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WEAK AND SMOOTH GLOBAL SOLUTION FOR LANDAU-LIFSHITZ-BLOCH-MAXWELL EQUATION[*,†]

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Abstract

This paper is devoted to investigate the existence and uniqueness of the solution of Landau-Lifshitz-Bloch-Maxwell equation. The Landau-Lifshitz-Bloch-Maxwell equation, which fits well for a wide range of temperature, is used to study the dynamics of magnetization vector in a ferromagnetic body. If the initial data is in (H^1, L^2, L^2), the existence of the global weak solution is established. If the initial data is in (H^{m+1}, H^m, H^m) $(m \geq 1)$, the existence and uniqueness of the global smooth solution are established.

Keywords Landau-Lifshitz-Bloch-Maxwell equation; global solution; paramagnetic-ferromagnetic transition; temperature-dependent magnetic theory; Landau-Lifshitz theory

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1 Introduction

In this paper, we consider the periodic initial value problem of the following equations

$$\frac{\partial Z}{\partial t} = \Delta Z + Z \times (\Delta Z + H) - k(1 + \mu|Z|^2)Z, \tag{1.1}$$

$$\frac{\partial E}{\partial t} + \sigma E = \nabla \times H, \tag{1.2}$$

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$$\frac{\partial H}{\partial t}+\beta\frac{\partial Z}{\partial t}=-\nabla\times E, \tag{1.3}$$
$$\nabla\cdot(H+\beta Z)=0,\quad \nabla\cdot E=0, \tag{1.4}$$
$$Z(x+2De_i,t)=Z(x,t),\quad H(x+2De_i,t)=H(x,t),\quad E(x+2De_i,t)=E(x,t), \tag{1.5}$$

with the initial conditions

$$Z(x,0)=Z_0(x),\quad H(x,0)=H_0(x),\quad E(x,0)=E_0(x),\quad x\in\mathbb{R}^d, \tag{1.6}$$

where σ,k,μ,β are positive constants, $Z\in\mathbb{R}^3$ is the spin polarization, $H(x,t)=(H_1,H_2,H_3)$ is the magnetic field, $E(x,t)=(E_1(x,t), E_2(x,t),E_3(x,t))$ is the electric field and $H^e=\Delta Z+H$ is the effective magnetic field. $x\in\Omega\subset\mathbb{R}^d$, $d=2,3$, $\Omega=\prod_{j=1}^{d}(-D,D)$, $t>0$.

Here the operator ∇ is defined as follows:

$$\nabla=\nabla_x=\begin{cases}(\partial_{x_1},\partial_{x_2},0), & d=2,\quad x=(x_1,x_2)\in\mathbb{R}^2,\\ (\partial_{x_1},\partial_{x_2},\partial_{x_3}), & d=3,\quad x=(x_1,x_2,x_3)\in\mathbb{R}^3.\end{cases}$$

System (1.1)-(1.4) was studied in [4] under the additional assumption that temperature is equal to constant. In [3, 4], Berti et. al. proposed a model for the study of the dynamics of magnetization vector in a ferromagnetic body. This model fits well for a wide range of temperature, and then can be used to establish a link between micromagnetics and the phase transition occurring from paramagnetic to ferromagnetic regimes.

System (1.1)-(1.4) generalizes some classical models for magnetically saturated bodies such as the well known Landau-Lifshitz equation. Landau-Lifshitz equation well describes the magnetization dynamics of ferromagnet at low temperature [22]. The Landau- Lifshitz-Gilbert equation is described as follows

$$M_t=M\times\Delta M-\lambda M\times(M\times\Delta M),\quad M\in\mathbb{S}^2, \tag{1.7}$$

where $M(x,t)=(M_1(x,t),M_2(x,t),M_3(x,t))$ is a magnetization vector, $\lambda>0$ is a Gilbert constant, "$\times$" denotes the vector outer product. Equation (1.7) has been investigated widely. Many important results have been obtained, see [6, 17, 20, 21] and therein references.

Equation (1.7) with $\lambda=0$ is so-called Schrödinger map [5]. This Schrödinger map has been widely studied in [1, 2, 5, 7, 8, 19, 21, 24–26] and therein references.

The model of [4] is very close to the Landau-Lifshitz-Bloch (LLB) equation. In order to describe the dynamics of magnetization vector Z in a ferromagnetic body for a wide range of temperature, Garanin et. al. [10–12] derived the Landau-Lifshitz-Bloch (LLB) equation from statistical mechanics with the mean field approximation.

At high temperature ($\theta \geq \theta_c$, θ_c-Curie value), the LLB model is commonly used to describe the dynamics of magnetic fields with non-constant modulus.

The LLB equation is given as follows

$$Z_t = -\gamma Z \times H^{\text{eff}} + \frac{L_1}{|Z|^2}(Z \cdot H^{\text{eff}})Z - \frac{L_2}{|Z|^2} Z \times (Z \times H^{\text{eff}}), \tag{1.8}$$

where $\gamma,\ L_1,\ L_2$ are constants, γ is the gyromagnetic ratio, H^{eff} is the effective field. By introducing the dimensionless field z, (1.8) can be rewritten as follows

$$z_t = -\gamma z \times H^{\text{eff}} + \frac{\gamma a_{\|}}{|z|^2}(H^{\text{eff}} \cdot z)z - \frac{\gamma a_{\perp}}{|z|^2} z \times (z \times H^{\text{eff}})$$

with $\gamma a_{\|} = L_1$, $\gamma a_{\perp} = L_2$. Here $a_{\|}$ and $a_{\perp}$ are dimensionless damping parameters that depend on the temperature, and are defined as follows [3]

$$a_{\|}(\theta) = \frac{2\theta}{3\theta_c}\lambda, \quad a_{\perp}(\theta) = \begin{cases} \lambda\Big(1 - \dfrac{\theta}{3\theta_c}\Big), & \text{if } \theta < \theta_c, \\ a_{\|}(\theta), & \text{if } \theta \geq \theta_c, \end{cases}$$

where $\lambda > 0$ is a constant. In [23], the author pointed that if $L_1 = L_2$, (1.8) can be reduced as follows

$$Z_t = k_1 \Delta Z + \gamma Z \times \Delta Z - k_2(1 + \mu |Z|^2)Z, \tag{1.9}$$

where $k_1 > 0,\ k_2 > 0,\ \gamma > 0,\ \mu > 0$. The existence of weak solution for equation (1.9) has been obtained in [23]. Equation (1.9) with $k = 0$ is used to discuss the hydrodynamics of Heisenberg paramagnet, see [9] and therein references. The existence and uniqueness of global smooth solution of equation (1.9) with $k = 0$ were established in [18].

Inspired by the ideas in [13–17], we will consider the existence of the global weak solution and the global smooth solution of Landau-Lifshitz-Bloch-Maxwell equation (1.1)-(1.6).

Note that system (1.1)-(1.4) is over-determined. Therefore we have to assume that the initial functions Z_0, H_0 and E_0 satisfy the following equations

$$\nabla \cdot (H_0 + \beta Z_0) = 0, \quad \nabla \cdot E_0 = 0 \tag{1.10}$$

in distribution.

Let $(Z_0, H_0, E_0) \in (H^{m+1}, H^m, H^m)$. For $m = 0$, the existence of global solutions of problem (1.1)-(1.6) is proved.

Theorem 1.1 *Assume that* $Z_0(x) \in H^1_{per}(\Omega)$, $H_0(x) \in L^2_{per}(\Omega)$, $E_0(x) \in L^2_{per}(\Omega)$, *and* (1.10) *is satisfied. The constants* k, σ, μ, β *are positive. Then the periodic initial value problem* (1.1)-(1.6) *has at least one global generalized solution* $(Z(x,t), H(x,t), E(x,t))$ *such that*

$$Z(x,t) \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap C^{(0,\frac{1}{3})}(0,T;L^2(\Omega)),$$
$$E(x,t), H(x,t) \in L^\infty(0,T;L^2_{per}(\Omega)) \cap C^{(0,\epsilon)}(0,T;H^{-\epsilon}(\Omega)), \quad for\ any\ \epsilon \in (0,1).$$

Theorem 1.2 *Assume that* $Z_0(x) \in H^1(\mathbb{R}^d)$, $H_0(x) \in L^2(\mathbb{R}^d)$, $E_0(x) \in L^2(\mathbb{R}^d)$, *and* (1.10) *is satisfied. The constants* k, σ, μ, β *are positive. Then there exists a global generalized solution* $(Z(x,t), H(x,t), E(x,t))$ *of the initial value problem* (1.1)-(1.4) (1.6) *such that*

$$Z(x,t) \in L^\infty(0,T;H^1(\mathbb{R}^d)) \cap L^2(0,T;H^2(\mathbb{R}^d)) \cap C^{(0,\frac{1}{3})}(0,T;L^2(\mathbb{R}^d)),$$
$$H(x,t), E(x,t) \in L^\infty(0,T;L^2(\mathbb{R}^d)) \cap C^{(0,\epsilon)}(0,T;H^{-\epsilon}(\mathbb{R}^d)), \quad for\ any\ \epsilon \in (0,1).$$

For more regular initial data, that is $m \geq 1$, not only the existence but also the uniqueness of global solution of problem (1.1)-(1.6) are proved.

Theorem 1.3 *Assume that* $d=2$, $\Omega \subset \mathbb{R}^2$, $k, \mu, \sigma, \beta > 0$, $(Z_0(x), H_0(x), E_0(x)) \in (H^{m+1}_{per}(\Omega),\ H^m_{per}(\Omega), H^m_{per}(\Omega))$, $m \geq 1$, *and* (Z_0, H_0, E_0) *satisfies* (1.10)*. Then there exists a unique global solution* (Z, H, E) *of the periodic initial value problem* (1.1)-(1.6)*. Moreover, we have*

$$Z(x,t) \in \bigcap_{s=0}^{[\frac{m+1}{2}]} W^s_\infty(0,T;H^{m+1-2s}_{per}(\Omega)),$$
$$H(x,t), E(x,t) \in \bigcap_{s=0}^{m} W^s_\infty(0,T;H^{m-s}_{per}(\Omega)), \quad for\ any\ T > 0.$$

Theorem 1.4 *Assume that* $d = 2$, $k, \mu, \sigma, \beta > 0$, $(Z_0(x), H_0(x), E_0(x)) \in (H^{m+1}(\mathbb{R}^2),\ H^m(\mathbb{R}^2),\ H^m(\mathbb{R}^2))$, $m \geq 1$, *and* (Z_0, H_0, E_0) *satisfies* (1.10)*. Then there exists a unique global solution* (Z, H, E) *of the initial value problem* (1.1)-(1.4) (1.6)*. Moreover, we have*

$$Z(x,t) \in \bigcap_{s=0}^{[\frac{m+1}{2}]} W^s_\infty(0,T;H^{m+1-2s}(\mathbb{R}^2)),$$
$$H(x,t), E(x,t) \in \bigcap_{s=0}^{m} W^s_\infty(0,T;H^{m-s}(\mathbb{R}^2)), \quad for\ any\ T > 0.$$

Theorem 1.5 *Assume that* $d=3$, $\Omega \subset \mathbb{R}^3$, $k, \mu, \sigma, \beta > 0$, $(Z_0(x), H_0(x), E_0(x)) \in (H^{m+1}_{per}(\Omega),\ H^m_{per}(\Omega), H^m_{per}(\Omega))$, $m \geq 1$, (Z_0, H_0, E_0) *satisfies* (1.10)*, and there exists a positive constant* $\delta_0 \ll 1$ *such that* $\|Z_0\|_{H^1_{per}(\Omega)} < \delta_0$*. Then there exists a unique global solution* (Z, H, E) *of the periodic initial value problem* (1.1)-(1.6)*. Moreover, we have*

$$Z(x,t) \in \bigcap_{s=0}^{[\frac{m+1}{2}]} W^s_\infty(0,T;H^{m+1-2s}_{per}(\Omega)),$$
$$H(x,t), E(x,t) \in \bigcap_{s=0}^{m} W^s_\infty(0,T;H^{m-s}_{per}(\Omega)), \quad for\ any\ T > 0.$$

Theorem 1.6 *Assume that* $d = 3$, $k, \mu, \sigma, \beta > 0$, $(Z_0(x), H_0(x), E_0(x)) \in (H^{m+1}(\mathbb{R}^3), H^m(\mathbb{R}^3), H^m(\mathbb{R}^3))$, $m \geq 1$, (Z_0, H_0, E_0) *satisfies* (1.10), *and there exists a positive constant* $\delta_0 \ll 1$ *such that* $\|Z_0\|_{H^1(\mathbb{R}^3)} < \delta_0$. *Then there exists a unique global solution* (Z, H, E) *of the initial value problem* (1.1)-(1.4) (1.6). *Moreover, we have*

$$
\begin{aligned}
&Z(x,t) \in \bigcap_{s=0}^{[\frac{m+1}{2}]} W^s_\infty(0,T; H^{m+1-2s}(\mathbb{R}^3)),\\
&H(x,t), E(x,t) \in \bigcap_{s=0}^{m} W^s_\infty(0,T; H^{m-s}(\mathbb{R}^3)), \quad \textit{for any } T > 0.
\end{aligned}
$$

Remark 1.1 In Theorems 1.5 and 1.6, we only assume that $\|Z_0\|_{H^1}$ is small enough, no matter the size of (H_0, E_0) and $\|\Delta Z_0\|_{H^m}$ $(m \geq 0)$. Indeed, Z_0 can be highly oscillatory initial value. For example,

$$
\begin{aligned}
&Z_0(x) = J^{-1-\alpha} \sin\Big(\frac{\pi J(x_1 + \cdots + x_d)}{D}\Big),\\
&\|Z_0\|_{L^2_{per}} = O\big(J^{-2-2\alpha}\big), \quad \text{as } J \to \infty,\\
&\|\nabla Z_0\|_{L^2_{per}} = O\big(J^{-2\alpha}\big), \quad \text{as } J \to \infty,\\
&\|\Delta Z_0\|_{L^2_{per}} = O\big(J^{2-2\alpha}\big), \quad \text{as } J \to \infty,\\
&\qquad \vdots\\
&\|\Delta Z_0\|_{H^m_{per}} = O\big(J^{2m+2-2\alpha}\big), \quad \text{as } J \to \infty.
\end{aligned}
$$

If $\alpha \in \big(0, 1/2\big)$, then Z_0 satisfies the condition of Theorem 1.5 with $J \gg 1$, that is $\|Z_0\|_{H^1_{per}} \ll 1$. But $\|\Delta Z_0\|_{H^m_{per}}$ $(m \geq 0)$ is large enough with $J \gg 1$.

In the proofs of Theorems 1.1-1.6, the main difficulties are that system (1.1)-(1.4) is over-determined and equation (1.1) is quasi-linear. The over-determination is overcame by using the conservation of Maxwell equation (1.2)-(1.4). Other main ingredients, which are applied to obtain a priori uniform estimates, consist of the symplectic structure of the term $Z \times \Delta Z$, the dissipation of the term $-k(1+\mu|Z|^2)Z$ and the regularity which is derived from the term ΔZ of equation (1.1).

In order to solve problem (1.1)-(1.6), let us introduce the following transformation

$$
w = H + \beta Z, \tag{1.11}
$$

and rewrite equations (1.1)-(1.6) as follows

$$
\frac{\partial Z}{\partial t} = \Delta Z + Z \times (\Delta Z + w) - k(1 + \mu|Z|^2)Z, \tag{1.12}
$$

$$
\frac{\partial E}{\partial t} + \sigma E = \nabla \times (w - \beta Z), \tag{1.13}
$$

$$\frac{\partial w}{\partial t} = -\nabla \times E, \tag{1.14}$$

$$\nabla \cdot w = 0, \quad \nabla \cdot E = 0, \tag{1.15}$$

$$Z(x+2De_i,t)=Z(x,t), \quad w(x+2De_i,t)=w(x,t), \quad E(x+2De_i,t)=E(x,t), \tag{1.16}$$

$$Z(x,0) = Z_0(x), \quad w(x,0) = w_0 = H_0(x) + \beta Z_0(x), \quad E(x,0) = E_0(x). \tag{1.17}$$

We first construct the solutions of equations (1.12)-(1.14) with (1.16) (1.17), and then prove that these solutions satisfy equation (1.15) under condition (1.10). Consequently, we get that there exist solutions of problem (1.12)-(1.17).

This paper is organized as follows. In Section 2, the Galerkin approximate solutions of problem (1.12)-(1.14) with (1.16) (1.17) are constructed, and a priori uniform estimate of these solutions in (H^1, L^2, L^2) is established. In Section 3, the existence of the generalized solutions of problem (1.12)- (1.17) is proved. Consequently, Theorems 1.1 and 1.2 are proved. In Section 4, a priori uniform estimates of the Galerkin approximate solutions in (H^{m+1}, H^m, H^m) $(m \geq 1)$ are established, and the existence and uniqueness of the global solution of problem (1.12)-(1.17) are proved. Consequently, Theorems 1.3-1.6 are obtained.

2 Approximate Solutions and A Priori Estimates for the Periodic Initial Value Problem

We use Galerkin method to solve equations (1.12)-(1.14) with (1.16) (1.17). In this section, we first establish a *priori* estimates for the approximate solutions of (1.12)-(1.14) with (1.16) (1.17).

Let $\omega_n(x) \in H^{\infty}_{per}(\Omega)$ $(n = 1, 2, \cdots)$ be the unit eigenfunctions satisfying the equation

$$\Delta\omega_n + \lambda_n\omega_n = 0 \tag{2.1}$$

with periodicity $\omega_n(x-De_i) = \omega_n(x+De_i)$ $(i = 1, 2, \cdots, d)$, and λ_n $(n = 1, 2, \cdots)$ be the corresponding eigenvalues which are different from each other. $\{\omega_n(x)\}$ consists of the orthogonal normal of $H^m_{per}(\Omega)$, $m = 0, 1, \cdots$.

Therefore

$$Z_0(x) = \sum_{s=1}^{\infty} \alpha_{0s}\omega_s(x), \quad w_0(x) = \sum_{s=1}^{\infty} \beta_{0s}\omega_s(x), \quad E_0(x) = \sum_{s=1}^{\infty} \gamma_{0s}\omega_s(x),$$

$$Z_{0N}(x) = \sum_{s=1}^{N} \alpha_{0s}\omega_s(x), \quad w_{0N}(x) = \sum_{s=1}^{N} \beta_{0s}\omega_s(x), \quad E_{0N}(x) = \sum_{s=1}^{N} \gamma_{0s}\omega_s(x),$$

where

$$\alpha_{0s} = \int_\Omega Z_0(x)\omega_s(x)\mathrm{d}x, \quad \beta_{0s} = \int_\Omega w_0(x)\omega_s(x)\mathrm{d}x, \quad \gamma_{0s} = \int_\Omega E_0(x)\omega_s(x)\mathrm{d}x.$$

If $(Z_0, w_0, E_0) \in \big(H^{m+1}_{per}(\Omega), H^m_{per}(\Omega), H^m_{per}(\Omega)\big)$ $(m \geq 0)$, then

$$\|Z_{0N} - Z_0\|_{H^{m+1}_{per}} \to 0, \quad \text{as } N \to \infty, \tag{2.2}$$

$$\|w_{0N} - w_0\|_{H^m_{per}} \to 0, \quad \text{as } N \to \infty, \tag{2.3}$$

$$\|E_{0N} - E_0\|_{H^m_{per}} \to 0, \quad \text{as } N \to \infty. \tag{2.4}$$

Denote the approximate solution of problem (1.12)-(1.14) by $Z_N(x,t)$, $w_N(x,t)$ and $E_N(x,t)$ defined in the following forms

$$\begin{aligned} &Z_N(x,t) = \sum_{s=1}^{N} \alpha_{sN}(t)\omega_s(x), \quad w_N(x,t) = \sum_{s=1}^{N} \beta_{sN}(t)\omega_s(x), \\ &E_N(x,t) = \sum_{s=1}^{N} \gamma_{sN}(t)\omega_s(x). \end{aligned} \tag{2.5}$$

Here $\alpha_{sN}(t)$, $\beta_{sN}(t)$ and $\gamma_{sN}(t)$ are three dimensional vector-valued functions, $s = 1, 2, \cdots, N$, $N = 1, 2, \cdots$, which satisfy the following system of ordinary differential equations of first order

$$\begin{aligned} \int_\Omega Z_{Nt}\omega_s(x)\mathrm{d}x = & -\int_\Omega \nabla Z_N \nabla \omega_s(x)\mathrm{d}x + \int_\Omega Z_N \times (\Delta Z_N + w_N)\omega_s(x)\mathrm{d}x \\ & - \int_\Omega k(1+\mu|Z_N|^2)Z_N w_s(x)\mathrm{d}x, \end{aligned} \tag{2.6}$$

$$\int_\Omega w_{Nt}\omega_s(x)\mathrm{d}x = -\int_\Omega (\nabla \times E_N)\omega_s(x)\mathrm{d}x, \tag{2.7}$$

$$\int_\Omega E_{Nt}\omega_s(x)\mathrm{d}x + \sigma \int_\Omega E_N\omega_s(x)\mathrm{d}x = \int_\Omega \big(\nabla \times (w_N - \beta Z_N)\big)\omega_s(x)\mathrm{d}x \tag{2.8}$$

with the initial conditions

$$\begin{aligned} &\alpha_{sN}(0) = \int_\Omega Z_N(x,0)\omega_s(x)\mathrm{d}x = \int_\Omega Z_0(x)\omega_s(x)\mathrm{d}x = \alpha_{0s}, \\ &\beta_{sN}(0) = \int_\Omega w_N(x,0)\omega_s(x)\mathrm{d}x = \int_\Omega w_0(x)\omega_s(x)\mathrm{d}x = \beta_{0s}, \\ &\gamma_{sN}(0) = \int_\Omega E_N(x,0)\omega_s(x)\mathrm{d}x = \int_\Omega E_0(x)\omega_s(x)\mathrm{d}x = \gamma_{0s}. \end{aligned} \tag{2.9}$$

Obviously, we have

$$\int_\Omega Z_{Nt}\omega_s(x)\mathrm{d}x = \alpha'_{sN}(t), \quad \int_\Omega w_{Nt}\omega_s(x)\mathrm{d}x = \beta'_{sN}(t), \quad \int_\Omega E_{Nt}\omega_s(x)\mathrm{d}x = \gamma'_{sN}(t). \tag{2.10}$$

For simplicity, we introduce a notation as follows

$$\|\cdot\|_{L^p_{per}(\Omega)} = \|\cdot\|_p, \quad p \geq 2. \tag{2.11}$$

Lemma 2.1 *Assume* $(Z_0(x), w_0(x), E_0(x)) \in (H^1_{per}(\Omega), L^2_{per}(\Omega), L^2_{per}(\Omega))$, *then for the solutions of the initial value problem* (2.6)-(2.9), *we have the following estimate*

$$\sup_{0\le t\le T}\big[\|Z_N(\cdot,t)\|_{H^1_{per}(\Omega)} + \|w_N(\cdot,t)\|_{L^2_{per}(\Omega)} + \|E_N(\cdot,t)\|_{L^2_{per}(\Omega)}\big] \le K_0, \tag{2.12}$$

$$\int_0^T \|\Delta Z_N\|^2_{L^2_{per}(\Omega)}\mathrm{d}t \le K_1, \quad \textit{for any } T \ge 0, \tag{2.13}$$

where K_0 *and* K_1 *are constants which are independent of* N *and* D.

Proof Multiplying (2.6) by $\alpha_{sN}(t)$, and summing up the results with respect to $s = 1, 2, \cdots, N$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Z_N(\cdot,t)\|_2^2 + \int_\Omega |\nabla Z_N(\cdot,t)|^2\mathrm{d}x + k\int_\Omega (1+\mu|Z_N|^2)|Z_N|^2\mathrm{d}x = 0.$$

Since $k, \mu > 0$, one has

$$\|Z_N(\cdot,t)\|_2^2 \le \|Z_N(\cdot,0)\|_2^2\mathrm{e}^{-2kt} \le \|Z_0(x)\|_2^2\mathrm{e}^{-2kt}, \tag{2.14}$$

$$\int_0^t \|\nabla Z_N(\cdot,t)\|_2^2\mathrm{d}t + k\int_0^t\int_\Omega (1+\mu|Z_N|^2)|Z_N|^2\mathrm{d}x\mathrm{d}t \le 2\|Z_0(x)\|_2^2. \tag{2.15}$$

Making the scalar product of $\beta_{sN}(t)$ with (2.7) and the scalar product of $\gamma_{sN}(t)$ with (2.8) respectively, adding the two obtained equalities, and then summing up the results with respect to $s = 1, 2, \cdots, N$, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|E_N(\cdot,t)\|_2^2 + \|w_N(\cdot,t)\|_2^2\big) + \sigma\|E_N(\cdot,t)\|_2^2\\ &= -\beta\int_\Omega (\nabla\times Z_N)\cdot E_N\mathrm{d}x \le \frac{\sigma}{2}\|E_N\|_2^2 + C\|\nabla Z_N\|_{L^2}^2.\end{aligned} \tag{2.16}$$

From (2.16) and (2.15), we have

$$\|E_N(\cdot,t)\|_2^2+\|w_N(\cdot,t)\|_2^2+\sigma\int_0^t \|E_N(\cdot,\tau)\|_2^2\mathrm{d}\tau \le \|E_0\|_2^2+\|w_0\|_2^2+C\|Z_0\|_{L^2}^2. \tag{2.17}$$

Making the scalar product of $-\lambda_s\alpha_{sN}(t)$ with (2.6), summing up the results with respect to $s = 1, 2, \cdots, N$, and noticing that

$$\Delta Z_N = -\sum_{s=1}^N \lambda_s\alpha_{sN}(t)w_s(x),$$

we have

$$\begin{aligned}\int_\Omega Z_{Nt}\cdot\Delta Z_N\mathrm{d}x = &\int_\Omega \Delta Z_N\cdot\Delta Z_N\mathrm{d}x - k\int_\Omega (1+\mu|Z_N|^2)Z_N\cdot\Delta Z_N\mathrm{d}x\\ &+\int_\Omega (Z_N\times w_N)\cdot\Delta Z_N\mathrm{d}x,\end{aligned}$$

$$-k\int_\Omega(1+\mu|Z_N|^2)Z_N\cdot\Delta Z_N\mathrm{d}x=k\|\nabla Z_N\|_2^2+\int_\Omega k\mu|Z_N|^2\nabla Z_N\cdot\nabla Z_N\mathrm{d}x$$
$$+\int_\Omega k\mu\nabla|Z_N|^2\cdot\nabla|Z_N|^2\mathrm{d}x,$$

and then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla Z_N(\cdot,t)\|_2^2+\|\Delta Z_N(\cdot,t)\|_2^2+k\|\nabla Z_N\|_2^2$$
$$+\int_\Omega k\mu|Z_N|^2|\nabla Z_N|^2\mathrm{d}x+\int_\Omega k\mu\big|\nabla|Z_N|^2\big|^2\mathrm{d}x=-\int_\Omega(Z_N\times w_N)\cdot\Delta Z_N\mathrm{d}x. \quad (2.18)$$

By Gagliardo-Nirenberg inequality and estimates (2.14) (2.17), we obtain

$$\left|-\int_\Omega(Z_N\times w_N)\cdot\Delta Z_N\mathrm{d}x\right|\le\|w_N\|_2\|Z_N\|_\infty\|\Delta Z_N\|_2\le C\|w_N\|_2\|Z_N\|_2^{1-\frac{d}{4}}\|\Delta Z_N\|_2^{1+\frac{d}{4}}$$
$$\le\frac{1}{2}\|\Delta Z_N\|_2^2+C(\|Z_0\|_2^2+\|w_0\|_2^2+\|E_0\|_2^2)^{\frac{4}{4-d}}\|Z_0\|_2^2\mathrm{e}^{-2kt}. \quad (2.19)$$

Applying (2.18) (2.19) and Gronwall inequality, we can prove estimates (2.12) and (2.13).

Lemma 2.2 *Assume that the conditions of Lemma* 2.1 *are satisfied. For the solution* $(Z_N(x,t),\ w_N(x,t),E_N(x,t))$ *of the initial value problem* (2.6)-(2.9)*, there is the following estimate*

$$\|Z_{Nt}(\cdot,t)\|_{H^{-2}(\Omega)}+\|E_{Nt}(\cdot,t)\|_{H^{-1}(\Omega)}+\|w_{Nt}(\cdot,t)\|_{H^{-1}(\Omega)}\le K_2,\quad \textit{for any } t\ge 0,$$

where K_2 *is independent of* N *and* D*,* $H^{-m}(\Omega)$ *is the dual space of the space* $H^m_{per}(\Omega)$*.*

Proof For any $\varphi\in H^2_{per}$, φ can be represented as

$$\varphi=\varphi_N+\overline{\varphi}_N,$$

where

$$\varphi_N=\sum_{s=1}^N\beta_s\omega_s(x),\quad \overline{\varphi}_N=\sum_{s=N+1}^\infty\beta_s\omega_s(x).$$

For $s\ge N+1$, it yields that

$$\int_\Omega Z_{Nt}\omega_s(x)\mathrm{d}x=0.$$

Then by Lemma 2.1, there are the following estimates

$$\int_\Omega Z_{Nt}\varphi\mathrm{d}x=\int_\Omega Z_{Nt}\varphi_N(x)\mathrm{d}x$$
$$=-\int_\Omega\nabla Z_N\nabla\varphi_N\mathrm{d}x+\int_\Omega Z_N\times(\Delta Z_N+w_N)\varphi_N(x)\mathrm{d}x-k\int_\Omega(1+\mu|Z_N|^2)Z_N\varphi_N\mathrm{d}x$$

$$
\begin{aligned}
&= -\int_\Omega \nabla Z_N \nabla \varphi_N \mathrm{d}x + \int_\Omega (\nabla Z_N \times Z_N) \cdot \nabla \varphi_N \mathrm{d}x \\
&\quad + \int_\Omega (Z_N \times w_N) \cdot \varphi_N \mathrm{d}x - k \int_\Omega (1 + \mu |Z_N|^2) Z_N \varphi_N \mathrm{d}x \\
&\le \|\nabla Z_N\|_2 \|\nabla \varphi_N\|_2 + \|\nabla Z_N\|_2 \|Z_N\|_4 \|\nabla \varphi_N\|_4 + \|Z_N\|_4 \|w_N\|_2 \|\varphi_N\|_4 \\
&\quad + C(\|Z_N\|_6^6 + \|Z_N\|_2^2)\|\varphi_N\|_2 \\
&\le C\|\varphi_N\|_{H^2_{per}(\Omega)} \le C\|\varphi\|_{H^2_{per}(\Omega)}.
\end{aligned}
$$

Similarly, for $s \ge N+1$, it yields that

$$
\int_\Omega w_{Nt} \omega_s(x) \mathrm{d}x = 0, \quad \int_\Omega E_{Nt} \omega_s(x) \mathrm{d}x = 0.
$$

Then by Lemma 2.1, there are the following estimates

$$
\begin{aligned}
&\int_\Omega E_{Nt} \varphi \mathrm{d}x = \int_\Omega E_{Nt} \varphi_N \mathrm{d}x \le C_1(\|E_N\|_2 + \|w_N\|_2 + \|Z_N\|_2)(\|\nabla \varphi_N\|_2 + \|\varphi_N\|_2) \le C_1 \|\varphi\|_{H^1_{per}}, \\
&\int_\Omega w_{Nt} \varphi \mathrm{d}x = \int_\Omega w_{Nt} \varphi_N \mathrm{d}x \le C_2 \|E_N\|_2 \|\nabla \varphi_N\|_2 \le C_2 \|\varphi\|_{H^1_{per}}.
\end{aligned}
$$

Thus the following estimate holds

$$
\|Z_{Nt}\|_{H^{-2}(\Omega)} + \|E_{Nt}\|_{H^{-1}(\Omega)} + \|w_{Nt}\|_{H^{-1}(\Omega)} \le K_2.
$$

This lemma is proved.

Lemma 2.3 *Assume that the conditions of Lemma* 2.1 *are satisfied. For the solution* $(Z_N(x,t),\, w_N(x,t), E_N(x,t))$ *of the initial value problem* (2.6)-(2.9), *there are the following estimates for any* $t_1, t_2 \ge 0$

$$
\begin{aligned}
&\|Z_N(\cdot,t_1) - Z_N(\cdot,t_2)\|_2 \le K_3 |t_1 - t_2|^{\frac{1}{3}}, \\
&\|w_N(\cdot,t_1) - w_N(\cdot,t_2)\|_{H^{-\epsilon}} + \|E_N(\cdot,t_1) - E_N(\cdot,t_2)\|_{H^{-\epsilon}} \le K_4 |t_1 - t_2|^{\epsilon}, \quad \textit{for any } \epsilon \in (0,1),
\end{aligned}
$$

where the constants K_3 *and* K_4 *are independent of* N *and* D.

Proof By the Sobolev interpolation inequality of negative order, it yields that

$$
\begin{aligned}
\|Z_N(\cdot,t_1) - Z_N(\cdot,t_2)\|_2 &\le C\|Z_N(\cdot,t_1) - Z_N(\cdot,t_2)\|_{H^{-2}(\Omega)}^{\frac{1}{3}} \|Z_N(\cdot,t_1) - Z_N(\cdot,t_2)\|_{H^1(\Omega)}^{\frac{2}{3}} \\
&\le C \left\| \int_{t_1}^{t_2} \frac{\partial Z_N}{\partial t} \mathrm{d}t \right\|_{H^{-2}(\Omega)}^{\frac{1}{3}} \le C|t_2 - t_1|^{\frac{1}{3}}.
\end{aligned}
$$

Similarly, for any $\epsilon \in (0,1)$, we have

$$
\begin{aligned}
&\|w_N(\cdot,t_1) - w_N(\cdot,t_2)\|_{H^{-\epsilon}} + \|E_N(\cdot,t_1) - E_N(\cdot,t_2)\|_{H^{-\epsilon}} \\
&\le C\|w_N(\cdot,t_1) - w_N(\cdot,t_2)\|_{H^{-1}}^{\epsilon} \|w_N(\cdot,t_1) - w_N(\cdot,t_2)\|_2^{1-\epsilon} \\
&\quad + C\|E_N(\cdot,t_1) - E_N(\cdot,t_2)\|_{H^{-1}}^{\epsilon} \|E_N(\cdot,t_1) - E_N(\cdot,t_2)\|_2^{1-\epsilon} \\
&\le C \left\| \int_{t_1}^{t_2} \frac{\partial w_N}{\partial t} \mathrm{d}t \right\|_{H^{-1}}^{\epsilon} + C \left\| \int_{t_1}^{t_2} \frac{\partial E_N}{\partial t} \mathrm{d}t \right\|_{H^{-1}}^{\epsilon} \le C|t_2 - t_1|^{\epsilon}.
\end{aligned}
$$

This lemma is proved.

By using the above estimates of the approximate solution, we have:

Lemma 2.4 *Under the conditions of Lemma* 2.1, *there exists a unique global solution* $\big(\alpha_{sN}(t), \beta_{sN}(t), \gamma_{sN}(t)\big)$ $(s = 1, 2, \cdots, N,\ t \in [0, T],\ \textit{for any}\ T > 0)$ *of the initial value problem for the ordinary differential equations* (2.6)-(2.9). *Moreover this solution is continuously differentiable.*

3 Existence of Generalized Solution

This section is devoted to prove the existence of the generalized solution for the periodic initial value problem (1.12)-(1.17). First we give the definition of the generalized solution.

Definition 3.1 A triple of three-dimensional vector functions $\big(Z(x,t), w(x,t), E(x,t)\big) \in \big(L^\infty(0,T;H^1_{per}(\Omega)), L^\infty(0,T;L^2_{per}(\Omega)), L^\infty(0,T;L^2_{per}(\Omega))\big)$ is called the generalized solution for the periodic initial value problem (1.12)-(1.17), if for any vector-valued test function $\varphi(x,t) \in C^1([0,T];H^2_{pre}(\Omega))$ with $\varphi(x,t)|_{t=T} = 0$, and for any scalar test function $\xi(x,t) \in C^1([0,T];C^1_{per}(\Omega))$, the following equations hold

$$\iint_{Q_T} Z\cdot\varphi_t \mathrm{d}x\mathrm{d}t - \iint_{Q_T} \nabla Z\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t - \iint_{Q_T} (Z\times\nabla Z)\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t$$
$$+\iint_{Q_T} (Z\times w)\cdot\varphi \mathrm{d}x\mathrm{d}t - k\iint_{Q_T} (1+\mu|Z|^2)Z\cdot\varphi \mathrm{d}x\mathrm{d}t + \int_\Omega Z_0\cdot\varphi(x,0)\mathrm{d}x = 0, \tag{3.1}$$

$$\iint_{Q_T} E\cdot\varphi_t(x,t)\mathrm{e}^{\sigma t}\mathrm{d}x\mathrm{d}t + \iint_{Q_T} \mathrm{e}^{\sigma t}(\nabla\times\varphi)\cdot(w-\beta Z)(x,t)\mathrm{d}x\mathrm{d}t$$
$$+\int_\Omega E_0(x)\varphi(x,0)\mathrm{d}x = 0, \tag{3.2}$$

$$\iint_{Q_T} w\cdot\varphi_t(x,t)\mathrm{d}x\mathrm{d}t - \iint_{Q_T} (\nabla\times\varphi)\cdot E(x,t)\mathrm{d}x\mathrm{d}t + \int_\Omega w_0(x)\cdot\varphi(x,0)\mathrm{d}x = 0, \tag{3.3}$$

$$\iint_{Q_T} \nabla\xi\cdot w\mathrm{d}x\mathrm{d}t = 0, \tag{3.4}$$

$$\iint_{Q_T} \nabla\xi\cdot E\mathrm{d}x\mathrm{d}t = 0, \tag{3.5}$$

$$Z(x,0) = Z_0(x),\quad w(x,0) = w_0(x),\quad E(x,0) = E_0(x),\quad x\in\Omega, \tag{3.6}$$

where $Q_T = \Omega\times[0,T]$.

Lemma 3.2 *Let the initial vector functions* $(Z_0(x), H_0(x), E_0(x))$ *satisfy the condition*

$$\int_\Omega \nabla\xi\cdot E_0(x)\mathrm{d}x = 0,\qquad \int_\Omega \nabla\xi\cdot(H_0(x)+\beta Z_0(x))\mathrm{d}x = 0, \tag{3.7}$$

for all $\xi(x) \in C^1_{per}(\Omega)$. *Then for any* $\xi(x,t) \in C^1([0,T];C^1_{per}(\Omega))$ *with* $\xi(x,T) = 0$

and $\xi_0=\xi(x,0)$, *we have from* (3.2) *and* (3.3) *that*

$$\iint_{Q_T}\nabla\xi\cdot E(x,t)\mathrm{d}x\mathrm{d}t=0,\quad \iint_{Q_T}\nabla\xi\cdot w(x,t)\mathrm{d}x\mathrm{d}t=0,$$

that is (3.4) *and* (3.5) *hold.*

Proof Take

$$\varphi(x,t)=\int_0^t \mathrm{e}^{-\sigma\tau}\nabla\xi(x,\tau)\mathrm{d}\tau-\int_0^T \mathrm{e}^{-\sigma\tau}\nabla\xi(x,\tau)\mathrm{d}\tau.$$

Noting that $\xi(x,t)\in C^1([0,T];C^1_{per}(\Omega))$, by (3.2) we get

$$\iint_{Q_T}E\cdot\nabla\xi\mathrm{d}x\mathrm{d}t+\int_0^T\mathrm{e}^{-\sigma\tau}\mathrm{d}\tau\int_\Omega\nabla\xi(x,\tau)\cdot E_0(x)\mathrm{d}x=0.$$

Since

$$\int_\Omega\nabla\xi(x,\tau)\cdot E_0(x)\mathrm{d}x=0,$$

we obtain

$$\iint_{Q_T}E\cdot\nabla\xi\mathrm{d}x\mathrm{d}t=0.$$

Letting

$$\varphi=\int_0^t\nabla\xi(x,\tau)\mathrm{d}\tau-\int_0^T\nabla\xi(x,\tau)\mathrm{d}\tau,$$

we have from (3.3)

$$\iint_{Q_T}w\cdot\nabla\xi\mathrm{d}x\mathrm{d}t-\int_0^T\int_\Omega(H_0+\beta Z_0(x))\cdot\nabla\xi\mathrm{d}x\mathrm{d}\tau=0.$$

From

$$\int_\Omega(H_0+\beta Z_0(x))\cdot\nabla\xi\mathrm{d}x=0,$$

it follows that

$$\iint_{Q_T}w\cdot\nabla\xi\mathrm{d}x\mathrm{d}t=\iint_{Q_T}(H+\beta Z)\cdot\nabla\xi\mathrm{d}x\mathrm{d}t=0.$$

This lemma is proved.

Theorem 3.1 *Assume that* $Z_0(x)\in H^1_{per}(\Omega)$, $H_0(x)\in L^2_{per}(\Omega)$, $E_0(x)\in L^2_{per}(\Omega)$, *and* (3.7) *is satisfied. The constants* k,σ,μ,β *are positive. Then the periodic initial value problem* (1.12)-(1.17) *has at least one global generalized solution* $(Z(x,t),w(x,t),E(x,t))$ *such that*

$$Z(x,t) \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap C^{(0,\frac{1}{3})}(0,T;L^2(\Omega)),$$
$$E(x,t), w(x,t) \in L^\infty(0,T;L^2_{per}(\Omega)) \cap C^{(0,\epsilon)}(0,T;H^{-\epsilon}(\Omega)), \quad for\ any\ \epsilon \in (0,1). \tag{3.8}$$

Moreover, we have

$$\sup_{0\le t\le T}\left[\|Z(\cdot,t)\|_{H^1_{per}(\Omega)} + \|w(\cdot,t)\|_{L^2_{per}(\Omega)} + \|E(\cdot,t)\|_{L^2_{per}(\Omega)}\right] \le K_0, \tag{3.9}$$

$$\int_0^t \|\Delta Z\|^2_{L^2_{per}(\Omega)}\mathrm{d}t \le K_1, \tag{3.10}$$

$$\|Z_t(\cdot,t)\|_{H^{-2}(\Omega)} + \|E_t(\cdot,t)\|_{H^{-1}(\Omega)} + \|w_t(\cdot,t)\|_{H^{-1}(\Omega)} \le K_2, \tag{3.11}$$

where K_j $(j=0,1,2)$ is a constant which is independent of D.

Proof For any vector-valued test function $\varphi(x,t) \in C^1([0,T];H^2_{pre}(\Omega))$ with $\varphi(x,t)|_{t=T}=0$, we define an approximate sequence

$$\varphi_N(x,t) = \sum_{n=1}^N a_n(t)\omega_n(x),$$

where

$$a_n(t) = \int_\Omega \varphi(x,t)\omega_n(x)\mathrm{d}x.$$

We know that φ_N is uniformly convergent to $\varphi(x,t)$ in $C^1([0,T];H^2_{pre}(\Omega))$, that is,

$$\|\varphi_N - \varphi\|_{C^1([0,T];H^2_{pre}(\Omega))} \to 0, \quad \text{as } N \to \infty. \tag{3.12}$$

From the uniform estimates of the solution $\{Z_N(x,t),\, w_N(x,t),\, E_N(x,t)\}$ in Lemmas 2.1 and 2.2, by the Sobolev imbedding theorem and Lions-Aubin lemma, it yields that there exists a subsequence, which is still denoted by $\{Z_N(x,t), w_N(x,t), E_N(x,t)\}$, such that

$$Z_N(x,t) \rightharpoonup Z(x,t) \ \text{ weak}^* \text{ in } L^\infty(0,T;H^1_{per}(\Omega)) \cap L^2(0,T;H^2_{per}(\Omega)), \tag{3.13}$$
$$Z_{Nt}(x,t) \rightharpoonup Z_t(x,t) \ \text{ weak}^* \text{ in } L^\infty(0,T;H^{-2}_{per}(\Omega)), \tag{3.14}$$
$$Z_N(x,t) \to Z(x,t) \ \text{ strongly in } L^q(0,T;W^{1,p}_{per}(\Omega)), \quad 2\le q<\infty,\ 2\le p<6, \tag{3.15}$$
$$Z_N(x,t) \to Z(x,t) \ \text{ strongly in } L^q(0,T;L^p_{per}(\Omega)), \quad 2\le q<\infty,\ 2\le p\le\infty, \tag{3.16}$$
$$w_N(x,t) \rightharpoonup w(x,t) \ \text{ weak}^* \text{ in } L^\infty(0,T;L^2_{per}(\Omega)), \tag{3.17}$$
$$E_N(x,t) \rightharpoonup E(x,t) \ \text{ weak}^* \text{ in } L^\infty(0,T;L^2_{per}(\Omega)). \tag{3.18}$$

Making the scaler product of $a_s(t)$ with (2.6), the scaler product of $a_s(t)$ with (2.7), the scaler product of $\mathrm{e}^{\sigma t}a_s$ with (2.8), and summing up the products with respect to $s=1,2,\cdots,N$, we get

$$\iint_{Q_T} Z_{Nt}\cdot\varphi_N\mathrm{d}x\mathrm{d}t-\iint_{Q_T}\Delta Z_N\cdot\varphi_N\mathrm{d}x\mathrm{d}t-\iint_{Q_T}(Z_N\times(\Delta Z_N+w_N))\cdot\varphi_N\mathrm{d}x\mathrm{d}t$$
$$+k\iint_{Q_T}(1+\mu|Z_N|^2)Z_N\cdot\varphi_N\mathrm{d}x\mathrm{d}t=0, \tag{3.19}$$
$$\iint_{Q_T} w_{Nt}\cdot\varphi_N(x,t)\mathrm{d}x\mathrm{d}t=-\iint_{Q_T}(\nabla\times E_N)\cdot\varphi_N(x)\mathrm{d}x\mathrm{d}t, \tag{3.20}$$
$$\iint_{Q_T}\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\sigma t}E_N)\cdot\varphi_N(x,t)\mathrm{d}x\mathrm{d}t=\iint_{Q_T}\mathrm{e}^{\sigma t}\big(\nabla\times(w_N-\beta Z_N)\big)\cdot\varphi_N\mathrm{d}x\mathrm{d}t. \tag{3.21}$$

Rewriting (3.19), we get

$$\iint_{Q_T} Z_N\cdot\varphi_{Nt}\mathrm{d}x\mathrm{d}t-\iint_{Q_T}\nabla Z_N\cdot\nabla\varphi_N\mathrm{d}x\mathrm{d}t-\iint_{Q_T}(Z_N\times\nabla Z_N)\cdot\nabla\varphi_N\mathrm{d}x\mathrm{d}t$$
$$+\iint_{Q_T}(Z_N\times w_N)\cdot\varphi_N\mathrm{d}x\mathrm{d}t-k\iint_{Q_T}(1+\mu|Z_N|^2)Z_N\cdot\varphi_N\mathrm{d}x\mathrm{d}t$$
$$+\int_\Omega Z_N(x,0)\cdot\varphi_N(x,0)\mathrm{d}x=0. \tag{3.22}$$

Rewriting (3.20), we get

$$\iint_{Q_T} w_N(x,t)\cdot\varphi_{Nt}\mathrm{d}x\mathrm{d}t-\iint_{Q_T}(\nabla\times\varphi_N)\cdot E_N(x,t)\mathrm{d}x\mathrm{d}t+\int_\Omega w_N(x,0)\cdot\varphi_N(x,0)\mathrm{d}x=0. \tag{3.23}$$

Rewriting (3.21), we get

$$\iint_{Q_T} E_N\cdot(\varphi_{Nt}\mathrm{e}^{\sigma t})\mathrm{d}x\mathrm{d}t+\iint_{Q_T}\mathrm{e}^{\sigma t}(\nabla\times\varphi_N)\cdot(w_N-\beta Z_N)(x,t)\mathrm{d}x\mathrm{d}t$$
$$+\int_\Omega E_N(\cdot,0)\cdot\varphi_N(\cdot,0)\mathrm{d}x=0. \tag{3.24}$$

(2.2)-(2.4) and (3.12) imply that

$$\int_\Omega Z_N(x,0)\cdot\varphi_N(x,0)\mathrm{d}x\to\int_\Omega Z_0(x)\cdot\varphi(x,0)\mathrm{d}x,\quad\text{as }\ N\to\infty,$$
$$\int_\Omega w_N(x,0)\cdot\varphi_N(x,0)\mathrm{d}x\to\int_\Omega w_0(x)\cdot\varphi(x,0)\mathrm{d}x,\quad\text{as }\ N\to\infty,$$
$$\int_\Omega E_N(x,0)\cdot\varphi_N(x,0)\mathrm{d}x\to\int_\Omega E_0(x)\cdot\varphi(x,0)\mathrm{d}x,\quad\text{as }\ N\to\infty.$$

Note that

$$\begin{aligned}\iint_{Q_T}(\nabla\times\varphi_N)\cdot E_N\mathrm{d}x\mathrm{d}t&=\iint_{Q_T}\nabla\times(\varphi_N-\varphi)\cdot E_N\mathrm{d}x\mathrm{d}t+\iint_{Q_T}\nabla\times\varphi\cdot E_N\mathrm{d}x\mathrm{d}t\\&=\iint_{Q_T}\nabla\times(\varphi_N-\varphi)\cdot E_N\mathrm{d}x\mathrm{d}t+\iint_{Q_T}(\nabla\times\varphi)\cdot E\mathrm{d}x\mathrm{d}t\\&\quad+\iint_{Q_T}(\nabla\times\varphi)\cdot(E_N-E)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

(3.12) means that

$$\left|\iint_{Q_T}\nabla\times(\varphi_N-\varphi)\cdot E_N\mathrm{d}x\mathrm{d}t\right|\le\left(\iint_{Q_T}|\nabla(\varphi_N-\varphi)|^2\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}\|E_N\|_{L^2(Q_T)}\to 0,$$

as $N\to\infty$. (3.18) implies that

$$\left|\iint_{Q_T}(\nabla\times\varphi)\cdot(E_N-E)\mathrm{d}x\mathrm{d}t\right|\to 0,\quad\text{as } N\to\infty.$$

Therefore we have

$$\iint_{Q_T}(\nabla\times\varphi_N)\cdot E_N\mathrm{d}x\mathrm{d}t\to\iint_{Q_T}(\nabla\times\varphi)\cdot E\mathrm{d}x\mathrm{d}t,\quad\text{as } N\to\infty.$$

Similarly we can prove that

$$\begin{aligned}
&\iint_{Q_T}w_N\cdot\varphi_{Nt}\mathrm{d}x\mathrm{d}t\to\iint_{Q_T}w\cdot\varphi_t\mathrm{d}x\mathrm{d}t,\\
&\iint_{Q_T}E_N\cdot(\varphi_{Nt}\mathrm{e}^{\sigma t})\mathrm{d}x\mathrm{d}t\to\iint_{Q_T}E\cdot(\varphi_t\mathrm{e}^{\sigma t})\mathrm{d}x\mathrm{d}t,\\
&\iint_{Q_T}\mathrm{e}^{\sigma t}(\nabla\times\varphi_N)\cdot(w_N-\beta Z_N)(x,t)\mathrm{d}x\mathrm{d}t\to\iint_{Q_T}\mathrm{e}^{\sigma t}(\nabla\times\varphi)\cdot(w-\beta Z)(x,t)\mathrm{d}x\mathrm{d}t,\\
&\iint_{Q_T}Z_N\cdot\varphi_{Nt}\mathrm{d}x\mathrm{d}t\to\iint_{Q_T}Z\cdot\varphi_t\mathrm{d}x\mathrm{d}t,\\
&\iint_{Q_T}\nabla Z_N\cdot\nabla\varphi_N\mathrm{d}x\mathrm{d}t\to\iint_{Q_T}\nabla Z\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

as $\to\infty$. (3.15) (3.16) and (3.12) imply that

$$\begin{aligned}
&Z_N\times\nabla Z_N\to Z\times\nabla Z\ \text{ strongly in } L^2(Q_T),\\
&(1+\mu|Z_N|^2)Z_N\to(1+\mu|Z|^2)Z\ \text{ strongly in } L^2(Q_T),\\
&Z_N\times\varphi_N\to Z\times\varphi\ \text{ strongly in } L^2(Q_T).
\end{aligned}$$

Thus we obtain that

$$\begin{aligned}
&\iint_{Q_T}(Z_N\times\nabla Z_N)\cdot\nabla\varphi_N\mathrm{d}x\mathrm{d}t\to\iint_{Q_T}(Z\times\nabla Z)\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t,\quad\text{as } N\to\infty,\\
&\iint_{Q_T}(Z_N\times w_N)\cdot\varphi_N\mathrm{d}x\mathrm{d}t\to\iint_{Q_T}(Z\times w)\cdot\varphi\mathrm{d}x\mathrm{d}t,\quad\text{as } N\to\infty,\\
&\iint_{Q_T}(1+\mu|Z_N|^2)Z_N\cdot\varphi_N\mathrm{d}x\mathrm{d}t\to\iint_{Q_T}(1+\mu|Z|^2)Z\cdot\varphi\mathrm{d}x\mathrm{d}t,\quad\text{as } N\to\infty.
\end{aligned}$$

Therefore, taking $N\to\infty$ in (3.22)-(3.24), we obtain that the limit functions $Z(x,t)$, $w(x,t)$, $E(x,t)$ satisfy the integral equalities (3.1)-(3.3). By Lemma 3.2, equations (3.4) (3.5) hold. Obviously (3.6) holds. The generalized global solution of the periodic initial value problem (1.12)-(1.17) is obtained.

Employing the proof of Lemma 2.3, we get

$$Z \in C^{(0,\frac{1}{3})}\big(0,T;L^2_{per}(\Omega)\big), \quad w, E \in C^{(0,\epsilon)}\big(0,T;H^{-\epsilon}(\Omega)\big), \quad \text{for any } \epsilon \in (0,1).$$

This theorem is proved.

Since a priori estimates (3.9)-(3.11) are uniform with respect to D, by using the diagonal method and letting $D \to \infty$, we can get the following result.

Theorem 3.2 *Assume that* $Z_0(x) \in H^1(\mathbb{R}^d)$, $H_0(x) \in L^2(\mathbb{R}^d)$, $E_0(x) \in L^2(\mathbb{R}^d)$, *and* (3.4) *is satisfied. The constants* k, σ, μ, β *are positive. Then there exists a global generalized solution* $(Z(x,t), w(x,t), E(x,t))$ *of the initial value problem* (1.12)-(1.15) (1.17) *such that*

$$Z(x,t) \in L^\infty(0,T;H^1(\mathbb{R}^d)) \cap C^{(0,\frac{1}{3})}(0,T;L^2(\mathbb{R}^d)),$$
$$w(x,t), E(x,t) \in L^\infty(0,T;L^2(\mathbb{R}^d)) \cap C^{(0,\epsilon)}(0,T;H^{-\epsilon}(\mathbb{R}^d)), \quad \textit{for any } \epsilon \in (0,1).$$

4 Regularity and Global Smooth Solution

In this section, we devote to proving the existence and uniqueness of global smooth solution to the periodic problem (1.12)-(1.17) and the initial value problem (1.12)-(1.15) (1.17). For this aim, we study the regularity of Galerkin approximate solution (Z_N, w_N, E_N) of the problem (2.6)-(2.9).

In Subsection 4.1, we consider the case of $d = 2$. For this case, the global smooth solution exists and is unique no matter the size of the initial data (Z_0, H_0, E_0).

In Subsection 4.2, we consider the case of $d = 3$. For this case, we only establish the existence and uniqueness of global smooth solution provided $\|Z_0\|_{H^1}$ is small enough.

The following Gagliardo-Nirenberg inequality will be used many times.

Lemma 4.1(Gagliardo-Nirenberg Inequality) *Assume that* $u \in L^q(\Omega)$, $D^m u \in L^r(\Omega)$, $\Omega \subset \mathbb{R}^n$, $1 \le q, r \le \infty$, $0 \le j \le m$. *Then*

$$\|D^j u\|_{L^p(\Omega)} \le C(j,m;p,r,q)\|u\|^a_{W^m_r(\Omega)}\|u\|^{1-a}_{L^q(\Omega)}, \tag{4.1}$$

where $C(j,m;p,r,q)$ *is a positive constant, and*

$$\frac{1}{p} = \frac{j}{n} + a\Big(\frac{1}{r} - \frac{m}{n}\Big) + (1-a)\frac{1}{q}, \quad \frac{j}{m} \le a \le 1.$$

4.1 Dimension $d = 2$

In this subsection, we consider the special case which $d = 2$ and $x \in \Omega \subset \mathbb{R}^2$. We first establish a priori estimates of Galerkin approximate solution (Z_N, w_N, E_N), and then prove the existence and uniqueness of the solution (Z, w, E).

Lemma 4.2 *Assume that* $Z_0(x) \in H^2_{per}(\Omega)$, $w_0(x), E_0(x) \in H^1_{per}(\Omega)$, $\Omega \subset \mathbb{R}^2$. *Then for the solutions of problem* (2.6)-(2.9), *we have*

$$\sup_{0\le t\le T}\big[\|\Delta Z_N\|_2^2+\|\nabla w_N(\cdot,t)\|_2^2+\|\nabla E_N(\cdot,t)\|_2^2\big]+\int_0^T\|\nabla\Delta Z_N\|_2^2\mathrm{d}t\le K_3,\ \textit{for any } T>0, \tag{4.2}$$

where the constant K_3 *depends only on* $\|Z_0(x)\|_{H^2_{per}(\Omega)}$, $\|H_0(x)\|_{H^1_{per}(\Omega)}$ *and* $\|E_0(x)\|_{H^1_{per}(\Omega)}$, *and is independent of* N *and* D.

Proof Making the scalar product of $\lambda_s^2\alpha_{sN}$ with (2.6), summing the resulting equality with respect to $s=1,2,\cdots,N$, and noting that

$$\Delta^2 Z_N=\sum_{s=1}^N\lambda_s^2\alpha_{sN}(t)\omega_s(x),$$

we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta Z_N\|_2^2+\|\nabla\Delta Z_N\|_2^2=&-\int_\Omega\big(\nabla Z_N\times(\Delta Z_N+w_N)\big)\cdot\nabla\Delta Z_N\mathrm{d}x\\&-\int_\Omega(Z_N\times\nabla w_N)\cdot\nabla\Delta Z_N\mathrm{d}x\\&-k\int_\Omega(1+\mu|Z_N|^2)Z_N\cdot\Delta^2 Z_N\mathrm{d}x.\end{aligned} \tag{4.3}$$

By Hölder inequality, we have

$$\left|\int_\Omega\big(\nabla Z_N\times(\Delta Z_N+w_N)\big)\cdot\nabla\Delta Z_N\mathrm{d}x\right|\le\|\nabla Z_N\|_{L^\infty}(\|\Delta Z_N\|_2+\|w_N\|_2)\|\nabla\Delta Z_N\|_2.$$

By Gagliardo-Nirenlerg inequality, we get

$$\|\nabla Z_N\|_{L^\infty}\le C\|\nabla Z_N\|_{H^2}^{\frac12}\|\nabla Z_N\|_2^{\frac12}. \tag{4.4}$$

Thus, we have

$$\begin{aligned}&\left|\int_\Omega\big(\nabla Z_N\times(\Delta Z_N+w_N)\big)\cdot\nabla\Delta Z_N\mathrm{d}x\right|\\&\le\frac16\|\nabla\Delta Z_N\|_2^2+C(\|\nabla Z_0\|_2,\|w_0\|_2,\|E_0\|_2)(1+\|\Delta Z_N\|_2^4),\end{aligned} \tag{4.5}$$

where we have used the estimate (2.12). Similarly, we get

$$\begin{aligned}\left|\int_\Omega(Z_N\times\nabla w_N)\cdot\nabla\Delta Z_N\mathrm{d}x\right|&\le\|Z_N\|_{L^\infty}\|\nabla\Delta Z_N\|_2\|\nabla w_N\|_2\\&\le\frac16\|\nabla\Delta Z_N\|_2^2+C\big(1+\|\Delta Z_N\|_2^2\big)\|\nabla w_N\|_2^2,\end{aligned} \tag{4.6}$$

$$\begin{aligned}\left|k\int_\Omega(1+\mu|Z_N|^2)Z_N\cdot\Delta^2 Z_N\mathrm{d}x\right|&=\left|k\int_\Omega\big(\nabla\Delta Z_N\cdot\nabla Z_N+\mu\nabla(|Z_N|^2Z_N)\cdot\nabla\Delta Z_N\big)\mathrm{d}x\right|\\&\le k\|\nabla\Delta Z_N\|_2\|\nabla Z_N\|_2\big(1+3\mu\|Z_N\|_{L^\infty}^2\big)\\&\le\frac16\|\nabla\Delta Z_N\|_2^2+C\big(1+\|\Delta Z_N\|_2^4\big),\end{aligned} \tag{4.7}$$

where we have used Sobolev imbedding theorem $H^2_{per}(\Omega) \subset L^\infty_{per}(\Omega)$. Thus inserting estimates (4.5)-(4.7) into (4.3), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta Z_N\|_2^2 + \|\nabla\Delta Z_N\|_2^2 \le C\big(1+\|\Delta Z_N\|_2^2\big)\big(1+\|\Delta Z_N\|_2^2+\|\nabla w_N\|_2^2\big). \tag{4.8}$$

Next making the scalar product of $\lambda_s\beta_s$ with (2.7) and the scalar product of $\lambda_s\gamma_s$ with (2.8) respectively, summing the resulting equalities with respect to $s = 1, 2, \cdots, N$, and noting that

$$-\Delta w_N = \sum_{s=1}^{N} \lambda_s \beta_{sN}(t)\omega_s(x), \quad -\Delta E_N = \sum_{s=1}^{N} \lambda_s \gamma_{sN}(t)\omega_s(x),$$

we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w_N\|_2^2 = -\sum_{j=1}^{2}\int_\Omega (\nabla\times\partial_j E_N)\cdot\partial_j w_N \mathrm{d}x, \tag{4.9}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla E_N\|_2^2 + \sigma\|\nabla E_N\|_2^2 = \sum_{j=1}^{2}\int_\Omega \big(\nabla\times\partial_j(w_N-\beta Z_N)\big)\cdot\partial_j E_N \mathrm{d}x. \tag{4.10}$$

Summing (4.9) with (4.10), we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\nabla w_N\|_2^2 + \|\nabla E_N\|_2^2\big) + \sigma\|\nabla E_N\|_2^2 &= -\beta\sum_{j=1}^{2}\int_\Omega \big(\nabla\times\partial_j Z_N\big)\cdot\partial_j E_N \mathrm{d}x\\ &\le \frac{\sigma}{2}\|\nabla E_N\|_2^2 + C\|\Delta Z_N\|_2^2, \end{aligned}\tag{4.11}$$

where the following fact has been used

$$\int_\Omega (\nabla\times\partial_j w_N)\cdot\partial_j E_N \mathrm{d}x - \int_\Omega (\nabla\times\partial_j E_N)\cdot\partial_j w_N \mathrm{d}x = 0.$$

Combining (4.8) and (4.11) we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\Delta Z_N\|_2^2 + \|\nabla w_N\|_2^2 + \|\nabla E_N\|_2^2\big) + \|\nabla\Delta Z_N\|_2^2 + \sigma\|\nabla E_N\|_2^2\\ \le\, & C\big(1+\|\Delta Z_N\|_2^2\big)\big(1+\|\Delta Z_N\|_2^2+\|\nabla w_N\|_2^2\big). \end{aligned}\tag{4.12}$$

By using estimates (2.13) (4.12) and applying Gronwall inequality, we can prove estimate (4.2).

By the induction for m, we can prove the following lemma.

Lemma 4.3 *Assume that* $(Z_0(x), w_0(x), E_0(x)) \in (H^{m+1}_{per}(\Omega), H^m_{per}(\Omega), H^m_{per}(\Omega))$ $(m \ge 0)$, *then the solutions* $(Z_N(x), w_N(x), E_N(x))$ *of the problem* (2.6)-(2.9) *satisfy the following estimate*

$$\sup_{t\in[0,T]}\left[\|Z_N(\cdot,t)\|^2_{H^{m+1}_{per}(\Omega)}+\|w_N(\cdot,t)\|^2_{H^m_{per}(\Omega)}+\|E_N(\cdot,t)\|^2_{H^m_{per}(\Omega)}\right]$$
$$+\int_0^T\|Z_N(\cdot,t)\|^2_{H^{m+2}_{per}(\Omega)}\mathrm{d}t\le K_{m+2},\quad \textit{for any } T>0, \tag{4.13}$$

where the constant K_{m+2} is independent of D and N.

Proof From Lemmas 2.1 and 4.2, the estimate (4.13) is verified for $m=0,1$.

Making the scalar product of $\lambda_s^3\alpha_{sN}$ with (2.6), summing the resulting equality with respect to $s=1,2,\cdots,N$, and noting that

$$\Delta^3 Z_N=-\sum_{s=1}^N\lambda_s^3\alpha_{sN}(t)\omega_s(x),$$

we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta Z_N\|_2^2+\|\Delta^2 Z_N\|_2^2=-\int_\Omega\{\Delta[Z_N\times(\Delta Z_N+w_N)]\}\cdot\Delta^2 Z_N\mathrm{d}x$$
$$-k\int_\Omega\{\Delta(Z_N+\mu|Z_N|^2Z_N)\}\cdot\Delta^2 Z_N\mathrm{d}x. \tag{4.14}$$

By Hölder inequality, we have

$$\left|\int_\Omega\{\Delta[Z_N\times(\Delta Z_N+w_N)]\}\cdot\Delta^2 Z_N\mathrm{d}x\right|$$
$$\le 2\int_\Omega\left|(\nabla Z_N\times\nabla\Delta Z_N)\cdot\Delta^2 Z_N\right|\mathrm{d}x+\sum_{j=0}^2\binom{2}{j}\int_\Omega\left|(\nabla^j Z_N\times\nabla^{2-j}w_N)\cdot\Delta^2 Z_N\right|\mathrm{d}x$$
$$\le C\left\{\|\nabla Z_N\|_{L^\infty}\|\nabla\Delta Z_N\|_2+\sum_{j=0}^1\|\nabla^j Z_N\|_{L^\infty}\|\nabla^{2-j}w_N\|_2+\|\nabla^2 Z_N\|_2\|w_N\|_{L^\infty}\right\}\|\Delta^2 Z_N\|_2.$$

Using Sobolev imbedding theorem and the estimate (4.2), we get

$$\left|\int_\Omega\{\Delta[Z_N\times(\Delta Z_N+w_N)]\}\cdot\Delta^2 Z_N\mathrm{d}x\right|$$
$$\le\frac{1}{4}\|\Delta^2 Z_N\|_2^2+C\big(1+\|\nabla\Delta Z_N\|_2^2+\|\nabla\Delta Z_N\|_2^4+\|\Delta w_N\|_2^2\big), \tag{4.15}$$

where we have used Sobolev imbedding theorem $H^2_{per}(\Omega)\subset L^\infty_{per}(\Omega)$. Similarly, we get

$$\left|k\int_\Omega\{\Delta(Z_N+\mu|Z_N|^2Z_N)\}\cdot\Delta^2 Z_N\mathrm{d}x\right|$$
$$\le k\|\Delta^2 Z_N\|_2\big(\|\Delta Z_N\|_2+C\|Z_N\|^2_{L^\infty}\|\Delta Z_N\|_2+C\|Z_N\|_{L^\infty}\|\nabla Z_N\|_4^2\big)$$
$$\le\frac{1}{4}\|\Delta^2 Z_N\|_2^2+C, \tag{4.16}$$

where we have used the estimate (4.2), and Sobolev imbedding theorems $H^2_{per}(\Omega) \subset L^\infty_{per}(\Omega)$ and $H^1_{per}(\Omega) \subset L^4_{per}(\Omega)$. Thus inserting estimates (4.15) (4.16) into (4.14), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta Z_N\|_2^2 + \|\Delta^2 Z_N\|_2^2 \le C\big(1 + \|\nabla\Delta Z_N\|_2^2 + \|\nabla\Delta Z_N\|_2^4 + \|\Delta w_N\|_2^2\big). \quad (4.17)$$

Making the scalar product of $\lambda_s^2\beta_s$ with (2.7) and the scalar product of $\lambda_s^2\gamma_s$ with (2.8) respectively, summing the resulting equalities with respect to $s = 1, 2, \cdots, N$, and noting that

$$\Delta^2 w_N = \sum_{s=1}^{N} \lambda_s^2 \beta_{sN}(t)\omega_s(x), \quad \Delta^2 E_N = \sum_{s=1}^{N} \lambda_s^2 \gamma_{sN}(t)\omega_s(x),$$

we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta w_N\|_2^2 = -\int_\Omega (\nabla \times \Delta E_N)\cdot \Delta w_N \mathrm{d}x, \quad (4.18)$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta E_N\|_2^2 + \sigma\|\Delta E_N\|_2^2 = \int_\Omega \big(\nabla \times \Delta(w_N - \beta Z_N)\big)\cdot \Delta E_N \mathrm{d}x. \quad (4.19)$$

Summing (4.18) with (4.19), we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\Delta w_N\|_2^2 + \|\Delta E_N\|_2^2\big) + \sigma\|\Delta E_N\|_2^2 \\ = {}& -\beta\int_\Omega \big(\nabla \times \Delta Z_N\big)\cdot \Delta E_N \mathrm{d}x \le \frac{\sigma}{2}\|\Delta E_N\|_2^2 + C\|\nabla\Delta Z_N\|_2^2, \end{aligned} \quad (4.20)$$

where the following fact has been used

$$\int_\Omega (\nabla \times \Delta w_N)\cdot \Delta E_N \mathrm{d}x - \int_\Omega (\nabla \times \Delta E_N)\cdot \Delta w_N \mathrm{d}x = 0.$$

Combining (4.17) and (4.20) we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\nabla\Delta Z_N\|_2^2 + \|\Delta w_N\|_2^2 + \|\Delta E_N\|_2^2\big) + \|\Delta^2 Z_N\|_2^2 + \sigma\|\Delta E_N\|_2^2 \\ \le {}& C\big(1 + \|\nabla\Delta Z_N\|_2^2 + \|\nabla\Delta Z_N\|_2^4 + \|\Delta w_N\|_2^2\big). \end{aligned} \quad (4.21)$$

Using estimates (4.2) (4.21) and applying Gronwall inequality, we get

$$\begin{aligned}&\sup_{t\in[0,T]}\big[\|Z_N(\cdot,t)\|^2_{H^3_{per}(\Omega)} + \|w_N(\cdot,t)\|^2_{H^2_{per}(\Omega)} + \|E_N(\cdot,t)\|^2_{H^2_{per}(\Omega)}\big] \\ &+ \int_0^T \|Z_N(\cdot,t)\|^2_{H^4_{per}(\Omega)}\mathrm{d}t \le K_4, \quad \text{for any } T > 0, \end{aligned} \quad (4.22)$$

where the constant K_4 is independent of N and D.

Now assume that the estimate (4.13) is valid for $m = M \geq 2$, that is, we have

$$\sup_{t\in[0,T]} \left[\|Z_N(\cdot,t)\|^2_{H^{M+1}_{per}(\Omega)} + \|w_N(\cdot,t)\|^2_{H^M_{per}(\Omega)} + \|E_N(\cdot,t)\|^2_{H^M_{per}(\Omega)}\right] + \int_0^T \|Z_N(\cdot,t)\|^2_{H^{M+2}_{per}(\Omega)}\mathrm{d}t \leq K_{M+2}, \quad \text{for any } T > 0, \tag{4.23}$$

where the constant K_{M+2} is independent of N and D. Next we will prove (4.13) with $m = M + 1$.

Making the scalar product of $\lambda_s^{M+2}\alpha_{sN}$ with (2.6), summing the resulting equality with respect to $s = 1, 2, \cdots, N$, and noting that

$$\Delta^{M+2} Z_N = (-1)^M \sum_{s=1}^{N} \lambda_s^{M+2}\alpha_{sN}(t)\omega_s(x),$$

we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^{M+2}Z_N\|_2^2 + \|\nabla^{M+3}Z_N\|_2^2 = -\int_\Omega \{\nabla^{M+1}[Z_N \times (\Delta Z_N + w_N)]\} \cdot \nabla^{M+3}Z_N \mathrm{d}x - k\int_\Omega \{\nabla^{M+1}(Z_N + \mu|Z_N|^2 Z_N)\} \cdot \nabla^{M+3}Z_N \mathrm{d}x. \tag{4.24}$$

By Hölder inequality, we have

$$\begin{aligned}
&\left|\int_\Omega \{\nabla^{M+1}[Z_N \times (\Delta Z_N + w_N)]\} \cdot \nabla^{M+3}Z_N \mathrm{d}x\right| \\
\leq &\sum_{j=1}^{M+1} \binom{M+1}{j} \int_\Omega \left|(\nabla^j Z_N \times \nabla^{M+3-j}Z_N) \cdot \nabla^{M+3}Z_N\right| \mathrm{d}x \\
&+ \sum_{j=0}^{M+1} \binom{M+1}{j} \int_\Omega \left|(\nabla^j Z_N \times \nabla^{M+1-j}w_N) \cdot \nabla^{M+3}Z_N\right| \mathrm{d}x \\
\leq &C\Bigg\{\sum_{j=1}^{2} \|\nabla^j Z_N\|_{L^\infty}\|\nabla^{M+3-j}Z_N\|_2 + \sum_{j=0}^{2} \|\nabla^j Z_N\|_{L^\infty}\|\nabla^{M+1-j}w_N\|_2 \\
&+\chi(M \geq 3)\sum_{j=3}^{M} \|\nabla^j Z_N\|_4(\|\nabla^{M+3-j}Z_N\|_4 + \|\nabla^{M+1-j}w_N\|_4) \\
&+\|\nabla^{M+1}Z_N\|_2\|w_N\|_{L^\infty}\Bigg\}\|\nabla^{M+3}Z_N\|_2,
\end{aligned}$$

where the characteristic function

$$\chi(M \geq 3) = \begin{cases} 1, & M \geq 3, \\ 0, & 0 \leq M \leq 2. \end{cases}$$

Using Sobolev imbedding theorems and the estimate (4.23), we get

$$\begin{aligned}&\Big|\int_\Omega\{\nabla^{M+1}[Z_N\times(\Delta Z_N+w_N)]\}\cdot\nabla^{M+3}Z_N\mathrm{d}x\Big|\\ \le&\frac14\|\nabla^{M+3}Z_N\|_2^2+C\big(1+\|\nabla^{M+2}Z_N\|_2^2+\|\nabla^{M+2}Z_N\|_2^4+\|\nabla^{M+1}w_N\|_2^2\big),\end{aligned}\tag{4.25}$$

where we have used Sobolev imbedding theorems

$$H^{M+2}_{per}(\Omega)\subset W^{2,\infty}_{per}(\Omega),\quad H^{M+1}_{per}(\Omega)\subset W^{j,\infty}_{per}(\Omega),\quad j=0,1,$$

and

$$H^{j+1}_{per}(\Omega)\subset W^{j,4}_{per}(\Omega),\quad j=0,1,\cdots,M.$$

Similarly, we have

$$\begin{aligned}&\Big|k\int_\Omega\{\nabla^{M+1}(Z_N+\mu|Z_N|^2Z_N)\}\cdot\nabla^{M+3}Z_N\mathrm{d}x\Big|\\ \le& k\|\nabla^{M+3}Z_N\|_2\|\nabla^{M+1}Z_N\|_2\\ &+C\sum_{j_1+j_2+j_3=M+1}\|\nabla^{j_1}Z_N\|_6\|\nabla^{j_2}Z_N\|_6\|\nabla^{j_3}Z_N\|_6\|\nabla^{M+3}Z_N\|_2\\ \le&\frac14\|\nabla^{M+3}Z_N\|_2^2+C\big(1+\|\nabla^{M+2}Z_N\|_2^2\big),\end{aligned}\tag{4.26}$$

where we have used the estimate (4.23) and Sobolev imbedding theorems

$$H^{j+1}_{per}(\Omega)\subset W^{j,6}_{per}(\Omega),\quad j=0,1,\cdots,M+1.$$

Thus inserting estimates (4.25) (4.26) into (4.24), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^{M+2}Z_N\|_2^2+\|\nabla^{M+3}Z_N\|_2^2\le C\big(1+\|\nabla^{M+2}Z_N\|_2^2+\|\nabla^{M+2}Z_N\|_2^4+\|\nabla^{M+1}w_N\|_2^2\big).\tag{4.27}$$

Making the scalar product of $\lambda_s^{M+1}\beta_s$ with (2.7) and the scalar product of $\lambda_s^{M+1}\gamma_s$ with (2.8) respectively, summing the resulting equalities with respect to $s=1,2,\cdots,N$, and noting that

$$\begin{aligned}\Delta^{M+1}w_N&=(-1)^{M+1}\sum_{s=1}^N\lambda_s^{M+1}\beta_{sN}(t)\omega_s(x),\\ \Delta^{M+1}E_N&=(-1)^{M+1}\sum_{s=1}^N\lambda_s^{M+1}\gamma_{sN}(t)\omega_s(x),\end{aligned}$$

we obtain

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^{M+1}w_N\|_2^2=-\sum_{j_1+j_2=M+1}\int_\Omega(\nabla\times\partial_1^{j_1}\partial_2^{j_2}E_N)\cdot\partial_1^{j_1}\partial_2^{j_2}w_N\mathrm{d}x,\tag{4.28}$$

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^{M+1}E_N\|_2^2+\sigma\|\nabla^{M+1}E_N\|_2^2\\=&\sum_{j_1+j_2=M+1}\int_\Omega\big(\nabla\times\partial_1^{j_1}\partial_2^{j_2}(w_N-\beta Z_N)\big)\cdot\partial_1^{j_1}\partial_2^{j_2}E_N\mathrm{d}x.\end{aligned}\tag{4.29}$$

Summing (4.28) with (4.29), we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\nabla^{M+1}w_N\|_2^2+\|\nabla^{M+1}E_N\|_2^2\big)+\sigma\|\nabla^{M+1}E_N\|_2^2\\=&-\beta\sum_{j_1+j_2=M+1}\int_\Omega\big(\nabla\times\partial_1^{j_1}\partial_2^{j_2}Z_N\big)\cdot\partial_1^{j_1}\partial_2^{j_2}E_N\mathrm{d}x\\\le&\frac{\sigma}{2}\|\nabla^{M+1}E_N\|_2^2+C\|\nabla^{M+2}Z_N\|_2^2,\end{aligned}\tag{4.30}$$

where the following fact has been used

$$\int_\Omega(\nabla\times\partial_1^{j_1}\partial_2^{j_2}w_N)\cdot\partial_1^{j_1}\partial_2^{j_2}E_N\mathrm{d}x-\int_\Omega(\nabla\times\partial_1^{j_1}\partial_2^{j_2}E_N)\cdot\partial_1^{j_1}\partial_2^{j_2}w_N\mathrm{d}x=0.$$

Combining (4.27) and (4.30) we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\nabla^{M+2}Z_N\|_2^2+\|\nabla^{M+1}w_N\|_2^2+\|\nabla^{M+1}E_N\|_2^2\big)+\|\nabla^{M+3}Z_N\|_2^2\\\le&C\big(1+\|\nabla^{M+2}Z_N\|_2^2+\|\nabla^{M+1}w_N\|_2^2+\|\nabla^{M+2}Z_N\|_2^4\big).\end{aligned}\tag{4.31}$$

Using estimates (4.23) (4.31) and applying Gronwall inequality, we establish the estimate (4.13) with $m=M+1$.

By the induction, this lemma is proved.

Thanks to that the estimate (4.13) is uniform with respect to N, let $N\to\infty$, we obtain the following result.

Theorem 4.1 *Assume that* $d=2$, $\Omega\subset\mathbb{R}^2$, $k,\mu,\sigma,\beta>0$, $(Z_0(x),H_0(x),E_0(x))\in(H_{per}^{m+1}(\Omega),\ H_{per}^m(\Omega),H_{per}^m(\Omega))$, $m\ge1$, $\nabla(H_0+\beta Z_0)=0$, $\nabla\cdot E_0=0$. *Then there exists a unique global solution* (Z,w,E) *of the periodic initial value problem* (1.12)-(1.17). *Moreover, we have*

$$\begin{aligned}&Z(x,t)\in\bigcap_{s=0}^{[\frac{m+1}{2}]}W_\infty^s(0,T;H_{per}^{m+1-2s}(\Omega)),\\&w(x,t),E(x,t)\in\bigcap_{s=0}^{m}W_\infty^s(0,T;H_{per}^{m-s}(\Omega)),\quad \textit{for any } T>0,\end{aligned}\tag{4.32}$$

and

$$\begin{aligned}&\sup_{t\in[0,T]}\big[\|Z(\cdot,t)\|^2_{H_{per}^{m+1}(\Omega)}+\|w(\cdot,t)\|^2_{H_{per}^m(\Omega)}+\|E(\cdot,t)\|^2_{H_{per}^m(\Omega)}\big]\\&+\int_0^T\|Z(\cdot,t)\|^2_{H_{per}^{m+2}(\Omega)}\mathrm{d}t\le K_{m+2},\quad \textit{for any } T>0,\end{aligned}\tag{4.33}$$

where the constant K_{m+2} *is independent of* D.

Proof Similar to the proof of Theorem 3.1, let $N \to \infty$, we can prove that there exists a global solution

$$(Z, w, E) \in \big(C(0,T; H_{per}^{m+1}(\Omega)), L^\infty(0,T; H_{per}^m(\Omega)), L^\infty(0,T; H_{per}^m(\Omega))\big)$$

of problem (1.12)-(1.17), and the estimate (4.33) holds. The result (4.32) can be proved by using equations (1.12)-(1.14).

Now we devote to proving the uniqueness. Assume that there exist two solutions (Z_j, w_j, E_j) $(j = 1, 2)$. Let $(\varphi, \psi, \xi) = (Z_1 - Z_2, w_1 - w_2, E_1 - E_2)$, then (φ, ψ, ξ) satisfies the following equations

$$\frac{\partial \varphi}{\partial t} - \Delta\varphi - \varphi \times (\Delta Z_1 + w_1) - Z_2 \times (\Delta\varphi + \psi) = -k(1 + \mu|Z_1|^2)\varphi - k\mu(Z_1 + Z_2) \cdot \varphi Z_2, \tag{4.34}$$

$$\frac{\partial \xi}{\partial t} + \sigma\xi = \nabla \times (\psi - \beta\varphi), \tag{4.35}$$

$$\frac{\partial \psi}{\partial t} = -\nabla \times \xi, \tag{4.36}$$

$$\nabla \cdot \psi = 0, \quad \nabla \cdot \xi = 0, \tag{4.37}$$

$$\varphi(x + 2De_i, t) = \varphi(x, t), \quad \psi(x + 2De_i, t) = \psi(x, t), \quad \xi(x + 2De_i, t) = \xi(x, t), \tag{4.38}$$

$$\varphi(x, 0) = 0, \quad \psi(x, 0) = 0, \quad \xi(x, 0) = 0. \tag{4.39}$$

Making the scalar product of equation (4.34) with $\varphi - \Delta\varphi$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{\|\varphi(\cdot,t)\|_2^2 + \|\nabla\varphi(\cdot,t)\|_2^2\} + \{\|\nabla\varphi(\cdot,t)\|_2^2 + \|\Delta\varphi(\cdot,t)\|_2^2\} \\
= & -\int_\Omega \{\varphi \times (\Delta Z_1 + w_1)\} \cdot \Delta\varphi \mathrm{d}x + \int_\Omega (Z_2 \times \Delta\varphi) \cdot \varphi \mathrm{d}x + \int_\Omega (Z_2 \times \psi) \cdot (\varphi - \Delta\varphi) \\
& -k\int_\Omega \{(1 + \mu|Z_1|^2)\varphi + \mu(Z_1 + Z_2) \cdot \varphi Z_2\} \cdot (\varphi - \Delta\varphi)\mathrm{d}x \\
\le & \, C\{(\|\Delta Z_1\|_2 + \|w_1\|_2)\|\varphi\|_{L^\infty}\|\Delta\varphi\|_2 + \|Z_2\|_{L^\infty}\|\varphi\|_2\|\Delta\varphi\|_2 \\
& + \|Z_2\|_{L^\infty}\|\psi\|_2(\|\varphi\|_2 + \|\Delta\varphi\|_2) + (1 + \|Z_1\|_{L^\infty}^2 + \|Z_2\|_{L^\infty}^2)\|\varphi\|_2(\|\varphi\|_2 + \|\Delta\varphi\|_2)\}.
\end{aligned}$$

By using Gagliardo-Nirenberg inequality, one has

$$\|\varphi\|_{L^\infty} \le C\|\varphi\|_2^{\frac{1}{2}}\|\varphi\|_{H_{per}^2}^{\frac{1}{2}}. \tag{4.40}$$

Applying the estimate (4.33) with $m \ge 1$ and inequality (4.40), we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{\|\varphi(\cdot,t)\|_2^2 + \|\nabla\varphi(\cdot,t)\|_2^2\} + \{\|\nabla\varphi(\cdot,t)\|_2^2 + \|\Delta\varphi(\cdot,t)\|_2^2\} \\
\le & \frac{1}{2}\|\Delta\varphi(\cdot,t)\|_2^2 + C(\|\varphi(\cdot,t)\|_2^2 + \|\psi(\cdot,t)\|_2^2).
\end{aligned} \tag{4.41}$$

Next making the scalar product of equation (4.35) with ψ, and the scalar product of equation (4.36) with ξ, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{\|\psi(\cdot,t)\|_2^2+\|\xi(\cdot,t)\|_2^2\}+\sigma\|\xi(\cdot,t)\|_2^2=-\beta\int_\Omega(\nabla\times\varphi)\xi\mathrm{d}x\le\frac{\sigma}{2}\|\xi(\cdot,t)\|_2^2+C\|\varphi(\cdot,t)\|_2^2. \tag{4.42}$$

Summing (4.41) with (4.42) and applying Gronwall inequality, we get

$$\|\varphi(\cdot,t)\|_2^2+\|\nabla\varphi(\cdot,t)\|_2^2+\|\psi(\cdot,t)\|_2^2+\|\xi(\cdot,t)\|_2^2=0. \tag{4.43}$$

Therefore the global solution (Z,w,E) is unique for $m\ge 1$.

This theorem is proved.

Since a priori estimate (4.33) is uniform with respect to D, by using the diagonal method and letting $D\to\infty$, we can get the following result.

Theorem 4.2 *Assume that* $d=2$, $k,\mu,\sigma,\beta>0$, $(Z_0(x),H_0(x),E_0(x))\in(H^{m+1}(\mathbb{R}^2),\ H^m(\mathbb{R}^2),\ H^m(\mathbb{R}^2))$, $m\ge 1$, $\nabla(H_0+\beta Z_0)=0$, $\nabla\cdot E_0=0$. *Then there exists a unique global solution* (Z,w,E) *of the initial value problem* (1.12)-(1.15) (1.17). *Moreover, we have*

$$Z(x,t)\in\bigcap_{s=0}^{[\frac{m+1}{2}]}W_\infty^s(0,T;H^{m+1-2s}(\mathbb{R}^2)),$$

$$w(x,t),E(x,t)\in\bigcap_{s=0}^{m}W_\infty^s(0,T;H^{m-s}(\mathbb{R}^2)),\quad \textit{for any } T>0 \tag{4.44}$$

and

$$\sup_{t\in[0,T]}\left[\|Z(\cdot,t)\|_{H^{m+1}(\mathbb{R}^2)}^2+\|w(\cdot,t)\|_{H^m(\mathbb{R}^2)}^2+\|E(\cdot,t)\|_{H^m(\mathbb{R}^2)}^2\right]$$
$$+\int_0^T\|Z(\cdot,t)\|_{H^{m+2}(\mathbb{R}^2)}^2\mathrm{d}t\le K_{m+2},\quad \textit{for any } T>0. \tag{4.45}$$

4.2 Dimension $d=3$

In this subsection, we consider the special case when $d=3$ and $x\in\Omega\subset\mathbb{R}^3$. We first establish a priori estimates of Galerkin approximate solution (Z_N,w_N,E_N), and then prove the existence and uniqueness of the solution (Z,w,E).

Lemma 4.4 *Assume that* $Z_0(x)\in H^2_{per}(\Omega)$, $w_0(x),E_0(x)\in H^1_{per}(\Omega)$, $\Omega\subset\mathbb{R}^3$, σ, β μ *and* k *are positive constants. Then for any* $T>0$, *there exists a positive constant* $\delta_0\ll 1$ *such that if*

$$\|Z_0\|_{H^1_{per}(\Omega)}<\delta_0, \tag{4.46}$$

the solutions of problem (2.6)-(2.9) *satisfy the following estimate*

$$\sup_{0\le t\le T}\left\{\|\Delta Z_N\|_2^2+\|\nabla w_N(\cdot,t)\|_2^2+\|\nabla E_N(\cdot,t)\|_2^2\right\}$$
$$\le\frac{9}{8}\left\{\|\Delta Z_0\|_2^2+\|\nabla w_0(\cdot,t)\|_2^2+\|\nabla E_0(\cdot,t)\|_2^2+3+\beta_k\right\},\quad \textit{for any } T>0, \tag{4.47}$$

where constant

$$\beta_k=\begin{cases}\dfrac{\beta^2}{\sigma}-2k, & \dfrac{\beta^2}{\sigma}>2k,\\ 0, & \dfrac{\beta^2}{\sigma}\le 2k.\end{cases}$$

Proof Repeating the proceeding of the proof of Lemma 4.2 and revising some estimates, we can prove this lemma. In fact, the estimates (2.14) and (2.19) imply that

$$\|\nabla Z_N(\cdot,t)\|_2^2+\int_0^t\|Z_N(\cdot,\tau)\|_{H^2}^2\mathrm{d}\tau\le C\|Z_0\|_{H^1}^2\big(1+\rho_0^4\big),\tag{4.48}$$

where $\rho_0=\|Z_0\|_2^2+\|w_0\|_2^2+\|E_0\|_2^2$. The estimate (4.4) is replaced by

$$\|\nabla Z_N\|_{L^\infty}\le C(0,2;\infty,2,2)\|\nabla Z_N\|_{H^2}^{\frac34}\|\nabla Z_N\|_2^{\frac14},\tag{4.49}$$

and the estimate (4.5) is reformulated by

$$\begin{aligned}&\left|\int_\Omega\big(\nabla Z_N\times(\Delta Z_N+w_N)\big)\cdot\nabla\Delta Z_N\mathrm{d}x\right|\\ \le{}&\frac16\|\nabla\Delta Z_N\|_2^2+C\|\nabla Z_N\|_2^2\big(\|\Delta Z_N\|_2^2+\|\Delta Z_N\|_2^8+\|w_N\|_2^2+\|w_N\|_2^8\big).\end{aligned}\tag{4.50}$$

Similarly, the estimate (4.6) is replaced by

$$\begin{aligned}\left|\int_\Omega(Z_N\times\nabla w_N)\cdot\nabla\Delta Z_N\mathrm{d}x\right|&\le\|Z_N\|_{L^\infty}\|\nabla\Delta Z_N\|_2\|\nabla w_N\|_2\\ &\le\frac16\|\nabla\Delta Z_N\|_2^2+C\|Z_N\|_{H^2}^2\|\nabla w_N\|_2^2,\end{aligned}\tag{4.51}$$

the estimate (4.7) is reformulated by

$$k\int_\Omega Z_N\cdot\Delta^2Z_N\mathrm{d}x=k\|\Delta Z_N\|_2^2,\tag{4.52}$$

$$\begin{aligned}\left|k\int_\Omega\mu|Z_N|^2Z_N\cdot\Delta^2Z_N\mathrm{d}x\right|&\le 3k\mu\|\nabla\Delta Z_N\|_2\|\nabla Z_N\|_2\|Z_N\|_{L^\infty}^2\\ &\le\frac16\|\nabla\Delta Z_N\|_2^2+C\|Z_0\|_{H^1}^2\big(1+\rho_0^4\big)\|Z_N\|_{H^2}^4,\end{aligned}\tag{4.53}$$

the estimate (4.8) is replaced by

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta Z_N\|_2^2+\|\nabla\Delta Z_N\|_2^2+2k\|\Delta Z_N\|_2^2\\ \le{}&C\|Z_N\|_{H^2}^2\|\nabla w_N\|_2^2+C\|Z_0\|_{H^1}^2\big(1+\rho_0^4\big)\|Z_N\|_{H^2}^4\\ &+C\|\nabla Z_N\|_2^2\big(\|\Delta Z_N\|_2^2+\|\Delta Z_N\|_2^8+\|w_N\|_2^2+\|w_N\|_2^8\big),\end{aligned}\tag{4.54}$$

the estimate (4.11) is replaced by

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\nabla w_N\|_2^2+\|\nabla E_N\|_2^2\big)\le\frac{\beta^2}{\sigma}\|\Delta Z_N\|_2^2,\tag{4.55}$$

and the estimate (4.12) is reformulated by

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\Delta Z_N\|_2^2+\|\nabla w_N\|_2^2+\|\nabla E_N\|_2^2\big)+\Big(2k-\frac{\beta^2}{\sigma}\Big)\|\Delta Z_N\|_2^2\\ \le\ & C_{d3}\|Z_N\|_{H^2}^2\|\nabla w_N\|_2^2+C_{d3}\|Z_0\|_{H^1}^2\big(1+\rho_0^4\big)\|Z_N\|_{H^2}^4\\ &+C_{d3}\|\nabla Z_N\|_2^2\big(\|\Delta Z_N\|_2^2+\|\Delta Z_N\|_2^8+\|w_N\|_2^2+\|w_N\|_2^8\big).\end{aligned}\tag{4.56}$$

Integrating (4.56) with respect to t from 0 to T, and using the estimate (2.15) (4.48), we have

$$\begin{aligned}&\max_{0\le t\le T}\big(\|\Delta Z_N\|_2^2+\|\nabla w_N\|_2^2+\|\nabla E_N\|_2^2\big)\\ \le\ &\|\Delta Z_0\|_2^2+\|\nabla w_0\|_2^2+\|\nabla E_0\|_2^2+C_{d3}\|Z_0\|_2^2\big(\rho_0+\rho_0^4+\max_{0\le t\le T}\|\Delta Z_N\|_2^2\big)\\ &+K_{d3}\|Z_0\|_{H^1}^2\big(1+\rho_0^4\big)\big\{\beta_k+C_{d3}\max_{0\le t\le T}\|\nabla w_N\|_2^2\\ &+C_{d3}\|Z_0\|_{H^1}^2\big(1+\rho_0^4\big)\big[\|Z_0\|_{H^1}^2+\max_{0\le t\le T}\|\Delta Z_N\|_2^2+\big(\max_{0\le t\le T}\|\Delta Z_N\|_2^2\big)^3\big]\big\}.\end{aligned}\tag{4.57}$$

Take $\delta_0>0$ small enough and let $\|Z_0\|_{H^1}^2\le\delta_0$ such that

$$\max\{K_{d3},1\}\max\{C_{d3},1\}\|Z_0\|_{H^1}^2\big(1+\rho_0^4\big)\le\frac{\epsilon}{2},\quad \epsilon\in(0,1),$$

where

$$\epsilon=\frac{-2+2\sqrt{1+r_0^2/9}}{r_0^2},\quad r_0=\frac{9}{8}(\alpha_0+3+\beta_k),\quad \alpha_0=\|\Delta Z_0\|_2^2+\|\nabla w_0(\cdot,t)\|_2^2+\|\nabla E_0(\cdot,t)\|_2^2.$$

Thus we get

$$\begin{aligned}&(1-\epsilon)\max_{0\le t\le T}\big(\|\Delta Z_N\|_2^2+\|\nabla w_N\|_2^2+\|\nabla E_N\|_2^2\big)\\ \le\ &\|\Delta Z_0\|_2^2+\|\nabla w_0\|_2^2+\|\nabla E_0\|_2^2+\frac{\epsilon}{2}\Big\{2+\beta_k+\frac{\epsilon^2}{4}\Big\}+\frac{\epsilon^2}{4}\big(\max_{0\le t\le T}\|\Delta Z_N\|_2^2\big)^3.\end{aligned}\tag{4.58}$$

Consider the polynomial

$$P(r)=\frac{\epsilon^2}{4}r^3-(1-\epsilon)r+\alpha_0+3+\beta_k.$$

We find the fact that

$$P(r)\big|_{r=\alpha_0}>0,\quad P(r)\big|_{r=r_0}=0.\tag{4.59}$$

Thanks to

$$\epsilon<\frac{-2+2\sqrt{1+3r_0^2}}{3r_0^2},$$

there exists an $r_1>0$ such that

$$P(r)\big|_{r=r_0+r_1}<0.\tag{4.60}$$

Since $\big(Z_N(x,t), w_N(x,t), E_N(x,t)\big)$ is continuous with respect to t, we have that

$$\max_{0\le t\le T}\{\|\Delta Z_N\|_2^2+\|\nabla w_N\|_2^2+\|\nabla E_N\|_2^2\}$$

is continuous with respect to T, and

$$P\big(\max_{0\le t\le T}\{\|\Delta Z_N\|_2^2+\|\nabla w_N\|_2^2+\|\nabla E_N\|_2^2\}\big)$$

is also continuous with respect to T. Therefore the facts (4.58) (4.59) (4.60) mean that the estimate (4.47) is valid.

This lemma is proved.

Using Lemma 4.4, Lemma 4.3 can be proved when $d=3$. Now repeating the proceeding of the proofs of Theorems 4.1 and 4.2, we obtain the following results.

Theorem 4.3 *Assume that* $d=3$, $\Omega\subset\mathbb{R}^3$, $k,\mu,\sigma,\beta>0$, $(Z_0(x),H_0(x),E_0(x))\in(H_{per}^{m+1}(\Omega),\ H_{per}^m(\Omega),H_{per}^m(\Omega))$, $m\ge 1$, $\nabla(H_0+\beta Z_0)=0$, $\nabla\cdot E_0=0$, *and there exists a positive constant* $\delta_0\ll 1$ *such that* $\|Z_0\|_{H_{per}^1(\Omega)}<\delta_0$. *Then there exists a unique global solution* (Z,w,E) *of the periodic initial value problem* (1.12)-(1.17). *Moreover, we have*

$$\begin{aligned}&Z(x,t)\in\bigcap_{s=0}^{[\frac{m+1}{2}]}W_\infty^s(0,T;H_{per}^{m+1-2s}(\Omega)),\\&w(x,t),E(x,t)\in\bigcap_{s=0}^{m}W_\infty^s(0,T;H_{per}^{m-s}(\Omega)),\quad \textit{for any } T>0\end{aligned}\tag{4.61}$$

and

$$\begin{aligned}&\sup_{t\in[0,T]}\big[\|Z(\cdot,t)\|^2_{H_{per}^{m+1}(\Omega)}+\|w(\cdot,t)\|^2_{H_{per}^m(\Omega)}+\|E(\cdot,t)\|^2_{H_{per}^m(\Omega)}\big]\\&+\int_0^T\|Z(\cdot,t)\|^2_{H_{per}^{m+2}(\Omega)}\mathrm{d}t\le K_{m+2},\quad \textit{for any } T>0,\end{aligned}\tag{4.62}$$

where the constant K_{m+2} *is independent of* D.

Theorem 4.4 *Assume that* $d=3$, $k,\mu,\sigma,\beta>0$, $(Z_0(x),H_0(x),E_0(x))\in(H^{m+1}(\mathbb{R}^3),\ H^m(\mathbb{R}^3),\ H^m(\mathbb{R}^3))$, $m\ge1$, $\nabla(H_0+\beta Z_0)=0$, $\nabla\cdot E_0=0$, *and there exists a positive constant* $\delta_0\ll1$ *such that* $\|Z_0\|_{H^1(\mathbb{R}^3)}<\delta_0$. *Then there exists a unique global solution* (Z,w,E) *of the initial value problem* (1.12)-(1.15) (1.17). *Moreover, we have*

$$\begin{aligned}&Z(x,t)\in\bigcap_{s=0}^{[\frac{m+1}{2}]}W_\infty^s(0,T;H^{m+1-2s}(\mathbb{R}^3)),\\&w(x,t),E(x,t)\in\bigcap_{s=0}^{m}W_\infty^s(0,T;H^{m-s}(\mathbb{R}^3)),\quad \textit{for any } T>0\end{aligned}\tag{4.63}$$

and

$$\sup_{t\in[0,T]}\left[\|Z(\cdot,t)\|^2_{H^{m+1}(\mathbb{R}^3)}+\|w(\cdot,t)\|^2_{H^m(\mathbb{R}^3)}+\|E(\cdot,t)\|^2_{H^m(\mathbb{R}^3)}\right]$$
$$+\int_0^T\|Z(\cdot,t)\|^2_{H^{m+2}(\mathbb{R}^3)}\mathrm{d}t\le K_{m+2},\quad \textit{for any } T>0. \tag{4.64}$$

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