

Boundedness Characterization of Maximal Commutators on Orlicz Spaces in the Dunkl Setting

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Abstract. On the real line, the Dunkl operators

$$D_\nu(f)(x) := \frac{df(x)}{dx} + (2\nu+1) \frac{f(x)-f(-x)}{2x}, \quad \forall x \in \mathbb{R}, \forall \nu \geq -\frac{1}{2}$$

are differential-difference operators associated with the reflection group \mathbb{Z}_2 on \mathbb{R} , and on the \mathbb{R}^d the Dunkl operators $\{D_{k,j}\}_{j=1}^d$ are the differential-difference operators associated with the reflection group \mathbb{Z}_2^d on \mathbb{R}^d . In this paper, in the setting \mathbb{R} we show that $b \in BMO(\mathbb{R}, dm_\nu)$ if and only if the maximal commutator $M_{b,\nu}$ is bounded on Orlicz spaces $L_\Phi(\mathbb{R}, dm_\nu)$. Also in the setting \mathbb{R}^d we show that $b \in BMO(\mathbb{R}^d, h_k^2(x)dx)$ if and only if the maximal commutator $M_{b,k}$ is bounded on Orlicz spaces $L_\Phi(\mathbb{R}^d, h_k^2(x)dx)$.

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1 Introduction

Norm inequalities for several classical operators of harmonic analysis have been widely studied in the context of Orlicz spaces. It is well known that many of such operators fail to have continuity properties when they act between certain Lebesgue spaces and, in some situations, the Orlicz spaces appear as adequate substitutes. For example, the Hardy-Littlewood maximal operator is bounded on L^p for $1 < p < \infty$, but not on L^1 , but

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using Orlicz spaces, we can investigate the boundedness of the maximal operator near $p=1$, see [13] and [4] for more precise statements.

Let T be the classical singular integral operator, the commutator $[b, T]$ generated by T and a suitable function b is given by

$$[b, T]f := bT(f) - T(bf). \quad (1.1)$$

A well-known result due to Coifman, *et al.* [3] (see also [11]) states that $b \in BMO(\mathbb{R}^n)$ if and only if the commutator $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

Maximal commutator of Hardy-Littlewood maximal operator M with a locally integrable function b is defined by

$$M_b f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |b(x) - b(y)| |f(y)| dy,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing x . We refer to [2] for a detailed investigation of the operators M_b and the commutator of the maximal operator $[b, M]$ and references therein. For the boundedness of these operators in Orlicz space $L^\Phi(\mathbb{R}^n)$ see for instance [5, 7].

In [9], Dunkl introduced a family of first order differential-difference operators which play the role of the usual partial differentiation for the reflection group structure. For a real parameter $\nu \geq -1/2$, we consider the Dunkl operator, associated with the reflection group \mathbb{Z}_2 on \mathbb{R} :

$$D_\nu(f)(x) := \frac{df(x)}{dx} + (2\nu + 1) \frac{f(x) - f(-x)}{2x}, \quad \forall x \in \mathbb{R}.$$

Note that $D_{-1/2} = d/dx$.

In the setting \mathbb{R}^d the Dunkl operators $\{D_{k,j}\}_{j=1}^d$, which are the differential-difference operators introduced by Dunkl in [9]. These operators are very important in pure mathematics and in physics. They provide useful tools in the study of special functions with root systems.

It is well known that maximal operators play an important role in harmonic analysis (see [21]). Harmonic analysis associated to the Dunkl transform and the Dunkl differential-difference operator gives rise to convolutions with a relevant generalized translation. In this paper, in the framework of this analysis in the setting \mathbb{R} , we study the boundedness of the maximal commutator $M_{b,\nu}$ and the commutator of the maximal operator, $[b, M_\nu]$, on Orlicz spaces $L_\Phi(\mathbb{R}, dm_\nu)$, when b belongs to the space $BMO(\mathbb{R}, dm_\nu)$, by which some new characterizations of the space $BMO(\mathbb{R}, dm_\nu)$ are given. Also in the setting \mathbb{R}^d we study the boundedness of the maximal commutator $M_{b,k}$ and the commutator of the maximal operator, $[b, M_k]$, on the Orlicz space $L_\Phi(\mathbb{R}^d, h_k^2(x) dx)$, when b belongs to the space $BMO(\mathbb{R}^d, h_k^2(x) dx)$, by which some new characterizations of the space $BMO(\mathbb{R}^d, h_k^2(x) dx)$ are given.

The organization of this paper is as follows. In Sections 2 and 3, we give some preliminaries in the Dunkl setting, respectively, on \mathbb{R} and \mathbb{R}^d . We then present the boundedness of maximal commutators associated with Dunkl operators in Orlicz spaces $L_\Phi(\mathbb{R}, dm_\nu)$ in Section 4 and the boundedness of maximal commutators associated with Dunkl operators in Orlicz spaces $L_\Phi(\mathbb{R}^d, h_k^2(x)dx)$ in Section 5.

Finally, we make some conventions on notation. By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries in the Dunkl setting on \mathbb{R}

Let $\nu > -1/2$ be a fixed number and m_ν be the *weighted Lebesgue measure* on \mathbb{R} given by

$$dm_\nu(x) := \left(2^{\nu+1}\Gamma(\nu+1)\right)^{-1} |x|^{2\nu+1} dx, \quad \forall x \in \mathbb{R}.$$

For any $x \in \mathbb{R}$ and $r > 0$, let $B(x,r) := \{y \in \mathbb{R} : |y| \in]\max\{0, |x| - r\}, |x| + r[\}$. Then $B(0,r) =]-r, r[$ and

$$m_\nu B(0,r) = c_\nu r^{2\nu+2},$$

where $c_\nu := [2^{\nu+1}(\nu+1)\Gamma(\nu+1)]^{-1}$.

The *maximal operator* M_ν associated with Dunkl operator on the real line is given by

$$M_\nu f(x) := \sup_{r>0} (m_\nu B(x,r))^{-1} \int_{B(x,r)} |f(y)| dm_\nu(y), \quad \forall x \in \mathbb{R}$$

and the *maximal commutator* $M_{b,\nu}$ associated with Dunkl operator on the real line and with a locally integrable function $b \in L_1^{\text{loc}}(\mathbb{R}, dm_\nu)$ is defined by

$$M_{b,\nu} f(x) := \sup_{r>0} (m_\nu B(x,r))^{-1} \int_{B(x,r)} |b(x) - b(y)| |f(y)| dm_\nu(y), \quad \forall x \in \mathbb{R}.$$

For a function b defined on \mathbb{R} , we let, for any $x \in \mathbb{R}$,

$$b^-(x) := \begin{cases} 0, & \text{if } b(x) \geq 0, \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, for any $x \in \mathbb{R}$, $b^+(x) - b^-(x) = b(x)$. The following relations between $[b, M_\nu]$ and $M_{b,\nu}$ are valid :

Let b be any non-negative locally integrable function. Then

$$|[b, M_\nu]f(x)| \leq M_{b,\nu}(f)(x), \quad \forall x \in \mathbb{R}$$

holds for all $f \in L_1^{\text{loc}}(\mathbb{R}, dm_\nu)$.

If b is any locally integrable function on \mathbb{R} , then

$$|[b, M_\nu]f(x)| \leq M_{b, \nu}(f)(x) + 2b^-(x)M_\nu f(x), \quad \forall x \in \mathbb{R} \quad (2.1)$$

holds for all $f \in L_1^{\text{loc}}(\mathbb{R}, dm_\nu)$ (see, for example, [2]).

Recall also that Orlicz space was first introduced by Orlicz in [15, 16] as a generalizations of Lebesgue spaces L^p . Since then this space has been one of important functional frames in the mathematical analysis, and especially in real and harmonic analysis. Orlicz space is also an appropriate substitute for L^1 space when the space L^1 does not work.

To introduce the notion of Orlicz spaces in the Dunkl setting on \mathbb{R} , we first recall the definition of Young functions.

Definition 2.1. A function $\Phi: [0, \infty) \rightarrow [0, \infty]$ is called a Young function if Φ is convex, left-continuous, $\lim_{r \rightarrow +0} \Phi(r) = \Phi(0) = 0$ and $\lim_{r \rightarrow \infty} \Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0, \infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for all $r \geq s$. The set of Young functions such that

$$0 < \Phi(r) < \infty \quad \text{for all} \quad 0 < r < \infty$$

is denoted by \mathcal{Y} . If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0, \infty)$ and bijective from $[0, \infty)$ to itself.

For a Young function Φ and $0 \leq s \leq \infty$, let

$$\Phi^{-1}(s) := \inf\{r \geq 0: \Phi(r) > s\}.$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ . It is well known that

$$r \leq \Phi^{-1}(r) \tilde{\Phi}^{-1}(r) \leq 2r \quad \text{for any } r \geq 0, \quad (2.2)$$

where $\tilde{\Phi}(r)$ is defined by

$$\tilde{\Phi}(r) := \begin{cases} \sup\{rs - \Phi(s) : s \in [0, \infty)\}, & r \in [0, \infty) \\ \infty, & r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2 -condition, denoted also as $\Phi \in \Delta_2$, if

$$\Phi(2r) \leq C\Phi(r), \quad \forall r > 0$$

for some $C > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2 -condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \leq \frac{1}{2C}\Phi(Cr), \quad \forall r \geq 0$$

for some $C > 1$. In what follows, for any subset E of \mathbb{R} , we use χ_E to denote its characteristic function.

Definition 2.2. (Orlicz Space). For a Young function Φ , the set

$$L_\Phi(\mathbb{R}, dm_\nu) := \left\{ f \in L_1^{\text{loc}}(\mathbb{R}, dm_\nu) : \int_{\mathbb{R}} \Phi(k|f(x)|) dm_\nu(x) < \infty \text{ for some } k > 0 \right\}$$

is called the Orlicz space. If $\Phi(r) := r^p$ for all $r \in [0, \infty)$, $1 \leq p < \infty$, then $L_\Phi(\mathbb{R}, dm_\nu) = L_p(\mathbb{R}, dm_\nu)$. If $\Phi(r) := 0$ for all $r \in [0, 1]$ and $\Phi(r) := \infty$ for all $r \in (1, \infty)$, then $L_\Phi(\mathbb{R}, dm_\nu) = L_\infty(\mathbb{R}, dm_\nu)$. The space $L_\Phi^{\text{loc}}(\mathbb{R}, dm_\nu)$ is defined as the set of all functions f such that $f\chi_B \in L_\Phi(\mathbb{R}, dm_\nu)$ for all balls $B \subset \mathbb{R}$.

$L_\Phi(\mathbb{R}, dm_\nu)$ is a Banach space with respect to the norm

$$\|f\|_{L_{\Phi,\nu}} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}} \Phi\left(\frac{|f(x)|}{\lambda}\right) dm_\nu(x) \leq 1 \right\}.$$

For a measurable function f on \mathbb{R} and $t > 0$, let

$$m(f, t)_\nu := m_\nu\{x \in \mathbb{R} : |f(x)| > t\}.$$

Definition 2.3. The weak Orlicz space

$$WL_\Phi(\mathbb{R}, dm_\nu) := \{f \in L_{1,\nu}^{\text{loc}}(\mathbb{R}) : \|f\|_{WL_{\Phi,\nu}} < \infty\}$$

is defined by the norm

$$\|f\|_{WL_{\Phi,\nu}} := \inf \left\{ \lambda > 0 : \sup_{t>0} \Phi(t) m\left(\frac{f}{\lambda}, t\right)_\nu \leq 1 \right\}.$$

We note that $\|f\|_{WL_{\Phi,\nu}} \leq \|f\|_{L_{\Phi,\nu}}$

$$\begin{aligned} \sup_{t>0} \Phi(t) m(f, t)_\nu &= \sup_{t>0} t m(f, \Phi^{-1}(t))_\nu = \sup_{t>0} t m(\Phi(|f|), t)_\nu, \\ \int_{\mathbb{R}} \Phi\left(\frac{|f(x)|}{\|f\|_{L_{\Phi,\nu}}}\right) dm_\nu(x) &\leq 1, \quad \sup_{t>0} \Phi(t) m\left(\frac{f}{\|f\|_{WL_{\Phi,\nu}}}, t\right)_\nu \leq 1. \end{aligned} \tag{2.3}$$

The following analogue of the Hölder inequality is well known (see, e.g., [17]).

Theorem 2.1. Let the functions f and g be measurable on \mathbb{R} . For a Young function Φ and its complementary function $\tilde{\Phi}$, the following inequality is valid

$$\int_{\mathbb{R}} |f(x)g(x)| dm_\nu(x) \leq 2\|f\|_{L_{\Phi,\nu}} \|g\|_{L_{\tilde{\Phi},\nu}}.$$

By elementary calculations we have the following property.

Lemma 2.1. Let Φ be a Young function and B be a ball in \mathbb{R} . Then

$$\|\chi_B\|_{L_{\Phi,\nu}} = \|\chi_B\|_{WL_\Phi(\mathbb{R}, dm_\nu)} = \frac{1}{\Phi^{-1}\left((m_\nu(B))^{-1}\right)}.$$

By Theorem 2.1, Lemma 2.1 and (2.2) we obtain the following estimate.

Lemma 2.2. *For a Young function Φ and for the ball B the following inequality is valid:*

$$\int_B |f(y)| dm_\nu(y) \leq 2m_\nu(B) \Phi^{-1} \left((m_\nu(B))^{-1} \right) \|f\chi_B\|_{L_{\Phi,\nu}}.$$

The known boundedness statement for M_ν in Orlicz spaces on spaces of homogeneous type runs as follows.

Theorem 2.2. ([6]) *Let Φ be any Young function. Then the maximal operator M_ν is bounded from $L_\Phi(\mathbb{R}, dm_\nu)$ to $WL_\Phi(\mathbb{R}, dm_\nu)$ and for $\Phi \in \nabla_2$ bounded in $L_\Phi(\mathbb{R}, dm_\nu)$.*

3 Preliminaries in the Dunkl setting on \mathbb{R}^d

We consider \mathbb{R}^d with the Euclidean scalar product $\langle \cdot, \cdot \rangle$ and its associated norm $\|x\| := \sqrt{\langle x, x \rangle}$ for any $x \in \mathbb{R}^d$. For any $v \in \mathbb{R}^d \setminus \{0\}$ let σ_v be the reflection in the hyperplane $H_v \subset \mathbb{R}^d$ orthogonal to v :

$$\sigma_v(x) := x - \left(\frac{2\langle x, v \rangle}{\|v\|^2} \right) v, \quad \forall x \in \mathbb{R}^d.$$

A finite set $R \subset \mathbb{R}^d \setminus \{0\}$ is called a *root system*, if $\sigma_v R = R$ for all $v \in R$. We assume that it is normalized by $\|v\|^2 = 2$ for all $v \in R$.

The finite group G generated by the reflections $\{\sigma_v\}_{v \in R}$ is called the *reflection group* (or the *Coxeter-Weyl group*) of the root system. Then, we fix a G -invariant function $k: R \rightarrow \mathbb{C}$ called the *multiplicity function of the root system* and we consider the family of commuting operators $D_{k,j}$ defined for any $f \in C^1(\mathbb{R}^d)$ and any $x \in \mathbb{R}^d$ by

$$D_{k,j}f(x) := \frac{\partial}{\partial x_j} f(x) + \sum_{v \in R_+} k_v \frac{f(x) - f(\sigma_v(x))}{\langle x, v \rangle} \langle v, e_j \rangle, \quad 1 \leq j \leq d,$$

where $C^1(\mathbb{R}^d)$ denotes the set of all functions $f: \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\left\{ \frac{\partial f}{\partial x_j} \right\}_{j=1}^d$ are continuous on \mathbb{R}^d , $\{e_i\}_{i=1}^d$ are the standard unit vectors of \mathbb{R}^d and R_+ is a positive subsystem. These operators, defined by Dunkl [9], are independent of the choice of the positive subsystem R_+ and are of fundamental importance in various areas of mathematics and mathematical physics.

Throughout this paper, we assume that $k_v \geq 0$ for all $v \in R$ and we denote by h_k the weight function on \mathbb{R}^d given by

$$h_k(x) := \prod_{v \in R_+} |\langle x, v \rangle|^{k_v}, \quad \forall x \in \mathbb{R}^d.$$

The function h_k is G -invariant and homogeneous of degree γ_k , where $\gamma_k := \sum_{v \in R_+} k_v$.

Closely related to them is the so-called intertwining operator V_κ (the subscript means that the operator depends on the parameters κ_i , except in the rank-one case where the subscript is then a single parameter). The *intertwining operator* V_κ is the unique linear isomorphism of $\oplus_{n \geq 0} P_n$ such that

$$V(P_n) = P_n, \quad V_k(1) = 1, \quad D_i V_k = V_k \frac{\partial}{\partial x_i} \quad \text{for any } i \in \{1, \dots, d\}$$

with P_n being the subspace of homogeneous polynomials of degree n in d variables. The explicit formula of V_k is not known in general (see [19]). For the group $G := \mathbb{Z}_2^d$ and $h_k(x) := \prod_{i=1}^d |x_i|^{k_i}$ for all $x \in \mathbb{R}^d$, it is an integral transform

$$V_k f(x) := b_k \int_{[-1,1]^d} f(x_1 t_1, \dots, x_d t_d) \prod_{i=1}^d (1+t_i) (1-t_i^2)^{k_i-1} dt, \quad \forall x \in \mathbb{R}^d. \quad (3.1)$$

Let $B(x,r) := \{y \in \mathbb{R}^d : |x-y| < r\}$ denote the ball in \mathbb{R}^d that centered in $x \in \mathbb{R}^d$ and having radius $r > 0$. Then having

$$|B(0,r)|_k = \int_{B(0,r)} h_k^2(x) dx = \left(\frac{a_k}{d+2\gamma_k} \right) r^{d+2\gamma_k},$$

where

$$a_k := \left(\int_{S^{d-1}} h_k^2(x) d\sigma(x) \right)^{-1},$$

S^{d-1} is the unit sphere on \mathbb{R}^d with the normalized surface measure $d\sigma$.

The *maximal operator* M_k associated with the Dunkl operator on \mathbb{R}^d is given by

$$M_k f(x) := \sup_{r>0} \left(|B(x,r)|_k \right)^{-1} \int_{B(x,r)} |f(y)| h_k^2(y) dy, \quad \forall x \in \mathbb{R}^d$$

and the *maximal commutator* $M_{b,k}$ associated with the Dunkl operator on \mathbb{R}^d and with a locally integrable function $b \in L_1^{\text{loc}}(\mathbb{R}^d, h_k^2(x) dx)$ is defined by

$$M_{b,k} f(x) := \sup_{r>0} \left(|B(x,r)|_k \right)^{-1} \int_{B(x,r)} |b(x) - b(y)| |f(y)| h_k^2(y) dy, \quad \forall x \in \mathbb{R}^d.$$

In what follows, for any subset E of \mathbb{R}^d , we use χ_E to denote its *characteristic function*. Now, we introduce the notion of Orlicz spaces in the Dunkl setting on \mathbb{R}^d as follows.

Definition 3.1. (*Orlicz Space*). For a Young function Φ , the set

$$\begin{aligned} &L_\Phi(\mathbb{R}^d, h_k^2(x) dx) \\ &:= \left\{ f \in L_1^{\text{loc}}(\mathbb{R}^d, h_k^2(x) dx) : \int_{\mathbb{R}^d} \Phi(\lambda |f(x)|) h_k^2(x) dx < \infty \text{ for some } \lambda > 0 \right\} \end{aligned}$$

is called the Orlicz space. If $\Phi(r) := r^p$ for all $r \in [0, \infty)$, $1 \leq p < \infty$, then

$$L_{\Phi}(\mathbb{R}^d, h_k^2(x)dx) = L_p(\mathbb{R}^d, h_k^2(x)dx).$$

If $\Phi(r) := 0$ for all $r \in [0, 1]$ and $\Phi(r) := \infty$ for all $r \in (1, \infty)$, then

$$L_{\Phi}(\mathbb{R}^d, h_k^2(x)dx) = L_{\infty}(\mathbb{R}^d, h_k^2(x)dx).$$

The space $L_{\Phi}^{\text{loc}}(\mathbb{R}^d, h_k^2(x)dx)$ is defined as the set of all functions f such that $f\chi_B \in L_{\Phi}(\mathbb{R}^d, h_k^2(x)dx)$ for all balls $B \subset \mathbb{R}^d$.

$L_{\Phi}(\mathbb{R}^d, h_k^2(x)dx)$ is a Banach space with respect to the norm

$$\|f\|_{L_{\Phi,k}} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^d} \Phi\left(\frac{|f(x)|}{\lambda}\right) h_k^2(x)dx \leq 1 \right\}.$$

For a measurable function f on \mathbb{R}^d and $t > 0$, let

$$m(f, t)_k := |\{x \in \mathbb{R}^d : |f(x)| > t\}|_k.$$

Definition 3.2. The weak Orlicz space

$$WL_{\Phi}(\mathbb{R}^d, h_k^2(x)dx) := \{f \in L_1^{\text{loc}}(\mathbb{R}^d, h_k^2(x)dx) : \|f\|_{WL_{\Phi,k}} < \infty\}$$

is defined by the norm

$$\|f\|_{WL_{\Phi,k}} := \inf \left\{ \lambda > 0 : \sup_{t>0} \Phi(t) m\left(\frac{f}{\lambda}, t\right)_k \leq 1 \right\}.$$

We note that $\|f\|_{WL_{\Phi,k}} \leq \|f\|_{L_{\Phi,k}}$

$$\begin{aligned} \sup_{t>0} \Phi(t) m(f, t)_k &= \sup_{t>0} t m(f, \Phi^{-1}(t))_k = \sup_{t>0} t m(\Phi(|f|), t)_k, \\ \int_{\mathbb{R}^d} \Phi\left(\frac{|f(x)|}{\|f\|_{L_{\Phi,k}}}\right) h_k^2(x)dx &\leq 1, \quad \sup_{t>0} \Phi(t) m\left(\frac{f}{\|f\|_{WL_{\Phi,k}}}, t\right)_k \leq 1. \end{aligned} \quad (3.2)$$

The following analogue of the Hölder inequality is well known (see, e.g., [17]).

Theorem 3.1. Let the functions f and g be measurable on \mathbb{R}^d . For a Young function Φ and its complementary function $\tilde{\Phi}$, the following inequality is valid

$$\int_{\mathbb{R}^d} |f(x)g(x)| h_k^2(x)dx \leq 2\|f\|_{L_{\Phi,k}}\|g\|_{L_{\tilde{\Phi},k}}.$$

By elementary calculations we have the following property.

Lemma 3.1. *Let Φ be a Young function and B be a ball in \mathbb{R}^d . Then*

$$\|\chi_B\|_{L_{\Phi,k}} = \|\chi_B\|_{WL_{\Phi}(\mathbb{R}^d, h_k^2(x)dx)} = \frac{1}{\Phi^{-1}\left(\left(|B|_k\right)^{-1}\right)}.$$

By Theorem 3.1, Lemma 3.1 and (2.2) we obtain the following estimate.

Lemma 3.2. *For a Young function Φ and for the ball B the following inequality is valid:*

$$\int_B |f(y)| h_k^2(x) dx \leq 2|B|_k \Phi^{-1}\left(\left(|B|_k\right)^{-1}\right) \|f\chi_B\|_{L_{\Phi,k}}.$$

The known boundedness statement for M_k in Orlicz spaces on spaces of homogeneous type runs as follows.

Theorem 3.2. ([6]) *Let Φ be any Young function. Then the maximal operator M_k is bounded from $L_{\Phi}(\mathbb{R}^d, h_k^2(x)dx)$ to $WL_{\Phi}(\mathbb{R}^d, h_k^2(x)dx)$ and for $\Phi \in \nabla_2$ bounded in $L_{\Phi}(\mathbb{R}^d, h_k^2(x)dx)$.*

4 Boundedness of maximal commutators associated with Dunkl operators in Orlicz spaces $L_{\Phi}(\mathbb{R}, dm_{\nu})$

In this section, we investigate the boundedness of the maximal commutator $M_{b,\nu}$ and the commutator of the maximal operator, $[b, M_{\nu}]$, in Orlicz spaces $L_{\Phi}(\mathbb{R}, dm_{\nu})$.

We recall the definition of the space $BMO(\mathbb{R}, dm_{\nu})$.

Definition 4.1. Suppose that $b \in L_1^{\text{loc}}(\mathbb{R}, dm_{\nu})$. Let

$$\|b\|_{BMO(\nu)} := \sup_{x \in \mathbb{R}, r > 0} \frac{1}{m_{\nu}B(x,r)} \int_{B(x,r)} |b(y) - b_{B(x,r)}(x)| dm_{\nu}(y),$$

where, for any $x \in \mathbb{R}$ and $r > 0$,

$$b_{B(x,r)} := \frac{1}{m_{\nu}B(x,r)} \int_{B(x,r)} b(y) dm_{\nu}(y).$$

Define

$$BMO(\mathbb{R}, dm_{\nu}) := \left\{ b \in L_1^{\text{loc}}(\mathbb{R}, dm_{\nu}) : \|b\|_{BMO(\nu)} < \infty \right\}.$$

Modulo constants, the space $BMO(\mathbb{R}, dm_{\nu})$ is a Banach space with respect to the norm $\|\cdot\|_{BMO(\nu)}$.

We will need the following properties of BMO -functions (see [11]):

$$\|b\|_{BMO(\nu)} \approx \sup_{x \in \mathbb{R}, r > 0} \left(\frac{1}{m_{\nu}B(x,r)} \int_{B(x,r)} |b(y) - b_{B(x,r)}|^p dm_{\nu}(y) \right)^{\frac{1}{p}}, \tag{4.1}$$

where $1 \leq p < \infty$ and the positive equivalence constants are independent of b , and

$$\left| b_{B(x,r)} - b_{B(x,t)} \right| \leq C \|b\|_{BMO(\nu)} \ln \frac{t}{r} \quad \text{for any } x \in \mathbb{R} \text{ and } 0 < 2r < t, \quad (4.2)$$

where the positive constant C is independent of b, x, r and t .

Next, we recall the notion of weights. Let w be a locally integrable and positive function on (\mathbb{R}, dm_ν) . The function w is called a *Muckenhoupt $A_1(\mathbb{R}, dm_\nu)$ weight* if there exists a positive constant C such that for any ball B

$$\frac{1}{m_\nu(B)} \int_B w(x) dm_\nu(x) \leq C \operatorname{ess\,inf}_{x \in B} w(x).$$

Lemma 4.1. ([6, Chapter 1]) *Let $\omega \in A_1(\mathbb{R}, dm_\nu)$. Then the reverse Hölder inequality holds, that is, there exist $q > 1$ and a positive constant C such that*

$$\left(\frac{1}{m_\nu(B)} \int_B w(x)^q dm_\nu(x) \right)^{\frac{1}{q}} \leq \frac{C}{m_\nu(B)} \int_B w(x) dm_\nu(x)$$

for all balls B .

Lemma 4.2. *Let Φ be a Young function with $\Phi \in \Delta_2$, B be a ball in \mathbb{R} and $f \in L_{\Phi, \nu}(B)$. Then we have*

$$\frac{1}{2m_\nu(B)} \int_B |f(x)| dm_\nu(x) \leq \Phi^{-1}(m_\nu(B)^{-1}) \|f\|_{L_{\Phi, \nu}} \leq C \left(\frac{1}{m_\nu(B)} \int_B |f(x)|^p dm_\nu(x) \right)^{\frac{1}{p}}$$

for some $1 < p < \infty$, where the positive constant C is independent of f and B .

Proof. The left-hand side inequality is just Lemma 2.2.

Next we prove the right-hand side inequality. Our idea is from [10]. Take $g \in L_{\tilde{\Phi}, \nu}$ with $\|g\|_{L_{\tilde{\Phi}}(\mathbb{R}, dm_\nu)} \leq 1$. Note that $\tilde{\Phi} \in \nabla_2$ since $\Phi \in \Delta_2$, therefore M_ν is bounded on $L_{\tilde{\Phi}}(\mathbb{R}, dm_\nu)$ from Theorem 2.2. Let $Q := \|M_\nu\|_{L_{\tilde{\Phi}, \nu} \rightarrow L_{\tilde{\Phi}, \nu}}$ and define a function

$$Rg(x) := \sum_{k=0}^{\infty} \frac{M_\nu^k g(x)}{(2Q)^k}, \quad \forall x \in \mathbb{R},$$

where

$$M_\nu^k g := \begin{cases} |g|, & k=0, \\ M_\nu g, & k=1, \\ M_\nu(M_\nu^{k-1} g), & k \geq 2. \end{cases}$$

For every $g \in L_{\tilde{\Phi}}(\mathbb{R}, dm_\nu)$ with $\|g\|_{L_{\tilde{\Phi}}(\mathbb{R}, dm_\nu)} \leq 1$, the function Rg has the following properties:

- $|g(x)| \leq Rg(x)$ for almost every $x \in \mathbb{R}$;

- $\|Rg\|_{L_{\Phi,\nu}} \leq 2\|g\|_{L_{\Phi,\nu}}$;
- $M_\nu(Rg)(x) \leq 2QRg(x)$ for all $x \in \mathbb{R}$, that is, Rg is a Muckenhoupt $A_1(\mathbb{R}, dm_\nu)$ weight with the A_1 constant less than or equal to $2Q$.

By Lemma 4.1, there exist positive constants $q > 1$ and C independent of g such that for all balls B ,

$$\left(\frac{1}{m_\nu(B)} \int_B Rg(x)^q dm_\nu(x) \right)^{\frac{1}{q}} \leq \frac{C}{m_\nu(B)} \int_B Rg(x) dm_\nu(x).$$

By Lemma 2.2, we obtain

$$\begin{aligned} \|Rg\|_{L_{q,\nu}(B)} &= m_\nu(B)^{1/q} \left(\frac{1}{m_\nu(B)} \int_B Rg(x)^q dm_\nu(x) \right)^{\frac{1}{q}} \\ &\leq m_\nu(B)^{1/q} \frac{C}{m_\nu(B)} \int_B Rg(x) dm_\nu(x) \\ &\leq C m_\nu(B)^{-1/q'} \frac{\|Rg\|_{L_{\Phi,\nu}}}{\Phi^{-1}(m_\nu(B)^{-1})} \leq C m_\nu(B)^{-1/q'} \frac{1}{\Phi^{-1}(m_\nu(B)^{-1})}. \end{aligned}$$

Thus, we have

$$\begin{aligned} \int_B |f(x)g(x)| dm_\nu(x) &\leq \int_B |f(x)| Rg(x) dm_\nu(x) \leq \|f\|_{L_{q',\nu}(B)} \|Rg\|_{L_{q,\nu}(B)} \\ &\leq C \left(\frac{1}{m_\nu(B)} \int_B |f(x)|^{q'} dm_\nu(x) \right)^{\frac{1}{q'}} \frac{1}{\Phi^{-1}(m_\nu(B)^{-1})}. \end{aligned}$$

Since the Luxembourg-Nakano norm is equivalent to the Orlicz norm we obtain

$$\begin{aligned} \|f\|_{L_{\Phi,\nu}} &\leq \sup \left\{ \left| \int_B f(x)g(x) dm_\nu(x) \right| : g \in L_{\Phi}(\mathbb{R}, dm_\nu), \|g\|_{L_{\Phi,\nu}} \leq 1 \right\} \\ &\leq C \left(\frac{1}{m_\nu(B)} \int_B |f(x)|^{q'} dm_\nu(x) \right)^{\frac{1}{q'}} \frac{1}{\Phi^{-1}(m_\nu(B)^{-1})}. \end{aligned}$$

Consequently, the right-hand side inequality follows with $p = q'$. □

We have the following result from (4.1) and Lemma 4.2.

Lemma 4.3. *Let $b \in BMO(\mathbb{R}, dm_\nu)$ and Φ be a Young function with $\Phi \in \Delta_2$. Then*

$$\|b\|_{BMO(\nu)} \approx \sup_{x \in \mathbb{R}, r > 0} \Phi^{-1}(\nu(B(x,r)^{-1})) \left\| b(\cdot) - b_{B(x,r)} \right\|_{L_{\Phi,\nu}(B(x,r))}, \tag{4.3}$$

where the positive equivalence constants are independent of b .

By Theorem 2.2 and Theorem 1.13 in [2] we obtain the following theorem.

Theorem 4.1. Let $b \in BMO(\mathbb{R}, dm_\nu)$ and $\Phi \in \nabla_2$. Then the operator $M_{b,\nu}$ is bounded on $L_\Phi(\mathbb{R}, dm_\nu)$, and the inequality

$$\|M_{b,\nu}f\|_{L_{\Phi,\nu}} \leq C_0 \|b\|_{BMO(\nu)} \|f\|_{L_{\Phi,\nu}} \quad (4.4)$$

holds with the positive constant C_0 independent of f .

The following theorem is valid.

Theorem 4.2. Let $b \in BMO(\mathbb{R}, dm_\nu)$ and Φ be a Young function. Then the condition $\Phi \in \nabla_2$ is necessary for the boundedness of $M_{b,\nu}$ on $L_\Phi(\mathbb{R}, dm_\nu)$.

Proof. Assume that (4.4) holds. For the particular symbol $b(\cdot) := \log|\cdot| \in BMO(\mathbb{R}, dm_\nu)$ and $f := \chi_{B(0,r)}$ for all $r > 0$, (4.4) becomes

$$\|M_{b,\nu}\chi_{B(0,r)}\|_{L_{\Phi,\nu}} \leq C_1 \|\chi_{B(0,r)}\|_{L_{\Phi,\nu}}, \quad (4.5)$$

where $r := (a_1 uv)^{-1/(2\nu+2)}$, $B := B(0,r)$, $a_r := m_\nu B(0,r)$, $u > 0$ and $v > 1$. By Lemma 2.1 and (2.2), we have

$$\begin{aligned} \|\chi_{B(0,r)}\|_{L_{\Phi,\nu}} &= \frac{1}{\Phi^{-1}((m_\nu B(0,r))^{-1})} \\ &= \frac{1}{\Phi^{-1}(r^{-2\nu-2}(m_\nu B(0,1))^{-1})} = \frac{1}{\Phi^{-1}(uv)} \leq \frac{1}{uv} \tilde{\Phi}^{-1}(uv). \end{aligned}$$

On the other hand, if $x \notin B(0,r)$ then $B(0,r) \subset B(x,2|x|)$ because for any $y \in B(0,r)$ we have

$$|x-y| \leq |x| + |y| \leq |x| + r \leq 2|x|.$$

Also for each $y \in B(0,r)$, we have

$$b(x) - b(y) \geq \log\left(\frac{|x|}{r}\right).$$

Therefore

$$\begin{aligned} M_{b,\nu}\chi_{B(0,r)}(x) &\geq \frac{1}{m_\nu B(x,2|x|)} \int_{B(x,2|x|) \cap B(0,r)} |b(x) - b(y)| dm_\nu(y) \\ &\geq \left(\frac{r}{2|x|}\right)^{2\nu+2} \log\left(\frac{|x|}{r}\right). \end{aligned}$$

Following the ideas of [14], for $g := \tilde{\Phi}^{-1}(u)\chi_{B(0,s)}$ with $s := (a_1 u)^{-1/(2\nu+2)}$ we obtain

$$\int_{\mathbb{R}} \tilde{\Phi}(|g(x)|) dm_\nu(x) \leq u m_\nu B(0,s) = u s^{2\nu+2} m_\nu B(0,1) = 1.$$

Since the Luxembourg-Nakano norm is equivalent to the Orlicz norm

$$\|f\|_{L_{\Phi, \nu}}^* := \sup \left\{ \int_{\mathbb{R}} |f(x)g(x)| dm_{\nu}(x) : \|g\|_{L_{\tilde{\Phi}, \nu}} \leq 1 \right\}$$

(more precisely, $\|f\|_{L_{\Phi, \nu}} \leq \|f\|_{L_{\Phi, \nu}}^* \leq 2\|f\|_{L_{\Phi, \nu}}$), it follows that

$$\begin{aligned} & \left\| M_{b, \nu} \chi_{B(0, r)} \right\|_{L_{\Phi, \nu}}^* \\ &= \sup \left\{ \int_{\mathbb{R}} |M_{b, \nu} \chi_{B(0, r)}(x)g(x)| dm_{\nu}(x) : \int_{\mathbb{R}} \tilde{\Phi}(|g(x)|) dm_{\nu}(x) \leq 1 \right\} \\ &\geq \tilde{\Phi}^{-1}(u) \int_{B(0, s)} M_{b, \nu} \chi_{B(0, r)}(x) dm_{\nu}(x) \\ &\geq \tilde{\Phi}^{-1}(u) \int_{B(0, s) \setminus B(0, r)} \left(\frac{r}{2|x|} \right)^{2\nu+2} \log \left(\frac{|x|}{r} \right) dm_{\nu}(x) \\ &= \frac{\tilde{\Phi}^{-1}(u)}{2^{2\nu+2} a_1 u \nu} \int_{B(0, s) \setminus B(0, r)} \frac{1}{|x|^{2\nu+2}} \log \left(\frac{|x|}{r} \right) dm_{\nu}(x) \\ &= \frac{\tilde{\Phi}^{-1}(u)}{2^{2\nu+3} a_1 u \nu} (2\nu+2) a_1 \left(\log \frac{s}{r} \right)^2 = \frac{\tilde{\Phi}^{-1}(u)}{2^{2\nu+3} (2\nu+2) u \nu} (\log v)^2. \end{aligned}$$

Hence, (4.5) implies that

$$\frac{\tilde{\Phi}^{-1}(u)}{2^{2\nu+3} (2\nu+2) u \nu} (\log v)^2 \leq \frac{2C_1}{u \nu} \tilde{\Phi}^{-1}(u \nu)$$

for $u > 0$ and $v > 1$. Thus, taking $v = \exp(\sqrt{(2\nu+2)C_1} \cdot 2^{\frac{2\nu+5}{2}})$ we obtain

$$2\tilde{\Phi}^{-1}(u) \leq \tilde{\Phi}^{-1}(u \exp(\sqrt{(2\nu+2)C_1} \cdot 2^{\frac{2\nu+5}{2}}))$$

for $u > 0$ or

$$\tilde{\Phi}(2t) \leq \exp(\sqrt{(2\nu+2)C_1} \cdot 2^{\frac{2\nu+5}{2}}) \tilde{\Phi}(t)$$

for every $t > 0$, and so $\tilde{\Phi}$ satisfies the Δ_2 condition. □

By Theorems 4.1 and 4.2 we have the following result.

Corollary 4.1. *Let $b \in BMO(\mathbb{R}, dm_{\nu})$ and $\Phi \in \mathcal{Y}$. Then the condition $\Phi \in \nabla_2$ is necessary and sufficient for the boundedness of $M_{b, \nu}$ on $L_{\Phi}(\mathbb{R}, dm_{\nu})$.*

Theorem 4.3. *Let $b \in L_1^{\text{loc}}(\mathbb{R}, dm_{\nu})$ and Φ be a Young function. The condition $b \in BMO(\mathbb{R}, dm_{\nu})$ is necessary for the boundedness of $M_{b, \nu}$ on $L_{\Phi}(\mathbb{R}, dm_{\nu})$.*

Proof. Suppose that $M_{b,\nu}$ is bounded from $L_\Phi(\mathbb{R}, dm_\nu)$ to $L_\Phi(\mathbb{R}, dm_\nu)$. Choose any ball B in \mathbb{R} ; by (2.2), we have

$$\begin{aligned} & \frac{1}{m_\nu B} \int_B |b(y) - b_B| dm_\nu(y) \\ & \leq \frac{1}{m_\nu B} \int_B \frac{1}{m_\nu B} \int_B |b(y) - b(z)| \chi_B(z) dm_\nu(z) dm_\nu(y) \\ & \leq \frac{1}{m_\nu B} \int_B M_{b,\nu}(\chi_B)(y) dm_\nu(y) \\ & \leq \frac{2}{m_\nu B} \|M_{b,\nu}(\chi_B)\|_{L_{\Phi,\nu}} \|1\|_{L_{\Phi,\nu}(B)} \\ & \leq \frac{2}{m_\nu B} \|\chi_B\|_{L_{\Phi,\nu}} \|\chi_B\|_{L_{\Phi,\nu}} \leq C. \end{aligned}$$

Thus, $b \in BMO(\mathbb{R}, dm_\nu)$. □

By Theorems 4.1 and 4.3 we have the following result.

Corollary 4.2. *Let Φ be a Young function with $\Phi \in \nabla_2$. Then the condition $b \in BMO(\mathbb{R}, dm_\nu)$ is necessary and sufficient for the boundedness of $M_{b,\nu}$ on $L_\Phi(\mathbb{R}, dm_\nu)$.*

From (2.1) and Corollary 4 we deduce the following conclusion.

Corollary 4.3. *Let Φ be a Young function with $\Phi \in \nabla_2$. Then the conditions $b^+ \in BMO(\mathbb{R}, dm_\nu)$ and $b^- \in L_\infty(\mathbb{R}, dm_\nu)$ are sufficient for the boundedness of $[b, M_\nu]$ on $L_\Phi(\mathbb{R}, dm_\nu)$.*

5 Boundedness of maximal commutators associated with Dunkl operators in Orlicz spaces $L_\Phi(\mathbb{R}^d, h_k^2(x) dx)$

In this section, we investigate the boundedness of the maximal commutator $M_{b,k}$ and the commutator of the maximal operator, $[b, M_k]$, in Orlicz spaces $L_\Phi(\mathbb{R}^d, h_k^2(x) dx)$. Indeed, these results and their proofs are similar to those presented in Section 4 with slight modifications. For the convenience of the reader, we give the details.

We recall the definition of the space $BMO(\mathbb{R}^d, h_k^2(x) dx)$.

Definition 5.1. *Suppose that $b \in L_1^{\text{loc}}(\mathbb{R}^d, h_k^2(x) dx)$. Let*

$$\|b\|_{BMO(k)} := \sup_{x \in \mathbb{R}^d, r > 0} \frac{1}{|B(x,r)|_k} \int_{B(x,r)} |b(y) - b_{B(x,r)}| h_k^2(y) dy,$$

where, for any $x \in \mathbb{R}^d$ and $r > 0$,

$$b_{B(x,r)} := \frac{1}{|B(x,r)|_k} \int_{B(x,r)} b(y) h_k^2(y) dy.$$

Define

$$BMO(\mathbb{R}^d, h_k^2(x)dx) := \left\{ b \in L_1^{loc}(\mathbb{R}^d, h_k^2(x)dx) : \|b\|_{BMO(k)} < \infty \right\}.$$

Modulo constants, the space $BMO(\mathbb{R}^d, h_k^2(x)dx)$ is a Banach space with respect to the norm $\|\cdot\|_{BMO(k)}$.

We will need the following properties of BMO -functions (see [11]):

$$\|b\|_{BMO(k)} \approx \sup_{x \in \mathbb{R}^d, r > 0} \left(\frac{1}{|B(x,r)|_k} \int_{B(x,r)} |b(y) - b_{B(x,r)}|^p h_k^2(y) dy \right)^{\frac{1}{p}}, \tag{5.1}$$

where $1 \leq p < \infty$ and the positive equivalence constants are independent of b , and

$$\left| b_{B(x,r)} - b_{B(x,t)} \right| \leq C \|b\|_{BMO(k)} \ln \frac{t}{r} \quad \text{for any } x \in \mathbb{R}^d \text{ and } 0 < 2r < t, \tag{5.2}$$

where the positive constant C is independent of b , x , r and t .

Next, we recall the notion of weights. Let w be a locally integrable and positive function on $(\mathbb{R}^d, h_k^2(x)dx)$. The function w is called a *Muckenhoupt* $A_1(\mathbb{R}^d, h_k^2(x)dx)$ weight if there exists a positive constant C such that for any ball B

$$\frac{1}{|B|_k} \int_B w(x) h_k^2(x) dx \leq C \operatorname{ess\,inf}_{x \in B} w(x).$$

Lemma 5.1. ([6, Chapter 1]) *Let $\omega \in A_1(\mathbb{R}^d, h_k^2(x)dx)$. Then the reverse Hölder inequality holds, that is, there exist $q > 1$ and a positive constant C such that*

$$\left(\frac{1}{|B|_k} \int_B w(x)^q h_k^2(x) dx \right)^{\frac{1}{q}} \leq \frac{C}{|B|_k} \int_B w(x) h_k^2(x) dx$$

for all balls B .

Lemma 5.2. *Let Φ be a Young function with $\Phi \in \Delta_2$, B be a ball in \mathbb{R}^d and $f \in L_{\Phi,k}(B)$. Then we have*

$$\begin{aligned} \frac{1}{2|B|_k} \int_B |f(x)| h_k^2(x) dx &\leq \Phi^{-1}(|B|_k^{-1}) \|f\|_{L_{\Phi,k}(B)} \\ &\leq C \left(\frac{1}{|B|_k} \int_B |f(x)|^p h_k^2(x) dx \right)^{\frac{1}{p}} \end{aligned}$$

for some $1 < p < \infty$, where the positive constant C is independent of f and B .

Proof. The left-hand side inequality is just Lemma 3.2.

Next we prove the right-hand side inequality. We use some ideas from [10]. Take $g \in L_{\tilde{\Phi}}(\mathbb{R}^d, h_k^2(x)dx)$ with $\|g\|_{L_{\tilde{\Phi},k}} \leq 1$. Note that $\tilde{\Phi} \in \nabla_2$ since $\Phi \in \Delta_2$, therefore M_k is bounded on $L_{\tilde{\Phi}}(\mathbb{R}^d, h_k^2(x)dx)$ from Theorem 3.2. Let $Q := \|M_\nu\|_{L_{\tilde{\Phi},k} \rightarrow L_{\tilde{\Phi},k}}$ and define a function

$$Rg(x) := \sum_{k=0}^{\infty} \frac{M_k^k g(x)}{(2Q)^k}, \quad \forall x \in \mathbb{R}^d,$$

where

$$M_k^k g := \begin{cases} |g|, & k=0, \\ M_k g, & k=1, \\ M_k(M_k^{k-1}g), & k \geq 2. \end{cases}$$

For every $g \in L_{\tilde{\Phi}}(\mathbb{R}^d, h_k^2(x)dx)$ with $\|g\|_{L_{\tilde{\Phi},k}} \leq 1$, the function Rg has the following properties:

- $|g(x)| \leq Rg(x)$ for almost every $x \in \mathbb{R}^d$;
- $\|Rg\|_{L_{\tilde{\Phi},k}} \leq 2\|g\|_{L_{\tilde{\Phi},k}}$;
- $M_k(Rg)(x) \leq 2QRg(x)$ for all $x \in \mathbb{R}^d$, that is, Rg is a Muckenhoupt $A_1(\mathbb{R}^d, h_k^2(x)dx)$ weight with the A_1 constant less than or equal to $2Q$.

By Lemma 4.1, there exist positive constants $q > 1$ and C independent of g such that for all balls B ,

$$\left(\frac{1}{|B|_k} \int_B Rg(x)^q h_k^2(x) dx \right)^{\frac{1}{q}} \leq \frac{C}{|B|_k} \int_B Rg(x) h_k^2(x) dx.$$

By Lemma 3.2, we obtain

$$\begin{aligned} \|Rg\|_{L_{q,k}(B)} &= |B|_k^{1/q} \left(\frac{1}{|B|_k} \int_B Rg(x)^q h_k^2(x) dx \right)^{\frac{1}{q}} \\ &\leq |B|_k^{1/q} \frac{C}{|B|_k} \int_B Rg(x) h_k^2(x) dx \\ &\leq C |B|_k^{-1/q'} \frac{\|Rg\|_{L_{\tilde{\Phi},k}}}{\Phi^{-1}(|B|_k^{-1})} \leq C |B|_k^{-1/q'} \frac{1}{\Phi^{-1}(|B|_k^{-1})}. \end{aligned}$$

Thus, we have

$$\begin{aligned} &\int_B |f(x)g(x)| h_k^2(x) dx \\ &\leq \int_B |f(x)| Rg(x) h_k^2(x) dx \leq \|f\|_{L_{q',k}(B)} \|Rg\|_{L_{q,k}(B)} \\ &\leq C \left(\frac{1}{|B|_k} \int_B |f(x)|^{q'} h_k^2(x) dx \right)^{\frac{1}{q'}} \frac{1}{\Phi^{-1}(|B|_k^{-1})}. \end{aligned}$$

Since the Luxembourgn-Nakano norm is equivalent to the Orlicz norm we obtain

$$\begin{aligned} \|f\|_{L_{\Phi,k}(B)} &\leq \sup \left\{ \left| \int_B f(x)g(x)h_k^2(x)dx \right| : g \in L_{\tilde{\Phi}}(\mathbb{R}^d, h_k^2(x)dx), \|g\|_{L_{\tilde{\Phi},k}} \leq 1 \right\} \\ &\leq C \left(\frac{1}{m_\nu(B)} \int_B |f(x)|^{q'} h_k^2(x)dx \right)^{\frac{1}{q'}} \frac{1}{\Phi^{-1}(|B|_k^{-1})}. \end{aligned}$$

Consequently, the right-hand side inequality follows with $p = q$. □

We have the following result from (5.1) and Lemma 4.2.

Lemma 5.3. *Let $b \in BMO(\mathbb{R}^d, h_k^2(x)dx)$ and Φ be a Young function with $\Phi \in \Delta_2$. Then*

$$\|b\|_{BMO(k)} \approx \sup_{x \in \mathbb{R}^d, r > 0} \Phi^{-1}(|B(x,r)|_k^{-1}) \left\| b(\cdot) - b_{B(x,r)} \right\|_{L_{\Phi,k}(B(x,r))}, \tag{5.3}$$

where the positive equivalence constants are independent of b .

By Theorem 3.2 and Theorem 1.13 in [2] we get the following theorem.

Theorem 5.1. *Let $b \in BMO(\mathbb{R}^d, h_k^2(x)dx)$ and $\Phi \in \nabla_2$. Then the operator $M_{b,k}$ is bounded on $L_\Phi(\mathbb{R}^d, h_k^2(x)dx)$, and the inequality*

$$\|M_{b,k}f\|_{L_{\Phi,k}} \leq C_0 \|b\|_{BMO(k)} \|f\|_{L_{\Phi,k}} \tag{5.4}$$

holds with the positive constant C_0 independent of f .

The following theorem is valid.

Theorem 5.2. *Let $b \in BMO(\mathbb{R}^d, h_k^2(x)dx)$ and Φ be a Young function. Then the condition $\Phi \in \nabla_2$ is necessary for the boundedness of $M_{b,k}$ on $L_\Phi(\mathbb{R}^d, h_k^2(x)dx)$.*

Proof. Assume that (5.4) holds. For the particular symbol $b(\cdot) := \log|\cdot| \in BMO(\mathbb{R}^d, h_k^2(x)dx)$ and $f := \chi_{B(0,r)}$ for all $r > 0$, (5.4) becomes

$$\left\| M_{b,k} \chi_{B(0,r)} \right\|_{L_{\Phi,k}} \leq C_1 \left\| \chi_{B(0,r)} \right\|_{L_{\Phi,k}}, \tag{5.5}$$

where $r := (a_1 uv)^{-1/(2v+2)}$, $B := B(0,r)$, $a_r := m_\nu B(0,r)$, $u > 0$ and $v > 1$. By Lemma 3.1 and (2.2), we have

$$\begin{aligned} \left\| \chi_{B(0,r)} \right\|_{L_{\Phi,k}} &= \frac{1}{\Phi^{-1}(|B(0,r)|_k^{-1})} \\ &= \frac{1}{\Phi^{-1}(r^{-2\gamma_k-d}(|B(0,1)|_k)^{-1})} = \frac{1}{\Phi^{-1}(uv)} \leq \frac{1}{uv} \tilde{\Phi}^{-1}(uv). \end{aligned}$$

On the other hand, if $x \notin B(0, r)$ then $B(0, r) \subset B(x, 2|x|)$ because for any $y \in B(0, r)$ we have

$$|x - y| \leq |x| + |y| \leq |x| + r \leq 2|x|.$$

Also for each $y \in B(0, r)$, we have

$$b(x) - b(y) \geq \log\left(\frac{|x|}{r}\right).$$

Therefore,

$$\begin{aligned} M_{b,k}\chi_{B(0,r)}(x) &\geq \frac{1}{|B(x, 2|x|)|_k} \int_{B(x, 2|x|) \cap B(0,r)} |b(x) - b(y)| h_k^2(y) dy \\ &\geq \left(\frac{r}{2|x|}\right)^{2\gamma_k+d} \log\left(\frac{|x|}{r}\right). \end{aligned}$$

Following the ideas of [14], for $g := \tilde{\Phi}^{-1}(u)\chi_{B(0,s)}$ with $s := (a_1 u)^{-1/(2\gamma_k+d)}$ we obtain

$$\int_{\mathbb{R}^d} \tilde{\Phi}(|g(x)|) h_k^2(x) dx \leq u |B(0,s)|_k = u s^{2\gamma_k+d} |B(0,1)|_k = 1.$$

Since the Luxembourgn-Nakano norm is equivalent to the Orlicz norm

$$\|f\|_{L_{\Phi,k}}^* := \sup \left\{ \int_{\mathbb{R}^d} |f(x)g(x)| h_k^2(x) dx : \|g\|_{L_{\tilde{\Phi},k}} \leq 1 \right\}$$

(more precisely, $\|f\|_{L_{\Phi,k}} \leq \|f\|_{L_{\Phi,k}}^* \leq 2\|f\|_{L_{\Phi,k}}$), it follows that

$$\begin{aligned} &\left\| M_{b,k}\chi_{B(0,r)} \right\|_{L_{\Phi,k}}^* \\ &= \sup \left\{ \int_{\mathbb{R}^d} |M_{b,k}\chi_{B(0,r)}(x)g(x)| h_k^2(x) dx : \int_{\mathbb{R}^d} \tilde{\Phi}(|g(x)|) h_k^2(x) dx \leq 1 \right\} \\ &\geq \tilde{\Phi}^{-1}(u) \int_{B(0,s)} M_{b,k}\chi_{B(0,r)}(x) h_k^2(x) dx \\ &\geq \tilde{\Phi}^{-1}(u) \int_{B(0,s) \setminus B(0,r)} \left(\frac{r}{2|x|}\right)^{2\gamma_k+d} \log\left(\frac{|x|}{r}\right) h_k^2(x) dx \\ &= \frac{\tilde{\Phi}^{-1}(u)}{2^{2\gamma_k+d} a_1 u v} \int_{B(0,s) \setminus B(0,r)} \frac{1}{|x|^{2\gamma_k+d}} \log\left(\frac{|x|}{r}\right) h_k^2(x) dx \\ &= \frac{\tilde{\Phi}^{-1}(u)}{2^{2\gamma_k+d+1} a_1 u v} (2\gamma_k+d) a_1 \left(\log \frac{s}{r}\right)^2 \\ &= \frac{\tilde{\Phi}^{-1}(u)}{2^{2\gamma_k+d+1} (2\gamma_k+d) u v} (\log v)^2. \end{aligned}$$

Hence, (5.5) implies that

$$\frac{\tilde{\Phi}^{-1}(u)}{2^{2\gamma_k+d+1}(2\gamma_k+d)uv}(\log v)^2 \leq \frac{2C_1}{uv} \tilde{\Phi}^{-1}(uv)$$

for $u > 0$ and $v > 1$. Thus, taking $v = \exp(\sqrt{(2\gamma_k+d)C_1} \cdot 2^{\frac{2\gamma_k+d+3}{2}})$ we obtain

$$2\tilde{\Phi}^{-1}(u) \leq \tilde{\Phi}^{-1}(u \exp(\sqrt{(2\gamma_k+d)C_1} \cdot 2^{\frac{2\gamma_k+d+3}{2}}))$$

for $u > 0$ or

$$\tilde{\Phi}(2t) \leq \exp(\sqrt{(2\gamma_k+d)C_1} \cdot 2^{\frac{2\gamma_k+d+3}{2}}) \tilde{\Phi}(t)$$

for every $t > 0$, and so $\tilde{\Phi}$ satisfies the Δ_2 condition. □

By Theorems 5.1 and 5.2 we have the following result.

Corollary 5.1. *Let $b \in BMO(\mathbb{R}^d, h_k^2(x)dx)$ and $\Phi \in \mathcal{Y}$. Then the condition $\Phi \in \nabla_2$ is necessary and sufficient for the boundedness of $M_{b,k}$ on $L_\Phi(\mathbb{R}^d, h_k^2(x)dx)$.*

Theorem 5.3. *Let $b \in L_1^{loc}(\mathbb{R}^d, h_k^2(x)dx)$ and Φ be a Young function. The condition $b \in BMO(\mathbb{R}^d, h_k^2(x)dx)$ is necessary for the boundedness of $M_{b,k}$ on $L_\Phi(\mathbb{R}^d, h_k^2(x)dx)$.*

Proof. Suppose that $M_{b,k}$ is bounded from $L_\Phi(\mathbb{R}^d, h_k^2(x)dx)$ to $L_\Phi(\mathbb{R}^d, h_k^2(x)dx)$. Choosing any ball B in \mathbb{R}^d , by (2.2), we obtain

$$\begin{aligned} & \frac{1}{|B|_k} \int_B |b(y) - b_B| h_k^2(y) dy \\ & \leq \frac{1}{|B|_k} \int_B \frac{1}{|B|_k} \int_B |b(y) - b(z)| \chi_B(z) h_k^2(z) dz h_k^2(y) dy \\ & \leq \frac{1}{|B|_k} \int_B M_{b,k}(\chi_B)(y) h_k^2(y) dy \leq \frac{2}{|B|_k} \|M_{b,k}(\chi_B)\|_{L_{\Phi,k}(B)} \|1\|_{L_{\tilde{\Phi},k}(B)} \\ & \leq \frac{2}{|B|_k} \|\chi_B\|_{L_{\Phi,k}} \|\chi_B\|_{L_{\tilde{\Phi},k}} \leq C. \end{aligned}$$

Thus, $b \in BMO(\mathbb{R}^d, h_k^2(x)dx)$. □

By Theorems 5.1 and 5.3 we have the following result.

Corollary 5.2. *Let Φ be a Young function with $\Phi \in \nabla_2$. Then the condition $b \in BMO(\mathbb{R}^d, h_k^2(x)dx)$ is necessary and sufficient for the boundedness of $M_{b,k}$ on $L_\Phi(\mathbb{R}^d, h_k^2(x)dx)$.*

From (2.1) and Corollary 5 we deduce the following conclusion.

Corollary 5.3. *Let Φ be a Young function with $\Phi \in \nabla_2$. Then the conditions $b^+ \in BMO(\mathbb{R}^d, h_k^2(x)dx)$ and $b^- \in L_\infty(\mathbb{R}^d, h_k^2(x)dx)$ are sufficient for the boundedness of $[b, M_k]$ on $L_\Phi(\mathbb{R}^d, h_k^2(x)dx)$.*

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