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An Equivalent Characterization of $CMO(\mathbb{R}^n)$ with A_p Weights

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Abstract. Let $1<p<\infty$ and $\omega\in A_p$. The space $CMO(\mathbb{R}^n)$ is the closure in $BMO(\mathbb{R}^n)$ of the set of $C_c^\infty(\mathbb{R}^n)$. In this paper, an equivalent characterization of $CMO(\mathbb{R}^n)$ with A_p weights is established.

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1 Introduction

The goal of this paper is to provide an equivalent characterization of $CMO(\mathbb{R}^n)$, which is useful in the study of compactness of commutators of singular integral operator and fractional integral operator.

The space $BMO(\mathbb{R}^n)$ is defined by the set of functions $f\in L^1_{loc}(\mathbb{R}^n)$ such that

$$\|f\|_{BMO(\mathbb{R}^n)}:=\sup_{Q\subset\mathbb{R}^n} M(f,Q)<\infty,$$

where

$$M(f,Q):=\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx,\quad f_Q:=\frac{1}{|Q|}\int_Q f(x)dx.$$

The space $CMO(\mathbb{R}^n)$ is the closure in $BMO(\mathbb{R}^n)$ of the set of $C_c^\infty(\mathbb{R}^n)$, which is a proper subspace of $BMO(\mathbb{R}^n)$.

In fact, it is known that $CMO(\mathbb{R}^n)=VMO_0(\mathbb{R}^n)$, where $VMO_0(\mathbb{R}^n)$ is the closure of $C_0(\mathbb{R}^n)$ in $BMO(\mathbb{R}^n)$, see [2,3,9]. Here $C_0(\mathbb{R}^n)$ is the set of continuous functions on $\mathbb{R}^n$ which vanish at infinity. Neri [8] gave a characterization of $CMO(\mathbb{R}^n)$ by Riesz transforms. Meanwhile, Neri proposed the following characterization of $CMO(\mathbb{R}^n)$ and its proof was established by Uchiyama in his remarkable work [11].

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Theorem 1.1. *Let $f \in BMO(\mathbb{R}^n)$. Then $f \in CMO(\mathbb{R}^n)$ if and only if f satisfies the following three conditions*

(a) $\lim\limits_{a\to 0} \sup\limits_{|Q|=a} M(f,Q)=0$;

(b) $\lim\limits_{a\to \infty} \sup\limits_{|Q|=a} M(f,Q)=0$;

(c) $\lim\limits_{|x|\to \infty} M(f,Q+x)=0$ *for each cube $Q\subset\mathbb{R}^n$, where $Q+x:=\{y+x : y\in Q\}$.*

Recently, Guo, Wu and Yang [6] established an equivalent characterization of space $CMO(\mathbb{R}^n)$ by local mean oscillations. Lots of works about space $CMO(\mathbb{R}^n)$ have been studied, see [4] for example. Muckenhoupt and Wheeden [7, Theorem 5] showed the norm of $BMO_\omega(\mathbb{R}^n)$(see Definition 1.2) is equivalent to the norm of $BMO(\mathbb{R}^n)$, where the weight function ω is Muckenhoupt A_p weight. So it is natural to consider equivalent characterizations of $CMO(\mathbb{R}^n)$ associated to A_p weights.

To state our main results, we first recall some relevant notions and notations.

The following class of A_p was introduced in [1,5].

Definition 1.1. *Let $\omega(x)\geq 0$ and $\omega(x)\in L^1_{loc}(\mathbb{R}^n)$. For $1<p<\infty$, we say that $\omega(x)\in A_p$ if there exists a constant $C>0$ such that for any cube Q,*

$$\Big(\frac{1}{|Q|}\int_Q \omega(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q \omega(x)^{-\frac{1}{p-1}}dx\Big)^{p-1}\leq C. \tag{1.1}$$

Also, for $p=1$, we say that $\omega(x)\in A_1$ if there is a constant $C>0$ such that

$$M\omega(x)\leq C\omega(x), \tag{1.2}$$

where M is the Hardy-Littlewood maximal operator. For $p\geq 1$, the smallest constant appearing in (1.1) *and* (1.2) *is called the A_p characteristic constant of ω and is denoted by $[\omega]_{A_p}$.*

Definition 1.2. *Let $\omega\in A_p$. For a cube Q in $\mathbb{R}^n$, we say a function $f\in L^1_{loc}(\mathbb{R}^n)$ is in $BMO_\omega(\mathbb{R}^n)$ if f satisfies*

$$\|f\|_{BMO_\omega(\mathbb{R}^n)}:=\sup_{Q\subset\mathbb{R}^n} M(f,Q)_\omega<\infty,$$

where

$$m(f,Q)_\omega:=\frac{1}{\omega(Q)}\int_Q f(x)\omega(x)dx,$$
$$M(f,Q)_\omega:=\frac{1}{\omega(Q)}\int_Q |f(x)-m(f,Q)_\omega|\omega(x)dx.$$

Let $\omega\in A_p(p\geq 1)$, $q>1$, $f\in L^1_{loc}(\mathbb{R}^n)$. Then $BMO_{\omega,q}(\mathbb{R}^n)$ is defined by

$$\|f\|_{BMO_{\omega,q}(\mathbb{R}^n)}:=\sup_{Q\subset\mathbb{R}^n} M(f,Q)_{\omega,q}<\infty,$$

where

$$M(f,Q)_{\omega,q}:=\Big(\frac{1}{\omega(Q)}\int_Q |f(x)-m(f,Q)_\omega|^q\omega(x)dx\Big)^{1/q}.$$

Now, we can formulate our main results as follows.

Theorem 1.2. *Let $p\geq 1$, $1<q<\infty$. Suppose $f\in BMO(\mathbb{R}^n)$ and $\omega\in A_p$. Then the following conditions are equivalent:*

(1) $f\in CMO(\mathbb{R}^n)$;

(2) *f satisfies the following three conditions:*

(i) $\lim\limits_{a\to 0}\sup\limits_{|Q|=a} M(f,Q)_{\omega,q}=0$,

(ii) $\lim\limits_{a\to\infty}\sup\limits_{|Q|=a} M(f,Q)_{\omega,q}=0$,

(iii) $\lim\limits_{|x|\to\infty} M(f,Q+x)_{\omega,q}=0$ *for each* $Q\subset\mathbb{R}^n$.

(3) *f satisfies the following three conditions:*

(i′) $\lim\limits_{a\to 0}\sup\limits_{|Q|=a} M(f,Q)_{\omega}=0$,

(ii′) $\lim\limits_{a\to\infty}\sup\limits_{|Q|=a} M(f,Q)_{\omega}=0$,

(iii′) $\lim\limits_{|x|\to\infty} M(f,Q+x)_{\omega}=0$ *for each* $Q\subset\mathbb{R}^n$.

Throughout this paper, the letter C, will stand for positive constants, not necessarily the same one at each occurrence, but independent of the essential variables. If $f\leq Cg$, we write $f\lesssim g$ or $g\gtrsim f$; and if $f\lesssim g\lesssim f$, we write $f\sim g$. A dyadic cube Q on $\mathbb{R}^n$ is a cube of the form

$$\Big\{x=(x_1,\cdots,x_n)\in\mathbb{R}^n: k_i2^j\leq x_i<(k_i+1)2^j, i=1,\cdots,n, k_i\in\mathbb{Z}, j\in\mathbb{Z}\Big\},$$

R_j means $\{x\in\mathbb{R}^n:|x_i|<2^j, i=1,2,\cdots,n\}$. For $\lambda>0$, λQ denotes the cube with the same center as Q and side-length λ times the side-length of Q.

2 The proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. To do this, we firstly recall some auxiliary lemmas. Note that [7, Theorem 3] impiles the following weighted John-Nirenberg inequalities, also see [1,10].

Lemma 2.1. ***(John-Nirenberg)*** *Let $p\in[1,\infty)$, $\omega\in A_p$ and $f\in BMO_\omega(\mathbb{R}^n)$. For every $\alpha>0$ and cube Q, there exist constants c_1 and c_2 such that*

$$\omega(\{x\in Q:|f(x)-f_Q|>\alpha\})<c_1e^{-\frac{\alpha}{c_2\|f\|_{BMO_\omega(\mathbb{R}^n)}}}\omega(Q).$$

Next, we recall some useful properties of A_p weights.

Lemma 2.2 ([5]). *Let $\omega\in A_p$ and $1\le p<\infty$.*

1. *There exist $0<\delta<1$ and $C>0$ that depending only on the dimension n, p, and $[\omega]_{A_p}$ such that for any cube Q and any measurable subset S of Q we have*
$$\frac{\omega(S)}{\omega(Q)}\le C\Big(\frac{|S|}{|Q|}\Big)^\delta. \tag{2.1}$$
2. *There exist constants C and $\gamma>0$ that depending only on the dimension n, p, and $[\omega]_{A_p}$ such that for every cube Q we have*
$$\Big(\frac{1}{|Q|}\int_Q\omega(x)^{1+\gamma}dx\Big)^{\frac{1}{1+\gamma}}\le\frac{C}{|Q|}\int_Q\omega(x)dx. \tag{2.2}$$
3. *For all $\lambda>1$, and all cubes Q,*
$$\omega(\lambda Q)\le\lambda^{np}[\omega]_{A_p}\omega(Q). \tag{2.3}$$

Now, we are in position to prove the Theorem 1.2.

Proof. To prove (1)$\Rightarrow$(2) in Theorem 1.2. Assume that $f\in CMO(\mathbb{R}^n)$. If $f\in C_c^\infty(\mathbb{R}^n)$, then (i)$-$(iii) hold. It is obvious that (i) holds for uniformly continuous functions f. Without loss of generality, we assume $supp(f)\subset Q_0$. Then for each $Q\subset\mathbb{R}^n$, there exists $h\in\mathbb{R}^n$, for $|x|>|h|$, we have $Q_0\cap(Q+x)=\emptyset$, (iii) holds.

Note that

$$\begin{aligned}&\Big(\frac{1}{\omega(Q)}\int_Q|f(x)-m(f,Q)_\omega|^q\omega(x)dx\Big)^{1/q}\\ \le&\Big(\frac{1}{\omega(Q)}\int_{\mathbb{R}^n}|f(x)-m(f,Q)_\omega|^q\omega(x)dx\Big)^{1/q}.\end{aligned}$$

For $f\in C_c^\infty(\mathbb{R}^n)$, we have

$$\Big(\int_{\mathbb{R}^n}|f(x)-m(f,Q)_\omega|^q\omega(x)dx\Big)^{1/q}<\infty.$$

On the other hand, $Q(0,r)$ denotes the closed cube centered at 0 with side-length r. For any $x_0\in Q(0,r)$, there exists a cube Q centered at x_0 such that $Q(0,r)\subset Q$, by (2.1), we get

$$\frac{1}{\omega(Q)}\int_Q|f(x)-m(f,Q)_\omega|^q\omega(x)dx\lesssim\frac{1}{\omega(Q(0,r))}\Big(\frac{|Q(0,r)|}{|Q|}\Big)^\delta,$$

which tends to 0 as $|Q|$ tends to $+\infty$, (ii) holds.

If $f\in CMO(\mathbb{R}^n)\setminus C_c^\infty(\mathbb{R}^n)$, for any given $\varepsilon>0$, there exists $f_\varepsilon\in C_c^\infty(\mathbb{R}^n)$ satisfying (i)$-$(iii) and $\|f-f_\varepsilon\|_{BMO(\mathbb{R}^n)}<\varepsilon$. Then by Lemma 2.1 and (2.2), for $\omega\in A_p$, $1<p<\infty$, it is easy to see

$$\|f-f_\varepsilon\|_{BMO_{\omega,q}(\mathbb{R}^n)}\lesssim\|f-f_\varepsilon\|_{BMO_\omega(\mathbb{R}^n)}\lesssim\|f-f_\varepsilon\|_{BMO(\mathbb{R}^n)}\lesssim\varepsilon. \tag{2.4}$$

The detailed proof of (2.4) also can be found in [1,7]. By (2.4) and the triangle inequality, we deduce that (i)$-$(iii) hold for f.

The proof of (2)$\Rightarrow$(3). By the Hölder inequality, we get

$$\begin{aligned}&\frac{1}{\omega(Q)}\int_Q|f(x)-m(f,Q)_\omega|\omega(x)dx\\ \lesssim&\frac{1}{\omega(Q)}\Big(\int_Q|f(x)-m(f,Q)_\omega|^q\omega(x)dx\Big)^{1/q}\Big(\int_Q\omega(x)dx\Big)^{1/q'}\\ =&\Big(\frac{1}{Q}\int_Q|f(x)-m(f,Q)_\omega|^q\omega(x)\Big)^{1/q},\end{aligned} \tag{2.5}$$

where $1/q+1/q'=1$.

It follows from (2.5) that if f satisfies (i)$-$(iii) then f satisfies (i$'$)$-$(iii$'$).

The proof of (3)$\Rightarrow$(1). Now we show that if f satisfies (i$'$)$-$(iii$'$) then for all $\varepsilon>0$, there exists $g_\varepsilon\in BMO(\mathbb{R}^n)$ such that

$$\inf_{h\in C_c^\infty(\mathbb{R}^n)}\|g_\varepsilon-h\|_{BMO_\omega(\mathbb{R}^n)}<C_n\varepsilon, \tag{2.6}$$

$$\|g_\varepsilon-f\|_{BMO_\omega(\mathbb{R}^n)}<C_n\varepsilon. \tag{2.7}$$

We prove (2.6) and (2.7) by the following two steps.

Step I By (i$'$) and (ii$'$), there exist i_ε and k_ε such that

$$\sup\{M(f,Q)_\omega:|Q|\le 2^{n(i_\varepsilon+8)}\}<\varepsilon, \tag{2.8}$$

$$\sup\{M(f,Q)_\omega:|Q|\ge 2^{nk_\varepsilon}\}<\varepsilon. \tag{2.9}$$

By (iii$'$), there exists $j_\varepsilon>k_\varepsilon$ such that

$$\sup\{M(f,Q)_\omega:Q\cap R_{j_\varepsilon}=\emptyset\}<\varepsilon. \tag{2.10}$$

Now for each $x\in R_{j_\varepsilon}$, we take dyadic cube Q_x with side-length 2^{i_ε} containing x; if $x\in R_m\setminus R_{m-1}(j_\varepsilon<m)$, Q_x means a dyadic cube of side-length $2^{i_\varepsilon+m-j_\varepsilon}$. Set $g_\varepsilon'(x)=m(f,Q_x)_\omega$, by (ii$'$), there exists $m_\varepsilon>j_\varepsilon$ such that

$$\sup\{|g_\varepsilon'(x)-g_\varepsilon'(y)|:x,y\in R_{m_\varepsilon}\setminus R_{m_\varepsilon-1}\}<\varepsilon. \tag{2.11}$$

To see this, by (ii$'$), let $m_\varepsilon > j_\varepsilon + k_\varepsilon - i_\varepsilon$ be large enough such that when $\omega(R_{m_\varepsilon}) \geq 2^{n(m_\varepsilon + i_\varepsilon - j_\varepsilon)}$,

$$M(f,R_{m_\varepsilon+1})_\omega < \frac{\varepsilon}{C_1(j_\varepsilon - i_\varepsilon + 1)} \tag{2.12}$$

for some positive constant C_1.

For $x \in R_{m_\varepsilon} \setminus R_{m_\varepsilon - 1}$, it is obvious that

$$2^{j_\varepsilon - i_\varepsilon} Q_x \subset R_{m_\varepsilon+1} \subset 8 \cdot 2^{j_\varepsilon - i_\varepsilon} Q_x.$$

This together with (2.12) and (2.3) imply that

$$\begin{aligned} &|m(f,2^{j_\varepsilon - i_\varepsilon}Q_x)_\omega - m(f,R_{m_\varepsilon+1})_\omega| \\ \lesssim &\frac{\omega(R_{m_\varepsilon+1})}{\omega(2^{j_\varepsilon - i_\varepsilon}Q_x)} M(f,R_{m_\varepsilon+1})_\omega \lesssim \frac{\varepsilon}{C_1(j_\varepsilon - i_\varepsilon + 1)} \lesssim \frac{\varepsilon}{8}. \end{aligned} \tag{2.13}$$

Since $Q_x \subset R_{m_\varepsilon} \setminus R_{m_\varepsilon - 1}$, by (2.3) and (2.12), we have

$$\begin{aligned} &|m(f,R_{m_\varepsilon+1})_\omega - m(f,R_{m_\varepsilon} \setminus R_{m_\varepsilon - 1})_\omega| \\ \lesssim &\frac{\omega(R_{m_\varepsilon+1})}{\omega(R_{m_\varepsilon} \setminus R_{m_\varepsilon - 1})} M(f,R_{m_\varepsilon+1})_\omega \\ \lesssim &\frac{\omega(8 \cdot 2^{j_\varepsilon - i_\varepsilon}Q_x)}{\omega(Q_x)} M(f,R_{m_\varepsilon+1})_\omega \lesssim \frac{\varepsilon}{8}. \end{aligned} \tag{2.14}$$

By (2.13), (2.14) and (2.12), we conclude that for any Q_x with $x \in R_{m_\varepsilon} \setminus R_{m_\varepsilon - 1}$,

$$\begin{aligned} &|m(f,Q_x)_\omega - m(f,R_{m_\varepsilon} \setminus R_{m_\varepsilon - 1})_\omega| \\ \lesssim &|m(f,2^{j_\varepsilon - i_\varepsilon}Q_x)_\omega - m(f,R_{m_\varepsilon} \setminus R_{m_\varepsilon - 1})_\omega| + \sum_{k=1}^{j_\varepsilon - i_\varepsilon} |m(f,2^k Q_x)_\omega - m(f,2^{k-1}Q_x)_\omega| \\ \lesssim &|m(f,2^{j_\varepsilon - i_\varepsilon}Q_x)_\omega - m(f,R_{m_\varepsilon+1})_\omega| \\ &\quad + |m(f,R_{m_\varepsilon+1})_\omega - m(f,R_{m_\varepsilon} \setminus R_{m_\varepsilon - 1})_\omega| + \sum_{k=1}^{j_\varepsilon - i_\varepsilon} 2^{np} \frac{\varepsilon}{C_1(j_\varepsilon - i_\varepsilon + 1)} \\ \lesssim &\frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{2^{np}}{C_1}\varepsilon \lesssim \frac{\varepsilon}{2}. \end{aligned} \tag{2.15}$$

For any $Q_x, Q_y \subset R_{m_\varepsilon} \setminus R_{m_\varepsilon - 1}$, by (2.15), we get

$$\begin{aligned} &|m(f,Q_x)_\omega - m(f,Q_y)_\omega| \\ \lesssim &|m(f,Q_x)_\omega - m(f,R_{m_\varepsilon} \setminus R_{m_\varepsilon - 1})_\omega| + |m(f,R_{m_\varepsilon} \setminus R_{m_\varepsilon - 1})_\omega - m(f,Q_y)_\omega| \\ \lesssim &\varepsilon. \end{aligned}$$

Step II Define $g_\varepsilon(x)=g_\varepsilon^{'}(x)$ when $x\in R_{m_\varepsilon}$ and $g_\varepsilon(x)=m(f,R_{m_\varepsilon}\backslash R_{m_\varepsilon-1})_\omega$ when $x\in R_{m_\varepsilon}^c$. Notice that

$$if\ \bar{Q}_x\cap\bar{Q}_y\neq\emptyset,\quad diam\ Q_x\leq 2diam\ Q_y. \tag{2.16}$$

By the definition of $i_\varepsilon,j_\varepsilon$ and m_ε, if $\bar{Q}_x\cap\bar{Q}_y\neq\emptyset$ or $x,y\in R_{m_\varepsilon-1}^c$, there exists $C_2>0$ such that

$$|g_\varepsilon(x)-g_\varepsilon(y)|<C_2\varepsilon. \tag{2.17}$$

In fact, assume that $|x|<|y|$. Firstly, we show that if $x,y\in R_{m_\varepsilon-1}^c$, then (2.17) holds. By noting that $x,y\in R_{m_\varepsilon}^c$, we get

$$g_\varepsilon(x)=g_\varepsilon(y)=m(f,R_{m_\varepsilon}\backslash R_{m_\varepsilon-1})_\omega$$

and (2.17) holds. Next, if $x,y\in R_{m_\varepsilon}\backslash R_{m_\varepsilon-1}$, we deduce from (2.11) that

$$|g_\varepsilon(x)-g_\varepsilon(y)|=|g_\varepsilon^{'}(x)-g_\varepsilon^{'}(y)|<\varepsilon.$$

Thirdly, if $x\in R_{m_\varepsilon}\backslash R_{m_\varepsilon-1}$ and $y\in R_{m_\varepsilon}^c$, (2.15) indicates that

$$|g_\varepsilon(x)-g_\varepsilon(y)|=|m(f,Q_x)_\omega-m(f,R_{m_\varepsilon}\backslash R_{m_\varepsilon-1})_\omega|<\varepsilon.$$

Now we show if $\bar{Q}_x\cap\bar{Q}_y\neq\emptyset$, then (2.17) holds. We assume that $Q_x\neq Q_y$ and let Q be the smallest cube containing Q_x and Q_y, then $Q\subset 4Q_x$. If $x,y\in R_{j_\varepsilon}$, then

$$Q_x,Q_y\subset R_{j_\varepsilon}\qquad\text{and}\qquad |Q|<2^{n(i_\varepsilon+4)},$$

by (2.8), (2.17) holds. Similarly, if $Q_x\subset R_{j_\varepsilon}$, $Q_y\subset R_{m_\varepsilon-1}$ and $\bar{Q}_x\cap\bar{Q}_y\neq\emptyset$, by (2.8), (2.17) also holds. If $x,y\in R_{m_\varepsilon-1}^c$, notice that $Q\cap R_{j_\varepsilon}=\emptyset$ and by (2.10),

$$|g_\varepsilon^{'}(x)-m(f,Q)_\omega|\lesssim\frac{\omega(Q)}{\omega(Q_x)}M(f,Q)_\omega\lesssim\varepsilon.$$

Similarly, we have

$$|g_\varepsilon^{'}(y)-m(f,Q)_\omega|\lesssim\varepsilon.$$

Hence

$$|g_\varepsilon^{'}(x)-g_\varepsilon^{'}(y)|\lesssim|g_\varepsilon^{'}(x)-m(f,Q)_\omega|+|m(f,Q)_\omega-g_\varepsilon^{'}(y)|\lesssim\varepsilon.$$

Combining these cases, (2.17) holds.

We turn to prove that g_ε satisfies (2.6). Set

$$\widetilde{h}_\varepsilon(x):=g_\varepsilon(x)-m(f,R_{m_\varepsilon}\backslash R_{m_\varepsilon-1})_\omega.$$

By the definition of g_ε, we get

$$\widetilde{h}_\varepsilon(x)=0\text{ for any }x\in R_{m_\varepsilon},\quad \|\widetilde{h}_\varepsilon-g_\varepsilon\|_{BMO_\omega(\mathbb{R}^n)}=0.$$

Moreover, if $\bar{Q}_x\cap\bar{Q}_y\neq\emptyset$ or $x,y\in R^c_{m_\varepsilon-1}$, by (2.17), we have

$$|\widetilde{h_\varepsilon}(x)-\widetilde{h_\varepsilon}(y)|=|g_\varepsilon(x)-g_\varepsilon(y)|<C_2\varepsilon.$$

Observe that $supp(\widetilde{h_\varepsilon})\subset R_{m_\varepsilon}$. Take a positive valued function $\varphi(x)\in C^\infty_c(\mathbb{R}^n)$ supported in $B(0,1)$ and $\int_{\mathbb{R}^n}\varphi(x)dx=1$. For $t>0$, set

$$\varphi_t(x)=\frac{1}{t^n}\varphi(\frac{x}{t}).$$

Select $t<2^{i_\varepsilon}$, then

$$\begin{aligned}|\varphi_t*\widetilde{h_\varepsilon}(x)-\widetilde{h_\varepsilon}(x)|&\lesssim\int_{\mathbb{R}^n}\varphi_t(y)|\widetilde{h_\varepsilon}(x-y)-\widetilde{h_\varepsilon}(x)|dy\\ &=\int_{\mathbb{R}^n}\varphi(u)|\widetilde{h_\varepsilon}(x-tu)-\widetilde{h_\varepsilon}(x)|du\lesssim\sup_{u\in\mathbb{R}^n}|\widetilde{h_\varepsilon}(x-tu)-\widetilde{h_\varepsilon}(x)|,\end{aligned}$$

where in the second inequality we make the change of variable $y=ut$.

Since $u\in B(0,1)$ and $t<2^{i_\varepsilon}$, $\forall x\in\mathbb{R}^n$,

$$|(x-tu)-x|=|tu|<2^{i_\varepsilon}.$$

By (2.17), if $x,x-tu\in R_{m_\varepsilon}$, $\bar{Q}_x\cap\bar{Q}_{x-tu}\neq\emptyset$, hence

$$|\widetilde{h_\varepsilon}(x-tu)-\widetilde{h_\varepsilon}(x)|<C_2\varepsilon.$$

If one of x and $x-tu$ in $R^c_{m_\varepsilon}$, the other must be in $R^c_{m_\varepsilon-1}$, we also have

$$|\widetilde{h_\varepsilon}(x-tu)-\widetilde{h_\varepsilon}(x)|<C_2\varepsilon.$$

Moreover, $\varphi_t*\widetilde{h_\varepsilon}(x)\in C^\infty_c(\mathbb{R}^n)$ and

$$\|\varphi_t*\widetilde{h_\varepsilon}-\widetilde{h_\varepsilon}\|_{BMO_\omega(\mathbb{R}^n)}\lesssim\|\varphi_t*\widetilde{h_\varepsilon}-\widetilde{h_\varepsilon}\|_{L^\infty(\mathbb{R}^n)}+\varepsilon.$$

Therefore

$$\begin{aligned}&\|\varphi_t*\widetilde{h_\varepsilon}-g_\varepsilon\|_{BMO_\omega(\mathbb{R}^n)}\\ \lesssim&\|\varphi_t*\widetilde{h_\varepsilon}-\widetilde{h_\varepsilon}\|_{BMO_\omega(\mathbb{R}^n)}+\|\widetilde{h_\varepsilon}-g_\varepsilon\|_{BMO_\omega(\mathbb{R}^n)}\\ \lesssim&\|\varphi_t*\widetilde{h_\varepsilon}-\widetilde{h_\varepsilon}\|_{L^\infty(\mathbb{R}^n)}+\varepsilon.\end{aligned}$$

We obtain that (2.6) holds.

Now we prove (2.7). By the definition i_ε and j_ε again, we obtain that for any $x\in R_{m_\varepsilon}$,

$$\int_{Q_x}|f(y)-g_\varepsilon(y)|\omega(y)dy\lesssim\omega(Q_x)\varepsilon. \tag{2.18}$$

Indeed,

$$\int_{Q_x}|f(y)-g_\varepsilon(y)|\omega(y)dy=\int_{Q_x}|f(y)-m(f,Q_x)_\omega|\omega(y)dy.$$

If $Q_x\cap R_{j_\varepsilon}=\emptyset$, by (2.10), (2.18) holds. If $Q_x\cap R_{j_\varepsilon}\neq\emptyset$, using (2.8), (2.18) holds.

Let Q be a arbitrary cube in $\mathbb{R}^n$. In order to prove (2.7) holds, it suffices to show

$$M(f-g_\varepsilon,Q)_\omega<\varepsilon. \tag{2.19}$$

We consider the following four cases:

Case(i): $Q\subset R_{m_\varepsilon}$ and $\max\{diam\ Q_x : Q_x\cap Q\neq\emptyset\}>4diam\ Q$, by (2.16), the number of $Q_x\cap Q\neq\emptyset$ is finite. If $Q_{x_i}\cap Q\neq\emptyset$ and $Q_{x_j}\cap Q\neq\emptyset$, $\bar{Q}_{x_i}\cap\bar{Q}_{y_j}\neq\emptyset$, by (2.17),

$$\begin{aligned}M(g_\varepsilon,Q)_\omega&\lesssim\frac{1}{\omega(Q)}\sum_{i:Q_{x_i}\cap Q\neq\emptyset}\int_{Q_{x_i}\cap Q}|g_\varepsilon(x)-m(g_\varepsilon,Q)_\omega|\omega(x)dx\\&\lesssim\frac{1}{\omega(Q)}\sum_{i:Q_{x_i}\cap Q\neq\emptyset}\int_{Q_{x_i}\cap Q}\frac{1}{\omega(Q)}\sum_{j:Q_{x_j}\cap Q\neq\emptyset}\int_{Q_{x_j}\cap Q}|g_\varepsilon(x)-g_\varepsilon(y)|\omega(y)dy\omega(x)dx\\&\lesssim\varepsilon.\end{aligned}$$

Moreover, if $Q\cap R_{j_\varepsilon}\neq\emptyset$, then $|Q|\leq 2^{n(i_\varepsilon+1)}$, by (2.8), we have $M(f,Q)_\omega<\varepsilon$; if $Q\cap R_{j_\varepsilon}=\emptyset$, by (2.10), we also obtain $M(f,Q)_\omega<\varepsilon$. Hence

$$M(f-g_\varepsilon,Q)_\omega\lesssim M(f,Q)_\omega+M(g_\varepsilon,Q)_\omega\lesssim\varepsilon.$$

Case(ii): $Q\subset R_{m_\varepsilon}$ and $\max\{diam\ Q_x : Q_x\cap Q\neq\emptyset\}\leq 4diam\ Q$, we have

$$\bigcup_{Q_{x_i}\cap Q\neq\emptyset}\supset Q\,,\qquad\sum_{Q_{x_i}\cap Q\neq\emptyset}\omega(Q_{x_i})\sim\omega(Q).$$

Invoking (2.18), we get

$$\begin{aligned}M(f-g_\varepsilon,Q)_\omega&\lesssim\frac{2}{\omega(Q)}\sum_{Q_{x_i}\cap Q\neq\emptyset}\int_{Q_{x_i}}|f(y)-g_\varepsilon(y)|\omega(y)dy\\&\lesssim\frac{2}{\omega(Q)}\sum_{Q_{x_i}\cap Q\neq\emptyset}\omega(Q_{x_i})\varepsilon\lesssim\varepsilon.\end{aligned}$$

Case(iii): $Q\subset R^c_{m_\varepsilon-1}$, then $Q\cap R_{j_\varepsilon}=\emptyset$ and $M(f,Q)_\omega<\varepsilon$. Using (2.17),

$$M(g_\varepsilon,Q)_\omega\lesssim\frac{1}{\omega(Q)}\int_Q\frac{1}{\omega(Q)}\int_Q|g_\varepsilon(x)-g_\varepsilon(y)|\omega(y)dy\omega(x)dx<\varepsilon.$$

Hence

$$M(f-g_\varepsilon,Q)_\omega \lesssim M(f,Q)_\omega + M(g_\varepsilon,Q)_\omega < \varepsilon.$$

Case(iv): $Q\cap R^c_{m_\varepsilon}\neq\emptyset$ and $Q\cap R_{m_\varepsilon-1}\neq\emptyset$. Let P_Q be a smallest positive number such that $Q\subset R_{P_Q}$. Then

$$M(f,Q)_\omega \lesssim M(f,R_{P_Q})_\omega.$$

Moreover,

$$\begin{aligned}
&M(f-g_\varepsilon,R_{P_Q})_\omega\omega(R_{P_Q}) \lesssim \int_{R_{P_Q}} |(f-g_\varepsilon)(x)-m(f-g_\varepsilon,R_{P_Q}\backslash R_{m_\varepsilon})_\omega|\omega(x)dx\\
\lesssim&\int_{R_{P_Q}} |f(x)-m(f,R_{P_Q}\backslash R_{m_\varepsilon})_\omega|\omega(x)dx+\int_{R_{P_Q}} |g_\varepsilon(x)-m(g_\varepsilon,R_{P_Q}\backslash R_{m_\varepsilon})_\omega|\omega(x)dx.
\end{aligned}$$

On the one hand, by (2.18), we have

$$\begin{aligned}
&\int_{R_{P_Q}} |f(x)-m(f,R_{P_Q}\backslash R_{m_\varepsilon})_\omega|\omega(x)dx\\
\lesssim&\int_{R_{P_Q}} |f(x)-m(f,R_{P_Q})_\omega|\omega(x)dx+|m(f,R_{P_Q})_\omega-m(f,R_{P_Q}\backslash R_{m_\varepsilon})_\omega|\omega(R_{P_Q})\\
\lesssim&\int_{R_{P_Q}} |f(x)-m(f,R_{P_Q})_\omega|\omega(x)dx\\
\lesssim&\,\omega(R_{P_Q})\varepsilon.
\end{aligned}$$

On the other hand, it is easy to prove that

$$\sum_{i:Q_{x_i}\subset R_{m_\varepsilon}} \omega(Q_{x_i}) \sim \omega(R_{m_\varepsilon}).$$

Combining with (2.9) and (2.18) and the fact that $g_\varepsilon(x)=g_\varepsilon(y)$ for any $x,y\in R^c_{m_\varepsilon}$, we obtain

$$\begin{aligned}
&\int_{R_{P_Q}} |g_\varepsilon(x)-m(g_\varepsilon,R_{P_Q}\backslash R_{m_\varepsilon})_\omega|\omega(x)dx\\
\lesssim&\frac{1}{\omega(R_{P_Q}\backslash R_{m_\varepsilon})}\int_{R_{P_Q}}\int_{R_{P_Q}\backslash R_{m_\varepsilon}} |g_\varepsilon(x)-g_\varepsilon(y)|\omega(y)dy\omega(x)dx\\
=&\frac{1}{\omega(R_{P_Q}\backslash R_{m_\varepsilon})}\int_{R_{m_\varepsilon}}\int_{R_{P_Q}\backslash R_{m_\varepsilon}} |g_\varepsilon(x)-g_\varepsilon(y)|\omega(y)dy\omega(x)dx\\
\lesssim&\frac{1}{\omega(R_{P_Q}\backslash R_{m_\varepsilon})}\int_{R_{m_\varepsilon}}\int_{R_{P_Q}\backslash R_{m_\varepsilon}} \big[|g_\varepsilon(x)-f(x)|\\
&\quad+|f(x)-m(f,R_{m_\varepsilon}\backslash R_{m_\varepsilon-1})|\big]\omega(y)dy\omega(x)dx\\
\lesssim&\frac{1}{\omega(R_{P_Q}\backslash R_{m_\varepsilon})}\int_{R_{P_Q}\backslash R_{m_\varepsilon}}\sum_{i:Q_{x_i}\subset R_{m_\varepsilon}}\int_{Q_{x_i}} |g_\varepsilon(x)-f(x)|\omega(x)dx\omega(y)
\end{aligned}$$

$$
\begin{aligned}
&\quad +\int_{R_{m_\varepsilon}}\left[|f(x)-m(f,R_{m_\varepsilon})_\omega|+|m(f,R_{m_\varepsilon})_\omega-m(f,R_{m_\varepsilon}\backslash R_{m_\varepsilon-1})_\omega|\right]\omega(x)dx\\
&\lesssim \frac{1}{\omega(R_{P_Q}\backslash R_{m_\varepsilon})}\int_{R_{P_Q}\backslash R_{m_\varepsilon}}\varepsilon\sum_{i:Q_{x_i}\subset R_{m_\varepsilon}}\omega(Q_{x_i})\omega(y)dy\\
&\quad +\int_{R_{m_\varepsilon}}|f(x)-m(f,R_{m_\varepsilon})_\omega|\omega(x)dx\\
&\lesssim \omega(R_{m_\varepsilon})\varepsilon\lesssim\omega(R_{P_Q})\varepsilon.
\end{aligned}
$$

This implies (2.7) and completes the proof of Theorem 1.2. □

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