

The Monge-Ampère Equation for Strictly $(n-1)$ -convex Functions with Neumann Condition

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Abstract. A C^2 function on \mathbb{R}^n is called strictly $(n-1)$ -convex if the sum of any $n-1$ eigenvalues of its Hessian is positive. In this paper, we establish a global C^2 estimates to the Monge-Ampère equation for strictly $(n-1)$ -convex functions with Neumann condition. By the method of continuity, we prove an existence theorem for strictly $(n-1)$ -convex solutions of the Neumann problems.

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain and $\nu(x)$ be the outer unit normal at $x \in \partial\Omega$. Suppose $f \in C^2(\Omega)$ is positive and $\phi \in C^3(\overline{\Omega})$. In this paper, we mainly consider the following equations of Monge-Ampère type with Neumann condition,

$$\begin{cases} \det(W) = f(x), & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = -u + \phi(x), & \text{in } \partial\Omega. \end{cases} \quad (1.1)$$

where the matrix $W = (w_{\alpha_1 \dots \alpha_m, \beta_1 \dots \beta_m})_{n \times n}$, for $m = n-1$, with the elements as follows,

$$w_{\alpha_1 \dots \alpha_m, \beta_1 \dots \beta_m} = \sum_{i=1}^m \sum_{j=1}^n u_{\alpha_i j} \delta_{\beta_1 \dots \beta_{i-1} \beta_i \beta_{i+1} \dots \beta_m}^{\alpha_1 \dots \alpha_{i-1} j \alpha_{i+1} \dots \alpha_m}, \quad (1.2)$$

a linear combination of u_{ij} , where $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$ and $\delta_{\beta_1 \dots \beta_{i-1} \beta_i \beta_{i+1} \dots \beta_m}^{\alpha_1 \dots \alpha_{i-1} j \alpha_{i+1} \dots \alpha_m}$ is the generalized Kronecker symbol. All indexes $i, j, \alpha_i, \beta_i, \dots$ come from 1 to n .

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For general $1 \leq m \leq n-1$, the matrix $W \in \mathbb{R}^{C_n^m \times C_n^m}$, $C_n^m = \frac{n!}{m!(n-m)!}$, comes from the following operator $U^{[m]}$ as in [2] and [10]. First, note that $(u_{ij})_{n \times n}$ induces an operator U on \mathbb{R}^n by

$$U(e_i) = \sum_{j=1}^n u_{ij}e_j, \quad \forall 1 \leq i \leq n,$$

where $\{e_1, e_2, \dots, e_n\}$ is the standard orthogonal basis of \mathbb{R}^n . We further extend U to act on the real vector space $\wedge^m \mathbb{R}^n$ by

$$U^{[m]}(e_{\alpha_1} \wedge \dots \wedge e_{\alpha_m}) = \sum_{i=1}^m e_{\alpha_1} \wedge \dots \wedge U(e_{\alpha_i}) \wedge \dots \wedge e_{\alpha_m},$$

where $\{e_{\alpha_1} \wedge \dots \wedge e_{\alpha_m} \mid 1 \leq \alpha_1 < \dots < \alpha_m \leq n\}$ is the standard basis for $\wedge^m \mathbb{R}^n$. Then W is the matrix of $U^{[m]}$ under this standard basis. It is convenient to denote the multi-index by $\bar{\alpha} = (\alpha_1 \dots \alpha_m)$. We only consider the increasing multi-index, that is, $1 \leq \alpha_1 < \dots < \alpha_m \leq n$. By the dictionary arrangement, we can arrange all increasing multi-indexes from 1 to C_n^m , and use $N_{\bar{\alpha}}$ denote the order number of the multi-index $\bar{\alpha} = (\alpha_1 \dots \alpha_m)$, i.e., $N_{\bar{\alpha}} = 1$ for $\bar{\alpha} = (12 \dots m)$, \dots . We also use $\bar{\alpha}$ denote the index set $\{\alpha_1, \dots, \alpha_m\}$ without confusion. It is not hard to see that

$$W_{N_{\bar{\alpha}}N_{\bar{\alpha}}} = w_{\bar{\alpha},\bar{\alpha}} = \sum_{i=1}^m u_{\alpha_i\alpha_i}, \tag{1.3}$$

$$W_{N_{\bar{\alpha}}N_{\bar{\beta}}} = w_{\bar{\alpha}\bar{\beta}} = (-1)^{|i-j|} u_{\alpha_i\beta_j}, \tag{1.4}$$

if the index set $\{\alpha_1, \dots, \alpha_m\} \setminus \{\alpha_i\}$ equals to the index set $\{\beta_1, \dots, \beta_m\} \setminus \{\beta_j\}$ but $\alpha_i \neq \beta_j$; and also

$$W_{N_{\bar{\alpha}}N_{\bar{\beta}}} = w_{\bar{\alpha}\bar{\beta}} = 0, \tag{1.5}$$

if the index sets $\{\alpha_1, \dots, \alpha_m\}$ and $\{\beta_1, \dots, \beta_m\}$ have more than one different element. Specifically, for $n=3, m=2$, we have

$$W = \begin{pmatrix} u_{11} + u_{22} & u_{23} & -u_{13} \\ u_{32} & u_{11} + u_{33} & u_{12} \\ -u_{31} & u_{21} & u_{22} + u_{33} \end{pmatrix}.$$

It follows that W is symmetrical and diagonal with $(u_{ij})_{n \times n}$ diagonal. The eigenvalues of W are the sums of eigenvalues of $(u_{ij})_{n \times n}$. Denoted by $\mu(D^2u) = (\mu_1, \dots, \mu_n)$ the eigenvalues of the Hessian and by $\lambda(W) = (\lambda_1, \lambda_2, \dots, \lambda_{C_n^m})$ the eigenvalues of W . Generally, for any $k=1, 2, \dots, C_n^m$, we define the k^{th} elementary symmetry function by

$$S_k(W) = S_k(\lambda(W)) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq C_n^m} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k},$$

We also set $S_0 = 1$. In particular, we have

$$\det(W) = S_{C_n^m}(W) = \lambda_1 \lambda_2 \cdots \lambda_{C_n^m} = \prod_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} (\mu_{i_1} + \mu_{i_2} + \cdots + \mu_{i_m}).$$

If $m = 1$, the equation (1.1) is known as Monge-Ampère equation.

Define the Gårding's cone in \mathbb{R}^n by

$$\Gamma_{k,n} = \{\mu \in \mathbb{R}^n \mid S_i(\mu) > 0, \forall 1 \leq i \leq k\}.$$

Then we define the generalized Gårding's cone by, for $1 \leq m \leq n$, $1 \leq k \leq C_n^m$,

$$\Gamma_{k,n}^{(m)} = \{\mu \in \mathbb{R}^n \mid \{\mu_{i_1} + \cdots + \mu_{i_m} \mid 1 \leq i_1 < \cdots < i_m \leq n\} \in \Gamma_{k,C_n^m}\}.$$

In the absence of ambiguity, we omit the subscript n for simplicity. Obviously, $\Gamma_k = \Gamma_k^{(1)}$ and $\Gamma_n \subset \Gamma_k^{(m)} \subset \Gamma_1$. Normally, we say a C^2 function u is convex if any eigenvalue of the Hessian is nonnegative, equivalently $\mu(D^2u) \in \overline{\Gamma_n}$. Similarly, we give the following definition of m -convexity.

Definition 1.1. We say a C^2 function u is strictly m -convex if $\mu(D^2u) \in \Gamma_{C_n^m}^{(m)}$, i.e., the sum of any m eigenvalues of the Hessian is positive. Furthermore, we say u is m -convex if $\mu(D^2u) \in \overline{\Gamma_{C_n^m}^{(m)}}$, i.e., the sum of any m eigenvalues of the Hessian is nonnegative.

In particular, if $\mu(D^2u) \in \Gamma_n^{(n-1)}$ for any $x \in \Omega$, then equivalently $\lambda(W) \in \Gamma_n$, such that the equation (1.1) is elliptic (see [2] or [18]). In addition, we say u is a strictly $(n-1)$ -convex solution if u is a solution of (1.1).

For the Dirichlet problem in \mathbb{R}^n , many results are known. For example, the Dirichlet problem of Laplace equation is studied in [8], Caffarelli-Nirenberg-Spruck [1], and Ivchikina [16] solved the Dirichlet problem of Monge-Ampère equation, and Caffarelli-Nirenberg-Spruck [2] solved the Dirichlet problem of general Hessian equations even including the case considered here. For the general Hessian quotient equation, the Dirichlet problem is solved by Trudinger in [31]. Finally, Guan [9] treated the Dirichlet problem for general fully nonlinear elliptic equation on the Riemannian manifolds.

Also, the Neumann or oblique derivative problem of partial differential equations was widely studied. For a priori estimates and the existence theorem of Laplace equation with Neumann boundary condition, we refer to the book [8]. Also, we can see the book written by Lieberman [17] for the Neumann or oblique derivative problem of linear and quasilinear elliptic equations. In 1987, Lions-Trudinger-Urbas solved the Neumann problem of Monge-Ampère equation in the celebrated paper [21]. For the the Neumann problem of k -Hessian equations, Trudinger [32] established the existence theorem when the domain is a ball, and he conjectured (in [32], page 305) that one can solve the problem in sufficiently smooth uniformly convex domains. Recently, Ma and Qiu [22] gave a positive answer to this problem and solved the the Neumann problem of k -Hessian equations

in uniformly convex domains. After their work, the research on the Neumann problem of other equations has made progress (see, e.g., [3, 4, 23, 33]).

Similarly to m -convexity for the Hessian (see Definition 1.1), we can formulate the notion of m -convexity for curvature operator and second fundamental forms of hypersurfaces. There are large amount literature in differential geometry on this subject. For example, Sha [25] and Wu [34] introduced the m -convexity of the sectional curvature of Riemannian manifolds and studied the topology for these manifolds. In a series interesting papers, Harvey and Lawson ([11-13]) introduce some generalized convexities on the solutions of the nonlinear elliptic Dirichlet problem, m -convexity is a special case. Han-Ma-Wu [10] obtained an existence theorem of m -convex starshaped hypersurface with prescribed mean curvature. More recently, in the complex space \mathbb{C}^n case, Tosatti and Weinkove ([29, 30]) solved the Monge-Ampère equation for $(n-1)$ -plurisubharmonic functions on a compact Kähler manifold, where the $(n-1)$ -plurisubharmonicity means the sum of any $n-1$ eigenvalues of the complex Hessian is nonnegative.

From the above geometry and analysis reasons, it is naturally to study the Neumann problem (1.1). In [6], the author considered the following Neumann problem for general fully nonlinear equations

$$\begin{cases} S_k(W) = f(x), & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = -u + \phi(x), & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

Eq. (1.1) is a special case of (1.6) when $m = n - 1, k = n$. Parallel to Definition 1.1, we give

Definition 1.2. We say u is k -admissible if $\mu(D^2u) \in \Gamma_k^{(m)}$. Particularly, if $k = C_n^m$, u is strictly m -convex.

For $k \leq C_{n-1}^{m-1} = \frac{m}{n} C_n^m$, we obtained an existence theorem of the k -admissible solution with less geometric restrictions to the boundary. For $m < \frac{n}{2}$ and $k = C_{n-1}^{m-1} + k_0 \leq \frac{n-m}{n} C_n^m$, we got an existence theorem if Ω is strictly (m, k_0) -convex, i.e., $\kappa \in \Gamma_{k_0}^{(m)}$, where $\kappa = (\kappa_1, \dots, \kappa_{n-1})$ denote the principal curvatures of $\partial\Omega$ with respect to its inner normal $-\nu$. We didn't prove the existence for strictly m -convex solution for the equation (1.6) in [6]. Particularly, for $m = n - 1$ (maybe the most interesting case except the case $m = 1$), we got the existence of the k -admissible solution for $k \leq n - 1$ only except that of the $(n - 1)$ -convex solution for $k = n$. In this paper, given a strong geometric restriction to the boundary, we can prove the existence of strictly $(n - 1)$ -convex solution to the Neumann problem (1.1).

We always denote $\kappa = (\kappa_1, \dots, \kappa_{n-1})$ the principal curvature and $H = \sum_{i=1}^{n-1} \kappa_i$ the mean curvature of the boundary. We now state the main result of this paper as follows.

Theorem 1.1. Suppose $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) is a bounded strictly convex domain with C^4 boundary. Denote $\kappa_{\max}(x)$ ($\kappa_{\min}(x)$) the maximum (minimum) principal curvature at $x \in \partial\Omega$ such that

$$\kappa_{\max} - \kappa_{\min} < \frac{H}{2(n-1)(n-2)}. \tag{1.7}$$

Let $f \in C^2(\Omega)$ be a positive function and $\phi \in C^3(\overline{\Omega})$. Then there exists a unique strictly $(n-1)$ -convex solution $u \in C^{3,\alpha}(\overline{\Omega})$ of the Neumann problem (1.1).

We may point out that the curvature condition (1.7) is only used to obtain the upper bound for the double normal derivative in Lemma 4.3. When the dimension n is large, it is easy to see that the domain Ω is almost a ball. As a special case, for $n=3$, $H=\kappa_{\max}+\kappa_{\min}$, we have

Corollary 1.1. Suppose $\Omega \subset \mathbb{R}^3$ is a bounded strictly convex domain with C^4 boundary. Denote $\kappa_{\max}(x)$ ($\kappa_{\min}(x)$) the maximum (minimum) principal curvature at $x \in \partial\Omega$ such that $\kappa_{\max} < \frac{5}{3}\kappa_{\min}$. Let $f \in C^2(\Omega)$ be a positive function and $\phi \in C^3(\overline{\Omega})$. Then there exists a unique strictly 2-convex solution $u \in C^{3,\alpha}(\overline{\Omega})$ of the Neumann problem (1.1).

The rest of this paper is arranged as follows. In Section 2, we give some basic properties of the elementary symmetric functions and some notations. In Section 3, we establish a priori C^0 estimates and global gradient estimates. In Section 4, we show the proof of the global estimates of second order derivatives. Finally, we can prove the existence theorem by the method of continuity in Section 5.

2 Preliminary

In this section, we give some basic properties of elementary symmetric functions and some notations. First, we denote by $S_k(\lambda|i)$ the symmetric function with $\lambda_i = 0$ and $S_k(\lambda|ij)$ the symmetric function with $\lambda_i = \lambda_j = 0$.

Proposition 2.1. Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ and $k = 1, \dots, n$, then

$$S_k(\lambda) = S_k(\lambda|i) + \lambda_i S_{k-1}(\lambda|i), \quad \forall 1 \leq i \leq n, \quad (2.1)$$

$$\sum_{i=1}^n \lambda_i S_{k-1}(\lambda|i) = k S_k(\lambda), \quad (2.2)$$

$$\sum_{i=1}^n S_k(\lambda|i) = (n-k) S_k(\lambda). \quad (2.3)$$

We also denote by $S_k(W|i)$ the symmetric function with W deleting the i -row and i -column and $S_k(W|ij)$ the symmetric function with W deleting the i, j -rows and i, j -columns. Then we have the following identities.

Proposition 2.2. Suppose $A = (a_{ij})_{n \times n}$ is diagonal and k is a positive integer, then

$$\frac{\partial S_k(A)}{\partial a_{ij}} = \begin{cases} S_{k-1}(A|i), & \text{if } i=j, \\ 0, & \text{if } i \neq j. \end{cases} \quad (2.4)$$

Furthermore, suppose $W = (w_{\bar{\alpha}\bar{\beta}})_{C_n^m \times C_n^m}$ defined as in (1.2) is diagonal, then

$$\frac{\partial S_k(W)}{\partial u_{ij}} = \begin{cases} \sum_{i \in \bar{\alpha}} S_{k-1}(W|N_{\bar{\alpha}}), & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{2.5}$$

Proof. For (2.4), see a proof in [18]. Note that

$$\frac{\partial S_k(W)}{\partial u_{ij}} = \sum_{\bar{\alpha}, \bar{\beta}} \frac{\partial S_k(W)}{\partial w_{\bar{\alpha}\bar{\beta}}} \frac{\partial w_{\bar{\alpha}\bar{\beta}}}{\partial u_{ij}}, \tag{2.6}$$

Using (1.3) and (1.4), then (1.5) and (2.5) are immediate consequences of (2.4). □

Recall that the Gårding's cone is defined by

$$\Gamma_k = \left\{ \lambda \in \mathbb{R}^n \mid S_i(\lambda) > 0, \quad \forall 1 \leq i \leq k \right\}.$$

Proposition 2.3. Let $\lambda \in \Gamma_k$ and $k \in \{1, \dots, n\}$. Suppose that $\lambda_1 \geq \dots \geq \lambda_k \geq \dots \geq \lambda_n$, then we have

$$S_{k-1}(\lambda|n) \geq \dots \geq S_{k-1}(\lambda|k) \geq \dots \geq S_{k-1}(\lambda|1) > 0, \tag{2.7}$$

$$\lambda_1 S_{k-1}(\lambda|1) \geq \frac{k}{n} S_k(\lambda), \tag{2.8}$$

$$S_k^{\frac{1}{k}}(\lambda) \text{ is concave in } \Gamma_k. \tag{2.9}$$

Proof. All the properties are well-known. For example, see [18] or [15] for a proof of (2.7), [5] or [14] for (2.8) and [2] for (2.9). □

The Newton-Maclaurin inequality is as follows:

Proposition 2.4. For $\lambda \in \Gamma_k$ and $k > l > 0$, we have

$$\left(\frac{S_k(\lambda)}{C_n^k} \right)^{\frac{1}{k}} \leq \left(\frac{S_l(\lambda)}{C_n^l} \right)^{\frac{1}{l}}, \tag{2.10}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$. Furthermore we have

$$\sum_{i=1}^n \frac{\partial S_k^{\frac{1}{k}}}{\partial \lambda_i} \geq [C_n^k]^{\frac{1}{k}}. \tag{2.11}$$

Proof. See [28] for a proof of (2.10). For (2.11), we use (2.10) and Proposition 2.1 to get

$$\sum_{i=1}^n \frac{\partial S_k^{\frac{1}{k}}(\lambda)}{\partial \lambda_i} = \frac{1}{k} S_k^{\frac{1}{k}-1} \sum_{i=1}^n S_{k-1}(\lambda|i) = \frac{n-k+1}{k} S_k^{\frac{1}{k}-1} S_{k-1}(\lambda) \geq [C_n^k]^{\frac{1}{k}}.$$

This completes the proof of the proposition. □

We define

$$d(x) = \text{dist}(x, \partial\Omega), \quad \Omega_\mu = \{x \in \Omega \mid d(x) < \mu\}. \quad (2.12)$$

It is well known that there exists a small positive universal constant μ_0 such that $d(x) \in C^4(\Omega_\mu)$, $\forall 0 < \mu \leq \mu_0$, provided $\partial\Omega \in C^4$. As in Simon-Spruck [27] or Lieberman [17] (p. 331), we can extend v by $v = -Dd$ in Ω_μ and note that v is a $C^3(\overline{\Omega_\mu})$ vector field. As mentioned in the book [17], we also have the following formulas

$$|Dv| + |D^2v| \leq C(n, \Omega), \quad \text{in } \Omega_\mu, \quad (2.13a)$$

$$\sum_{i=1}^n v^i D_j v^i = \sum_{i=1}^n v^i D_i v^j = \sum_{i=1}^n d_i d_{ij} = 0, \quad \text{in } \Omega_\mu, \quad (2.13b)$$

$$|v| = |Dd| = 1, \quad \text{in } \Omega_\mu. \quad (2.13c)$$

3 The zero-order and first-order estimates

As proved in [6], we have the following theorem.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) be a bounded domain with C^3 boundary, and $f \in C^1(\Omega)$ be a positive function and $\phi \in C^3(\overline{\Omega})$. Suppose that $u \in C^2(\overline{\Omega}) \cap C^3(\Omega)$ is a k -admissible solution of the Neumann problem (1.6). Then there exists a constant C_1 depending only on $k, m, n, |f|_{C^1}, |\phi|_{C^3}$ and Ω , such that*

$$\sup_{\overline{\Omega}} (|u| + |Du|) \leq C_1. \quad (3.1)$$

Proof. See Theorem 3.1 in [6] for the zero-order estimate. See Theorem 4.2 and Theorem 4.4 in [6] for the first-order estimate. The proof of the gradient estimates could also be found in [7]. \square

4 Global second order derivatives estimates

Generally, the double normal estimates are the most important and hardest parts for the Neumann problem. As in [21] and [22], we construct sub and super barrier functions to give lower and upper bounds for u_{vv} on the boundary. Then we give the global second order estimates.

In this section, we establish the following global second order estimate.

Theorem 4.1. *Suppose $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) is a bounded strictly convex domain with C^4 boundary. Denote $\kappa_{\max}(x)$ ($\kappa_{\min}(x)$) the maximum (minimum) principal curvature at $x \in \partial\Omega$ such that*

$$\kappa_{\max} - \kappa_{\min} < \frac{\gamma H}{2(n-1)(n-2)} \quad \forall \gamma \in \left[\frac{1}{2}, 1\right).$$

Let $f(x, z) \in C^2(\Omega \times \mathbb{R})$ be a positive function and $\phi(x, z) \in C^3(\overline{\Omega} \times \mathbb{R})$ be decreasing with respect to z . If $u \in C^{3,\alpha}(\overline{\Omega})$ is a strictly $(n-1)$ -convex solution of the Neumann problem

$$\begin{cases} \det(W) = f(x, u), & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \phi(x, u), & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Then we have

$$\sup_{\overline{\Omega}} |D^2u| \leq C, \tag{4.2}$$

where C depends only on $n, \gamma, |u|_{C^1(\overline{\Omega})}, |f|_{C^2(\overline{\Omega} \times [-M_0, M_0])}, \min f, |\phi|_{C^3(\overline{\Omega} \times [-M_0, M_0])}$ and Ω , with $M_0 = \sup_{\Omega} |u|$.

Throughout the rest of this paper, we always admit the Einstein's summation convention. All repeated indices come from 1 to n . We will always denote $F(D^2u) = \det(W)$ and

$$F^{ij} = \frac{\partial F(D^2u)}{\partial u_{ij}} = \frac{\partial \det(W)}{\partial w_{\bar{\alpha}\bar{\beta}}} \frac{\partial w_{\bar{\alpha}\bar{\beta}}}{\partial u_{ij}}. \tag{4.3}$$

From (1.3) and (2.5) in Proposition 2.2 we have, for any $1 \leq i \leq n$,

$$F^{ii} = \sum_{i \in \bar{\alpha}} \frac{\partial \det(W)}{\partial w_{\bar{\alpha}\bar{\alpha}}} = \sum_{i \in \bar{\alpha}} S_{n-1}(W|N_{\bar{\alpha}}). \tag{4.4}$$

Throughout the rest of the paper, we will also denote

$$\mathcal{F} = \sum_{i=1}^n F^{ii} = (n-1) \sum_{N_{\bar{\alpha}}=1}^n S_{n-1}(W|N_{\bar{\alpha}})$$

for simplicity.

4.1 Reduce the global second derivative estimates into double normal derivatives estimates on boundary

Using the method of Lions-Trudinger-Urbas [21], we can reduce the second derivative estimates of the solution into the boundary double normal estimates.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded strictly convex domain with C^4 boundary. Assume $f(x, z) \in C^2(\overline{\Omega} \times \mathbb{R})$ is positive and $\phi(x, z) \in C^3(\overline{\Omega} \times \mathbb{R})$ is decreasing with respect to z . If u is a strictly $(n-1)$ -convex solution of the Neumann problem (4.1), denote $N = \sup_{\partial\Omega} |u_{\nu\nu}|$, then we have*

$$\sup_{\overline{\Omega}} |D^2u| \leq C_0(1+N), \tag{4.5}$$

where C_0 depends only on n , $|u|_{C^1(\bar{\Omega})}$, $|f|_{C^2(\bar{\Omega} \times [-M_0, M_0])}$, $\min f$, $|\phi|_{C^3(\bar{\Omega} \times [-M_0, M_0])}$ and Ω , with $M_0 = \sup_{\Omega} |u|$.

Proof. Write Eq. (4.1) in the form of

$$\begin{cases} \det(W)^{\frac{1}{n}} = \tilde{f}(x, u), & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \phi(x, u), & \text{on } \partial\Omega, \end{cases} \quad (4.6)$$

where $\tilde{f} = f^{\frac{1}{n}}$. Since $\lambda(W) \in \Gamma_n \subset \Gamma_2$ in \mathbb{R}^n , we have

$$\sum_{i \neq j} |u_{ij}| \leq c(n) S_1(W) = c(n) S_1(D^2u), \quad (4.7)$$

where $c(n)$ is a universal number independent of u . It is sufficiently to prove (4.5) for any direction $\xi \in \mathbb{S}^{n-1}$, that is

$$u_{\xi\xi} \leq C_0(1+N). \quad (4.8)$$

We consider the following auxiliary function in $\Omega \times \mathbb{S}^{n-1}$,

$$v(x, \xi) = u_{\xi\xi} - v'(x, \xi) + K_1|x|^2 + K_2|Du|^2, \quad (4.9)$$

where

$$v'(x, \xi) = a^l u_l + b = 2(\xi \cdot \nu) \xi^l \cdot (D_x \phi + \phi_z Du - u_l D\nu^l),$$

with $\xi^l = \xi - (\xi \cdot \nu) \nu$ and

$$a^l = 2(\xi \cdot \nu) (\xi^l \phi_z - \xi^{li} D_i \nu^l), \quad b = 2(\xi \cdot \nu) \xi^l \phi_{x_l}.$$

K_1, K_2 are positive constants to be determined. By direct computations, we have

$$v_i = u_{\xi\xi i} - D_i a^l u_l - a^l u_{li} - D_i b + 2K_1 x_i + 2K_2 u_l u_{li}, \quad (4.10)$$

$$\begin{aligned} v_{ij} = & u_{\xi\xi ij} - D_{ij} a^l u_l - D_i a^l u_{lj} - D_j a^l u_{li} - a^l u_{lij} - D_{ij} b \\ & + 2K_1 \delta_{ij} + 2K_2 u_{li} u_{lj} + 2K_2 u_l u_{lij}. \end{aligned} \quad (4.11)$$

Denote $\tilde{F}(D^2u) = \det(W)^{\frac{1}{n}}$, and

$$\tilde{F}^{ij} = \frac{\partial \tilde{F}}{\partial u_{ij}} = \frac{1}{n} \det(W)^{\frac{1-n}{n}} \frac{\partial \det(W)}{\partial w_{\bar{\alpha}\bar{\beta}}} \frac{\partial w_{\bar{\alpha}\bar{\beta}}}{\partial u_{ij}}, \quad (4.12)$$

and

$$\tilde{F}^{pq,rs} = \frac{\partial^2 \tilde{F}}{\partial u_{pq} \partial u_{rs}} = \frac{1}{n} \det(W)^{\frac{1-n}{n}} \frac{\partial^2 \det(W)}{\partial w_{\bar{\alpha}\bar{\beta}} \partial w_{\bar{\eta}\bar{\zeta}}} \frac{\partial w_{\bar{\alpha}\bar{\beta}}}{\partial u_{pq}} \frac{\partial w_{\bar{\eta}\bar{\zeta}}}{\partial u_{rs}}, \quad (4.13)$$

since $w_{\bar{\alpha}\bar{\beta}}$ is a linear combination of u_{ij} , $1 \leq i, j \leq n$. Differentiating Eq. (4.6) twice, we have

$$\tilde{F}^{ij}u_{ijl} = D_l \tilde{f}, \tag{4.14}$$

$$\tilde{F}^{pq,rs}u_{pq\zeta}u_{rs\zeta} + \tilde{F}^{ij}u_{ij\zeta\zeta} = D_{\zeta\zeta} \tilde{f}. \tag{4.15}$$

By the concavity of $\det(W)^{\frac{1}{n}}$ operator with respect to W , we have

$$D_{\zeta\zeta} \tilde{f} = \tilde{F}^{pq,rs}u_{pq\zeta}u_{rs\zeta} + \tilde{F}^{ij}u_{ij\zeta\zeta} \leq \tilde{F}^{ij}u_{ij\zeta\zeta}. \tag{4.16}$$

Now we contract (4.11) with \tilde{F}^{ij} to get, using (4.14)-(4.16),

$$\begin{aligned} \tilde{F}^{ij}v_{ij} &= \tilde{F}^{ij}u_{ij\zeta\zeta} - \tilde{F}^{ij}D_{ij}a^l u_l - 2\tilde{F}^{ij}D_i a^l u_{lj} - \tilde{F}^{ij}u_{ijl}a^l \\ &\quad - \tilde{F}^{ij}D_{ij}b + 2K_1 \tilde{\mathcal{F}} + 2K_2 \tilde{F}^{ij}u_{il}u_{jl} + 2K_2 \tilde{F}^{ij}u_{ijl}u_l \\ &\geq D_{\zeta\zeta} \tilde{f} - \tilde{F}^{ij}D_{ij}a^l u_l - 2\tilde{F}^{ij}D_i a^l u_{lj} - a^l D_l \tilde{f} - \tilde{F}^{ij}D_{ij}b \\ &\quad + 2K_1 \tilde{\mathcal{F}} + 2K_2 \tilde{F}^{ij}u_{il}u_{jl} + 2K_2 u_l D_l \tilde{f}, \end{aligned} \tag{4.17}$$

where $\tilde{\mathcal{F}} = \sum_{i=1}^n \tilde{F}^{ii}$. Note that

$$\begin{aligned} D_{\zeta\zeta} \tilde{f} &= \tilde{f}_{\zeta\zeta} + 2\tilde{f}_{\zeta z}u_{\zeta} + \tilde{f}_z u_{\zeta\zeta}, \quad D_{ij}a^l = 2(\zeta \cdot \nu)\zeta^l \phi_{zz}u_{ij} + r_{ij}^l, \\ D_{ij}b &= 2(\zeta \cdot \nu)\zeta^l \phi_{x_l z}u_{ij} + r_{ij}, \end{aligned}$$

with $|r_{ij}^l|, |r_{ij}| \leq C(|u|_{C^1}, |\phi|_{C^3}, |\partial\Omega|_{C^4})$. At the maximum point $x_0 \in \Omega$ of v , we can assume $(u_{ij})_{n \times n}$ is diagonal. It follows that, by the Cauchy-Schwartz inequality,

$$\begin{aligned} \tilde{F}^{ij}v_{ij} &\geq -C(\tilde{\mathcal{F}} + K_2 + 1) - C\tilde{F}^{ii}|u_{ii}| + \tilde{f}_z u_{\zeta\zeta} + 2K_1 \tilde{\mathcal{F}} + 2K_2 \tilde{F}^{ii}u_{ii}^2 \\ &\geq -C(\tilde{\mathcal{F}} + K_2 + 1) + \tilde{f}_z u_{\zeta\zeta} + 2K_1 \tilde{\mathcal{F}} + (2K_2 - 1)\tilde{F}^{ii}u_{ii}^2, \end{aligned} \tag{4.18}$$

where $C = C(|u|_{C^1}, |\phi|_{C^3}, |\partial\Omega|_{C^4}, |f|_{C^2})$.

Assume $u_{11} \geq u_{22} \cdots \geq u_{nn}$, and denote $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ the eigenvalues of the matrix $(w_{\bar{\alpha}\bar{\beta}})_{n \times n}$. It is easy to see

$$\lambda_1 = u_{11} + \sum_{i=2}^{n-1} u_{ii} \leq (n-1)u_{11}.$$

Then we have, by (2.5) in Proposition 2.2 and (2.9) in Proposition 2.3,

$$\begin{aligned} \tilde{F}^{11}u_{11}^2 &= \sum_{1 \in \bar{\alpha}} \frac{1}{n} \det(W)^{\frac{1-n}{n}} S_{n-1}(\lambda|N_{\bar{\alpha}})u_{11}^2 \\ &\geq \frac{1}{(n-1)n} \det(W)^{\frac{1-n}{n}} S_{n-1}(\lambda|1)\lambda_1 u_{11} \\ &= \frac{1}{(n-1)n} \det(W)^{\frac{1}{n}} u_{11} = \frac{\tilde{f}}{(n-1)n} u_{11}. \end{aligned} \tag{4.19}$$

We can assume $u_{\xi\xi} \geq 0$, otherwise we have (4.8). Plug (4.19) into (4.18) and use the Cauchy-Schwartz inequality, then

$$\tilde{F}^{ii}v_{ii} \geq (K_2 - 1) \sum_{i=1}^n \tilde{F}^{ii}u_{ii}^2 + \left(\frac{K_2\tilde{f}}{(n-1)n} + \tilde{f}_z\right)u_{\xi\xi} + (2K_1 - C)\tilde{\mathcal{F}} - C(K_2 + 1). \tag{4.20}$$

Choosing $K_2 = \frac{(n-1)\max|f_z|}{\min f} + 1$ and $K_1 = C(K_2 + 2) + 1$, it follows that

$$\tilde{F}^{ii}v_{ii} \geq (2K_1 - C)\tilde{\mathcal{F}} - C(K_2 + 1) > 0, \tag{4.21}$$

since we have $\tilde{\mathcal{F}} \geq 1$ from (2.11). This implies that $v(x, \xi)$ attains its maximum on the boundary by the maximum principle. Now we assume $(x_0, \xi_0) \in \partial\Omega \times S^{n-1}$ is the maximum point of $v(x, \xi)$ in $\bar{\Omega} \times S^{n-1}$. Then we consider two cases as follows:

Case1. ξ_0 is a tangential vector at $x_0 \in \partial\Omega$.

We directly have $\xi_0 \cdot \nu = 0$, $\nu = -Dd$, $v'(x_0, \xi_0) = 0$, and $u_{\xi_0, \xi_0}(x_0) > 0$. As in [17], we define

$$c^{ij} = \delta_{ij} - \nu^i \nu^j, \quad \text{in } \Omega_\mu, \tag{4.22}$$

and it is easy to see that $c^{ij}D_j$ is a tangential direction on $\partial\Omega$. We compute at (x_0, ξ_0) . From the boundary condition, we have

$$u_{li}v^l = (c^{ij} + \nu^i \nu^j)v^l u_{lj} = c^{ij}u_j \phi_z + c^{ij} \phi_{x_j} - c^{ij}u_l D_j v^l + \nu^i \nu^j v^l u_{lj}. \tag{4.23}$$

It follows that

$$\begin{aligned} u_{lip}v^l &= [c^{pq} + \nu^p \nu^q]u_{liq}v^l \\ &= c^{pq}D_q(c^{ij}u_j \phi_z + c^{ij} \phi_{x_j} - c^{ij}u_l D_j v^l + \nu^i \nu^j v^l u_{lj}) - c^{pq}u_{li}D_q v^l + \nu^p \nu^q v^l u_{liq}, \end{aligned}$$

then we obtain

$$\begin{aligned} u_{\xi_0 \xi_0} v &= \sum_{i, l, p=1}^n \xi_0^i \xi_0^p u_{lip} v^l \\ &= \sum_{i=1}^n \xi_0^i \xi_0^q \left(D_q(c^{ij}u_j \phi_z + c^{ij} \phi_{x_j} - c^{ij}u_l D_j v^l + \nu^i \nu^j v^l u_{lj}) - u_{li}D_q v^l \right) \\ &\leq -2\xi_0^i \xi_0^q u_{li}D_q v^l + C(1 + |u_{\nu\nu}|). \end{aligned} \tag{4.24}$$

We use $\phi_z \leq 0$ in the last inequality. We assume $\xi_0 = e_1$, it is easy to get the bound for $u_{1i}(x_0)$ for $i > 1$ from the maximum of $v(x, \xi)$ in the ξ_0 direction. In fact, we can assume $\xi(t) = \frac{(1, t, 0, \dots, 0)}{\sqrt{1+t^2}}$. Then we have

$$0 = \frac{dv(x_0, \xi(t))}{dt} \Big|_{t=0} = 2u_{12}(x_0) - 2v^2(\phi_z u_1 - u_l D_l v^l),$$

consequently,

$$|u_{12}|(x_0) \leq C + C|Du|. \tag{4.25}$$

Similarly, we have for $\forall i > 1$,

$$|u_{1i}|(x_0) \leq C + C|Du|. \tag{4.26}$$

Thus we have, by $D_1\nu^1 \geq \kappa_{\min} > 0$,

$$\begin{aligned} u_{\xi_0 \bar{\xi}_0 \nu} &\leq -2D_1\nu^1 u_{11} + C(1 + |u_{\nu\nu}|) \\ &\leq -2\kappa_{\min} u_{\xi_0 \bar{\xi}_0} + C(1 + |u_{\nu\nu}|). \end{aligned}$$

On the other hand, we have from the Hopf lemma, (4.10) and (4.26),

$$\begin{aligned} 0 &\leq v_\nu(x_0, \xi_0) \\ &= u_{\xi_0 \bar{\xi}_0 \nu} - D_\nu a^l u_l - a^l u_{\nu\nu} - D_\nu b + 2K_1 x_i \nu^i + 2K_2 u_l u_{l\nu} \\ &\leq -2\kappa_{\min} u_{\xi_0 \bar{\xi}_0} + C(1 + |u_{\nu\nu}|). \end{aligned}$$

Then we get

$$u_{\xi_0 \bar{\xi}_0}(x_0) \leq C(1 + |u_{\nu\nu}|). \tag{4.27}$$

Case2. ξ_0 is non-tangential.

We can find a tangential vector τ , such that $\xi_0 = \alpha\tau + \beta\nu$, with $\alpha^2 + \beta^2 = 1$. Then we have

$$\begin{aligned} u_{\xi_0 \bar{\xi}_0}(x_0) &= \alpha^2 u_{\tau\tau}(x_0) + \beta^2 u_{\nu\nu}(x_0) + 2\alpha\beta u_{\tau\nu}(x_0) \\ &= \alpha^2 u_{\tau\tau}(x_0) + \beta^2 u_{\nu\nu}(x_0) + 2(\xi_0 \cdot \nu) \xi_0' \cdot (\phi_z Du - u_l D\nu^l). \end{aligned}$$

By the definition of $v(x_0, \xi_0)$,

$$v(x_0, \xi_0) = \alpha^2 v(x_0, \tau) + \beta^2 v(x_0, \nu) \leq \alpha^2 v(x_0, \xi_0) + \beta^2 v(x_0, \nu).$$

Thus, $v(x_0, \xi_0) = v(x_0, \nu)$ and

$$u_{\xi_0 \bar{\xi}_0}(x_0) \leq |u_{\nu\nu}| + C. \tag{4.28}$$

In conclusion, we have (4.8) in both cases. □

First, we denote $d(x) = \text{dist}(x, \partial\Omega)$, and define

$$h(x) = -d(x) + K_3 d^2(x). \tag{4.29}$$

The constant K_3 will be determined later. Then we give the following key lemma.

Lemma 4.2. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded strictly convex domain with C^2 boundary. Denote $\kappa_{\max}(x)$ ($\kappa_{\min}(x)$) the maximum (minimum) principal curvature at $x \in \partial\Omega$. Let $u \in C^2(\bar{\Omega})$ is strictly $(n-1)$ -convex and $h(x)$ is defined as in (4.29). Then, for any $\gamma \in [\frac{1}{2}, 1)$, there exists K_3 , a sufficiently large number depending only on $n, \gamma, \min f$ and Ω , such that,

$$F^{ij}h_{ij} \geq \gamma\kappa_0(1+\mathcal{F}), \quad \text{in } \Omega_\mu \ (0 < \mu \leq \tilde{\mu}), \quad (4.30)$$

where

$$\kappa_0 = \frac{H}{n-1} \geq \kappa_{\min} \quad \text{and} \quad \tilde{\mu} = \min\left\{\frac{1}{4K_3}, \frac{2-\gamma}{2K_3}, \frac{1}{2\kappa_{\min}}, \mu_0\right\}.$$

F^{ij} is defined by (4.3), and μ_0 is mentioned in (2.13). As γ tends to 1, K_3 tends to infinity.

Proof. For $x_0 \in \Omega_\mu$, there exists $y_0 \in \partial\Omega$ such that $|x_0 - y_0| = d(x_0)$. Then, in terms of the principal coordinate system at y_0 , we have (see [8], Lemma 14.17)

$$[D^2d(x_0)] = -\text{diag}\left[\frac{\kappa_1}{1-\kappa_1d}, \dots, \frac{\kappa_{n-1}}{1-\kappa_{n-1}d}, 0\right], \quad (4.31)$$

$$Dd(x_0) = -\nu(x_0) = (0, \dots, 0, -1). \quad (4.32)$$

Observe that

$$[D^2h(x_0)] = \text{diag}\left[\frac{((1-2K_3d)\kappa_1}{1-\kappa_1d}, \dots, \frac{(1-2K_3d)\kappa_{n-1}}{1-\kappa_{n-1}d}, 2K_3\right]. \quad (4.33)$$

Denote $\mu_i = \frac{(1-K_3d)\kappa_i}{1-\kappa_id} > 0, \forall 1 \leq i \leq n-1$, and $\mu_n = 2K_3$ for simplicity. Then we define

$$\lambda(D^2h) = \{\mu_{i_1} + \dots + \mu_{i_{n-1}} \mid 1 \leq i_1 < \dots < i_{n-1} \leq n\}$$

and assume $\lambda_1 \geq \dots \geq \lambda_{n-1} \geq \lambda_n$, it is easy to see that

$$\lambda_{n-1} \geq 2K_3 + \sum_{l=1}^{m-1} \mu_{i_l} \geq K_3, \quad (4.34)$$

if we choose K_3 sufficiently large and $\mu \leq \frac{1}{4K_3}$. It is also easy to see that h is strictly convex.

We now consider the function $w = h - \frac{1}{2}\gamma\kappa_0|x|^2$. As above, we define $\tilde{\mu}(D^2w) = (\tilde{\mu}_1, \dots, \tilde{\mu}_n)$ the eigenvalues of the Hessian D^2w , and

$$\tilde{\lambda} = \{\tilde{\mu}_{i_1} + \dots + \tilde{\mu}_{i_{n-1}} \mid 1 \leq i_1 < \dots < i_{n-1} \leq n\}$$

with $\tilde{\lambda}_1 \geq \dots \geq \tilde{\lambda}_n$. For any $\gamma \in [\frac{1}{2}, 1)$, assume $\mu \leq \min\left\{\frac{1}{4K_3}, \frac{2-\gamma}{2K_3}, \frac{1}{2\kappa_{\min}}\right\}$, we have

$$\frac{1-K_3d}{1-\kappa_id} > \gamma, \quad \forall i = 1, \dots, n-1.$$

Set $\delta = \frac{1}{2} \left(\frac{1-K_3d}{1-\kappa_{\min}d} - \gamma \right)$ independent of K_3 , recalling $H = \sum_{i=1}^{n-1} \kappa_i$, it follows that

$$\tilde{\lambda}_n = \sum_{i=1}^{n-1} \mu_i - (n-1)\gamma\kappa_0 \geq (n-1)\delta\kappa_0. \tag{4.35}$$

By the concavity of $\tilde{F}(D^2u) = \det(W)^{\frac{1}{n}}$, denote $\tilde{F}^{ij} = \frac{\partial \tilde{F}}{\partial u_{ij}}$, we have

$$\begin{aligned} \tilde{F}^{ij}w_{ij} &\geq \tilde{F}[D^2u + D^2w] - \tilde{F}[D^2u] \geq \tilde{F}[D^2w] \\ &\geq K_3^{n-1}((n-1)\delta\kappa_0) \geq K_3, \end{aligned} \tag{4.36}$$

for a large enough $K_3 \geq \frac{1}{(n-1)\delta\kappa_0}$. Then we get

$$\tilde{F}^{ij}h_{ij} = \tilde{F}^{ij} \left(h - \frac{1}{2}\gamma\kappa_0|x|^2 + \frac{1}{2}\gamma\kappa_0|x|^2 \right)_{ij} \geq K_3 + \gamma\kappa_0\tilde{\mathcal{F}}. \tag{4.37}$$

If we choose $K_3 \geq \frac{\gamma\kappa_0 \max f^{\frac{1}{n}}}{n \min f}$, then we have

$$F^{ij}h_{ij} \geq \gamma\kappa_0(1 + \mathcal{F}). \tag{4.38}$$

This completes the proof of the lemma. □

Following the line of Qiu-Ma [22] and Chen-Ma-Zhang [4], we construct the sub barrier function as

$$P(x) = g(x)(Du \cdot v - \phi(x, u)) - G(x), \tag{4.39}$$

with

$$\begin{aligned} v(x) &= -Dd(x), \quad g(x) = 1 - \beta h(x), \\ G(x) &= (A + \sigma N)h(x), \end{aligned}$$

where A, σ , and β are positive constants to be determined. We have the following lemma. The curvature condition in Theorem 4.1 is only used here.

Lemma 4.3. *Fix σ , if we select β large, μ small, A large, and assume N large, then*

$$P \geq 0, \quad \text{in } \Omega_\mu. \tag{4.40}$$

Furthermore, we have

$$\sup_{\partial\Omega} u_{vv} \leq C + \sigma N, \tag{4.41}$$

where C depends only on $n, |u|_{C^1}, |\partial\Omega|_{C^2}, |f|_{C^2}$ and $|\phi|_{C^2}$.

Proof. We assume $P(x)$ attains its minimum point x_0 in the interior of Ω_μ . Differentiate P twice to obtain

$$P_i = g_i(u_l v^l - \phi) + g(u_{li} v^l + u_l D_i v^l - D_i \phi) - G_i, \tag{4.42}$$

$$\begin{aligned} P_{ij} = & g_{ij}(u_l v^l - \phi) + g_i(u_{lj} v^l + u_l D_j v^l - D_j \phi) \\ & + g_j(u_{li} v^l + u_l D_i v^l - D_i \phi) + g(u_{lij} v^l + u_{li} D_j v^l \\ & + u_{lj} D_i v^l + u_l D_{ij} v^l - D_{ij} \phi) - G_{ij}. \end{aligned} \tag{4.43}$$

By a rotation of coordinates, we may assume that $(u_{ij})_{n \times n}$ is diagonal at x_0 , so are W and $(F^{ij})_{n \times n}$, with F^{ij} defined by (4.3).

We choose $\mu < \min\{\tilde{\mu}, \frac{2\epsilon}{\beta}, \frac{\epsilon}{2K_3}\}$, where $\tilde{\mu}$ is defined in Lemma 4.2 and $\epsilon \in (0, \frac{1}{2})$ is a small positive number to be determined, such that $|\beta h| \leq \beta \frac{\mu}{2} \leq \epsilon$. It follows that

$$1 \leq g \leq 1 + \epsilon. \tag{4.44}$$

Since $h_i = -(1 - 2K_3 d)d_i$, we also have

$$(1 - \epsilon)|d_i| \leq |h_i| \leq |d_i|. \tag{4.45}$$

By a straight computation, using Lemma 4.2, we obtain

$$\begin{aligned} F^{ij} P_{ij} = & F^{ii} g_{ii}(u_l v^l - \phi) + 2F^{ii} g_i(u_{ii} v^i + u_l D_i v^l - D_i \phi) \\ & + g F^{ii}(u_{l ii} v^l + 2u_{ii} D_i v^i + u_l D_{ii} v^l - D_{ii} \phi) - (A + \sigma N) F^{ii} h_{ii} \\ \leq & (\beta C_1 - (A + \sigma N)\gamma \kappa_0)(\mathcal{F} + 1) - 2\beta F^{ii} u_{ii} h_i v^i + 2g F^{ii} u_{ii} D_i v^i, \end{aligned} \tag{4.46}$$

where $C_1 = C_1(|u|_{C^1}, |\partial\Omega|_{C^3}, |\phi|_{C^2}, |f|_{C^1}, n)$.

We divide indexes $I = \{1, 2, \dots, n\}$ into two sets in the following way,

$$B = \{i \in I \mid |\beta d_i^2| < \epsilon \kappa_{\min}\}, \quad G = I \setminus B = \{i \in I \mid |\beta d_i^2| \geq \epsilon \kappa_{\min}\},$$

where κ_{\min} (κ_{\max}) is the minimum (maximum) principal curvature of the boundary. For $i \in G$, by $P_i(x_0) = 0$, we get

$$u_{ii} = (1 - 2K_3 d) \left(\frac{(A + \sigma N)}{g} + \frac{\beta(u_l v^l - \phi)}{g} \right) + \frac{u_l D_i v^l - D_i \phi}{d_i}. \tag{4.47}$$

Because $|d_i^2| \geq \frac{\epsilon \kappa_{\min}}{\beta}$, (4.44) and (4.45), we have

$$\left| \frac{(1 - 2K_3 d)\beta(u_l v^l - \phi)}{g} + \frac{u_l D_i v^l - D_i \phi}{d_i} \right| \leq \beta C_2 (\epsilon^{-1}, |u|_{C^1}, |\partial\Omega|_{C^2}, |\psi|_{C^1}).$$

Then let $A \geq 3\beta C_2$, we have

$$\frac{A}{3} + \frac{1 - \epsilon}{1 + \epsilon} \sigma N \leq u_{ii} \leq \frac{4A}{3} + \sigma N, \tag{4.48}$$

for $\forall i \in G$. We choose $\beta \geq 2n\epsilon\kappa_{\min} + 1$ to let $|d_i^2| \leq \frac{1}{2n}$ for $i \in B$. Because $|Dd| = 1$, there is a $i_0 \in G$, say $i_0 = 1$, such that

$$d_1^2 \geq \frac{1}{n}. \tag{4.49}$$

We have

$$\begin{aligned} -2\beta \sum_{i \in I} F^{ii} u_{ii} h_i v^i &= -2\beta \sum_{i \in G} F^{ii} u_{ii} h_i v^i - 2\beta \sum_{i \in B} F^{ii} u_{ii} h_i v^i \\ &\leq -2(1-\epsilon)\beta F^{11} u_{11} d_1^2 - 2\beta \sum_{i \in B, u_{ii} < 0} F^{ii} u_{ii} d_i^2 \\ &\leq -\frac{\beta F^{11} u_{11}}{n} - 2\epsilon\kappa_{\min} \sum_{u_{ii} < 0} F^{ii} u_{ii}, \end{aligned} \tag{4.50}$$

$$\begin{aligned} 2g \sum_{i \in I} F^{ii} u_{ii} D_i v^i &= 2g \sum_{u_{ii} \geq 0} F^{ii} u_{ii} D_i v^i + 2g \sum_{u_{ii} < 0} F^{ii} u_{ii} D_i v^i \\ &\leq 2\kappa_{\max} \sum_{u_{ii} \geq 0} F^{ii} u_{ii} + 2\kappa_{\min} \sum_{u_{ii} < 0} F^{ii} u_{ii}. \end{aligned} \tag{4.51}$$

Plug (4.50) and (4.51) into (4.46) to get

$$\begin{aligned} F^{ij} P_{ij} &\leq \left(\beta C_1 - (A + \sigma N) \gamma \kappa_0 \right) (\mathcal{F} + 1) - \frac{\beta}{2n} F^{11} u_{11} \\ &\quad + 2(1-\epsilon)\kappa_{\min} \sum_{u_{ii} < 0} F^{ii} u_{ii} + 2\kappa_{\max} \sum_{u_{ii} \geq 0} F^{ii} u_{ii}. \end{aligned} \tag{4.52}$$

Denote $u_{22} \geq \dots \geq u_{nn}$, and

$$\lambda_1 = \max_{1 \in \bar{\alpha}} \{w_{\bar{\alpha}\bar{\alpha}}\} = u_{11} + \sum_{i=2}^{n-1} u_{ii}, \quad \lambda_{m_1} = \min_{1 \in \bar{\alpha}} \{w_{\bar{\alpha}\bar{\alpha}}\} = u_{11} + \sum_{i=3}^n u_{ii},$$

and $\lambda_2 \geq \dots \geq \lambda_n > 0$ the eigenvalues of the matrix W . Assume $N > 1$, from (4.5) we see that

$$u_{ii} \leq 2C_0 N, \quad \forall i \in I. \tag{4.53}$$

Then

$$\lambda_i \leq 2(n-1)C_0 N, \quad \forall 1 \leq i \leq n. \tag{4.54}$$

If $u_{11} \leq u_{22}$, we see that $\lambda_{m_1} = \lambda_n$. Then

$$F^{11} > S_{n-1}(\lambda|n) \geq \frac{1}{n(n-1)} \mathcal{F}, \tag{4.55}$$

it follows that

$$F^{ij}P_{ij} \leq (\beta C_1 - (A + \sigma N)\gamma\kappa_0)(\mathcal{F} + 1) + 2C_0\kappa_{\max}N\mathcal{F} - \frac{\beta}{2n^2(n-1)}\left(\frac{A}{3} + \frac{1-\epsilon}{1+\epsilon}\sigma N\right)\mathcal{F} < 0. \tag{4.56}$$

if we choose $\beta > 12n^2(n-1)\kappa_{\max}C_0/\sigma$ and $A > \frac{\beta C_1}{\gamma\kappa_0}$.
 In the following cases, we always assume $u_{11} > u_{22}$.

Case1. $u_{nn} \geq 0$. It follows from

$$nf = \sum_{i=1}^n F^{ii}u_{ii} = \sum_{u_{ii} \geq 0} F^{ii}u_{ii}$$

and (4.80) that

$$F^{ij}P_{ij} \leq (\beta C_1 - (A + \sigma N)\gamma\kappa_0)(\mathcal{F} + 1) + 2\kappa_{\max}nf < 0, \tag{4.57}$$

if we choose $A > \beta C_1 + 2\kappa_{\max}n \max f / \gamma\kappa_0$.

Case2. $B = \sum_{u_{ii} < 0} u_{ii} > -(n-2)\sigma N - \epsilon N$ and $\lambda_n \leq \epsilon N$. It follows from $\lambda_n = \sum_{i=2}^n u_{ii}$ that

$$\begin{aligned} & 2\kappa_{\max} \sum_{u_{ii} \geq 0} F^{ii}u_{ii} + (2-\epsilon)\kappa_{\min} \sum_{u_{ii} < 0} F^{ii}u_{ii} \\ & \leq 2\kappa_{\max}F^{11}u_{11} + 2[\kappa_{\max}(\epsilon N - B) + (1-\epsilon)\kappa_{\min}B]\mathcal{F} \\ & \leq 2(n-2)[\kappa_{\max} - (1-\epsilon)\kappa_{\min}]\sigma N\mathcal{F} + 4\epsilon\kappa_{\max}N\mathcal{F} + 2\kappa_{\max}F^{11}u_{11}. \end{aligned} \tag{4.58}$$

Since $\kappa_{\max} - \kappa_{\min} < \frac{\gamma H}{2(n-1)(n-2)}$, we have

$$(n-1)\gamma\kappa_0 = \gamma H\kappa_{\min} > 2(n-1)(n-2)(\kappa_{\max} - \kappa_{\min}). \tag{4.59}$$

We can choose a sufficiently small $\epsilon = \epsilon(n, \gamma, \kappa_{\max}, \kappa_{\min})$ to get

$$2\kappa_{\max} \sum_{u_{ii} \geq 0} F^{ii}u_{ii} + (2-\epsilon)\kappa_{\min} \sum_{u_{ii} < 0} F^{ii}u_{ii} \leq \gamma\kappa_0\sigma N\mathcal{F} + 2\kappa_{\max}F^{11}u_{11}.$$

We now choose $A > \beta C_1 + 1$ and $\beta \geq 4n\kappa_{\max}$ to get

$$F^{ij}P_{ij} < 0. \tag{4.60}$$

Case3. $B = \sum_{u_{ii} < 0} u_{ii} > -(n-2)\sigma N - \epsilon N$ and $\lambda_n > \epsilon N$. It is easy to see, by (4.54), that,

$$\begin{aligned} F^{11} & > S_{n-1}(\lambda|1) = \lambda_2 \cdots \lambda_n \\ & \geq \epsilon^{n-1}N^{n-1} = \left(\frac{\epsilon}{2(n-1)C_0}\right)^{n-1} [2(n-1)C_0N]^{n-1} \\ & \geq \frac{1}{n} \left(\frac{\epsilon}{2(n-1)C_0}\right)^{n-1} S_{n-1}(\lambda). \end{aligned} \tag{4.61}$$

Similarly, if we choose $\beta > \frac{2^{n+1}3n^2(n-1)^n\kappa_{\max}(C_0)^n}{\sigma\epsilon^{n-1}}$ and $A > \frac{\beta C_1}{\gamma\kappa_0}$, then

$$F^{ij}P_{ij} < 0. \tag{4.62}$$

Case4. $B = \sum_{u_{ii} < 0} u_{ii} \leq -(n-2)\sigma N - \epsilon N$. We have

$$\lambda_n = u_{22} + \sum_{i=3}^n u_{ii} > 0.$$

It follows that

$$u_{22} \geq \frac{|B|}{n-2} \geq \left(\sigma + \frac{\epsilon}{n-2}\right)N > u_{11},$$

if we assume $N > \frac{4(n-2)A}{3\epsilon}$. This contradicts to that $u_{11} > u_{22}$.

In conclusion, we choose a small $\epsilon = \epsilon(n, \gamma, \kappa_{\max}, \kappa_{\min})$,

$$\beta = \max \left\{ 4n\kappa_{\max} + 1, \frac{2^{n+1}3n^2(n-1)^n\kappa_{\max}(C_0)^n}{\sigma\epsilon^{n-1}} \right\}$$

and $\mu = \min \left\{ \tilde{\mu}, \frac{2\epsilon}{\beta}, \frac{\epsilon}{2K_3} \right\}$. If $A > \max \left\{ 3\beta C_2, \frac{\beta C_1 + 2\kappa_{\max} n \max f}{\gamma\kappa_0} \right\}$ and $N > \frac{4(n-2)A}{3\epsilon}$, we obtain $F^{ii}P_{ij} < 0$, which contradicts to that P attains its minimum in the interior of Ω_μ . This implies that P attains its minimum on the boundary $\partial\Omega_\mu$.

On $\partial\Omega$, it is easy to see

$$P = 0. \tag{4.63}$$

On $\partial\Omega_\mu \cap \Omega$, we have

$$P \geq -C_3(|u|_{C^1}, |\phi|_{C^0}) + (A + \sigma N) \frac{\mu}{2} \geq 0, \tag{4.64}$$

if we take $A = \max \left\{ \frac{2C_3}{\mu}, 3\beta C_2, \frac{\beta C_1 + 2\kappa_{\max} n \max f}{\gamma\kappa_0} \right\}$. Finally the maximum principle tells us that

$$P \geq 0, \text{ in } \Omega_\mu. \tag{4.65}$$

Suppose $u_{vv}(y_0) = \sup_{\partial\Omega} u_{vv} > 0$, we have

$$\begin{aligned} 0 &\geq P_v(y_0) \\ &\geq (u_{vv} + u_l D_i v^l v^i - D_v \phi) - (A + \sigma N) h_v \\ &\geq u_{vv}(y_0) - C(|u|_{C^1}, |\partial\Omega|_{C^2}, |\phi|_{C^2}) - (A + \sigma N). \end{aligned}$$

Then we get

$$\sup_{\partial\Omega} u_{vv} \leq C + \sigma N. \tag{4.66}$$

This completes the proof of the lemma. □

In a similar way, we construct the super barrier function as

$$\bar{P}(x) := g(x)(Du \cdot v - \phi(x, u)) + G(x). \tag{4.67}$$

We also have the following lemma.

Lemma 4.4. Fix σ , if we select β large, μ small, A large, then

$$\bar{P} \leq 0, \quad \text{in } \Omega_\mu. \tag{4.68}$$

Furthermore, we have

$$\inf_{\partial\Omega} u_{vv} \geq -C - \sigma N, \tag{4.69}$$

where C depends only on $n, |u|_{C^1}, |\partial\Omega|_{C^2}, |f|_{C^2}$ and $|\phi|_{C^2}$.

Proof. We assume $\bar{P}(x)$ attains its maximum point x_0 in the interior of Ω_μ . Differentiate \bar{P} twice to obtain

$$\bar{P}_i = g_i(u_l v^l - \phi) + g(u_l v^l + u_l D_i v^l - D_i \phi) + G_i, \tag{4.70}$$

$$\begin{aligned} \bar{P}_{ij} = & g_{ij}(u_l v^l - \phi) + g_i(u_{lj} v^l + u_l D_j v^l - D_j \phi) \\ & + g_j(u_{li} v^l + u_l D_i v^l - D_i \phi) + g(u_{lij} v^l + u_{li} D_j v^l \\ & + u_{lj} D_i v^l + u_l D_{ij} v^l - D_{ij} \phi) + G_{ij}. \end{aligned} \tag{4.71}$$

As before we assume that (u_{ij}) is diagonal at x_0 , so are W and (F^{ij}) , with F^{ij} defined by (4.3). We choose μ small enough such that $|\beta h| \leq \beta \frac{\mu}{2} \leq \frac{1}{2}$. It follows that

$$1 \leq g \leq \frac{3}{2}. \tag{4.72}$$

Recall that $h_i = -(1 - 2K_3 d)d_i$, we also have

$$\frac{1}{2}|d_i| \leq |h_i| \leq |d_i|. \tag{4.73}$$

By a straight computation, using Lemma 4.7, we obtain

$$\begin{aligned} F^{ij} \bar{P}_{ij} = & F^{ii} g_{ii}(u_l v^l - \phi) + 2F^{ii} g_i(u_{ii} v^i + u_l D_i v^l - D_i \phi) \\ & + g F^{ii}(u_{l ii} v^l + 2u_{ii} D_i v^i + u_l D_{ii} v^l - D_{ii} \phi) + (A + \sigma N) F^{ii} h_{ii} \\ \geq & ((A + \sigma N) \gamma \kappa_0 - \beta C_1) (\mathcal{F} + 1) - 2\beta F^{ii} u_{ii} h_i v^i + 2g F^{ii} u_{ii} D_i v^i \end{aligned} \tag{4.74}$$

where $C_1 = C_1(|u|_{C^1}, |\partial\Omega|_{C^3}, |\phi|_{C^2}, |f|_{C^1}, n)$.

We divide indexes $I = \{1, 2, \dots, n\}$ into two sets in the following way,

$$B = \left\{ i \in I \mid |\beta d_i^2| < \frac{1}{2} \kappa_{\min} \right\}, \quad G = I \setminus B = \left\{ i \in I \mid |\beta d_i^2| \geq \frac{1}{2} \kappa_{\min} \right\},$$

where κ_{\min} (κ_{\max}) is the minimum (maximum) principal curvature of the boundary. For $i \in G$, by $\bar{P}_i(x_0) = 0$, we get

$$u_{ii} = (1 - 2K_3d) \left(\frac{-(A + \sigma N)}{g} + \frac{\beta(u_1 v^l - \phi)}{g} \right) + \frac{u_1 D_i v^l - D_i \phi}{d_i}. \tag{4.75}$$

Because $|d_i^2| \geq \frac{\kappa_{\min}}{2\beta}$, by (4.72) and (4.73), we have

$$\left| \frac{(1 - 2K_3d)\beta(u_1 v^l - \phi)}{g} + \frac{u_1 D_i v^l - D_i \phi}{d_i} \right| \leq \beta C_2 (|u|_{C^1}, |\partial\Omega|_{C^2}, |\psi|_{C^1}).$$

Then let $A \geq 3\beta C_2$, we have

$$-\frac{4A}{3} - \sigma N \leq u_{ii} \leq -\frac{A}{3} - \frac{1}{3}\sigma N, \tag{4.76}$$

for $\forall i \in G$. We choose $\beta \geq n\kappa_{\min} + 1$ to let $|d_i^2| \leq \frac{1}{2n}$ for $i \in B$. Because $|Dd| = 1$, there is a $i_0 \in G$, say $i_0 = 1$, such that

$$d_1^2 \geq \frac{1}{n}. \tag{4.77}$$

We have

$$\begin{aligned} -2\beta \sum_{i \in I} F^{ii} u_{ii} h_i v^i &= -2\beta \sum_{i \in G} F^{ii} u_{ii} h_i v^i - 2\beta \sum_{i \in B} F^{ii} u_{ii} h_i v^i \\ &\geq -\beta F^{11} u_{11} d_1^2 - 2\beta \sum_{i \in B, u_{ii} > 0} F^{ii} u_{ii} d_i^2 \\ &\geq -\frac{\beta F^{11} u_{11}}{n} - \kappa_{\min} \sum_{u_{ii} > 0} F^{ii} u_{ii}, \end{aligned} \tag{4.78}$$

$$\begin{aligned} 2g \sum_{i \in I} F^{ii} u_{ii} D_i v^i &= 2g \sum_{u_{ii} > 0} F^{ii} u_{ii} D_i v^i + 2g \sum_{u_{ii} \leq 0} F^{ii} u_{ii} D_i v^i \\ &\geq 2\kappa_{\min} \sum_{u_{ii} > 0} F^{ii} u_{ii} + 2\kappa_{\max} \sum_{u_{ii} \leq 0} F^{ii} u_{ii}. \end{aligned} \tag{4.79}$$

Plug (4.78) and (4.79) into (4.74) to get

$$\begin{aligned} F^{ii} \bar{P}_{ij} &\geq ((A + \sigma N - \beta C_1) \gamma \kappa_0) (\mathcal{F} + 1) - \frac{\beta}{2n} F^{11} u_{11} \\ &\quad + \kappa_{\min} \sum_{u_{ii} > 0} F^{ii} u_{ii} + 2\kappa_{\max} \sum_{u_{ii} \leq 0} F^{ii} u_{ii}. \end{aligned} \tag{4.80}$$

Denote $u_{22} \geq \dots \geq u_{nn}$, and

$$\lambda_{m_1} = \min_{1 \in \bar{\alpha}} \{w_{\bar{\alpha}\bar{\alpha}}\} = u_{11} + \sum_{i=3}^n u_{ii},$$

and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0$ the eigenvalues of the matrix W . Assume $N > 1$, from (4.5) we see that

$$u_{ii} \leq 2C_0N, \quad \forall i \in I. \tag{4.81}$$

Then

$$\lambda_i \leq 2(n-1)C_0N, \quad \forall 1 \leq i \leq n. \tag{4.82}$$

Since $u_{11} \leq u_{22}$, we see that $\lambda_{m_1} = \lambda_n$. Then

$$F^{11} > S_{n-1}(\lambda|n) \geq \frac{1}{n(n-1)}\mathcal{F}, \tag{4.83}$$

it follows that

$$\begin{aligned} F^{ij}\bar{P}_{ij} \geq & ((A + \sigma N)\gamma\kappa_0 - \beta C_1)(\mathcal{F} + 1) - 2C_0\kappa_{\max}N\mathcal{F} \\ & + \frac{\beta}{2n^2(n-1)}\left(\frac{A}{3} + \frac{1}{3}\sigma N\right)\mathcal{F} > 0. \end{aligned} \tag{4.84}$$

if we choose $\beta = 12n^2(n-1)\kappa_{\max}C_0/\sigma + n\kappa_{\min} + 1$ and $A > \frac{\beta C_1}{\gamma\kappa_0}$. This contradicts to that \bar{P} attains its maximum in the interior of Ω_μ . This contradiction implies that \bar{P} attains its maximum on the boundary $\partial\Omega_\mu$.

On $\partial\Omega$, it is easy to see $\bar{P} = 0$. On $\partial\Omega_\mu \cap \Omega$, we have

$$\bar{P} \leq C_3(|u|_{C^1}, |\phi|_{C^0}) - (A + \sigma N)\frac{\mu}{2} \leq 0,$$

if we take $A = \frac{2C_3}{\mu} + \frac{\beta C_1}{k_3} + 1$. Finally the maximum principle tells us that

$$\bar{P} \leq 0, \quad \text{in } \Omega_\mu. \tag{4.85}$$

Suppose $u_{vv}(y_0) = \inf_{\partial\Omega} u_{vv}$, we have

$$\begin{aligned} 0 & \leq P_v(y_0) \\ & \leq (u_{vv} + u_l D_i v^l v^i - D_v \phi) + (A + \sigma N)h_v \\ & \leq u_{vv}(y_0) + C(|u|_{C^1}, |\partial\Omega|_{C^2}, |\phi|_{C^2}) + (A + \sigma N). \end{aligned} \tag{4.86}$$

Then we get

$$\inf_{\partial\Omega} u_{vv} \geq -C - \sigma N. \tag{4.87}$$

This completes the proof of the lemma. □

Then we prove Theorem 4.1 immediately.

Proof of Theorem 4.1. We choose $\sigma = \frac{1}{2}$ in Lemmas 4.3 and 4.4, then

$$\sup_{\partial\Omega} |u_{vv}| \leq C. \tag{4.88}$$

Combining (4.88) with (4.5) in Lemma 4.1, we obtain (4.2). □

5 Existence of the Neumann boundary problem

We use the method of continuity to prove the existence theorem for the Neumann problem (1.1).

Proof of Theorem 1.1. Consider a family of equations with parameter t ,

$$\begin{cases} \det(W) = tf + (1-t)(n-1)^n, & \text{in } \Omega, \\ u_\nu = -u + t\phi + (1-t)(x \cdot \nu + \frac{1}{2}|x|^2), & \text{on } \partial\Omega. \end{cases} \quad (5.1)$$

From Theorems 3.1 and 4.1, we get a global C^2 estimate independent of t for Eq. (5.1). It follows that Eq. (5.1) is uniformly elliptic. Due to the concavity of $\det^{\frac{1}{n}}(W)$ with respect to D^2u (see [2]), we can get the global Hölder estimates of second derivatives following the discussions in [19], that is, we can get

$$|u|_{C^{2,\alpha}} \leq C, \quad (5.2)$$

where C depends only on $n, |u|_{C^2}, |f|_{C^2}, \min f, |\phi|_{C^3}$ and Ω . It is easy to see that $\frac{1}{2}|x|^2$ is a strictly $(n-1)$ -convex solution to (5.1) for $t=0$. Applying the method of continuity (see [8], Theorem 17.28), the existence of the classical solution holds for $t=1$. By the standard regularity theory of uniformly elliptic partial differential equations, we can obtain the higher regularity. The uniqueness is easy to get from maximum principle. \square

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