = 2$ and

$$
\begin{aligned}
&a_1(t) = \frac{1}{4}\sin t, \quad a_2(t) = \frac{1}{2}te^{-t},\\
&K_0(t,s) = e^{-(t+s)}, \quad K_1(t,s) = e^{-(t-s)}, \quad K_2(t,s) = e^{t-s},\\
&f(t) = te^{-t} - \frac{1}{4}q_1 t\sin te^{-q_1 t} - \frac{1}{2}q_2 t^2 e^{-q_2 t - t} - \frac{1}{2}q_1^2 t^2 e^{-t} - \frac{1}{4}e^{-t}\\
&\qquad\quad + \frac{1}{2}te^{-3t} + \frac{1}{4}e^{-3t} - \frac{1}{4}e^{t} + \frac{1}{4}e^{(t-2q_2 t)} + \frac{1}{2}q_2 te^{(t-2q_2 t)}.
\end{aligned}
$$

Then the exact solution is $u(t) = te^{-t}$, for $t \in [0,1]$.

Firstly, for a special case with $q_1 = q_2$, we use the piecewise quadratic space $S_2^{(-1)}(I_h)$ with the collocation parameters

$$
C_1 = \Big(\frac{5-\sqrt{15}}{10}, \frac{1}{2}, \frac{5+\sqrt{15}}{10}\Big), \quad q = 0.1 \quad \text{and} \quad C_2 = \Big(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\Big), \quad q = 0.9,
$$

respectively. The results are presented in Fig. 4. It is easy to see that the method is order of three. The result of this experiment, obtained by collocation schemes with $p = 2$, is consistent with the result of single proportional delay (see [6]).

Next, we consider the VIFEMPDs (1.2) with $q_1 \neq q_2$. In our numerical implementation, we use the space $S_2^{(-1)}(I_h)$ with the collocation points

$$
C = ((5-\sqrt{15})/10, 1/2, (5+\sqrt{15})/10),
$$

and parameters $(q_1, q_2) = (0.5, 0.1)$, $(q_1, q_2) = (0.99, 0.8)$ respectively. The results are shown in Fig. 5. It is easy to see that the convergence order is three. At the collocation points where

$$
e_h^{it}(t) = e_h(t),
$$

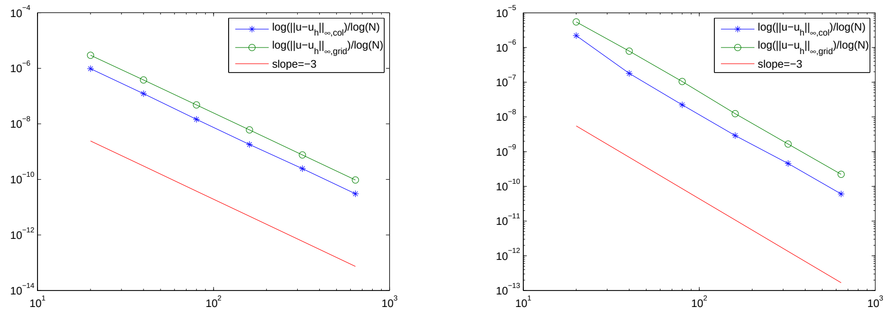

Figure 4: Example 5.1(a): the errors for $S_2^{(-1)}(I_h)$, left is by choice $q=0.1$ and right is by $q=0.9$.

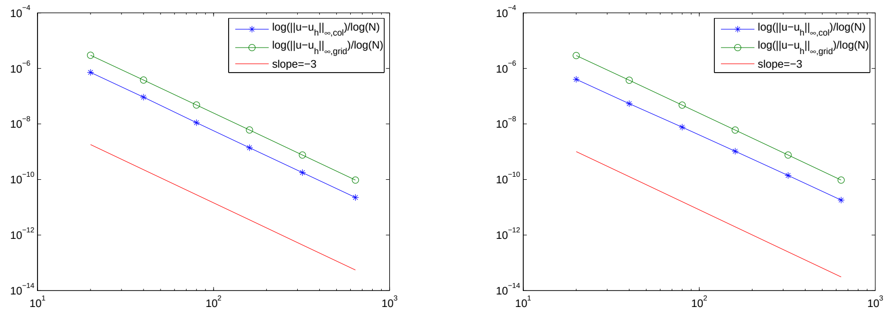

Figure 5: Example 5.1(b): the errors for $S_2^{(-1)}(I_h)$, left is by choice $(q_1,q_2)=(0.5,0.1)$ and right is by $(q_1,q_2)=(0.99,0.8)$.

the convergence order can only reach three.

Example 5.2. We consider Example 5.1 again, but with $a_1(t)=a_2(t)=0$, and

$$f(t)=te^{-t}-\frac{1}{2}q_1^2t^2e^{-t}-\frac{1}{4}e^{-t}+\frac{1}{2}te^{-3t}+\frac{1}{4}e^{-3t}-\frac{1}{4}e^{t}+\frac{1}{4}e^{(t-2q_2t)}+\frac{1}{2}q_2te^{(t-2q_2t)}.$$

Then the exact solution is still $u(t)=te^{-t}$.

In our simulations, we use the piecewise quadratic space $S_2^{(-1)}(I_h)$ with the collocation parameters

$$C_1=\Big(\frac{1}{4},\frac{1}{2},\frac{3}{4}\Big),\quad (q_1,q_2)=(0.5,0.1)\quad \text{and}\quad C_2=\Big(\frac{1}{3},\frac{1}{2},\frac{2}{3}\Big),\quad (q_1,q_2)=(0.99,0.8),$$

respectively. Here, the collocation parameters C_1 and C_2 both satisfy the orthogonality condition in Theorem 4.4. The results presented in Fig. 6 clearly exhibit the theoretical superconvergence, order of four. Comparing the results for Example 5.1 and Example 5.2, we can see that e_h^{it} can not achieve a higher order convergence due to the existence of the q-difference term $\{u(q_kt)\}_{k=1}^{p}$.

Example 5.3. Consider the first kind Volterra integral functional equation

$$\int_{q_1t}^{t}\widetilde{K_1}(t,s)u(s)ds+\int_{q_2t}^{t}\widetilde{K_2}(t,s)u(s)ds=g(t),\quad t\in[0,1],\quad g(0)=0, \tag{5.1}$$

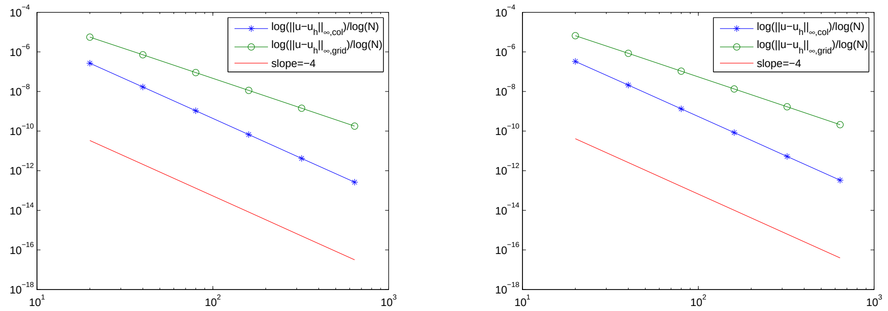

Figure 6: Example 5.2: the errors for $S_2^{(-1)}(I_h)$, left is by choice $C_1 = (1/4,1/2,3/4)$ and right is by $C_2 = (1/3,1/2,2/3)$.

where

$$\widetilde{K_1}(t,s) = e^{-10(t-s)}, \quad \widetilde{K_2}(t,s) = e^{-(t-s)},$$
$$g(t) = \frac{e^t - e^{(11q_1-10)t}}{11} + \frac{e^t - e^{(2q_2-1)t}}{2}.$$

Then the exact solution is $u(t) = e^t$. Eq. (5.1) can be rewritten in the form of (1.2) by

$$\begin{aligned}&\big(\widetilde{K_1}(t,t) + \widetilde{K_2}(t,t)\big)u(t)\\ =&q_1\widetilde{K_1}(t,q_1t)u(q_1t) + q_2\widetilde{K_2}(t,q_2t)u(q_2t) + g'(t)\\ &-\int_{q_1t}^{t}\frac{\partial\widetilde{K_1}}{\partial t}(t,s)u(s)ds - \int_{q_2t}^{t}\frac{\partial\widetilde{K_2}}{\partial t}(t,s)u(s)ds.\end{aligned} \tag{5.2}$$

Note that $\widetilde{K_1}(t,t) = \widetilde{K_2}(t,t) = 1$. Then we have

$$\begin{aligned}&a_1(t) = \frac{1}{2}q_1\widetilde{K_1}(t,q_1t) = \frac{1}{2}q_1e^{-10(1-q_1)t}, \qquad a_2(t) = \frac{1}{2}q_2\widetilde{K_2}(t,q_2t) = \frac{1}{2}q_2e^{-(1-q_2)t},\\ &f(t) = \frac{1}{2}g'(t) = \frac{1}{22}[e^t - (11q_1-10)e^{(11q_1-10)t}] + \frac{1}{4}[e^t - (2q_2-1)e^{(2q_2-1)t}],\\ &K_0(t,s) = -\frac{1}{2}\Big(\frac{\partial\widetilde{K_1}}{\partial t}(t,s) + \frac{\partial\widetilde{K_2}}{\partial t}(t,s)\Big) = 5e^{-10(t-s)} + \frac{1}{2}e^{-(t-s)},\\ &K_1(t,s) = \frac{1}{2}\frac{\partial\widetilde{K_1}}{\partial t}(t,s) = -5e^{-10(t-s)}, \qquad K_2(t,s) = \frac{1}{2}\frac{\partial\widetilde{K_2}}{\partial t}(t,s) = -\frac{1}{2}e^{-(t-s)}.\end{aligned}$$

We will use the piecewise quadratic space with the collocation points

$$C = ((5-\sqrt{15})/10, 1/2, (5+\sqrt{15})/10),$$

and the choices of the delay parameters $(q_1,q_2) = (0.9,0.2)$ and $(q_1,q_2) = (0.75,0.5)$. The result is shown in Fig. 7, and the numerical result is consistent with theoretical order, i.e., order of $m = 3$.

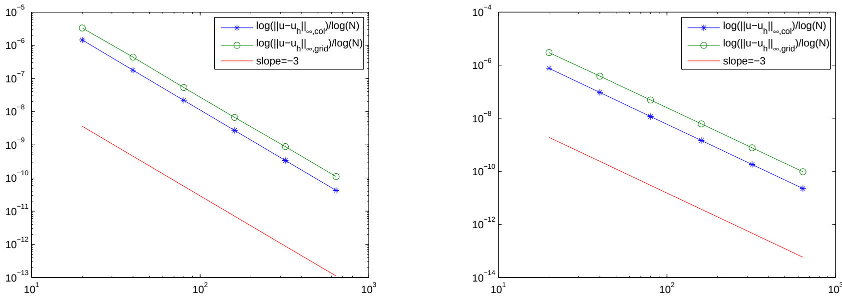

Figure 7: Example 5.3: the errors for $S_2^{(-1)}(I_h)$, left is by choice $(q_1,q_2) = (0.9,0.2)$ and right is by $(q_1,q_2) = (0.75,0.5)$.

Example 5.4. Consider the VIFEMPDs (1.2) with $p = 3$, i.e., with three proportional delays. Let

$$\begin{aligned} &a_1(t) = \frac{1}{8}\sin t, \quad a_2(t) = \frac{1}{4}te^{-t}, \quad a_3(t) = \frac{1}{2}e^{-t}, \\ &f(t) = e^t - \frac{1}{8}\sin te^{q_1 t} - \frac{1}{4}te^{q_2 t - t} - \frac{1}{2}e^{q_3 t - t} - te^{-t} - \frac{1}{2}e^{2q_1 t - t} \\ &\qquad\quad + \frac{1}{2}e^{-t} - q_2 te^t - \frac{1}{11}e^{11 q_3 t - 10t} + \frac{1}{11}e^{-10t}, \\ &K_0(t,s) = e^{-(t+s)}, \quad K_1(t,s) = e^{-(t-s)}, \quad K_2(t,s) = e^{t-s}, \quad K_3(t,s) = e^{-10(t-s)}, \end{aligned}$$

then the exact solution is $u(t) = e^t (t \in [0,1])$. In this experiment, we use space $S_2^{(-1)}(I_h)$ with the collocation parameters C=$((5-\sqrt{15})/10, 1/2, (5+\sqrt{15})/10)$, delay parameters $(q_1,q_2,q_3) = (0.99,0.9,0.5)$ and $(q_1,q_2,q_3) = (0.9,0.75,0.5)$. The result presented in Fig. 8 implies an order of three.

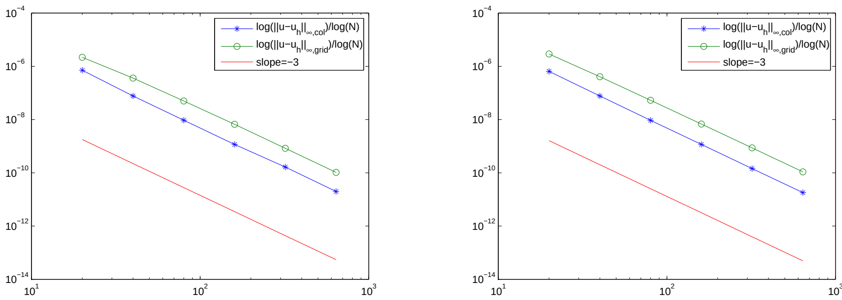

Figure 8: Example 5.4: the errors for $S_2^{(-1)}(I_h)$, left is by choice $(q_1,q_2,q_3) = (0.99,0.9,0.5)$ and right is by $(q_1,q_2,q_3) = (0.9,0.75,0.5)$.

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