Ann. of Appl. Math.
35:4(2019), 392-448

PROPERTIES OF SOLUTIONS OF n-DIMENSIONAL INCOMPRESSIBLE NAVIER-STOKES EQUATIONS*

Linghai Zhang†

(*Dept. of Math., Lehigh University, 17 East Memorial Drive, Bethlehem, Pennsylvania 18015, USA*)

Abstract

Consider the n-dimensional incompressible Navier-Stokes equations

$$\frac{\partial}{\partial t}\mathbf{u} - \alpha\triangle\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}(\mathbf{x}, t), \quad \nabla\cdot\mathbf{u} = 0, \nabla\cdot\mathbf{f} = 0,$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \nabla\cdot\mathbf{u}_0 = 0.$$

There exists a global weak solution under some assumptions on the initial function and the external force. It is well known that the global weak solutions become sufficiently small and smooth after a long time. Here are several very interesting questions about the global weak solutions of the Cauchy problems for the n-dimensional incompressible Navier-Stokes equations.

- Can we establish better decay estimates with sharp rates not only for the global weak solutions but also for all order derivatives of the global weak solutions?
- Can we accomplish the exact limits of all order derivatives of the global weak solutions in terms of the given information?
- Can we use the global smooth solution of the linear heat equation, with the same initial function and the external force, to approximate the global weak solutions of the Navier-Stokes equations?
- If we drop the nonlinear terms in the Navier-Stokes equations, will the exact limits reduce to the exact limits of the solutions of the linear heat equation?
- Will the exact limits of the derivatives of the global weak solutions of the Navier-Stokes equations and the exact limits of the derivatives of the global smooth solution of the heat equation increase at the same rate as the order m of the derivative increases? In another word, will the ratio of the exact limits for the derivatives of the global weak solutions of the Navier-Stokes equations be the same as the ratio of the exact limits for the derivatives of the global smooth solutions for the linear heat equation?

*Manuscript received June 10, 2019

†Corresponding author. E-mail: liz5@lehigh.edu

The positive solutions to these questions obtained in this paper will definitely help us to better understand the properties of the global weak solutions of the incompressible Navier-Stokes equations and hopefully to discover new special structures of the Navier-Stokes equations.

Keywords the n-dimensional incompressible Navier-Stokes equations; decay estimates with sharp rates; exact limits; appropriate coupling of existing ideas and results; Fourier transformation; Parseval's identity; Lebesgue's dominated convergence theorem; Gagliardo-Nirenberg's interpolation inequality

2000 Mathematics Subject Classification 35Q20

1 Introduction

1.1 Mathematical model equations

Consider the Cauchy problem for the n-dimensional incompressible Navier-Stokes equations

$$\frac{\partial}{\partial t}\mathbf{u} - \alpha\triangle\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}(\mathbf{x},t), \quad \nabla\cdot\mathbf{u} = 0, \nabla\cdot\mathbf{f} = 0, \tag{1}$$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \nabla\cdot\mathbf{u}_0 = 0. \tag{2}$$

The real vector valued function $\mathbf{u} = \mathbf{u}(\mathbf{x},t)$ represents the velocity of the fluid at position $\mathbf{x}$ and time t. The real scalar function $p = p(\mathbf{x},t)$ represents the pressure of the fluid at $\mathbf{x}$ and t. The positive constant $\alpha > 0$ represents the diffusion coefficient. See Leray [7], Temam [14] and [15].

Consider the Cauchy problem for the linear heat equation

$$\frac{\partial}{\partial t}\mathbf{v} - \alpha\triangle\mathbf{v} = \mathbf{f}(\mathbf{x},t), \quad \nabla\cdot\mathbf{v} = 0, \nabla\cdot\mathbf{f} = 0, \tag{3}$$

$$\mathbf{v}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \nabla\cdot\mathbf{u}_0 = 0. \tag{4}$$

Can we use the solution of the linear heat equation to approximate the solution of the Navier-Stokes equation? Theorems 2 and 4 given in Subsection 1.5 not only provide a positive solution but also demonstrate how well it approximates.

1.2 Previous related results

Let us review several well known results about the n-dimensional incompressible Navier-Stokes equations (1)-(2).

The existence and regularity of the global weak solutions: First of all, let us consider the case $n = 2$. If the initial function $\mathbf{u}_0 \in \mathcal{S}(\mathbb{R}^2)$ and the external force $\mathbf{f} \in L^1(\mathbb{R}^+, L^2(\mathbb{R}^2)) \cap C^\infty(\mathbb{R}^n \times \mathbb{R}^+)$, then there exists a unique global smooth solution

$$\mathbf{u} \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+).$$

See Leray [7], Temam [14] and [15]. Secondly, let us consider the case $n \geq 3$ (basically, n=3 or n=4). Suppose that the initial function $\mathbf{u}_0 \in L^2(\mathbb{R}^n)$ and the external force $\mathbf{f}\in L^1(\mathbb{R}^+, L^2(\mathbb{R}^n))$. Then there exists a global weak solution $\mathbf{u}\in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^n))$, such that $\nabla\mathbf{u} \in L^2(\mathbb{R}^+, L^2(\mathbb{R}^n))$. Moreover, there holds the following elementary uniform energy estimate

$$\begin{aligned}&\left\{\int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} + 2\alpha \int_0^t\int_{\mathbb{R}^n} |\nabla\mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x}\mathrm{d}\tau\right\}^{1/2}\\ \le &\left\{\int_{\mathbb{R}^n} |\mathbf{u}_0(\mathbf{x})|^2 \mathrm{d}\mathbf{x}\right\}^{1/2} + \int_0^\infty \left\{\int_{\mathbb{R}^n} |\mathbf{f}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\right\}^{1/2} \mathrm{d}t.\end{aligned}$$

Additionally, there holds the following formal representation for the global weak solution

$$\begin{aligned}\mathbf{u}(\mathbf{x},t) = &\frac{1}{(4\pi\alpha t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\alpha t}\right) \mathbf{u}_0(\mathbf{y})\mathrm{d}\mathbf{y}\\ &+ \int_0^t \left\{\frac{1}{[4\pi\alpha(t-\tau)]^{n/2}} \int_{\mathbb{R}^n} \exp\left[-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\alpha(t-\tau)}\right] \mathbf{f}(\mathbf{y},\tau)\mathrm{d}\mathbf{y}\right\} \mathrm{d}\tau\\ &- \int_0^t \left\{\frac{1}{[4\pi\alpha(t-\tau)]^{n/2}} \int_{\mathbb{R}^n} \exp\left[-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\alpha(t-\tau)}\right] (\mathbf{u}\cdot\nabla)\mathbf{u}(\mathbf{y},\tau)\mathrm{d}\mathbf{y}\right\} \mathrm{d}\tau\\ &- \int_0^t \left\{\frac{1}{[4\pi\alpha(t-\tau)]^{n/2}} \int_{\mathbb{R}^n} \exp\left[-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\alpha(t-\tau)}\right] \nabla p(\mathbf{y},\tau)\mathrm{d}\mathbf{y}\right\} \mathrm{d}\tau,\end{aligned}$$

for all $\mathbf{x} \in \mathbb{R}^n$ and $t > 0$, where

$$p(\mathbf{x},t) = (-\triangle)^{-1} \sum_{k=1}^n \sum_{l=1}^n \frac{\partial^2}{\partial x_k \partial x_l}[u_k(\mathbf{x},t)u_l(\mathbf{x},t)].$$

Furthermore, if $\mathbf{u}_0 \in H^{2m+1}(\mathbb{R}^n)$ and $\mathbf{f} \in L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)) \cap L^2(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n))$, then there exists a unique local smooth solution $\mathbf{u} \in L^\infty(0,\varepsilon; H^{2m+1}(\mathbb{R}^n))$, such that $\nabla\mathbf{u} \in L^2(0,\varepsilon; H^{2m+1}(\mathbb{R}^n))$, where $\varepsilon > 0$ is an appropriate positive constant. Additionally, the global weak solution coincides with the local smooth solution before a possible singularity occurs at a finite time. See Leray [7], Temam [14] and [15]. If the initial function $\mathbf{u}_0 \in L^2(\mathbb{R}^n) \cap L^n(\mathbb{R}^n)$ and the external force $\mathbf{f} \in L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)) \cap L^1(\mathbb{R}^+, L^n(\mathbb{R}^n))$ are large, then after a long time, the global weak solutions $\mathbf{u} \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^n))$, such that $\nabla\mathbf{u} \in L^2(\mathbb{R}^+, L^2(\mathbb{R}^n))$, become sufficiently small and therefore becomes sufficiently smooth: $\mathbf{u} \in L^\infty((T,\infty), H^{2m+1}(\mathbb{R}^n))$, such that $\nabla\mathbf{u} \in L^2((T,\infty), H^{2m+1}(\mathbb{R}^n))$, for all positive integers $m \ge 1$, where $T \gg 1$ is a sufficiently large positive constant. See Leray [7] and Miyakawa and Sohr [11]. See also Heywood [3] and Kato [4]. For the partial regularity of suitable weak solutions, Caffarelli, Kohn and Nirenberg [1] proved that the one-dimensional Hausdorff measure of the set consisting of all space-time singular points is zero. Later, Lin [8] improved and simplified their proof and obtained the same partial regularity result.

Necas, Ruzicka and Sverak [16] proved that there exists no self-similar solution of the form

$$\mathbf{u}(\mathbf{x},t)=\mathbf{U}\Big(\frac{\mathbf{x}}{\sqrt{T-t}}\Big).$$

A very interesting, natural question arises: When the global weak solutions become a global smooth solution? Under appropriate conditions, there are positive solutions to this important question. First of all, if the initial function $\mathbf{u}_0\in\mathcal{S}(\mathbb{R}^n)$ and the external force $\mathbf{f}\in L^1(\mathbb{R}^+,L^2(\mathbb{R}^n))\cap C^\infty(\mathbb{R}^n\times\mathbb{R}^+)$ are reasonably small, then there exists a unique global smooth solution $\mathbf{u}\in C^\infty(\mathbb{R}^n\times\mathbb{R}^+)$. See Fujita and Kato [2] and Lei and Lin [5]. If $\mathbf{u}_0\in\mathcal{S}(\mathbb{R}^n)$ and $\mathbf{f}\in L^1(\mathbb{R}^+,L^2(\mathbb{R}^n))\cap C^\infty(\mathbb{R}^n\times\mathbb{R}^+)$, and if there exist positive constants $\lambda>n\geq 3$ and $\mu>2$, with $n/\lambda+2/\mu\leq 1$, such that

$$\int_0^\infty\Big\{\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^\lambda\mathrm{d}\mathbf{x}\Big\}^{\mu/\lambda}\mathrm{d}t<\infty,$$

then the global weak solutions become the global smooth solution

$$\mathbf{u}\in C^\infty(\mathbb{R}^n\times\mathbb{R}^+).$$

See Serrin [10].

Very recently, Lei and Lin [5], Lei, Lin and Zhou [6], Peng and Zhou [12] established very interesting results for (1)-(2), including the existence of large global smooth solution with special initial function for $n=3$. In particular, Lei, Lin and Zhou derived new uniform energy estimate by virtue of a special structure of helicity, which is crucial with respect to the natural scalings for (1). The new energy functional is coercive for a class of initial functions. Then they constructed a family of large global smooth solutions of (1)-(2) with finite energy based on the uniform energy estimate in [6].

Decay estimates with sharp rates: Suppose that the initial function $\mathbf{u}_0\in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ and the external force $\mathbf{f}\in L^1(\mathbb{R}^n\times\mathbb{R}^+)\cap L^1(\mathbb{R}^+,L^2(\mathbb{R}^n))$. Then

$$(1+t)^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\leq C_0,$$

for all time $t>0$. Moreover, there hold the following the decay estimates with sharp rates

$$\begin{aligned}
&(1+t)^{2+n/2}\int_{\mathbb{R}^n}|\nabla\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\leq C_1,\\
&(1+t)^{3+n/2}\int_{\mathbb{R}^n}|\triangle\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\leq C_2,\\
&(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\leq C_3,\\
&(1+t)^{2m+2+n/2}\int_{\mathbb{R}^n}|\nabla\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\leq C_4,
\end{aligned}$$

for all positive integers $m \geq 1$ and for all time $t \geq T$, where $C_0 > 0$, $C_1 > 0$, $C_2 > 0$, $C_3 > 0$ and $C_4 > 0$ are positive constants, independent of $\mathbf{u}$ and $(\mathbf{x}, t)$. See Oliver and Titi [9] and Schonbek and Wiegner [13].

1.3 Main motivations, main purposes, main difficulties and main strategies

Motivation: Consider the Cauchy problem for the homogeneous heat equation

$$\frac{\partial}{\partial t}\mathbf{v} - \alpha \triangle \mathbf{v} = 0, \quad \mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}),$$

where $\alpha > 0$ is a positive constant. Let the initial function $\mathbf{v}_0 \in L^1(\mathbb{R}^n)$. There exists a global smooth solution, given explicitly by

$$\mathbf{v}(\mathbf{x}, t) = \frac{1}{(4\pi\alpha t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\alpha t}\right) \mathbf{v}_0(\mathbf{y}) \mathrm{d}\mathbf{y}.$$

It is very easy to derive the representation of the Fourier transformation

$$\widehat{\mathbf{v}}(\xi, t) = \exp(-\alpha|\xi|^2 t)\widehat{\mathbf{v}}_0(\xi),$$

for all $(\xi, t) \in \mathbb{R}^n \times \mathbb{R}^+$.

Let $m \geq 0$ be any integer. Then by using Parseval's identity, the representation of the Fourier transformation, Lebesgue's dominated convergence theorem and a simple change of variables, we have the following exact limit

$$\begin{aligned}
t^{2m+n/2} \int_{\mathbb{R}^n} |\triangle^m \mathbf{v}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} &= \frac{t^{2m+n/2}}{(2\pi)^n} \int_{\mathbb{R}^n} |\xi|^{4m} |\widehat{\mathbf{v}}(\xi, t)|^2 \mathrm{d}\xi \\
&= \frac{t^{2m+n/2}}{(2\pi)^n} \int_{\mathbb{R}^n} |\xi|^{4m} \exp(-2\alpha|\xi|^2 t) |\widehat{\mathbf{v}}_0(\xi)|^2 \mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m} \exp(-2\alpha|\eta|^2) |\widehat{\mathbf{v}}_0(t^{-1/2}\eta)|^2 \mathrm{d}\eta \\
&\to \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m} \exp(-2\alpha|\eta|^2) \mathrm{d}\xi \left\{\int_{\mathbb{R}^n} \mathbf{v}_0(\mathbf{x}) \mathrm{d}\mathbf{x}\right\}^2,
\end{aligned}$$

as $t \to \infty$.

Very similarly, we have

$$\begin{aligned}
t^{2m+1+n/2} \int_{\mathbb{R}^n} |\nabla \triangle^m \mathbf{v}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} &= \frac{t^{2m+1+n/2}}{(2\pi)^n} \int_{\mathbb{R}^n} |\xi|^{4m+2} |\widehat{\mathbf{v}}(\xi, t)|^2 \mathrm{d}\xi \\
&= \frac{t^{2m+1+n/2}}{(2\pi)^n} \int_{\mathbb{R}^n} |\xi|^{4m+2} \exp(-2\alpha|\xi|^2 t) |\widehat{\mathbf{v}}_0(\xi)|^2 \mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) |\widehat{\mathbf{v}}_0(t^{-1/2}\eta)|^2 \mathrm{d}\eta \\
&\to \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \left\{\int_{\mathbb{R}^n} \mathbf{v}_0(\mathbf{x}) \mathrm{d}\mathbf{x}\right\}^2,
\end{aligned}$$

as $t \to \infty$. These limits are optimal if

$$\int_{\mathbb{R}^n} \mathbf{v}_0(\mathbf{x})\mathrm{d}\mathbf{x} \neq \mathbf{0}.$$

These results may be improved if $\int_{\mathbb{R}^n} \mathbf{v}_0(\mathbf{x})\mathrm{d}\mathbf{x} = \mathbf{0}$.

Motivated by the exact limits of the global smooth solution of the homogeneous heat equation, we will establish the exact limits of the global weak solutions of the Cauchy problem for the n-dimensional incompressible Navier-Stokes equations. Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ represent the global weak solutions of (1)-(2) corresponding to $\mathbf{u}_0 = \mathbf{u}_0(\mathbf{x})$ and $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$, and let $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ represent the global smooth solution of (3)-(4) corresponding to $\mathbf{v}_0 = \mathbf{u}_0(\mathbf{x})$ and $\mathbf{r} = \mathbf{f}(\mathbf{x}, t)$. The first purpose is to establish the following estimates

$$t^{2m+1+n/2} \int_{\mathbb{R}^n} |\triangle^m \mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \leq \mathcal{A}(m, n, \alpha, \delta, \varepsilon) + \mathcal{B}(m, n, \alpha, \delta, \varepsilon) t^{-n},$$

$$t^{2m+2+n/2} \int_{\mathbb{R}^n} |\nabla \triangle^m \mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \leq \mathcal{A}\Big(m + \frac{1}{2}, n, \alpha, \delta, \varepsilon\Big) + \mathcal{B}\Big(m + \frac{1}{2}, n, \alpha, \delta, \varepsilon\Big) t^{-n},$$

and

$$t^{2m+1+n/2} \int_{\mathbb{R}^n} |\triangle^m [\mathbf{u}(\mathbf{x}, t) - \mathbf{v}(\mathbf{x}, t)]|^2 \mathrm{d}\mathbf{x} \leq \mathcal{C}(m, n, \alpha, \delta, \varepsilon) + \mathcal{D}(m, n, \alpha, \delta, \varepsilon) t^{-n},$$

$$t^{2m+2+n/2} \int_{\mathbb{R}^n} |\nabla \triangle^m [\mathbf{u}(\mathbf{x}, t) - \mathbf{v}(\mathbf{x}, t)]|^2 \mathrm{d}\mathbf{x} \leq \mathcal{D}\Big(m + \frac{1}{2}, n, \alpha, \delta, \varepsilon\Big) + \mathcal{D}\Big(m + \frac{1}{2}, n, \alpha, \delta, \varepsilon\Big) t^{-n},$$

for all positive integers $m \geq 1$ and for all time $t \geq T$, where $\mathcal{A}(m, n, \alpha, \delta, \varepsilon)$, $\mathcal{B}(m, n, \alpha, \delta, \varepsilon)$, $\mathcal{C}(m, n, \alpha, \delta, \varepsilon)$, $\mathcal{D}(m, n, \alpha, \delta, \varepsilon)$ are positive constants, independent of time t, $0 < \delta < 1$ and $0 < \varepsilon < 1$ are small positive constants.

The main purpose of this paper is to accomplish the following exact limits

$$\lim_{t \to \infty} \left\{ (1 + t)^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \right\},$$

$$\lim_{t \to \infty} \left\{ (1 + t)^{2+n/2} \int_{\mathbb{R}^n} |\nabla \mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \right\},$$

$$\lim_{t \to \infty} \left\{ (1 + t)^{2m+1+n/2} \int_{\mathbb{R}^n} |\triangle^m \mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \right\},$$

$$\lim_{t \to \infty} \left\{ (1 + t)^{2m+2+n/2} \int_{\mathbb{R}^n} |\nabla \triangle^m \mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \right\},$$

and

$$\lim_{t \to \infty} \left\{ (1 + t)^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x}, t) - \mathbf{v}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \right\},$$

$$\lim_{t \to \infty} \left\{ (1 + t)^{2+n/2} \int_{\mathbb{R}^n} |\nabla [\mathbf{u}(\mathbf{x}, t) - \mathbf{v}(\mathbf{x}, t)]|^2 \mathrm{d}\mathbf{x} \right\},$$

$$\lim_{t\to\infty}\left\{(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\},$$
$$\lim_{t\to\infty}\left\{(1+t)^{2m+2+n/2}\int_{\mathbb{R}^n}|\nabla\triangle^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\},$$

for all positive integers $m\geq 1$, in terms of the following integrals

$$\int_{\mathbb{R}^n}\phi_{ij}(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\psi_{ij}(\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}t,$$
$$\int_{\mathbb{R}^n}\phi_{ij}(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\psi_{ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}^n}u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,$$
$$\int_0^\infty\int_{\mathbb{R}^n}u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,$$

where $\phi_{ij}=\phi_{ij}(\mathbf{x})$ are functions related to the initial function and $\psi_{ij}=\psi_{ij}(\mathbf{x},t)$ are functions related to the external force, for all $i=1,2,3,\cdots,n$ and $j=1,2,3,\cdots,n$.

Here are more motivations to study the exact limits:

(1) to study the influence of the nonlinear functions $(\mathbf{u}\cdot\nabla)\mathbf{u}$ and ∇p on the limits;

(2) to study the influence of the initial function $\mathbf{u}_0$ and the external force $\mathbf{f}$ on the limits;

(3) to study the influence of a special structure of the n-dimensional incompressible Navier-Stokes equations on the limits;

(4) to study the precise asymptotic behaviours of the energies $\int_{\mathbb{R}^n}|\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}$ and $\int_{\mathbb{R}^n}|\nabla\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}$ on (T,∞) to obtain more uniform energy estimates, where $T\gg 1$ is a sufficiently large positive constant.

One technical difficulty to study the exact limits is that the integral

$$\int_{\mathbb{R}^n}\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\mathrm{d}\eta$$

of the exponential function $\exp[-\alpha|\eta|^2(1-\frac{\tau}{t})]$ with respect to η over $\mathbb{R}^n$ is divergent at $\tau=t$, for all $t>0$. One of the main technical advances is to make use of the decay estimates with sharp rates for the derivatives of the global weak solutions and the following particular integral

$$\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{|\eta|^{n+\delta}}\mathrm{d}\eta$$

to control the integral

$$\int_{\mathbb{R}^n}|\eta|^{4m}\left|\int_{(1-\varepsilon)t}^t\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\big\{\widehat{(\mathbf{u}\cdot\nabla)}\mathbf{u}(t^{-1/2}\eta,\tau)+\widehat{\nabla p}(t^{-1/2}\eta,\tau)\big\}\mathrm{d}\tau\right|^2\mathrm{d}\eta,$$

for all positive integers $m \geq 1$ and for all sufficiently large time $t \geq T$, where $0 < \delta < 1$ and $0 < \varepsilon < 1$ are small positive constants.

The uniform energy estimates of all order derivatives of the global weak solutions for the general case (when both the initial function and the external force are large) have been open. Therefore, the existence and uniqueness of the global smooth solution $\mathbf{u} \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}^+)$ of (1)-(2) have been open. For the main purposes, we only need the existence and decay estimates with sharp rates of the local smooth solution on (T, ∞). These results are well known.

Here are the main strategies to establish the exact limits of

$$t^{2m+1+n/2} \int_{\mathbb{R}^n} |\triangle^m \mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \quad \text{and} \quad t^{2m+2+n/2} \int_{\mathbb{R}^n} |\nabla \triangle^m \mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x}, \quad \text{as } t \to \infty.$$

First of all, we will couple together the Fourier transformation, Parseval's identity, appropriate decomposition of the temporal interval $[0, t]$, Lebesgue's dominated convergence theorem and Gagliardo-Nirenberg's interpolation inequality in a new way to establish the exact limits for the global smooth solution of the Cauchy problem for the heat equation. Then we will couple together the special structure of the n-dimensional incompressible Navier-Stokes equations, some uniform energy estimates, the decay estimates with sharp rates of the derivatives of the global weak solutions for all sufficiently large time and the exact limits for the linear heat equation to accomplish the exact limits of the global weak or smooth solution of the Cauchy problem for the n-dimensional incompressible Navier-Stokes equations (1)-(2).

1.4 Mathematical assumptions

Let us make the following assumptions for the Cauchy problem for the n-dimensional incompressible Navier-Stokes equations.

(A1) Suppose that the initial function $\mathbf{u}_0 \in L^1(\mathbb{R}^n) \cap H^{2m+1}(\mathbb{R}^n)$ and the external force $\mathbf{f} \in L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)) \cap L^2(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n))$, for all positive integers $m \geq 1$.

(A2) Suppose that there exist real scalar functions $\phi_{ij} \in C^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $\psi_{ij} \in C^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^n \times \mathbb{R}^+)$, for all $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$, such that

$$\mathbf{u}_0(\mathbf{x}) = \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \phi_{1j}(\mathbf{x}), \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \phi_{2j}(\mathbf{x}), \cdots, \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \phi_{nj}(\mathbf{x}) \right),$$
$$\mathbf{f}(\mathbf{x}, t) = \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \psi_{1j}(\mathbf{x}, t), \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \psi_{2j}(\mathbf{x}, t), \cdots, \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \psi_{nj}(\mathbf{x}, t) \right),$$

for all $(\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R}^+$.

Remark 1 The assumptions in (A2) are motivated by the incompressible conditions $\nabla \cdot \mathbf{u}_0 = 0$ and $\nabla \cdot \mathbf{f} = 0$, by the integrals

$$\int_{\mathbb{R}^n} \mathbf{u}_0(\mathbf{x}) \mathrm{d}\mathbf{x} = \mathbf{0}, \quad \int_{\mathbb{R}^n} \mathbf{f}(\mathbf{x}, t) \mathrm{d}\mathbf{x} = \mathbf{0},$$

for all $t > 0$, and by the special structure of the nonlinear functions $(\mathbf{u} \cdot \nabla)\mathbf{u}$ and ∇p.

(A3) Suppose that there exist the following integrals

$$\int_0^\infty (1+t)^{2+n/2} \int_{\mathbb{R}^n} |\mathbf{f}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t < \infty,$$
$$\int_0^\infty (1+t)^{2m+2+n/2} \int_{\mathbb{R}^n} |\triangle^m \mathbf{f}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t < \infty,$$

for all positive integers $m \geq 1$.

(A4) Let $0 < \delta < 1$ and $0 < \varepsilon < 1$ be small positive constants. Suppose that there exist the following limits

$$\lim_{t\to\infty} \sum_{i=1}^n \sum_{j=1}^n \left\{ t^{(n+2+\delta)/4} \sup_{(1-\varepsilon)t \leq \tau \leq t} \int_{\mathbb{R}^n} |(-\triangle)^{(n-2+\delta)/4} \psi_{ij}(\mathbf{x}, \tau)| \mathrm{d}\mathbf{x} \right\}^2 = 0,$$
$$\lim_{t\to\infty} \sum_{i=1}^n \sum_{j=1}^n \left\{ t^{m+(n+2+\delta)/4} \sup_{(1-\varepsilon)t \leq \tau \leq t} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \psi_{ij}(\mathbf{x}, \tau)| \mathrm{d}\mathbf{x} \right\}^2 = 0,$$

for all positive integers $m \geq 1$.

Motivation for (A5). Let the initial function $\mathbf{u}_0 \in L^2(\mathbb{R}^n) \cap L^n(\mathbb{R}^n)$ and the external force $\mathbf{f} \in L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)) \cap L^1(\mathbb{R}^+, L^n(\mathbb{R}^n))$. Then there exists a global weak solution $\mathbf{u} \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^n))$, such that $\nabla \mathbf{u} \in L^2(\mathbb{R}^+, L^2(\mathbb{R}^n))$. It is well known that after a long time, the global weak solutions become sufficiently small and sufficiently smooth and there hold the decay estimates with sharp rates for the global weak solution, for all sufficiently large $t \geq T$, see Leray [7]. Without loss of generality, for the purposes of establishing the exact limits, we may assume that there exists a global smooth solution, that is, $\mathbf{u} \in L^\infty(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n))$, such that $\nabla \mathbf{u} \in L^2(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n))$, for all positive integers $m \geq 1$. However, we will not use the existence or the uniqueness of the global smooth solution in the mathematical analysis.

(A5) Suppose that there exists a unique global smooth solution to the Cauchy problem for the n-dimensional incompressible Navier-Stokes equations: $\mathbf{u} \in L^\infty(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n))$, such that $\nabla \mathbf{u} \in L^2(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n))$, for all positive integers $m \geq 1$. Moreover, suppose that there hold the following decay estimates with sharp rates

$$
\begin{aligned}
&(1+t)^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le C_0,\\
&(1+t)^{2+n/2}\int_{\mathbb{R}^n}|\nabla\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le C_1,\\
&(1+t)^{3+n/2}\int_{\mathbb{R}^n}|\triangle\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le C_2,\\
&(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le C_3,\\
&(1+t)^{2m+2+n/2}\int_{\mathbb{R}^n}|\nabla\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le C_4,
\end{aligned}
$$

for all positive integers $m\ge 1$ and for all time $t>0$, where $C_0>0$, $C_1>0$, $C_2>0$, $C_3>0$ and $C_4>0$ are positive constants, independent of $\mathbf{u}$ and $(\mathbf{x},t)$, they depend on the following integrals of the initial function and the external force:

$$
\begin{aligned}
&\int_{\mathbb{R}^n}|\mathbf{u}_0(\mathbf{x})|\mathrm{d}\mathbf{x}, && \int_0^\infty\!\!\int_{\mathbb{R}^n}|\mathbf{f}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t,\\
&\int_{\mathbb{R}^n}|\mathbf{u}_0(\mathbf{x})|^2\mathrm{d}\mathbf{x}, && \int_0^\infty\Big[\int_{\mathbb{R}^n}|\mathbf{f}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\Big]^{1/2}\mathrm{d}t,\\
&\int_{\mathbb{R}^n}|\nabla\triangle^m\mathbf{u}_0(\mathbf{x})|^2\mathrm{d}\mathbf{x}, && \int_0^\infty\Big[\int_{\mathbb{R}^n}|\triangle^m\mathbf{f}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\Big]^{1/2}\mathrm{d}t.
\end{aligned}
$$

We do not assume that the initial function $\mathbf{u}_0$ and the external force $\mathbf{f}$ are small.

1.5 Main results

Definition 1 Let $\rho>0$ be a positive constant. Define the fractional order derivative $(-\triangle)^\rho\phi$ by using the Fourier transformation

$$\widehat{(-\triangle)^\rho}\phi(\xi)\overset{\text{def}}{=}|\xi|^{2\rho}\widehat{\phi}(\xi).$$

Let $\mathbf{u}=\mathbf{u}(\mathbf{x},t)$ represent the global smooth solution of the Cauchy problem for the n-dimensional incompressible Navier-Stokes equations corresponding to the initial function $\mathbf{u}_0=\mathbf{u}_0(\mathbf{x})$ and the external force $\mathbf{f}=\mathbf{f}(\mathbf{x},t)$. Let $\mathbf{v}=\mathbf{v}(\mathbf{x},t)$ represent the global smooth solution of (3)-(4) corresponding to the initial function $\mathbf{v}_0=\mathbf{u}_0(\mathbf{x})$ and the external force $\mathbf{r}=\mathbf{f}(\mathbf{x},t)$. Let the assumptions (A1), (A2), (A3), (A4) and (A5) hold. Let $0<\delta<1$ and $0<\varepsilon<1$ be small positive constants.

Theorem 1 *There hold the following estimates for the global smooth solution of* (1)-(2)

$$t^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le\mathcal{A}(m,n,\alpha,\delta,\varepsilon)+\mathcal{B}(m,n,\alpha,\delta,\varepsilon)t^{-n},$$

and

$$t^{2m+2+n/2}\int_{\mathbb{R}^n}|\nabla\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le\mathcal{A}\Big(m+\frac{1}{2},n,\alpha,\delta,\varepsilon\Big)+\mathcal{B}\Big(m+\frac{1}{2},n,\alpha,\delta,\varepsilon\Big)t^{-n},$$

for all positive integers $m \geq 1$ and for all time $t > 0$, where

$$\begin{aligned}
&\mathcal{A}(m,n,\alpha,\delta,\varepsilon)\\
=&\frac{7}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\int_{\mathbb{R}^n}|\phi_{ij}(\mathbf{x})|\mathrm{d}\mathbf{x}\right\}^2\\
&+\frac{7}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\int_0^\infty\int_{\mathbb{R}^n}|\psi_{ij}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{14}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{7}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{|\eta|^{n+\delta}}\mathrm{d}\eta\\
&\cdot\sum_{i=1}^{n}\sum_{j=1}^{n}\sup_{t>0}\left\{t^{m+(n+2+\delta)/4}\sup_{(1-\varepsilon)t\leq\tau\leq t}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\psi_{ij}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right\}^2,\\
&\mathcal{B}(m,n,\alpha,\delta,\varepsilon)\\
=&\frac{14}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{|\eta|^{n+\delta}}\mathrm{d}\eta\\
&\cdot\sum_{i=1}^{n}\sum_{j=1}^{n}\sup_{t>0}\left\{t^{m+(3n+2+\delta)/4}\sup_{(1-\varepsilon)t\leq\tau\leq t}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}[u_i(\mathbf{x},\tau)u_j(\mathbf{x},\tau)]|\mathrm{d}\mathbf{x}\right\}^2.
\end{aligned}$$

Theorem 2 *There hold the following estimates*

$$\begin{aligned}
&t^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\\
&\leq\mathcal{C}(m,n,\alpha,\delta,\varepsilon)+\mathcal{D}(m,n,\alpha,\delta,\varepsilon)t^{-n},
\end{aligned}$$

and

$$\begin{aligned}
&t^{2m+2+n/2}\int_{\mathbb{R}^n}|\nabla\triangle^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\\
&\leq\mathcal{C}\Big(m+\frac{1}{2},n,\alpha,\delta,\varepsilon\Big)+\mathcal{D}\Big(m+\frac{1}{2},n,\alpha,\delta,\varepsilon\Big)t^{-n},
\end{aligned}$$

for all positive integers $m \geq 1$ and for all time $t > 0$, where

$$\begin{aligned}
\mathcal{C}(m,n,\alpha,\delta,\varepsilon)&=\frac{8}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,\\
\mathcal{D}(m,n,\alpha,\delta,\varepsilon)&=\frac{8}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{|\eta|^{n+\delta}}\mathrm{d}\eta\\
&\cdot\sum_{i=1}^{n}\sum_{j=1}^{n}\sup_{t>0}\left\{t^{m+(3n+2+\delta)/4}\sup_{(1-\varepsilon)t\leq\tau\leq t}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}[u_i(\mathbf{x},\tau)u_j(\mathbf{x},\tau)]|\mathrm{d}\mathbf{x}\right\}^2.
\end{aligned}$$

Remark 2 The estimates in both theorems are true for all $t > 0$ if there exists a global smooth solution, and the estimates are true for all $t \geq T \gg 1$ if there exists a global weak solution. The limits of the last terms in all of the above estimates are equal to zero, as $t \to \infty$. Note that there hold the following estimates for the global smooth solution

$$\begin{aligned}
&(1+t)^{m+(3n+2+\delta)/4} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4}[u_i(\mathbf{x},t)u_j(\mathbf{x},t)]|\mathrm{d}\mathbf{x} \\
\leq\ & C_1 \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{(1+t)^{m+1+n/2} \int_{\mathbb{R}^n} |\triangle^m [u_i(\mathbf{x},t)u_j(\mathbf{x},t)]|\mathrm{d}\mathbf{x}\right\}^{(3n+2-\delta)/(4n)} \\
&\cdot \left\{(1+t)^{m+n+1+n/2} \int_{\mathbb{R}^n} |\triangle^{m+n} [u_i(\mathbf{x},t)u_j(\mathbf{x},t)]|\mathrm{d}\mathbf{x}\right\}^{(n-2+\delta)/(4n)} \\
\leq\ & C_2 \left\{(1+t)^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\right\}^{1/2} \\
&\cdot \left\{(1+t)^{2m+1+n/2} \int_{\mathbb{R}^n} |\triangle^m \mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\right\}^{(3n+2-\delta)/(8n)} \\
&\cdot \left\{(1+t)^{2m+2n+1+n/2} \int_{\mathbb{R}^n} |\triangle^{m+n} \mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\right\}^{(n-2+\delta)/(8n)} \leq C_3,
\end{aligned}$$

for all positive integers $m \geq 1$ and for all time $t > 0$, where $C_1 > 0$, $C_2 > 0$ and $C_3 > 0$ are positive constants, independent of $\mathbf{u}$ and $(\mathbf{x}, t)$.

Definition 2 Let the assumptions (A1), (A2), (A3), (A4) and (A5) hold. Let

$$\begin{aligned}
\alpha_{ij} &\stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \phi_{ij}(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^\infty \int_{\mathbb{R}^n} \psi_{ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t, \\
\lambda_{ij} &\stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \phi_{ij}(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^\infty \int_{\mathbb{R}^n} \psi_{ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}^n} u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t, \\
\mu_{ij} &\stackrel{\text{def}}{=} \int_0^\infty \int_{\mathbb{R}^n} u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,
\end{aligned}$$

for all positive integers $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$.

Theorem 3 *There hold the following exact limits*

$$\begin{aligned}
&\lim_{t\to\infty} \left\{(1+t)^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\right\} \\
=\ & \frac{1}{n(n+2)(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^2 \exp(-2\alpha|\eta|^2)\mathrm{d}\eta \\
&\cdot \left\{ n \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij}^2 + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij}^2 + \left[\sum_{i=1}^{n} \alpha_{ii}\right]^2 - \left[\sum_{i=1}^{n} \lambda_{ii}\right]^2 \right\},
\end{aligned}$$

$$\begin{aligned}&\lim_{t\to\infty}\left\{(1+t)^{2+n/2}\int_{\mathbb{R}^n}|\nabla\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\=&\frac{1}{n(n+2)(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^4\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\\&\cdot\left\{n\sum_{i=1}^n\sum_{j=1}^n\lambda_{ij}^2+2\sum_{i=1}^n\sum_{j=1}^n\alpha_{ij}^2+\left[\sum_{i=1}^n\alpha_{ii}\right]^2-\left[\sum_{i=1}^n\lambda_{ii}\right]^2\right\},\\&\lim_{t\to\infty}\left\{(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\=&\frac{1}{n(n+2)(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\\&\cdot\left\{n\sum_{i=1}^n\sum_{j=1}^n\lambda_{ij}^2+2\sum_{i=1}^n\sum_{j=1}^n\alpha_{ij}^2+\left[\sum_{i=1}^n\alpha_{ii}\right]^2-\left[\sum_{i=1}^n\lambda_{ii}\right]^2\right\},\\&\lim_{t\to\infty}\left\{(1+t)^{2m+2+n/2}\int_{\mathbb{R}^n}|\nabla\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\=&\frac{1}{n(n+2)(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+4}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\\&\cdot\left\{n\sum_{i=1}^n\sum_{j=1}^n\lambda_{ij}^2+2\sum_{i=1}^n\sum_{j=1}^n\alpha_{ij}^2+\left[\sum_{i=1}^n\alpha_{ii}\right]^2-\left[\sum_{i=1}^n\lambda_{ii}\right]^2\right\},\end{aligned}$$

for all positive integers $m\geq 1$.

Theorem 4 *There hold the following exact limits*

$$\begin{aligned}&\lim_{t\to\infty}\left\{(1+t)^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\=&\frac{1}{n(n+2)(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^2\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\cdot\left\{n\sum_{i=1}^n\sum_{j=1}^n\mu_{ij}^2-\left[\sum_{i=1}^n\mu_{ii}\right]^2\right\},\\&\lim_{t\to\infty}\left\{(1+t)^{2+n/2}\int_{\mathbb{R}^n}|\nabla[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\}\\=&\frac{1}{n(n+2)(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^4\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\cdot\left\{n\sum_{i=1}^n\sum_{j=1}^n\mu_{ij}^2-\left[\sum_{i=1}^n\mu_{ii}\right]^2\right\},\\&\lim_{t\to\infty}\left\{(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\}\\=&\frac{1}{n(n+2)(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\cdot\left\{n\sum_{i=1}^n\sum_{j=1}^n\mu_{ij}^2-\left[\sum_{i=1}^n\mu_{ii}\right]^2\right\},\end{aligned}$$

$$\lim_{t\to\infty}\left\{(1+t)^{2m+2+n/2}\int_{\mathbb{R}^n}|\nabla\triangle^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\}$$
$$=\frac{1}{n(n+2)(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+4}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\cdot\left\{n\sum_{i=1}^n\sum_{j=1}^n\mu_{ij}^2-\left[\sum_{i=1}^n\mu_{ii}\right]^2\right\},$$

for all positive integers $m\geq 1$.

Note that we may make these results more explicit by using the precise values

$$\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta=\left(\frac{\pi}{2\alpha}\right)^{n/2}\frac{1}{(4\alpha)^{2m+1}}n(n+2)\cdots(n+4m),$$
$$\int_{\mathbb{R}^n}|\eta|^{4m+4}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta=\left(\frac{\pi}{2\alpha}\right)^{n/2}\frac{1}{(4\alpha)^{2m+2}}n(n+2)\cdots(n+4m+2).$$

2 Mathematical Analysis and Proofs of the Main Results

2.1 Linear analysis

The main purpose of this subsection is to establish the exact limits of the global smooth solution of the heat equation. Let the initial function $\mathbf{v}_0\in L^1(\mathbb{R}^n)$ and the external force $\mathbf{r}\in L^1(\mathbb{R}^n\times\mathbb{R}^+)$. Then there exists a unique global strong solution $\mathbf{v}\in C(\mathbb{R}^n\times\mathbb{R}^+)$, given explicitly by

$$\mathbf{v}(\mathbf{x},t)=\frac{1}{(4\pi\alpha t)^{n/2}}\int_{\mathbb{R}^n}\exp\left(-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\alpha t}\right)\mathbf{v}_0(\mathbf{y})\mathrm{d}\mathbf{y}$$
$$+\int_0^t\left\{\frac{1}{[4\pi\alpha(t-\tau)]^{n/2}}\int_{\mathbb{R}^n}\exp\left[-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\alpha(t-\tau)}\right]\mathbf{r}(\mathbf{y},\tau)\mathrm{d}\mathbf{y}\right\}\mathrm{d}\tau,$$

for all $\mathbf{x}\in\mathbb{R}^n$ and $t>0$. If $\mathbf{r}\in C^\infty(\mathbb{R}^n\times\mathbb{R}^+)$, then the solution $\mathbf{v}\in C^\infty(\mathbb{R}^n\times\mathbb{R}^+)$ is smooth.

Theorem 5 *Let the initial function $\mathbf{v}_0\in L^1(\mathbb{R}^n)$ and the external force $\mathbf{r}\in C^\infty(\mathbb{R}^n\times\mathbb{R}^+)\cap L^1(\mathbb{R}^n\times\mathbb{R}^+)$. Suppose that*

$$\lim_{t\to\infty}\left\{t^{(n+\delta)/4}\sup_{(1-\varepsilon)t\leq\tau\leq t}\int_{\mathbb{R}^n}|(-\triangle)^{(n-4+\delta)/4}\mathbf{r}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right\}=0,$$
$$\lim_{t\to\infty}\left\{t^{(n+2+\delta)/4}\sup_{(1-\varepsilon)t\leq\tau\leq t}\int_{\mathbb{R}^n}|(-\triangle)^{(n-2+\delta)/4}\mathbf{r}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right\}=0,$$
$$\lim_{t\to\infty}\left\{t^{m+(n+\delta)/4}\sup_{(1-\varepsilon)t\leq\tau\leq t}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-4+\delta)/4}\mathbf{r}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right\}=0,$$
$$\lim_{t\to\infty}\left\{t^{m+(n+2+\delta)/4}\sup_{(1-\varepsilon)t\leq\tau\leq t}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{r}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right\}=0.$$

Then there hold the following exact limits for the global smooth solution of the Cauchy problem for the heat equation

$$\begin{aligned}
&\lim_{t\to\infty}\left\{(1+t)^{n/2}\int_{\mathbb{R}^n}|\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\
=&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_{\mathbb{R}^n}\mathbf{v}_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\!\!\int_{\mathbb{R}^n}\mathbf{r}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,\\
&\lim_{t\to\infty}\left\{(1+t)^{1+n/2}\int_{\mathbb{R}^n}|\nabla\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\
=&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^2\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_{\mathbb{R}^n}\mathbf{v}_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\!\!\int_{\mathbb{R}^n}\mathbf{r}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,\\
&\lim_{t\to\infty}\left\{(1+t)^{2m+n/2}\int_{\mathbb{R}^n}|\triangle^m\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\
=&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_{\mathbb{R}^n}\mathbf{v}_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\!\!\int_{\mathbb{R}^n}\mathbf{r}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,\\
&\lim_{t\to\infty}\left\{(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\nabla\triangle^m\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\
=&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_{\mathbb{R}^n}\mathbf{v}_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\!\!\int_{\mathbb{R}^n}\mathbf{r}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2.
\end{aligned}$$

Theorem 6 *Let the initial function $\mathbf{v}_0\in L^1(\mathbb{R}^n)$ and the external force $\mathbf{r}\in C^\infty(\mathbb{R}^n\times\mathbb{R}^+)\cap L^1(\mathbb{R}^n\times\mathbb{R}^+)$. Suppose that there exist real scalar functions $\phi_{ij}\in C^1(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$ and $\psi_{ij}\in C^\infty(\mathbb{R}^n\times\mathbb{R}^+)\cap L^1(\mathbb{R}^n\times\mathbb{R}^+)$, for all $i=1,2,3,\cdots,n$ and $j=1,2,3,\cdots,n$, such that*

$$\mathbf{v}_0(\mathbf{x})=\left(\sum_{j=1}^n\frac{\partial}{\partial x_j}\phi_{1j}(\mathbf{x}),\sum_{j=1}^n\frac{\partial}{\partial x_j}\phi_{2j}(\mathbf{x}),\cdots,\sum_{j=1}^n\frac{\partial}{\partial x_j}\phi_{nj}(\mathbf{x})\right),$$

$$\mathbf{r}(\mathbf{x},t)=\left(\sum_{j=1}^n\frac{\partial}{\partial x_j}\psi_{1j}(\mathbf{x},t),\sum_{j=1}^n\frac{\partial}{\partial x_j}\psi_{2j}(\mathbf{x},t),\cdots,\sum_{j=1}^n\frac{\partial}{\partial x_j}\psi_{nj}(\mathbf{x},t)\right).$$

Let $0<\delta<1$ and $0<\varepsilon<1$ be small positive constants, such that

$$\lim_{t\to\infty}\sum_{i=1}^n\sum_{j=1}^n\left\{t^{(n+2+\delta)/4}\sup_{(1-\varepsilon)t\le\tau\le t}\int_{\mathbb{R}^n}|(-\triangle)^{(n-2+\delta)/4}\psi_{ij}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right\}^2=0,$$

$$\lim_{t\to\infty}\sum_{i=1}^n\sum_{j=1}^n\left\{t^{(n+4+\delta)/4}\sup_{(1-\varepsilon)t\le\tau\le t}\int_{\mathbb{R}^n}|(-\triangle)^{(n+\delta)/4}\psi_{ij}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right\}^2=0,$$

$$\lim_{t\to\infty}\sum_{i=1}^n\sum_{j=1}^n\left\{t^{m+(n+2+\delta)/4}\sup_{(1-\varepsilon)t\le\tau\le t}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\psi_{ij}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right\}^2=0,$$

$$\lim_{t\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{t^{m+(n+4+\delta)/4}\sup_{(1-\varepsilon)t\le\tau\le t}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n+\delta)/4}\psi_{ij}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right\}^2=0.$$

Then there hold the following exact limits for the solution of the Cauchy problem for the heat equation

$$\begin{aligned}
&\lim_{t\to\infty}\left\{(1+t)^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\
=&\frac{1}{n(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^2\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\cdot\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\int_{\mathbb{R}^n}\phi_{ij}(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\psi_{ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right]^2\right\},\\
&\lim_{t\to\infty}\left\{(1+t)^{2+n/2}\int_{\mathbb{R}^n}|\nabla\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\
=&\frac{1}{n(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^4\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\cdot\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\int_{\mathbb{R}^n}\phi_{ij}(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\psi_{ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right]^2\right\},\\
&\lim_{t\to\infty}\left\{(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\
=&\frac{1}{n(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\cdot\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\int_{\mathbb{R}^n}\phi_{ij}(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\psi_{ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right]^2\right\},\\
&\lim_{t\to\infty}\left\{(1+t)^{2m+2+n/2}\int_{\mathbb{R}^n}|\nabla\triangle^m\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\
=&\frac{1}{n(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+4}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\cdot\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\int_{\mathbb{R}^n}\phi_{ij}(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\psi_{ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right]^2\right\}.
\end{aligned}$$

The proofs of Theorems 5 and 6 consist of the following four lemmas.

Remark 3 Let $\mathbf{v}_0\in L^1(\mathbb{R}^n)$ and $\mathbf{r}\in L^1(\mathbb{R}^n\times\mathbb{R}^+)$. For the heat equation, there hold the following representations

$$\begin{aligned}
\widehat{\mathbf{v}}(\xi,t)&=\exp(-\alpha|\xi|^2t)\widehat{\mathbf{v}}_0(\xi)+\int_0^t\exp[-\alpha|\xi|^2(t-\tau)]\widehat{\mathbf{r}}(\xi,\tau)\mathrm{d}\tau\\
&=\exp(-\alpha|\eta|^2)\widehat{\mathbf{v}}_0(t^{-1/2}\eta)+\int_0^t\exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,
\end{aligned}$$

where $\eta=t^{1/2}\xi$ and $t>0$. Let us consider the exponential function $\exp[-\alpha|\eta|^2(1-\frac{\tau}{t})]$ of τ on the closed interval $[0,t]$. Obviously, there exists a positive lower bound for $1-\frac{\tau}{t}$ on the interval $[0,(1-\varepsilon)t]$. However, there is no positive lower bound for $1-\frac{\tau}{t}$ on $[(1-\varepsilon)t,t]$. This means that we will treat the exponential function $\exp[-\alpha|\eta|^2(1-\frac{\tau}{t})]$ very differently on the intervals $[0,(1-\varepsilon)t]$ and $[(1-\varepsilon)t,t]$.

Lemma 1 *Let* $\mathbf{v}_0 \in L^1(\mathbb{R}^n)$ *and* $\mathbf{r} \in L^1(\mathbb{R}^n \times \mathbb{R}^+)$*. Then there hold the following exact limits*

$$\lim_{t\to\infty}\int_{\mathbb{R}^n}\left|\exp(-\alpha|\eta|^2)\widehat{\mathbf{v}}_0(t^{-1/2}\eta)+\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta$$
$$=\int_{\mathbb{R}^n}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_{\mathbb{R}^n}\mathbf{v}_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\mathbf{r}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,$$

$$\lim_{t\to\infty}\int_{\mathbb{R}^n}|\eta|^2\left|\exp(-\alpha|\eta|^2)\widehat{\mathbf{v}}_0(t^{-1/2}\eta)+\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta$$
$$=\int_{\mathbb{R}^n}|\eta|^2\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_{\mathbb{R}^n}\mathbf{v}_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\mathbf{r}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,$$

$$\lim_{t\to\infty}\int_{\mathbb{R}^n}|\eta|^{4m}\left|\exp(-\alpha|\eta|^2)\widehat{\mathbf{v}}_0(t^{-1/2}\eta)+\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta$$
$$=\int_{\mathbb{R}^n}|\eta|^{4m}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_{\mathbb{R}^n}\mathbf{v}_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\mathbf{r}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,$$

$$\lim_{t\to\infty}\int_{\mathbb{R}^n}|\eta|^{4m+2}\left|\exp(-\alpha|\eta|^2)\widehat{\mathbf{v}}_0(t^{-1/2}\eta)+\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta$$
$$=\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_{\mathbb{R}^n}\mathbf{v}_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\mathbf{r}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2.$$

Proof We will apply Lebesgue's dominated convergence theorem to establish these limits. Clearly, there holds the following estimate

$$\left|\exp(-\alpha|\eta|^2)\widehat{\mathbf{v}}_0(t^{-1/2}\eta)+\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|$$
$$\le\exp(-\alpha|\eta|^2)\int_{\mathbb{R}^n}|\mathbf{v}_0(\mathbf{x})|\mathrm{d}\mathbf{x}+\exp(-\alpha\varepsilon|\eta|^2)\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{r}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t,$$

for all $\eta \in \mathbb{R}^n$ and for all $t > 0$. Note that the Fourier transformations $\widehat{\mathbf{v}}_0(t^{-1/2}\eta)$ and $\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)$ are continuous functions of η and t, for each fixed τ. Now

$$\lim_{t\to\infty}\left\{\exp(-\alpha|\eta|^2)\widehat{\mathbf{v}}_0(t^{-1/2}\eta)+\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right\}$$
$$=\exp(-\alpha|\eta|^2)\widehat{\mathbf{v}}_0(\mathbf{0})+\exp(-\alpha|\eta|^2)\int_0^\infty\widehat{\mathbf{r}}(\mathbf{0},\tau)\mathrm{d}\tau$$
$$=\exp(-\alpha|\eta|^2)\left\{\int_{\mathbb{R}^n}\mathbf{v}_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\mathbf{r}(\mathbf{x},\tau)\mathrm{d}\mathbf{x}\mathrm{d}\tau\right\}.$$

The proof of Lemma 1 is finished.

Lemma 2 *Let $\mathbf{r} \in L^1(\mathbb{R}^n \times \mathbb{R}^+)$. Then there hold the following limits*

$$\lim_{t\to\infty}\int_{\mathbb{R}^n}\left|\int_{(1-\varepsilon)t}^{t}\exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta = 0,$$

$$\lim_{t\to\infty}\int_{\mathbb{R}^n}|\eta|^2\left|\int_{(1-\varepsilon)t}^{t}\exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta = 0,$$

$$\lim_{t\to\infty}\int_{\mathbb{R}^n}|\eta|^{4m}\left|\int_{(1-\varepsilon)t}^{t}\exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta = 0,$$

$$\lim_{t\to\infty}\int_{\mathbb{R}^n}|\eta|^{4m+2}\left|\int_{(1-\varepsilon)t}^{t}\exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta = 0.$$

Proof The proof consists of several steps. First of all, for all $\eta \in \mathbb{R}^n$ and $t > 0$, we have

$$\alpha|\eta|^2\int_{(1-\varepsilon)t}^{t}\exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right]\mathrm{d}\tau = [1-\exp(-\alpha\varepsilon|\eta|^2)]t.$$

There exists the following integral

$$\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{n+\delta}}\mathrm{d}\eta < \infty,$$

due to the existence of the following limit

$$\lim_{|\eta|\to 0}\frac{1-\exp(-\alpha\varepsilon|\eta|^2)}{\alpha\varepsilon|\eta|^2} = 1.$$

Note that

$$\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{n+\delta}}\mathrm{d}\eta = \frac{(\alpha\varepsilon)^{\delta/2}}{\alpha^2}\int_{\mathbb{R}^n}\frac{[1-\exp(-|\eta|^2)]^2}{|\eta|^{n+\delta}}\mathrm{d}\eta.$$

Let $\eta = t^{1/2}\xi$. Then there hold the following estimates

$$\begin{aligned}|\eta|^{2m+(n-4+\delta)/2}t\big|\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\big| &= \big|t^{m+(n+\delta)/4}\widehat{(-\triangle)^{m+(n-4+\delta)/4}\mathbf{r}}(\xi,\tau)\big|\\ &\le t^{m+(n+\delta)/4}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-4+\delta)/4}\mathbf{r}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x},\end{aligned}$$

and if $\eta \neq \mathbf{0}$, then

$$\begin{aligned}&|\eta|^{4m}\left|\int_{(1-\varepsilon)t}^{t}\exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\\ \le\ &\frac{1}{|\eta|^{n+\delta}}\left|\int_{(1-\varepsilon)t}^{t}\left\{|\eta|^2\exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right]\right\}\left\{|\eta|^{2m+(n-4+\delta)/2}\big|\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\big|\right\}\mathrm{d}\tau\right|^2\\ \le\ &\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{n+\delta}}\left\{t^{m+(n+\delta)/4}\sup_{(1-\varepsilon)t\le\tau\le t}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-4+\delta)/4}\mathbf{r}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right\}^2.\end{aligned}$$

Therefore, there holds the following estimate

$$\int_{\mathbb{R}^n} |\eta|^{4m} \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right] \widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta$$
$$\le \int_{\mathbb{R}^n} \frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{n+\delta}}\mathrm{d}\eta \cdot \left\{ t^{m+(n+\delta)/4} \sup_{(1-\varepsilon)t\le\tau\le t} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-4+\delta)/4}\mathbf{r}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x} \right\}^2.$$

Now we have

$$\lim_{t\to\infty} \int_{\mathbb{R}^n} |\eta|^{4m} \left| \sup_{(1-\varepsilon)t\le\tau\le t} \exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right] \widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta = 0.$$

Very similarly

$$\lim_{t\to\infty} \int_{\mathbb{R}^n} |\eta|^{4m+2} \left| \sup_{(1-\varepsilon)t\le\tau\le t} \exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right] \widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta = 0.$$

The proof of Lemma 2 is finished.

Lemma 3 *Let $\mathbf{v}_0 \in L^1(\mathbb{R}^n)$ and $\mathbf{r} \in L^1(\mathbb{R}^n \times \mathbb{R}^+)$. Then there hold the following limits*

$$\lim_{t\to\infty}\int_{\mathbb{R}^n}\left|\exp(-\alpha|\eta|^2)\widehat{\mathbf{v}}_0(t^{-1/2}\eta)+\int_0^t\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta$$
$$=\lim_{t\to\infty}\int_{\mathbb{R}^n}\left|\exp(-\alpha|\eta|^2)\widehat{\mathbf{v}}_0(t^{-1/2}\eta)+\int_0^{(1-\varepsilon)t}\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta,$$
$$\lim_{t\to\infty}\int_{\mathbb{R}^n}|\eta|^2\left|\exp(-\alpha|\eta|^2)\widehat{\mathbf{v}}_0(t^{-1/2}\eta)+\int_0^t\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\right|^2\mathrm{d}\eta$$
$$=\lim_{t\to\infty}\int_{\mathbb{R}^n}|\eta|^2\left|\exp(-\alpha|\eta|^2)\widehat{\mathbf{v}}_0(t^{-1/2}\eta)+\int_0^{(1-\varepsilon)t}\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\right|^2\mathrm{d}\eta,$$
$$\lim_{t\to\infty}\int_{\mathbb{R}^n}|\eta|^{4m}\left|\exp(-\alpha|\eta|^2)\widehat{\mathbf{v}}_0(t^{-1/2}\eta)+\int_0^t\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta$$
$$=\lim_{t\to\infty}\int_{\mathbb{R}^n}|\eta|^{4m}\left|\exp(-\alpha|\eta|^2)\widehat{\mathbf{v}}_0(t^{-1/2}\eta)+\int_0^{(1-\varepsilon)t}\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta,$$
$$\lim_{t\to\infty}\int_{\mathbb{R}^n}|\eta|^{4m+2}\left|\exp(-\alpha|\eta|^2)\widehat{\mathbf{v}}_0(t^{-1/2}\eta)+\int_0^t\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta$$
$$=\lim_{t\to\infty}\int_{\mathbb{R}^n}|\eta|^{4m+2}\left|\exp(-\alpha|\eta|^2)\widehat{\mathbf{v}}_0(t^{-1/2}\eta)+\int_0^{(1-\varepsilon)t}\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta.$$

Proof We will couple together the exact limits in Lemmas 1 and 2 to finish the proof of Lemma 3. Define the following complex, vector valued, auxiliary functions

$$\mathbf{c}(\eta,t)=\exp(-\alpha|\eta|^2)\widehat{\mathbf{v}}_0(t^{-1/2}\eta)+\int_0^t \exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,$$
$$\mathbf{d}(\eta,t)=\exp(-\alpha|\eta|^2)\widehat{\mathbf{v}}_0(t^{-1/2}\eta)+\int_0^{(1-\varepsilon)t} \exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,$$

for all $\eta\in\mathbb{R}^n$ and $t>0$. Then

$$\mathbf{c}(\eta,t)-\mathbf{d}(\eta,t)=\int_{(1-\varepsilon)t}^t \exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau.$$

Therefore, there hold the following estimates and limit

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}|\eta|^{4m}|\mathbf{c}(\eta,t)|^2\mathrm{d}\eta-\int_{\mathbb{R}^n}|\eta|^{4m}|\mathbf{d}(\eta,t)|^2\mathrm{d}\eta\right|\\
=&\left|\int_{\mathbb{R}^n}|\eta|^{4m}[|\mathbf{c}(\eta,t)|^2-|\mathbf{d}(\eta,t)|^2]\mathrm{d}\eta\right|\\
=&\left|\int_{\mathbb{R}^n}|\eta|^{4m}\{|\mathbf{c}(\eta,t)-\mathbf{d}(\eta,t)|^2+[\mathbf{c}(\eta,t)-\mathbf{d}(\eta,t)]\cdot\overline{\mathbf{d}}(\eta,t)+\mathbf{d}(\eta,t)\cdot[\overline{\mathbf{c}}(\eta,t)-\overline{\mathbf{d}}(\eta,\tau)]\}\mathrm{d}\eta\right|\\
\le&\int_{\mathbb{R}^n}|\eta|^{4m}|\mathbf{c}(\eta,t)-\mathbf{d}(\eta,t)|^2\mathrm{d}\eta\\
&+2\left\{\int_{\mathbb{R}^n}|\eta|^{4m}|\mathbf{c}(\eta,t)-\mathbf{d}(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2}\left\{\int_{\mathbb{R}^n}|\eta|^{4m}|\mathbf{d}(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2}\\
\to&0,
\end{aligned}$$

as $t\to\infty$, by using the results of Lemmas 1 and 2. Therefore

$$\lim_{t\to\infty}\int_{\mathbb{R}^n}|\eta|^{4m}|\mathbf{c}(\eta,t)|^2\mathrm{d}\eta=\lim_{t\to\infty}\int_{\mathbb{R}^n}|\eta|^{4m}|\mathbf{d}(\eta,t)|^2\mathrm{d}\eta.$$

Very similarly,

$$\lim_{t\to\infty}\int_{\mathbb{R}^n}|\eta|^{4m+2}|\mathbf{c}(\eta,t)|^2\mathrm{d}\eta=\lim_{t\to\infty}\int_{\mathbb{R}^n}|\eta|^{4m+2}|\mathbf{d}(\eta,t)|^2\mathrm{d}\eta.$$

The proof of Lemma 3 is finished.

Lemma 4 *Let $\mathbf{v}_0\in L^1(\mathbb{R}^n)$ and $\mathbf{r}\in L^1(\mathbb{R}^n\times\mathbb{R}^+)$. Then there hold the following exact limits*

$$\begin{aligned}
&\lim_{t\to\infty}\left\{(1+t)^{n/2}\int_{\mathbb{R}^n}|\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\
=&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_{\mathbb{R}^n}\mathbf{v}_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\mathbf{r}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,\\
&\lim_{t\to\infty}\left\{(1+t)^{1+n/2}\int_{\mathbb{R}^n}|\nabla\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\
=&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^2\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_{\mathbb{R}^n}\mathbf{v}_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\mathbf{r}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,
\end{aligned}$$

$$\begin{aligned}
&\lim_{t\to\infty}\left\{(1+t)^{2m+n/2}\int_{\mathbb{R}^n}|\triangle^m\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\
=&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_{\mathbb{R}^n}\mathbf{v}_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\!\!\int_{\mathbb{R}^n}\mathbf{r}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,\\
&\lim_{t\to\infty}\left\{(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\nabla\triangle^m\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\
=&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_{\mathbb{R}^n}\mathbf{v}_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\!\!\int_{\mathbb{R}^n}\mathbf{r}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2.
\end{aligned}$$

Proof The Fourier transformation of the global smooth solution of the linear heat equation $\mathbf{v}_t-\alpha\triangle\mathbf{v}=\mathbf{r}(\mathbf{x},t)$ may be represented as

$$\widehat{\mathbf{v}}(\xi,t)=\exp(-\alpha|\xi|^2t)\widehat{\mathbf{v}}_0(\xi)+\int_0^t\exp[-\alpha|\xi|^2(t-\tau)]\widehat{\mathbf{r}}(\xi,\tau)\mathrm{d}\tau,$$

for all $\xi\in\mathbb{R}^n$ and for all $t>0$. By using Parseval's identity, the representation of the Fourier transformation and the change of variables $\eta=t^{1/2}\xi$, we have

$$\begin{aligned}
&t^{2m+n/2}\int_{\mathbb{R}^n}|\triangle^m\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\\
=&\frac{t^{2m+n/2}}{(2\pi)^n}\int_{\mathbb{R}^n}|\xi|^{4m}|\widehat{\mathbf{v}}(\xi,t)|^2\mathrm{d}\xi\\
=&\frac{t^{2m+n/2}}{(2\pi)^n}\int_{\mathbb{R}^n}|\xi|^{4m}\left|\exp(-\alpha|\xi|^2t)\widehat{\mathbf{v}}_0(\xi)+\int_0^t\exp[-\alpha|\xi|^2(t-\tau)]\widehat{\mathbf{r}}(\xi,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\xi\\
=&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\left|\exp(-\alpha|\eta|^2)\widehat{\mathbf{v}}_0(t^{-1/2}\eta)+\int_0^t\exp\left[-\alpha|\eta|^2\left(1-\frac{\tau}{t}\right)\right]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\\
\to&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_{\mathbb{R}^n}\mathbf{v}_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\!\!\int_{\mathbb{R}^n}\mathbf{r}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,
\end{aligned}$$

as $t\to\infty$, by using the results of Lemmas 1-3. Very similarly, we have

$$\begin{aligned}
&t^{2m+1+n/2}\int_{\mathbb{R}^n}|\nabla\triangle^m\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\\
=&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\left|\exp(-\alpha|\eta|^2)\widehat{\mathbf{v}}_0(t^{-1/2}\eta)+\int_0^t\exp\left[-\alpha|\eta|^2\left(1-\frac{\tau}{t}\right)\right]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\\
\to&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_{\mathbb{R}^n}\mathbf{v}_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\!\!\int_{\mathbb{R}^n}\mathbf{r}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,
\end{aligned}$$

as $t\to\infty$. The proof of Lemma 4 is finished.

Proof of Theorem 5 The proof is finished by coupling together the four lemmas.

Proof of Theorem 6 It is very similar to the proof of Theorem 5 and it is omitted.

Remark 4 For the following slightly more general function

$$\widehat{\mathbf{v}}_P(t^{-1/2}\eta,t)=\exp(-\alpha|\eta|^2)\widehat{\mathbf{v}}_0(t^{-1/2}\eta)+P(\eta)\int_0^t \exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right]\widehat{\mathbf{r}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,$$

where P is a homogeneous function, that is, $P(r\eta)=P(\eta)$, for all $\eta\in\mathbb{R}^n$ and $r\in\mathbb{R}$, we may establish very similar exact limits.

2.2 Nonlinear analysis

The main purpose of this subsection is to make use of the special structure of the n-dimensional incompressible Navier-Stokes equations and the linear results to accomplish the exact limits of the global smooth solution of the Cauchy problem. There exists a special structure in the Navier-Stokes equations. Note that

$$\begin{aligned}
&(\mathbf{u}\cdot\nabla)\mathbf{u}(\mathbf{x},t)+\nabla p(\mathbf{x},t)\\
=&\left(\sum_{j=1}^n\frac{\partial}{\partial x_j}[u_1(\mathbf{x},t)u_j(\mathbf{x},t)]+(-\triangle)^{-1}\frac{\partial}{\partial x_1}\left\{\sum_{k=1}^n\sum_{l=1}^n\frac{\partial^2}{\partial x_k\partial x_l}[u_k(\mathbf{x},t)u_l(\mathbf{x},t)]\right\},\right.\\
&\sum_{j=1}^n\frac{\partial}{\partial x_j}[u_2(\mathbf{x},t)u_j(\mathbf{x},t)]+(-\triangle)^{-1}\frac{\partial}{\partial x_2}\left\{\sum_{k=1}^n\sum_{l=1}^n\frac{\partial^2}{\partial x_k\partial x_l}[u_k(\mathbf{x},t)u_l(\mathbf{x},t)]\right\},\cdots,\\
&\left.\sum_{j=1}^n\frac{\partial}{\partial x_j}[u_n(\mathbf{x},t)u_j(\mathbf{x},t)]+(-\triangle)^{-1}\frac{\partial}{\partial x_n}\left\{\sum_{k=1}^n\sum_{l=1}^n\frac{\partial^2}{\partial x_k\partial x_l}[u_k(\mathbf{x},t)u_l(\mathbf{x},t)]\right\}\right)\\
=&\left(\sum_{j=1}^n\frac{\partial}{\partial x_j}\kappa_{1j}(\mathbf{x},t),\sum_{j=1}^n\frac{\partial}{\partial x_j}\kappa_{2j}(\mathbf{x},t),\cdots,\sum_{j=1}^n\frac{\partial}{\partial x_j}\kappa_{nj}(\mathbf{x},t)\right),
\end{aligned}$$

where

$$p(\mathbf{x},t)=(-\triangle)^{-1}\sum_{k=1}^n\sum_{l=1}^n\frac{\partial^2}{\partial x_k\partial x_l}[u_k(\mathbf{x},t)u_l(\mathbf{x},t)],$$

$$\kappa_{ij}(\mathbf{x},t)=u_i(\mathbf{x},t)u_j(\mathbf{x},t)+(-\triangle)^{-1}\sum_{k=1}^n\frac{\partial^2}{\partial x_i\partial x_k}[u_k(\mathbf{x},t)u_j(\mathbf{x},t)].$$

Below, we regard

$$\frac{\xi_i\xi_k\xi_l}{|\xi|^2}=0,\ \text{at }\xi=\mathbf{0},\qquad \frac{\eta_i\eta_k\eta_l}{|\eta|^2}=0,\ \text{at }\eta=\mathbf{0},$$

for all $i=1,2,3,\cdots,n$, $k=1,2,3,\cdots,n$ and $l=1,2,3,\cdots,n$.

Proof of Theorem 1 Taking the divergence of the Navier-Stokes equations yields

$$-\triangle p=\sum_{k=1}^n\sum_{l=1}^n\frac{\partial^2}{\partial x_k\partial x_l}(u_ku_l).$$

Performing the Fourier transformation to this equation and applying some simple properties lead to

$$\widehat{\nabla p}(\xi,t) = -\frac{\mathrm{i}\xi}{|\xi|^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\xi_k\xi_l\widehat{u_k u_l}(\xi,t).$$

First of all, let us rigorously derive the representation of the Fourier transformation $\widehat{\mathbf{u}}(\xi,t)$. Performing the Fourier transformation to the incompressible Navier-Stokes equations and multiplying the result by the integrating factor $\exp(\alpha|\xi|^2 t)$, integrating the result with respect to time t and then rearranging terms lead to the representation

$$\begin{aligned}\widehat{\mathbf{u}}(\xi,t) = &\exp(-\alpha|\xi|^2 t)\widehat{\mathbf{u}}_0(\xi) + \int_0^t \exp[-\alpha|\xi|^2(t-\tau)]\widehat{\mathbf{f}}(\xi,\tau)\mathrm{d}\tau\\ &-\int_0^t \exp[-\alpha|\xi|^2(t-\tau)]\widehat{(\mathbf{u}\cdot\nabla)\mathbf{u}}(\xi,\tau)\mathrm{d}\tau\\ &+\int_0^t \exp[-\alpha|\xi|^2(t-\tau)]\frac{\mathrm{i}\xi}{|\xi|^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\xi_k\xi_l\widehat{u_k u_l}(\xi,\tau)\mathrm{d}\tau.\end{aligned}$$

Let us make the change of variables $\eta = t^{1/2}\xi$, where $t>0$. Then

$$\begin{aligned}\widehat{\mathbf{u}}(t^{-1/2}\eta,t) = &\exp(-\alpha|\eta|^2)\widehat{\mathbf{u}}_0(t^{-1/2}\eta) + \int_0^t \exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\widehat{\mathbf{f}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\ &-\int_0^t \exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\widehat{(\mathbf{u}\cdot\nabla)\mathbf{u}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\ &+\int_0^t \exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]t^{-1/2}\frac{\mathrm{i}\eta}{|\eta|^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\eta_k\eta_l\widehat{u_k u_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau.\end{aligned}$$

Recall that the initial function and the external force are given by

$$\begin{aligned}\mathbf{u}_0(\mathbf{x}) &= \left(\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\phi_{1j}(\mathbf{x}),\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\phi_{2j}(\mathbf{x}),\cdots,\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\phi_{nj}(\mathbf{x})\right),\\ \mathbf{f}(\mathbf{x},t) &= \left(\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\psi_{1j}(\mathbf{x},t),\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\psi_{2j}(\mathbf{x},t),\cdots,\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\psi_{nj}(\mathbf{x},t)\right).\end{aligned}$$

Performing the Fourier transformation to these functions leads to

$$\begin{aligned}\widehat{\mathbf{u}}_0(\xi) &= \mathrm{i}\left(\sum_{j=1}^{n}\xi_j\widehat{\phi}_{1j}(\xi),\sum_{j=1}^{n}\xi_j\widehat{\phi}_{2j}(\xi),\cdots,\sum_{j=1}^{n}\xi_j\widehat{\phi}_{nj}(\xi)\right),\\ \widehat{\mathbf{f}}(\xi,t) &= \mathrm{i}\left(\sum_{j=1}^{n}\xi_j\widehat{\psi}_{1j}(\xi,t),\sum_{j=1}^{n}\xi_j\widehat{\psi}_{2j}(\xi,t),\cdots,\sum_{j=1}^{n}\xi_j\widehat{\psi}_{nj}(\xi,t)\right).\end{aligned}$$

Let us multiply the i-th component of $\widehat{\mathbf{u}}(t^{-1/2}\eta,t)$ by $t^{1/2}$. We have

$$\begin{aligned}t^{1/2}\widehat{u}_i(t^{-1/2}\eta,t)=&\,\mathrm{i}\exp(-\alpha|\eta|^2)\sum_{j=1}^{n}\eta_j\widehat{\phi}_{ij}(t^{-1/2}\eta)\\&+\mathrm{i}\int_0^t\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\sum_{j=1}^{n}\eta_j\widehat{\psi}_{ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\&-\mathrm{i}\int_0^t\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\sum_{j=1}^{n}\eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\&+\mathrm{i}\int_0^t\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau.\end{aligned}$$

By using Parseval's identity and the above representation of the Fourier transformation $\widehat{\mathbf{u}}(t^{-1/2}\eta,t)$, we have the following elementary estimates

$$\begin{aligned}&t^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\\=&\frac{t^{2m+1+n/2}}{(2\pi)^n}\int_{\mathbb{R}^n}|\xi|^{4m}|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi\\=&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}\Bigg|\exp(-\alpha|\eta|^2)\sum_{j=1}^{n}\eta_j\widehat{\phi}_{ij}(t^{-1/2}\eta)\\&+\int_0^t\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\sum_{j=1}^{n}\eta_j\widehat{\psi}_{ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\&-\int_0^t\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\sum_{j=1}^{n}\eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\&+\int_0^t\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Bigg|^2\mathrm{d}\eta\\=&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}\Bigg|\exp(-\alpha|\eta|^2)\sum_{j=1}^{n}\eta_j\widehat{\phi}_{ij}(t^{-1/2}\eta)\\&+\int_0^{(1-\varepsilon)t}\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\sum_{j=1}^{n}\eta_j\widehat{\psi}_{ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\&-\int_0^{(1-\varepsilon)t}\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\sum_{j=1}^{n}\eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\&+\int_0^{(1-\varepsilon)t}\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\end{aligned}$$

$$
\begin{aligned}
&+\int_{(1-\varepsilon)t}^{t}\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\sum_{j=1}^{n}\eta_j\widehat{\psi}_{ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&-\int_{(1-\varepsilon)t}^{t}\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\sum_{j=1}^{n}\eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&+\int_{(1-\varepsilon)t}^{t}\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Bigg|^2\mathrm{d}\eta\\
\le&\frac{7}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}\Bigg|\exp(-\alpha|\eta|^2)\sum_{j=1}^{n}\eta_j\widehat{\phi}_{ij}(t^{-1/2}\eta)\Bigg|^2\mathrm{d}\eta\\
&+\frac{7}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}\Bigg|\int_0^{(1-\varepsilon)t}\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\sum_{j=1}^{n}\eta_j\widehat{\psi}_{ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Bigg|^2\mathrm{d}\eta\\
&+\frac{7}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}\Bigg|\int_0^{(1-\varepsilon)t}\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\sum_{j=1}^{n}\eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Bigg|^2\mathrm{d}\eta\\
&+\frac{7}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}\Bigg|\int_0^{(1-\varepsilon)t}\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Bigg|^2\mathrm{d}\eta\\
&+\frac{7}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}\Bigg|\int_{(1-\varepsilon)t}^{t}\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\sum_{j=1}^{n}\eta_j\widehat{\psi}_{ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Bigg|^2\mathrm{d}\eta\\
&+\frac{7}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}\Bigg|\int_{(1-\varepsilon)t}^{t}\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\sum_{j=1}^{n}\eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Bigg|^2\mathrm{d}\eta\\
&+\frac{7}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}\Bigg|\int_{(1-\varepsilon)t}^{t}\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Bigg|^2\mathrm{d}\eta,
\end{aligned}
$$

for all $t>0$. Let us make use of the Cauchy-Schwartz's inequality to make estimates for these integrals one by one.

First of all, we have

$$
\begin{aligned}
\sum_{i=1}^{n}\Bigg|\exp(-\alpha|\eta|^2)\sum_{j=1}^{n}\eta_j\widehat{\phi}_{ij}(t^{-1/2}\eta)\Bigg|^2&\le\exp(-2\alpha|\eta|^2)\sum_{i=1}^{n}\sum_{j=1}^{n}|\eta_j|^2\sum_{j=1}^{n}\big|\widehat{\phi}_{ij}(t^{-1/2}\eta)\big|^2\\
&\le|\eta|^2\exp(-2\alpha|\eta|^2)\sum_{i=1}^{n}\sum_{j=1}^{n}\Big\{\int_{\mathbb{R}^n}|\phi_{ij}(\mathbf{x})|\mathrm{d}\mathbf{x}\Big\}^2.
\end{aligned}
$$

Therefore, there holds the following estimate for the first integral

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^{n} \left| \exp(-\alpha|\eta|^2) \sum_{j=1}^{n} \eta_j \widehat{\phi}_{ij}(t^{-1/2}\eta) \right|^2 \mathrm{d}\eta \\
&\le \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ \int_{\mathbb{R}^n} |\phi_{ij}(\mathbf{x})| \mathrm{d}\mathbf{x} \right\}^2.
\end{aligned}
$$

Secondly, it is easy to show that

$$
\begin{aligned}
&\sum_{i=1}^{n} \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right] \sum_{j=1}^{n} \eta_j \widehat{\psi}_{ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \\
&\le \exp(-2\alpha\varepsilon|\eta|^2) \sum_{i=1}^{n} \sum_{j=1}^{n} |\eta_j|^2 \sum_{j=1}^{n} \left\{ \int_0^{(1-\varepsilon)t} \big|\widehat{\psi}_{ij}(t^{-1/2}\eta,\tau)\big| \mathrm{d}\tau \right\}^2 \\
&\le |\eta|^2 \exp(-2\alpha\varepsilon|\eta|^2) \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ \int_0^{\infty}\!\!\int_{\mathbb{R}^n} |\psi_{ij}(\mathbf{x},t)| \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2.
\end{aligned}
$$

Therefore, there holds the following estimate for the second integral

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^{n} \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right] \sum_{j=1}^{n} \eta_j \widehat{\psi}_{ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
&\le \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^2) \mathrm{d}\eta \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ \int_0^{\infty}\!\!\int_{\mathbb{R}^n} |\psi_{ij}(\mathbf{x},t)| \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2.
\end{aligned}
$$

Thirdly, note that

$$
\begin{aligned}
\left| \sum_{j=1}^{n} \eta_j \widehat{u_i u_j}(t^{-1/2}\eta,\tau) \right|^2 &\le \sum_{j=1}^{n} |\eta_j|^2 \sum_{j=1}^{n} \big|\widehat{u_i u_j}(t^{-1/2}\eta,\tau)\big|^2 \\
&\le |\eta|^2 \sum_{j=1}^{n} \int_{\mathbb{R}^n} |u_i(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \int_{\mathbb{R}^n} |u_j(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \\
&\le |\eta|^2 \int_{\mathbb{R}^n} |u_i(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x},
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&\sum_{i=1}^{n} \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right] \sum_{j=1}^{n} \eta_j \widehat{u_i u_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \\
&\le |\eta|^2 \exp(-2\alpha\varepsilon|\eta|^2) \sum_{i=1}^{n} \left\{ \int_0^{(1-\varepsilon)t} \left[\int_{\mathbb{R}^n} |u_i(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x}\right]^{1/2} \left[\int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x}\right]^{1/2} \mathrm{d}\tau \right\}^2
\end{aligned}
$$

$$
\begin{aligned}
&\le |\eta|^2 \exp(-2\alpha\varepsilon|\eta|^2) \left\{ \int_0^{(1-\varepsilon)t} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x}\mathrm{d}\tau \right\}^2 \\
&\le |\eta|^2 \exp(-2\alpha\varepsilon|\eta|^2) \left\{ \int_0^{\infty} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2 .
\end{aligned}
$$

Therefore, there holds the following estimate for the third integral

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right] \sum_{j=1}^n \eta_j \widehat{u_i u_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
\le& \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta \left\{ \int_0^{\infty} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2 .
\end{aligned}
$$

Fourthly, similar to before, note that

$$
\begin{aligned}
\left| \sum_{k=1}^n \sum_{l=1}^n \eta_k\eta_l \widehat{u_k u_l}(t^{-1/2}\eta,\tau) \right|^2 &\le \sum_{k=1}^n \sum_{l=1}^n |\eta_k\eta_l|^2 \sum_{k=1}^n \sum_{l=1}^n \left|\widehat{u_k u_l}(t^{-1/2}\eta,\tau)\right|^2 \\
&\le |\eta|^4 \sum_{k=1}^n \sum_{l=1}^n \int_{\mathbb{R}^n} |u_k(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \int_{\mathbb{R}^n} |u_l(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \\
&= |\eta|^4 \left\{ \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right\}^2 .
\end{aligned}
$$

Thus

$$
\frac{1}{|\eta|^2} \left| \sum_{k=1}^n \sum_{l=1}^n \eta_k\eta_l \widehat{u_k u_l}(t^{-1/2}\eta,\tau) \right| \le \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x}.
$$

Now we get

$$
\begin{aligned}
&\sum_{i=1}^n \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n \sum_{l=1}^n \eta_k\eta_l \widehat{u_k u_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \\
\le& |\eta|^2 \exp(-2\alpha\varepsilon|\eta|^2) \left\{ \int_0^{(1-\varepsilon)t} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x}\mathrm{d}\tau \right\}^2 \\
\le& |\eta|^2 \exp(-2\alpha\varepsilon|\eta|^2) \left\{ \int_0^{\infty} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2 .
\end{aligned}
$$

Therefore, there holds the estimate for the fourth integral

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n \sum_{l=1}^n \eta_k\eta_l \widehat{u_k u_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
\le& \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta \left\{ \int_0^{\infty} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2 .
\end{aligned}
$$

Recall that $\eta = t^{1/2}\xi$, where $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^n$ and $t > 0$, we have

$$\alpha|\eta|^2 \int_{(1-\varepsilon)t}^{t} \exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\,\mathrm{d}\tau = [1-\exp(-\alpha\varepsilon|\eta|^2)]t,$$

and

$$\begin{aligned}&\sup_{(1-\varepsilon)t\le\tau\le t}\big\{|\eta|^{2m+(n-2+\delta)/2}\big|\widehat{\psi}_{ij}(t^{-1/2}\eta,\tau)\big|\big\}^2\\ &\le\left\{t^{m+(n-2+\delta)/4}\sup_{(1-\varepsilon)t\le\tau\le t}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\psi_{ij}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right\}^2.\end{aligned}$$

Very similarly

$$\begin{aligned}&\sup_{(1-\varepsilon)t\le\tau\le t}\big\{|\eta|^{2m+(n-2+\delta)/2}\big|\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\big|\big\}^2\\ &\le\left\{t^{m+(n-2+\delta)/4}\sup_{(1-\varepsilon)t\le\tau\le t}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}[u_i(\mathbf{x},\tau)u_j(\mathbf{x},\tau)]|\mathrm{d}\mathbf{x}\right\}^2,\end{aligned}$$

and

$$\begin{aligned}&\sup_{(1-\varepsilon)t\le\tau\le t}\left\{|\eta|^{2m+(n-2+\delta)/2}\left|\frac{1}{|\eta|^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\right|\right\}^2\\ &\le\sum_{k=1}^{n}\sum_{l=1}^{n}\sup_{(1-\varepsilon)t\le\tau\le t}\big||\eta|^{2m+(n-2+\delta)/2}\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\big|^2\\ &\le\sum_{k=1}^{n}\sum_{l=1}^{n}\left\{t^{m+(n-2+\delta)/4}\sup_{(1-\varepsilon)t\le\tau\le t}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}[u_k(\mathbf{x},\tau)u_l(\mathbf{x},\tau)]|\mathrm{d}\mathbf{x}\right\}^2.\end{aligned}$$

By using these estimates, we arrive at the following estimates

$$\begin{aligned}&\sum_{i=1}^{n}\left|\int_{(1-\varepsilon)t}^{t}|\eta|^{2m+(n+\delta)/2}\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\sum_{j=1}^{n}\eta_j\widehat{\psi}_{ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\\ &=\sum_{i=1}^{n}\left|\int_{(1-\varepsilon)t}^{t}|\eta|^2\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\left\{|\eta|^{2m-2+(n+\delta)/2}\sum_{j=1}^{n}\eta_j\widehat{\psi}_{ij}(t^{-1/2}\eta,\tau)\right\}\mathrm{d}\tau\right|^2\\ &\le\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2}\\ &\quad\cdot\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{t^{m+(n+2+\delta)/4}\sup_{(1-\varepsilon)t\le\tau\le t}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\psi_{ij}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right\}^2,\end{aligned}$$

$$
\begin{aligned}
&\sum_{i=1}^{n}\left|\int_{(1-\varepsilon)t}^{t}|\eta|^{2m+(n+\delta)/2}\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\sum_{j=1}^{n}\eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\\
\le&\ \frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2}\\
&\cdot\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{t^{m+(n+2+\delta)/4}\sup_{(1-\varepsilon)t\le\tau\le t}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}[u_i(\mathbf{x},\tau)u_j(\mathbf{x},\tau)]|\mathrm{d}\mathbf{x}\right\}^2
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{i=1}^{n}\left|\int_{(1-\varepsilon)t}^{t}|\eta|^{2m+(n+\delta)/2}\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\\
\le&\ \frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2}\\
&\cdot\sum_{k=1}^{n}\sum_{l=1}^{n}\left\{t^{m+(n+2+\delta)/4}\sup_{(1-\varepsilon)t\le\tau\le t}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}[u_k(\mathbf{x},\tau)u_l(\mathbf{x},\tau)]|\mathrm{d}\mathbf{x}\right\}^2.
\end{aligned}
$$

Therefore, there hold the following estimates for the fifth, the sixth and the seventh integrals

$$
\begin{aligned}
&\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}\left|\int_{(1-\varepsilon)t}^{t}\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\sum_{j=1}^{n}\eta_j\widehat{\psi}_{ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\\
\le&\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{n+\delta}}\mathrm{d}\eta\\
&\cdot\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{t^{m+(n+2+\delta)/4}\sup_{(1-\varepsilon)t\le\tau\le t}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\psi_{ij}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right\}^2,\\
&\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}\left|\int_{(1-\varepsilon)t}^{t}\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\sum_{j=1}^{n}\eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\\
\le&\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{n+\delta}}\mathrm{d}\eta\\
&\cdot\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{t^{m+(n+2+\delta)/4}\sup_{(1-\varepsilon)t\le\tau\le t}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}u_i(\mathbf{x},\tau)u_j(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right\}^2
\end{aligned}
$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^{n} \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^{n}\sum_{l=1}^{n} \eta_k\eta_l \widehat{u_k u_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
\le &\int_{\mathbb{R}^n} \frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{n+\delta}}\mathrm{d}\eta \\
&\cdot \sum_{k=1}^{n}\sum_{l=1}^{n} \left\{ t^{m+(n+2+\delta)/4} \sup_{(1-\varepsilon)t\le\tau\le t} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} u_k(\mathbf{x},\tau) u_l(\mathbf{x},\tau)|\mathrm{d}\mathbf{x} \right\}^2 .
\end{aligned}$$

Finally, by coupling together these estimates, the proof of the estimate for

$$t^{2m+1+n/2}\int_{\mathbb{R}^n} |\triangle^m \mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}$$

is finished. The proof of the estimate for

$$t^{2m+2+n/2}\int_{\mathbb{R}^n} |\nabla\triangle^m \mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}$$

is very similar and it is omitted. The proof of Theorem 1 is finished.

Proof of Theorem 2 As before, let $\mathbf{u}=\mathbf{u}(\mathbf{x},t)$ represent the global smooth solution of the Cauchy problem for the n-dimensional incompressible Navier-Stokes equations. Let $\mathbf{v}=\mathbf{v}(\mathbf{x},t)$ represent the global smooth solution to the linear heat equation. Let $\mathbf{w}=\mathbf{w}(\mathbf{x},t)=\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)$. Then

$$\begin{gathered}
\frac{\partial}{\partial t}\mathbf{w}-\alpha\triangle\mathbf{w}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=\mathbf{0}, \quad \nabla\cdot\mathbf{w}=0, \\
\mathbf{w}(\mathbf{x},0)=\mathbf{0}.
\end{gathered}$$

Similar to the above, there holds the following representation for the Fourier transformation

$$\begin{aligned}
\widehat{\mathbf{w}}(\xi,t)=&-\int_0^t \exp[-\alpha|\xi|^2(t-\tau)]\widehat{(\mathbf{u}\cdot\nabla)\mathbf{u}}(\xi,\tau)\mathrm{d}\tau \\
&+\int_0^t \exp[-\alpha|\xi|^2(t-\tau)]\frac{\mathrm{i}\xi}{|\xi|^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\xi_k\xi_l\widehat{u_k u_l}(\xi,\tau)\mathrm{d}\tau,
\end{aligned}$$

for all $\xi\in\mathbb{R}^n$ and $t>0$. Let $\eta=t^{1/2}\xi$. Then

$$\begin{aligned}
\widehat{\mathbf{w}}(t^{-1/2}\eta,t)=&-\int_0^t \exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right]\widehat{(\mathbf{u}\cdot\nabla)\mathbf{u}}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&+\int_0^t \exp\left[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\right] t^{-1/2}\frac{\mathrm{i}\eta}{|\eta|^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\eta_k\eta_l\widehat{u_k u_l}\mathbf{u}(t^{-1/2}\eta,\tau)\mathrm{d}\tau.
\end{aligned}$$

Let us multiply the i-th component of $\widehat{\mathbf{w}}$ by $t^{1/2}$ to get

$$t^{1/2}\widehat{w}_i(t^{1/2}\eta,t)=-\mathrm{i}\int_0^t \exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\sum_{j=1}^n \eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$
$$+\mathrm{i}\int_0^t \exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n \eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau.$$

The rest of the proof of Theorem 2 is very similar to that of Theorem 1 and it is omitted.

Definition 3 Let $0<\varepsilon<1$ be a small positive constant. Define

$$\begin{aligned}\Gamma_i(\eta,t)\stackrel{\text{def}}{=}\ &\exp(-\alpha|\eta|^2)\sum_{j=1}^n \eta_j\widehat{\phi}_{ij}(t^{-1/2}\eta)\\&+\int_0^{(1-\varepsilon)t} \exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\sum_{j=1}^n \eta_j\widehat{\psi}_{ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\&-\int_0^{(1-\varepsilon)t} \exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\sum_{j=1}^n \eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\&+\int_0^{(1-\varepsilon)t} \exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n \eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,\end{aligned}$$

$$\begin{aligned}\Lambda_i(\eta,t)\stackrel{\text{def}}{=}\ &\int_{(1-\varepsilon)t}^{t} \exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\sum_{j=1}^n \eta_j\widehat{\psi}_{ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\&-\int_{(1-\varepsilon)t}^{t} \exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\sum_{j=1}^n \eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\&+\int_{(1-\varepsilon)t}^{t} \exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n \eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,\end{aligned}$$

for all $\eta\in\mathbb{R}^n$ and $t>0$, and for all $i=1,2,3,\cdots,n$.

Definition 4 Let $0<\varepsilon<1$ be a small positive constant. Define

$$\begin{aligned}\Gamma_i'(\eta,t)\stackrel{\text{def}}{=}\ &-\int_0^{(1-\varepsilon)t} \exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\sum_{j=1}^n \eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\&+\int_0^{(1-\varepsilon)t} \exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n \eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,\end{aligned}$$

$$\begin{aligned}\Lambda_i'(\eta,t)\stackrel{\text{def}}{=}\ &-\int_{(1-\varepsilon)t}^{t} \exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\sum_{j=1}^n \eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\&+\int_{(1-\varepsilon)t}^{t} \exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n \eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,\end{aligned}$$

for all $\eta\in\mathbb{R}^n$ and $t>0$, and for all $i=1,2,3,\cdots,n$.

It is easy to see that

$$t^{1/2}\widehat{u}_i(t^{-1/2}\eta,t)=\mathrm{i}[\Gamma_i(\eta,t)+\Lambda_i(\eta,t)],$$

and

$$t^{1/2}\widehat{w}_i(t^{-1/2}\eta,t)=\mathrm{i}\Gamma_i'(\eta,t)+\mathrm{i}\Lambda_i'(\eta,t).$$

for all $\eta\in\mathbb{R}^n$ and $t>0$.

Lemma 5 *There holds the following result*

$$\lim_{t\to\infty}\Gamma_i(\eta,t)=\exp(-\alpha|\eta|^2)\sum_{j=1}^n\lambda_{ij}\eta_j+\exp(-\alpha|\eta|^2)\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\mu_{kl}\eta_k\eta_l.$$

Proof First of all, there holds the following estimate

$$\begin{aligned}|\Gamma_i(\eta,t)|\le{}&\exp(-\alpha|\eta|^2)\sum_{j=1}^n|\eta_j|\int_{\mathbb{R}^n}|\phi_{ij}(\mathbf{x})|\mathrm{d}\mathbf{x}\\&+\exp(-\alpha\varepsilon|\eta|^2)\sum_{j=1}^n|\eta_j|\int_0^\infty\int_{\mathbb{R}^n}|\psi_{ij}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\\&+\exp(-\alpha\varepsilon|\eta|^2)\sum_{j=1}^n|\eta_j|\int_0^\infty\int_{\mathbb{R}^n}|u_i(\mathbf{x},t)u_j(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\\&+\exp(-\alpha\varepsilon|\eta|^2)|\eta_i|\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t,\end{aligned}$$

for all $\eta\in\mathbb{R}^n$ and $t>0$. By using Lebesgue's dominated convergence theorem and the definition, as $t\to\infty$, we have the limit

$$\begin{aligned}&\lim_{t\to\infty}\Gamma_i(\eta,t)\\={}&\exp(-\alpha|\eta|^2)\sum_{j=1}^n\eta_j\widehat{\phi}_{ij}(\mathbf{0})+\exp(-\alpha|\eta|^2)\sum_{j=1}^n\eta_j\int_0^\infty\widehat{\psi}_{ij}(\mathbf{0},t)\mathrm{d}t\\&-\exp(-\alpha|\eta|^2)\sum_{j=1}^n\eta_j\int_0^\infty\widehat{u_iu_j}(\mathbf{0},t)\mathrm{d}t+\exp(-\alpha|\eta|^2)\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\eta_k\eta_l\int_0^\infty\widehat{u_ku_l}(\mathbf{0},t)\mathrm{d}t\\={}&\exp(-\alpha|\eta|^2)\sum_{j=1}^n\eta_j\left\{\int_{\mathbb{R}^n}\phi_{ij}(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\psi_{ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}^n}u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}\\&+\exp(-\alpha|\eta|^2)\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\eta_k\eta_l\int_0^\infty\int_{\mathbb{R}^n}u_k(\mathbf{x},t)u_l(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\\={}&\exp(-\alpha|\eta|^2)\sum_{j=1}^n\lambda_{ij}\eta_j+\exp(-\alpha|\eta|^2)\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\mu_{kl}\eta_k\eta_l.\end{aligned}$$

The proof of Lemma 5 is finished.

Lemma 5′ *There holds the following limit*

$$\lim_{t\to\infty}\Gamma_i'(\eta,t)=-\exp(-\alpha|\eta|^2)\sum_{j=1}^n\mu_{ij}\eta_j+\exp(-\alpha|\eta|^2)\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\mu_{kl}\mu_k\mu_l,$$

for all $i=1,2,3,\cdots,n$.

Proof There holds the following estimate

$$|\Gamma_i'(\eta,t)|\le\sum_{j=1}^n|\eta_j|\int_0^\infty\int_{\mathbb{R}^n}|u_i(\mathbf{x},t)u_j(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t+|\eta_i|\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t,$$

for all $\eta\in\mathbb{R}^n$ and $t>0$. As $t\to\infty$, similar to the above, we have the following limit

$$\begin{aligned}&-\int_0^{(1-\varepsilon)t}\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\sum_{j=1}^n\eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\&+\int_0^{(1-\varepsilon)t}\exp\Big[-\alpha|\eta|^2\Big(1-\frac{\tau}{t}\Big)\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\\to&-\exp(-\alpha|\eta|^2)\sum_{j=1}^n\eta_j\int_0^\infty\int_{\mathbb{R}^n}u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\\&+\exp(-\alpha|\eta|^2)\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\eta_k\eta_l\int_0^\infty\int_{\mathbb{R}^n}u_k(\mathbf{x},t)u_l(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\\=&-\exp(-\alpha|\eta|^2)\sum_{j=1}^n\mu_{ij}\eta_j+\exp(-\alpha|\eta|^2)\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\mu_{kl}\eta_k\eta_l,\end{aligned}$$

for all $i=1,2,3,\cdots,n$. The proof of Lemma 5′ is finished.

Lemma 6 (1) *There hold the following estimates*

$$\begin{aligned}&t^{m+(n+2+\delta)/4}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}[u_i(\mathbf{x},t)u_j(\mathbf{x},t)]|\mathrm{d}\mathbf{x}\\\le&Ct^{-n}\Big\{(1+t)^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\Big\}^{1/2}\\&\cdot\Big\{(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\Big\}^{(3n+2-\delta)/(8n)}\\&\cdot\Big\{(1+t)^{2m+2n+1+n/2}\int_{\mathbb{R}^n}|\triangle^{m+n}\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\Big\}^{(n-2+\delta)/(8n)},\end{aligned}$$

for all positive integers $m\ge1$ *and for all time* $t>0$, *where* $C>0$ *is a positive constant, independent of* $\mathbf{u}$ *and* $(\mathbf{x},t)$.

(2) *There hold the following limits*

$$\lim_{t\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{t^{(n+2+\delta)/4}\sup_{(1-\varepsilon)t\le\tau\le t}\int_{\mathbb{R}^n}|(-\triangle)^{(n-2+\delta)/4}[u_i(\mathbf{x},\tau)u_j(\mathbf{x},\tau)]|\mathrm{d}\mathbf{x}\right\}=0,$$

$$\lim_{t\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{t^{(n+4+\delta)/4}\sup_{(1-\varepsilon)t\le\tau\le t}\int_{\mathbb{R}^n}|(-\triangle)^{(n+\delta)/4}[u_i(\mathbf{x},\tau)u_j(\mathbf{x},\tau)]|\mathrm{d}\mathbf{x}\right\}=0,$$

$$\lim_{t\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{t^{m+(n+2+\delta)/4}\sup_{(1-\varepsilon)t\le\tau\le t}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}[u_i(\mathbf{x},\tau)u_j(\mathbf{x},\tau)]|\mathrm{d}\mathbf{x}\right\}=0,$$

$$\lim_{t\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{t^{m+(n+4+\delta)/4}\sup_{(1-\varepsilon)t\le\tau\le t}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n+\delta)/4}[u_i(\mathbf{x},\tau)u_j(\mathbf{x},\tau)]|\mathrm{d}\mathbf{x}\right\}=0.$$

Proof (1) First of all, by using the decay estimate with sharp rate of $\mathbf{u}$ and the decay estimates with sharp rates of the derivatives $\triangle^m\mathbf{u}$ and $\triangle^{m+n}\mathbf{u}$,

$$(1+t)^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le C_0,$$
$$(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le C_1,$$
$$(1+t)^{2m+2n+1+n/2}\int_{\mathbb{R}^n}|\triangle^{m+n}\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le C_2,$$

for all positive integers $m\ge 1$ and for all time $t>0$, where $C_0>0$, $C_1>0$ and $C_2>0$ are positive constants, independent of $\mathbf{u}$ and $(\mathbf{x},t)$, we have the following estimates

$$\begin{aligned}&(1+t)^{m+1+n/2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{\mathbb{R}^n}|\triangle^m[u_i(\mathbf{x},t)u_j(\mathbf{x},t)]|\mathrm{d}\mathbf{x}\\
\le{}& C_3(1+t)^{m+1+n/2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\int_{\mathbb{R}^n}|\triangle^m u_i(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{1/2}\left\{\int_{\mathbb{R}^n}|u_j(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{1/2}\\
&+C_4(1+t)^{m+1+n/2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\int_{\mathbb{R}^n}|u_i(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{1/2}\left\{\int_{\mathbb{R}^n}|\triangle^m u_j(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{1/2}\\
\le{}& C_5\left\{(1+t)^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{1/2}\left\{(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{1/2}\\
\le{}& C_6,\end{aligned}$$

and

$$\begin{aligned}&(1+t)^{m+n+1+n/2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{\mathbb{R}^n}|\triangle^{m+n}[u_i(\mathbf{x},t)u_j(\mathbf{x},t)]|\mathrm{d}\mathbf{x}\\
\le{}& C_7(1+t)^{m+n+1+n/2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\int_{\mathbb{R}^n}|\triangle^{m+n}u_i(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{1/2}\left\{\int_{\mathbb{R}^n}|u_j(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{1/2}\end{aligned}$$

$$+C_8(1+t)^{m+n+1+n/2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\int_{\mathbb{R}^n}|u_i(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{1/2}\left\{\int_{\mathbb{R}^n}|\triangle^{m+n}u_j(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{1/2}$$

$$\leq C_9\left\{(1+t)^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{1/2}\left\{(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^{m+n}\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{1/2}$$

$$\leq C_{10},$$

where $C_3>0$, $C_4>0$, $\cdots$, $C_9>0$ and $C_{10}>0$ are positive constants, independent of $\mathbf{u}$ and $(\mathbf{x},t)$.

Now by using the Gagliardo-Nirenberg's interpolation inequality, we have the following estimates

$$(1+t)^{m+(3n+2+\delta)/4}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}[u_i(\mathbf{x},t)u_j(\mathbf{x},t)]|\mathrm{d}\mathbf{x}$$

$$\leq C_{11}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{(1+t)^{m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m[u_i(\mathbf{x},t)u_j(\mathbf{x},t)]|\mathrm{d}\mathbf{x}\right\}^{(3n+2-\delta)/(4n)}$$

$$\cdot\left\{(1+t)^{m+n+1+n/2}\int_{\mathbb{R}^n}|\triangle^{m+n}[u_i(\mathbf{x},t)u_j(\mathbf{x},t)]|\mathrm{d}\mathbf{x}\right\}^{(n-2+\delta)/(4n)}$$

$$\leq C_{12}\left\{(1+t)^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{1/2}$$

$$\cdot\left\{(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{(3n+2-\delta)/(8n)}$$

$$\cdot\left\{(1+t)^{2m+2n+1+n/2}\int_{\mathbb{R}^n}|\triangle^{m+n}\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{(n-2+\delta)/(8n)}$$

$$\leq C_{13},$$

where $C_{11}>0$, $C_{12}>0$ and $C_{13}>0$ are positive constants, independent of $\mathbf{u}$ and $(\mathbf{x},t)$.

(2) The exact limits follow from the estimates. The proof of Lemma 6 is completely finished.

Definition 5 First of all, define the following real scalar functions

$$r_{ij}(\mathbf{x},t)\overset{\text{def}}{=}\psi_{ij}(\mathbf{x},t)-u_i(\mathbf{x},t)u_j(\mathbf{x},t)-(-\triangle)^{-1}\sum_{k=1}^{n}\frac{\partial^2}{\partial x_i\partial x_k}[u_k(\mathbf{x},t)u_j(\mathbf{x},t)],$$

for all $i=1,2,3,\cdots,n$ and $j=1,2,3,\cdots,n$. Then define the real vector valued function

$$\mathbf{r}(\mathbf{x},t)\overset{\text{def}}{=}\mathbf{f}(\mathbf{x},t)-(\mathbf{u}\cdot\nabla)\mathbf{u}(\mathbf{x},t)-\nabla(-\triangle)^{-1}\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{\partial^2}{\partial x_k\partial x_l}[u_k(\mathbf{x},t)u_l(\mathbf{x},t)].$$

Now we have

$$\mathbf{r}(\mathbf{x},t)=\left(\sum_{j=1}^{n}\frac{\partial}{\partial x_j}r_{1j}(\mathbf{x},t),\sum_{j=1}^{n}\frac{\partial}{\partial x_j}r_{2j}(\mathbf{x},t),\cdots,\sum_{j=1}^{n}\frac{\partial}{\partial x_j}r_{nj}(\mathbf{x},t)\right).$$

Let $m\geq 1$ be a positive integer. We find

$$\triangle^m\mathbf{r}(\mathbf{x},t)=\triangle^m\mathbf{f}(\mathbf{x},t)-\triangle^m[(\mathbf{u}\cdot\nabla)\mathbf{u}(\mathbf{x},t)]+\nabla\triangle^{m-1}\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{\partial^2}{\partial x_k\partial x_l}[u_k(\mathbf{x},t)u_l(\mathbf{x},t)].$$

Moreover, we may rewrite

$$\frac{\partial}{\partial t}\mathbf{u}-\alpha\triangle\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=\mathbf{f}(\mathbf{x},t),\quad \nabla\cdot\mathbf{u}=0,\nabla\cdot\mathbf{f}=0,$$
$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}),\quad \nabla\cdot\mathbf{u}_0=0,$$

as

$$\frac{\partial}{\partial t}\mathbf{u}-\alpha\triangle\mathbf{u}=\mathbf{r}(\mathbf{x},t),\quad \nabla\cdot\mathbf{u}=0,\nabla\cdot\mathbf{r}=0,$$
$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}),\quad \nabla\cdot\mathbf{u}_0=0.$$

Now we may apply the results for the linear heat equation developed in Subsection 2.1 and the results of Lemmas 5 and 6.

2.3 Precise values of several special integrals

In the proofs of Theorems 3 and 4, we will need the precise values of the following special integrals.

Lemma 7 *There hold the following results, for all positive integers $m\geq 1$, $i=1,2,3,\cdots,n$ and $j=1,2,3,\cdots,n$,*

(1)

$$\int_{\mathbb{R}^n}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta=\left(\frac{\pi}{2\alpha}\right)^{n/2},$$
$$\int_{\mathbb{R}^n}|\eta|^{4m}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta=\left(\frac{\pi}{2\alpha}\right)^{n/2}\frac{1}{(4\alpha)^{2m}}n(n+2)(n+4)\cdots(n+4m-2),$$
$$\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta=\left(\frac{\pi}{2\alpha}\right)^{n/2}\frac{1}{(4\alpha)^{2m+1}}n(n+2)(n+4)\cdots(n+4m),$$

(2)

$$\int_{\mathbb{R}^n}|\eta_i|^2\exp(-2\alpha|\eta|^2)\mathrm{d}\eta=\left(\frac{\pi}{2\alpha}\right)^{n/2}\frac{1}{4\alpha},$$
$$\int_{\mathbb{R}^n}|\eta_i|^2|\eta|^{4m}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta=\left(\frac{\pi}{2\alpha}\right)^{n/2}\frac{1}{(4\alpha)^{2m+1}}(n+2)(n+4)(n+6)\cdots(n+4m),$$
$$\int_{\mathbb{R}^n}|\eta_i|^2|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta=\left(\frac{\pi}{2\alpha}\right)^{n/2}\frac{1}{(4\alpha)^{2m+2}}(n+2)(n+4)(n+6)\cdots(n+4m+2),$$

(3)

$$\int_{\mathbb{R}^n} |\eta_i|^4 \exp(-2\alpha|\eta|^2)\mathrm{d}\eta = \Big(\frac{\pi}{2\alpha}\Big)^{n/2} \frac{3}{(4\alpha)^2},$$

$$\int_{\mathbb{R}^n} |\eta_i|^4 |\eta|^{4m} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta$$
$$= \Big(\frac{\pi}{2\alpha}\Big)^{n/2} \frac{3}{(4\alpha)^{2m+2}}(n+4)(n+6)(n+8)\cdots(n+4m+2),$$

$$\int_{\mathbb{R}^n} |\eta_i|^4 |\eta|^{4m+2} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta$$
$$= \Big(\frac{\pi}{2\alpha}\Big)^{n/2} \frac{3}{(4\alpha)^{2m+3}}(n+4)(n+6)(n+8)\cdots(n+4m+4),$$

(4) *for* $i \neq j$,

$$\int_{\mathbb{R}^n} |\eta_i|^2|\eta_j|^2 \exp(-2\alpha|\eta|^2)\mathrm{d}\eta = \Big(\frac{\pi}{2\alpha}\Big)^{n/2} \frac{1}{(4\alpha)^2},$$

$$\int_{\mathbb{R}^n} |\eta_i|^2|\eta_j|^2|\eta|^{4m} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta$$
$$= \Big(\frac{\pi}{2\alpha}\Big)^{n/2} \frac{1}{(4\alpha)^{2m+2}}(n+4)(n+6)(n+8)\cdots(n+4m+2),$$

$$\int_{\mathbb{R}^n} |\eta_i|^2|\eta_j|^2|\eta|^{4m+2} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta$$
$$= \Big(\frac{\pi}{2\alpha}\Big)^{n/2} \frac{1}{(4\alpha)^{2m+3}}(n+4)(n+6)(n+8)\cdots(n+4m+4).$$

Proof The ideas in the proof of Lemma 7 are easy but the details may be tedious. The details are omitted.

From Lemma 7, we can immediately obtain the following lemma.

Lemma 8 *There hold the following results, for all positive integers* $m \geq 1$, $i = 1, 2, 3, \cdots, n$ *and* $j = 1, 2, 3, \cdots, n$, *with* $i \neq j$,

(1)

$$\int_{\mathbb{R}^n} |\eta_i|^2 |\eta|^{4m-2} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta = \frac{1}{n}\int_{\mathbb{R}^n} |\eta|^{4m} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta$$
$$= \Big(\frac{\pi}{2\alpha}\Big)^{n/2} \frac{1}{(4\alpha)^{2m}}(n+2)(n+4)(n+6)\cdots(n+4m-2),$$

$$\int_{\mathbb{R}^n} |\eta_i|^2|\eta_j|^2|\eta|^{4m-2} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta$$
$$= \frac{1}{n(n+2)}\int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta$$
$$= \Big(\frac{\pi}{2\alpha}\Big)^{n/2} \frac{1}{(4\alpha)^{2m+1}}(n+4)(n+6)(n+8)\cdots(n+4m),$$

$$\int_{\mathbb{R}^n} |\eta_i|^4 |\eta|^{4m-2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta = \frac{3}{n(n+2)} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta$$
$$= \left(\frac{\pi}{2\alpha}\right)^{n/2} \frac{3}{(4\alpha)^{2m+1}} (n+4)(n+6)(n+8)\cdots(n+4m),$$

(2)

$$\int_{\mathbb{R}^n} |\eta_i|^2 |\eta|^{4m} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta = \frac{1}{n} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta$$
$$= \left(\frac{\pi}{2\alpha}\right)^{n/2} \frac{1}{(4\alpha)^{2m+1}} (n+2)(n+4)(n+6)\cdots(n+4m),$$
$$\int_{\mathbb{R}^n} |\eta_i|^2 |\eta_j|^2 |\eta|^{4m} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta = \frac{1}{n(n+2)} \int_{\mathbb{R}^n} |\eta|^{4m+4} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta$$
$$= \left(\frac{\pi}{2\alpha}\right)^{n/2} \frac{1}{(4\alpha)^{2m+2}} (n+4)(n+6)(n+8)\cdots(n+4m+2),$$
$$\int_{\mathbb{R}^n} |\eta_i|^4 |\eta|^{4m} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta = \frac{3}{n(n+2)} \int_{\mathbb{R}^n} |\eta|^{4m+4} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta$$
$$= \left(\frac{\pi}{2\alpha}\right)^{n/2} \frac{3}{(4\alpha)^{2m+2}} (n+4)(n+6)(n+8)\cdots(n+4m+2).$$

There are some kind of symmetry in these integrals because the results are independent of the positive integers $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$.

Lemma 9 *Let the positive integer $n \geq 2$. Let $\{\lambda_{ij} : i = 1, 2, 3, \cdots, n, j = 1, 2, 3, \cdots, n\}$, $\{\mu_{ij} : i = 1, 2, 3, \cdots, n, j = 1, 2, 3, \cdots, n\}$ be two sets of real constants, such that $\mu_{ij} = \mu_{ji}$, for all positive integers $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$. Then there hold the following results*

(1)

$$\int_{\mathbb{R}^n} \left\{\sum_{i=1}^n \lambda_{ii}|\eta_i|^2\right\} \left\{\sum_{k=1}^n \mu_{kk}|\eta_k|^2\right\} |\eta|^{4m-2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta$$
$$= \frac{1}{n(n+2)} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \left\{2\sum_{i=1}^n \lambda_{ii}\mu_{ii} + \sum_{i=1}^n \sum_{k=1}^n \lambda_{ii}\mu_{kk}\right\},$$

(2)

$$\int_{\mathbb{R}^n} \left\{\sum_{\substack{i=1\\ i\neq j}}^n \sum_{j=1}^n \lambda_{ij}\eta_i\eta_j\right\} \left\{\sum_{\substack{k=1\\ k\neq l}}^n \sum_{l=1}^n \mu_{kl}\eta_k\eta_l\right\} |\eta|^{4m-2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta$$
$$= \frac{2}{n(n+2)} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \left\{\sum_{\substack{i=1\\ i\neq j}}^n \sum_{j=1}^n \lambda_{ij}\mu_{ij}\right\},$$

(3)

$$\int_{\mathbb{R}^n}\left\{\sum_{i=1}^n\lambda_{ii}|\eta_i|^2\right\}\left\{\sum_{\substack{k=1\\k\neq l}}^n\sum_{l=1}^n\mu_{kl}\eta_k\eta_l\right\}|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta=0,$$

(4)

$$\int_{\mathbb{R}^n}\left\{\sum_{\substack{i=1\\i\neq j}}^n\sum_{j=1}^n\lambda_{ij}\eta_i\eta_j\right\}\left\{\sum_{k=1}^n\mu_{kk}|\eta_k|^2\right\}|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta=0.$$

Proof First of all, we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\left\{\sum_{i=1}^n\lambda_{ii}|\eta_i|^2\right\}\left\{\sum_{k=1}^n\mu_{kk}|\eta_k|^2\right\}|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\\
=&\int_{\mathbb{R}^n}\left\{\sum_{i=1}^n\lambda_{ii}\mu_{ii}|\eta_i|^4+\sum_{\substack{i=1\\i\neq k}}^n\sum_{k=1}^n\lambda_{ii}\mu_{kk}|\eta_i|^2|\eta_k|^2\right\}|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\\
=&\frac{1}{n(n+2)}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{3\sum_{i=1}^n\lambda_{ii}\mu_{ii}+\sum_{\substack{i=1\\i\neq k}}^n\sum_{k=1}^n\lambda_{ii}\mu_{kk}\right\}\\
=&\frac{1}{n(n+2)}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{2\sum_{i=1}^n\lambda_{ii}\mu_{ii}+\sum_{i=1}^n\sum_{k=1}^n\lambda_{ii}\mu_{kk}\right\}.\end{aligned}$$

Secondly, we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\left\{\sum_{\substack{i=1\\i\neq j}}^n\sum_{j=1}^n\lambda_{ij}\eta_i\eta_j\right\}\left\{\sum_{\substack{k=1\\k\neq l}}^n\sum_{l=1}^n\mu_{kl}\eta_k\mu_l\right\}|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\\
=&\int_{\mathbb{R}^n}\left\{\sum_{\substack{i=1\\i\neq j}}^n\sum_{j=1}^n\lambda_{ij}\eta_i\eta_j\right\}\left\{\sum_{\substack{k=1\\k\neq l}}^n\sum_{l=1}^n\underset{(k,l)=(i,j)\text{or}(k,l)=(j,i)}{\mu_{kl}\eta_k\eta_l}+\sum_{\substack{k=1\\k\neq l}}^n\sum_{l=1}^n\underset{(k,l)\neq(i,j),(k,l)\neq(j,i)}{\mu_{kl}\eta_k\eta_l}\right\}\\
&\cdot|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\\
=&\int_{\mathbb{R}^n}\left\{\sum_{\substack{i=1\\i\neq j}}^n\sum_{j=1}^n\lambda_{ij}\eta_i\eta_j\right\}\left\{\sum_{\substack{k=1\\k\neq l}}^n\sum_{l=1}^n\underset{(k,l)=(i,j)}{\mu_{kl}\eta_k\eta_l}+\sum_{\substack{k=1\\k\neq l}}^n\sum_{l=1}^n\underset{(k,l)=(j,i)}{\mu_{kl}\eta_k\eta_l}\right\}|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\end{aligned}$$

$$=\frac{2}{n(n+2)}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\sum_{\substack{i=1\\i\neq j}}^{n}\sum_{j=1}^{n}\lambda_{ij}\mu_{ij}\right\},$$

where

$$\int_{\mathbb{R}^n}\left\{\sum_{\substack{i=1\\i\neq j}}^{n}\sum_{j=1}^{n}\lambda_{ij}\eta_i\eta_j\right\}\left\{\sum_{\substack{k=1\\k\neq l}}^{n}\sum_{l=1}^{n}\mu_{kl}\eta_k\eta_l{}_{(k,l)\neq(i,j),(k,l)\neq(j,i)}\right\}|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta=0.$$

Note that

$$\int_{\mathbb{R}^n}\eta_k\eta_l|\eta_i|^2|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta=0,$$
$$\int_{\mathbb{R}^n}\eta_i\eta_j|\eta_k|^2|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta=0,$$

for all $i=1,2,3,\cdots,n$, $j=1,2,3,\cdots,n$, $k=1,2,3,\cdots,n$ and $l=1,2,3,\cdots,n$, with $i\neq j$ and $k\neq l$. Now the proofs of (3) and (4) are very easy. The proof of Lemma 9 is completely finished now.

Lemma 10 *Let*

$$\{\lambda_{ij}:i=1,2,3,\cdots,n,j=1,2,3,\cdots,n\},$$
$$\{\mu_{ij}:i=1,2,3,\cdots,n,j=1,2,3,\cdots,n\}$$

be two sets of real constants, such that $\mu_{ij}=\mu_{ji}$, for all positive integers $i=1,2,3,\cdots,n$ and $j=1,2,3,\cdots,n$. Then there hold the following results, for all positive integers $m\geq 1$,

(1)

$$\int_{\mathbb{R}^n}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{ij}\eta_i\eta_j\right\}\left\{\sum_{k=1}^{n}\sum_{l=1}^{n}\mu_{kl}\eta_k\eta_l\right\}|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta$$
$$=\frac{1}{n(n+2)}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{ii}\mu_{jj}+2\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{ij}\mu_{ij}\right\},$$

(2)

$$\int_{\mathbb{R}^n}\left\{\sum_{k=1}^{n}\sum_{l=1}^{n}\mu_{kl}\eta_k\eta_l\right\}^2|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta$$
$$=\frac{1}{n(n+2)}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\left[\sum_{k=1}^{n}\mu_{kk}\right]^2+2\sum_{k=1}^{n}\sum_{l=1}^{n}\mu_{kl}^2\right\},$$

(3)
$$\int_{\mathbb{R}^n}\sum_{i=1}^{n}\left\{\sum_{j=1}^{n}\lambda_{ij}\eta_j\right\}^2|\eta|^{4m}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta$$
$$=\frac{1}{n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{ij}^2\right\}.$$

Proof First of all, by using very simple computations, we have

$$\int_{\mathbb{R}^n}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{ij}\eta_i\eta_j\right\}\left\{\sum_{k=1}^{n}\sum_{l=1}^{n}\mu_{kl}\eta_k\eta_l\right\}|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta$$
$$=\int_{\mathbb{R}^n}\left\{\sum_{i=1}^{n}\lambda_{ii}|\eta_i|^2+\sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n}\lambda_{ij}\eta_i\eta_j\right\}$$
$$\cdot\left\{\sum_{k=1}^{n}\mu_{kk}|\eta_k|^2+\sum_{\substack{k=1\\ k\neq l}}^{n}\sum_{l=1}^{n}\mu_{kl}\eta_k\eta_l\right\}|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta$$
$$=\int_{\mathbb{R}^n}\left\{\sum_{i=1}^{n}\lambda_{ii}|\eta_i|^2\right\}\left\{\sum_{k=1}^{n}\mu_{kk}|\eta_k|^2\right\}|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta$$
$$+\int_{\mathbb{R}^n}\left\{\sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n}\lambda_{ij}\eta_i\eta_j\right\}\left\{\sum_{\substack{k=1\\ k\neq l}}^{n}\sum_{l=1}^{n}\mu_{kl}\eta_k\eta_l\right\}|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta$$
$$+\int_{\mathbb{R}^n}\left\{\sum_{i=1}^{n}\lambda_{ii}|\eta_i|^2\right\}\left\{\sum_{\substack{k=1\\ k\neq l}}^{n}\sum_{l=1}^{n}\mu_{kl}\eta_k\eta_l\right\}|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta$$
$$+\int_{\mathbb{R}^n}\left\{\sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n}\lambda_{ij}\eta_i\eta_j\right\}\left\{\sum_{k=1}^{n}\mu_{kk}|\eta_k|^2\right\}|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta.$$

Then, by using Lemma 9, we have the following computations

$$\int_{\mathbb{R}^n}\left\{\sum_{i=1}^{n}\lambda_{ii}|\eta_i|^2\right\}\left\{\sum_{k=1}^{n}\mu_{kk}|\eta_k|^2\right\}|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta$$
$$=\frac{1}{n(n+2)}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{2\sum_{i=1}^{n}\lambda_{ii}\mu_{ii}+\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{ii}\mu_{jj}\right\},$$

$$\int_{\mathbb{R}^n}\left\{\sum_{\substack{i=1\\i\neq j}}^{n}\sum_{j=1}^{n}\lambda_{ij}\eta_i\eta_j\right\}\left\{\sum_{\substack{k=1\\k\neq l}}^{n}\sum_{l=1}^{n}\mu_{kl}\eta_k\eta_l\right\}|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta$$

$$=\frac{2}{n(n+2)}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\sum_{\substack{i=1\\i\neq j}}^{n}\sum_{j=1}^{n}\lambda_{ij}\mu_{ij}\right\},$$

$$\int_{\mathbb{R}^n}\left\{\sum_{i=1}^{n}\lambda_{ii}|\eta_i|^2\right\}\left\{\sum_{\substack{k=1\\k\neq l}}^{n}\sum_{l=1}^{n}\mu_{kl}\eta_k\eta_l\right\}|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta=0,$$

and

$$\int_{\mathbb{R}^n}\left\{\sum_{\substack{i=1\\i\neq j}}^{n}\sum_{j=1}^{n}\lambda_{ij}\eta_i\eta_j\right\}\left\{\sum_{k=1}^{n}\mu_{kk}|\eta_k|^2\right\}|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta=0.$$

Therefore, by coupling together these results, we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{ij}\eta_i\eta_j\right\}\left\{\sum_{k=1}^{n}\sum_{l=1}^{n}\mu_{kl}\eta_k\eta_l\right\}|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\\
=&\frac{1}{n(n+2)}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{2\sum_{i=1}^{n}\lambda_{ii}\mu_{ii}+\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{ii}\mu_{jj}\right\}\\
&+\frac{2}{n(n+2)}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\sum_{\substack{i=1\\i\neq j}}^{n}\sum_{j=1}^{n}\lambda_{ij}\mu_{ij}\right\}\\
=&\frac{1}{n(n+2)}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{ii}\mu_{jj}+2\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{ij}\mu_{ij}\right\}.\end{aligned}$$

The second result follows from the first result. Now let us prove the third result. We have

$$\begin{aligned}&\int_{\mathbb{R}^n}\sum_{i=1}^{n}\left\{\sum_{j=1}^{n}\lambda_{ij}\eta_j\right\}^2|\eta|^{4m}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\\
=&\int_{\mathbb{R}^n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\lambda_{ij}\lambda_{ik}\eta_j\eta_k|\eta|^{4m}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_{ij}^2 |\eta_j|^2 |\eta|^{4m} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta \\
&\quad + \int_{\mathbb{R}^n} \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq k}}^{n}\sum_{k=1}^{n} \lambda_{ij}\lambda_{ik}\eta_j\eta_k |\eta|^{4m} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta \\
&= \frac{1}{n}\int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta \left\{ \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_{ij}^2 \right\},
\end{aligned}
$$

where

$$
\int_{\mathbb{R}^n} \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq k}}^{n}\sum_{k=1}^{n} \lambda_{ij}\lambda_{ik}\eta_j\eta_k |\eta|^{4m} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta = 0.
$$

The proof of Lemma 10 is finished.

2.4 Completion of the proofs of the main theorems

The main purpose of this subsection is to make complete use of the technical lemmas to finish the proofs of the exact limits of the global smooth solution of the Cauchy problem for the n-dimensional incompressible Navier-Stokes equations.

First of all, let us prove Theorem 3. As before, define the following three sets of real numbers

$$
\begin{aligned}
&\{\alpha_{ij} : i = 1,2,3,\cdots,n, j = 1,2,3,\cdots,n\},\\
&\{\lambda_{ij} : i = 1,2,3,\cdots,n, j = 1,2,3,\cdots,n\},\\
&\{\mu_{ij} : i = 1,2,3,\cdots,n, j = 1,2,3,\cdots,n\},
\end{aligned}
$$

by

$$
\begin{aligned}
\alpha_{ij} &= \int_{\mathbb{R}^n} \phi_{ij}(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^\infty\int_{\mathbb{R}^n} \psi_{ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,\\
\lambda_{ij} &= \int_{\mathbb{R}^n} \phi_{ij}(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^\infty\int_{\mathbb{R}^n} \psi_{ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t - \int_0^\infty\int_{\mathbb{R}^n} u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,\\
\mu_{ij} &= \int_0^\infty\int_{\mathbb{R}^n} u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,
\end{aligned}
$$

for all positive integers $i = 1,2,3,\cdots,n$ and $j = 1,2,3,\cdots,n$. Obviously, $\alpha_{ij} = \lambda_{ij} + \mu_{ij}$ and $\mu_{ij} = \mu_{ji}$.

Recall that there holds the following decay estimate

$$
(1+t)^{1+n/2}\int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x} \le C,
$$

for all time $t > 0$, where $C > 0$ is a positive constant, independent of $\mathbf{u}$ and $(\mathbf{x},t)$. Therefore, the existence of the following integrals are guaranteed

$$\int_0^\infty\int_{\mathbb{R}^n} u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,$$

for all $i=1,2,3,\cdots,n$ and $j=1,2,3,\cdots,n$.

Lemma 11 *Let $m\geq 1$ be a positive integer. There holds the following result*

$$\begin{aligned}&\int_{\mathbb{R}^n}\sum_{i=1}^n\left\{\sum_{j=1}^n\lambda_{ij}\eta_j+\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\mu_{kl}\eta_k\eta_l\right\}^2|\eta|^{4m}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\\ =&\frac{1}{n(n+2)}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\\ &\cdot\left\{n\sum_{i=1}^n\sum_{j=1}^n\lambda_{ij}^2+2\sum_{i=1}^n\sum_{j=1}^n\alpha_{ij}^2+\left[\sum_{i=1}^n\alpha_{ii}\right]^2-\left[\sum_{i=1}^n\lambda_{ii}\right]^2\right\}.\end{aligned}$$

Proof By using Lemma 10, we have the following computations

$$\begin{aligned}&\int_{\mathbb{R}^n}\sum_{i=1}^n\left\{\sum_{j=1}^n\lambda_{ij}\eta_j+\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\mu_{kl}\eta_k\eta_l\right\}^2|\eta|^{4m}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\\ =&\int_{\mathbb{R}^n}\sum_{i=1}^n\left\{\sum_{j=1}^n\lambda_{ij}\eta_j\right\}^2|\eta|^{4m}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\\ &+\int_{\mathbb{R}^n}\left\{\sum_{k=1}^n\sum_{l=1}^n\mu_{kl}\eta_k\eta_l\right\}^2|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\\ &+2\int_{\mathbb{R}^n}\left\{\sum_{i=1}^n\sum_{j=1}^n\lambda_{ij}\eta_i\eta_j\right\}\left\{\sum_{k=1}^n\sum_{l=1}^n\mu_{kl}\eta_k\eta_l\right\}|\eta|^{4m-2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\\ =&\frac{1}{n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\sum_{i=1}^n\sum_{j=1}^n\lambda_{ij}^2\right\}\\ &+\frac{1}{n(n+2)}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\left[\sum_{k=1}^n\mu_{kk}\right]^2+2\sum_{k=1}^n\sum_{l=1}^n\mu_{kl}^2\right\}\\ &+\frac{2}{n(n+2)}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\sum_{i=1}^n\sum_{j=1}^n\lambda_{ii}\mu_{jj}+2\sum_{i=1}^n\sum_{j=1}^n\lambda_{ij}\mu_{ij}\right\}\\ =&\frac{1}{n(n+2)}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\\ &\cdot\left\{(n+2)\sum_{i=1}^n\sum_{j=1}^n\lambda_{ij}^2+\left[\sum_{k=1}^n\mu_{kk}\right]^2+2\sum_{k=1}^n\sum_{l=1}^n\mu_{kl}^2\right.\\ &\left.+2\sum_{i=1}^n\sum_{j=1}^n\lambda_{ii}\mu_{jj}+4\sum_{i=1}^n\sum_{j=1}^n\lambda_{ij}\mu_{ij}\right\}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n(n+2)} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \\
&\quad \cdot \left\{ n \sum_{i=1}^n \sum_{j=1}^n \lambda_{ij}^2 + 2 \sum_{i=1}^n \sum_{j=1}^n (\lambda_{ij} + \mu_{ij})^2 + \left[\sum_{i=1}^n (\lambda_{ii} + \mu_{ii}) \right]^2 - \left[\sum_{i=1}^n \lambda_{ii} \right]^2 \right\} \\
&= \frac{1}{n(n+2)} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \\
&\quad \cdot \left\{ n \sum_{i=1}^n \sum_{j=1}^n \lambda_{ij}^2 + 2 \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij}^2 + \left[\sum_{i=1}^n \alpha_{ii} \right]^2 - \left[\sum_{i=1}^n \lambda_{ii} \right]^2 \right\}.
\end{aligned}
$$

The proof of Lemma 11 is finished.

Lemma 12 *There holds the following result*

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \sum_{i=1}^n \left\{ -\sum_{j=1}^n \mu_{ij}\eta_j + \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n \sum_{l=1}^n \mu_{kl}\eta_k\eta_l \right\}^2 |\eta|^{4m} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \\
&= \frac{1}{n(n+2)} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \left\{ n \sum_{i=1}^n \sum_{j=1}^n \mu_{ij}^2 - \left[\sum_{i=1}^n \mu_{ii} \right]^2 \right\}.
\end{aligned}
$$

Proof As before, by using Lemma 10, we have the following computations

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \sum_{i=1}^n \left\{ -\sum_{j=1}^n \mu_{ij}\eta_j + \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n \sum_{l=1}^n \mu_{kl}\eta_k\eta_l \right\}^2 |\eta|^{4m} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \\
&= \int_{\mathbb{R}^n} \sum_{i=1}^n \left\{ \sum_{j=1}^n \mu_{ij}\eta_j \right\}^2 |\eta|^{4m} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \\
&\quad + \int_{\mathbb{R}^n} \left\{ \sum_{k=1}^n \sum_{l=1}^n \mu_{kl}\eta_k\mu_l \right\}^2 |\eta|^{4m-2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \\
&\quad -2 \int_{\mathbb{R}^n} \left\{ \sum_{i=1}^n \sum_{j=1}^n \mu_{ij}\eta_i\eta_j \right\} \left\{ \sum_{k=1}^n \sum_{l=1}^n \mu_{kl}\eta_k\eta_l \right\} |\eta|^{4m-2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \\
&= \frac{1}{n} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \left\{ \sum_{i=1}^n \sum_{j=1}^n \mu_{ij}^2 \right\} \\
&\quad + \frac{1}{n(n+2)} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \left\{ \left[\sum_{i=1}^n \mu_{ii} \right]^2 + 2 \sum_{i=1}^n \sum_{j=1}^n \mu_{ij}^2 \right\} \\
&\quad - \frac{2}{n(n+2)} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \left\{ \left[\sum_{i=1}^n \mu_{ii} \right]^2 + 2 \sum_{i=1}^n \sum_{j=1}^n \mu_{ij}^2 \right\} \\
&= \frac{1}{n(n+2)} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \left\{ n \sum_{i=1}^n \sum_{j=1}^n \mu_{ij}^2 - \left[\sum_{i=1}^n \mu_{ii} \right]^2 \right\}.
\end{aligned}
$$

The proof of Lemma 12 is finished.

Proof of Theorem 3 By coupling together the results of Lemma 5, Lemma 6, Lemma 11 and Theorem 6, the proof is completed.

Proof of Theorem 4 By coupling together the results of Lemma 5′, Lemma 6, Lemma 12 and Theorem 6, the proof is finished.

Therefore, the proofs of the exact limits in Theorems 3 and 4 are finished.

3 Conclusions and Remarks

3.1 Summary

Consider the n-dimensional incompressible Navier-Stokes equations. Let the initial function $\mathbf{u}_0 \in L^1(\mathbb{R}^n) \cap L^n(\mathbb{R}^n) \cap H^{2m+1}(\mathbb{R}^n)$ and the external force $\mathbf{f} \in L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)) \cap L^1(\mathbb{R}^+, L^n(\mathbb{R}^n)) \cap L^2(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n))$. Then there exists a global weak solution. The global weak solutions become sufficiently small and sufficiently smooth after a long time, see [7]. Therefore, the mathematical analysis and results are rigorously true for all large time $t \geq T$.

Without loss of generality, suppose that there hold the following uniform energy estimates

$$\int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \leq C_0,$$

$$\int_{\mathbb{R}^n} |\nabla \mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \leq C_1,$$

$$\int_{\mathbb{R}^n} |\triangle \mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \leq C_2,$$

$$\int_{\mathbb{R}^n} |\triangle^m \mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \leq C_3$$

$$\int_{\mathbb{R}^n} |\nabla \triangle^m \mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \leq C_4,$$

for all $t > 0$, where $C_0 > 0$, $C_1 > 0$, $C_2 > 0$, $C_3 > 0$ and $C_4 > 0$ are positive constants, independent of $\mathbf{u}$ and $(\mathbf{x}, t)$.

Then there hold the decay estimates with sharp rates

$$(1+t)^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \leq C_0,$$

$$(1+t)^{2+n/2} \int_{\mathbb{R}^n} |\nabla \mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \leq C_1,$$

$$(1+t)^{3+n/2} \int_{\mathbb{R}^n} |\triangle \mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \leq C_2,$$

$$(1+t)^{2m+1+n/2} \int_{\mathbb{R}^n} |\triangle^m \mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \leq C_3,$$

$$(1+t)^{2m+2+n/2} \int_{\mathbb{R}^n} |\nabla \triangle^m \mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \leq C_4,$$

for all time $t > 0$, where $C_0 > 0$, $C_1 > 0$ $C_2 > 0$, $C_3 > 0$ and $C_4 > 0$ are positive constants, independent of $\mathbf{u}$ and $(\mathbf{x}, t)$.

By coupling together several important traditional ideas (Fourier transformation, Parseval's identity, Lebesgue's dominated convergence theorem, Gagliardo-Nirenberg's interpolation inequality), existing results (the existence of the global weak solutions, the existence of the local smooth solution and the decay estimates with sharp rates) and new idea (more appropriate coupling of existing methods and results), we have developed a new method to establish the estimates

$$t^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le\mathcal{A}(m,n,\alpha,\delta,\varepsilon)+\mathcal{B}(m,n,\alpha,\delta,\varepsilon)t^{-n},$$
$$t^{2m+2+n/2}\int_{\mathbb{R}^n}|\nabla\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le\mathcal{A}\Big(m+\frac{1}{2},n,\alpha,\delta,\varepsilon\Big)+\mathcal{B}\Big(m+\frac{1}{2},n,\alpha,\delta,\varepsilon\Big)t^{-n},$$

and

$$\begin{aligned}&t^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\\ \le\,&\mathcal{C}(m,n,\alpha,\delta,\varepsilon)+\mathcal{D}(m,n,\alpha,\delta,\varepsilon)t^{-n},\\ &t^{2m+2+n/2}\int_{\mathbb{R}^n}|\nabla\triangle^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\\ \le\,&\mathcal{C}\Big(m+\frac{1}{2},n,\alpha,\delta,\varepsilon\Big)+\mathcal{D}\Big(m+\frac{1}{2},n,\alpha,\delta,\varepsilon\Big)t^{-n},\end{aligned}$$

for all positive integers $m \ge 1$ and for all sufficiently large time $t \ge T$, and to accomplish the exact limits

$$\lim_{t\to\infty}\left\{(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\},$$
$$\lim_{t\to\infty}\left\{(1+t)^{2m+2+n/2}\int_{\mathbb{R}^n}|\nabla\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\},$$
$$\lim_{t\to\infty}\left\{(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\},$$
$$\lim_{t\to\infty}\left\{(1+t)^{2m+2+n/2}\int_{\mathbb{R}^n}|\nabla\triangle^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\},$$

in terms of α, m, n and the following integrals

$$\alpha_{ij}=\int_{\mathbb{R}^n}\phi_{ij}(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\psi_{ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,$$
$$\lambda_{ij}=\int_{\mathbb{R}^n}\phi_{ij}(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\psi_{ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}^n}u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,$$
$$\mu_{ij}=\int_0^\infty\int_{\mathbb{R}^n}u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,$$

for all positive integers $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$.

3.2 Remarks

There are many remarks to the main results.

Remark 5 If

$$\alpha_{ij} = \lambda_{ij}, \quad \mu_{ij} = 0,$$

for all $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$, then

$$\begin{aligned} & n\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{ij}^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}^2 + \left[\sum_{i=1}^{n}\alpha_{ii}\right]^2 - \left[\sum_{i=1}^{n}\lambda_{ii}\right]^2 \\ = & n\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{ij}^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}^2 = (n+2)\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}^2. \end{aligned}$$

That means that the exact limits of the global smooth solution to the Navier-Stokes equations reduce to those of the global smooth solution to the linear heat equation, if we drop the nonlinear terms $(\mathbf{u} \cdot \nabla)\mathbf{u}$ and ∇p.

Remark 6 The exact limits are increasing functions of the order m of the derivative and the dimension n, they are decreasing functions of the diffusion coefficient α. Let us consider more carefully the contributions made by various terms to the exact limit of the energy

$$(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}, \quad \text{as } t \to \infty.$$

The contributions made by the initial function and the external force are represented by α_{ij}, the contributions made by the nonlinear functions $(\mathbf{u} \cdot \nabla)\mathbf{u}$ and ∇p are represented by μ_{ij}. Let

$$P = n\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{ij}^2 - \left[\sum_{i=1}^{n}\lambda_{ii}\right]^2, \quad Q = 2\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}^2 + \left[\sum_{i=1}^{n}\alpha_{ii}\right]^2.$$

The exact limits are increasing functions of P and Q. Compared with the heat equation, the nonlinear functions $(\mathbf{u} \cdot \nabla)\mathbf{u}$ and ∇p make the exact limits larger.

As the order of the derivative increases, the value of the corresponding exact limit may also increase. This motivates the next definition.

Definition 6 Define the following ratios

$$\begin{aligned} R_{m,1} \stackrel{\text{def}}{=} & \left\{\lim_{t\to\infty}(1+t)^{2m+2+n/2}\int_{\mathbb{R}^n}|\nabla\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\} \\ & \cdot\left\{\lim_{t\to\infty}(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{-1}, \\ R_{m,2} \stackrel{\text{def}}{=} & \left\{\lim_{t\to\infty}(1+t)^{2m+3+n/2}\int_{\mathbb{R}^n}|\triangle^{m+1}\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\} \\ & \cdot\left\{\lim_{t\to\infty}(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{-1}. \end{aligned}$$

They measure how fast the values of the exact limits increase as the order of the derivative increases. Note that

$$\int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta = \Big(\frac{\pi}{2\alpha}\Big)^{n/2} \frac{1}{(4\alpha)^{2m+1}} n(n+2)\cdots(n+4m),$$
$$\int_{\mathbb{R}^n} |\eta|^{4m+4} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta = \Big(\frac{\pi}{2\alpha}\Big)^{n/2} \frac{1}{(4\alpha)^{2m+2}} n(n+2)\cdots(n+4m+2).$$

Remark 7 For the global smooth solution of the linear heat equation, we have

$$\begin{aligned}
R_{m,1} =& \Big\{\lim_{t\to\infty}(1+t)^{2m+2+n/2}\int_{\mathbb{R}^n}|\nabla\triangle^m \mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\Big\} \\
&\cdot\Big\{\lim_{t\to\infty}(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m \mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\Big\}^{-1} \\
=& \Big\{\int_{\mathbb{R}^n}|\eta|^{4m+4}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\Big\}\Big\{\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\Big\}^{-1} \\
=& \frac{n+4m+2}{4\alpha}, \\
R_{m,2} =& \Big\{\lim_{t\to\infty}(1+t)^{2m+3+n/2}\int_{\mathbb{R}^n}|\triangle^{m+1} \mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\Big\} \\
&\cdot\Big\{\lim_{t\to\infty}(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m \mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\Big\}^{-1} \\
=& \Big\{\int_{\mathbb{R}^n}|\eta|^{4m+6}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\Big\}\Big\{\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\Big\}^{-1} \\
=& \frac{(n+4m+2)(n+4m+4)}{(4\alpha)^2}.
\end{aligned}$$

Remark 8 For the global smooth solution to the Navier-Stokes equations, we also have

$$\begin{aligned}
R_{m,1} =& \Big\{\lim_{t\to\infty}(1+t)^{2m+2+n/2}\int_{\mathbb{R}^n}|\nabla\triangle^m \mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\Big\} \\
&\cdot\Big\{\lim_{t\to\infty}(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m \mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\Big\}^{-1} \\
=& \Big\{\int_{\mathbb{R}^n}|\eta|^{4m+4}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\Big\}\Big\{\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\Big\}^{-1} \\
=& \frac{n+4m+2}{4\alpha},
\end{aligned}$$

$$
\begin{aligned}
R_{m,2} &= \left\{ \lim_{t\to\infty} (1+t)^{2m+3+n/2} \int_{\mathbb{R}^n} |\triangle^{m+1}\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\} \\
&\quad \cdot \left\{ \lim_{t\to\infty} (1+t)^{2m+1+n/2} \int_{\mathbb{R}^n} |\triangle^{m}\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\}^{-1} \\
&= \left\{ \int_{\mathbb{R}^n} |\eta|^{4m+6} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta \right\} \left\{ \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta \right\}^{-1} \\
&= \frac{(n+4m+2)(n+4m+4)}{(4\alpha)^2}.
\end{aligned}
$$

Remark 9 Even though we did not find new structures in the Navier-Stokes equations, we did find a very interesting fact. That is, the ratios of the exact limits for both the heat equation and the Navier-Stokes equations are the same, as evidenced by the following computations. Note that the ratios are

$$
\begin{aligned}
R_{m,1} &= \left\{ \lim_{t\to\infty} (1+t)^{2m+2+n/2} \int_{\mathbb{R}^n} |\nabla\triangle^{m}\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\} \\
&\quad \cdot \left\{ \lim_{t\to\infty} (1+t)^{2m+1+n/2} \int_{\mathbb{R}^n} |\triangle^{m}\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\}^{-1} \\
&= \left\{ \lim_{t\to\infty} (1+t)^{2m+2+n/2} \int_{\mathbb{R}^n} |\nabla\triangle^{m}\mathbf{v}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\} \\
&\quad \cdot \left\{ \lim_{t\to\infty} (1+t)^{2m+1+n/2} \int_{\mathbb{R}^n} |\triangle^{m}\mathbf{v}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\}^{-1} \\
&= \left\{ \int_{\mathbb{R}^n} |\eta|^{4m+4} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta \right\} \left\{ \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta \right\}^{-1} \\
&= \frac{n+4m+2}{4\alpha},
\end{aligned}
$$

and

$$
\begin{aligned}
R_{m,2} &= \left\{ \lim_{t\to\infty} (1+t)^{2m+3+n/2} \int_{\mathbb{R}^n} |\triangle^{m+1}\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\} \\
&\quad \cdot \left\{ \lim_{t\to\infty} (1+t)^{2m+1+n/2} \int_{\mathbb{R}^n} |\triangle^{m}\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\}^{-1} \\
&= \left\{ \lim_{t\to\infty} (1+t)^{2m+3+n/2} \int_{\mathbb{R}^n} |\triangle^{m+1}\mathbf{v}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\} \\
&\quad \cdot \left\{ \lim_{t\to\infty} (1+t)^{2m+1+n/2} \int_{\mathbb{R}^n} |\triangle^{m}\mathbf{v}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\}^{-1} \\
&= \left\{ \int_{\mathbb{R}^n} |\eta|^{4m+6} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta \right\} \left\{ \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta \right\}^{-1} \\
&= \frac{(n+4m)(n+4m+2)}{(4\alpha)^2}.
\end{aligned}
$$

Remark 10 For the difference $\mathbf{u}-\mathbf{v}$, we have the same ratios as well

$$\begin{aligned}
R_{m,1} &= \left\{\lim_{t\to\infty}(1+t)^{2m+2+n/2}\int_{\mathbb{R}^n}|\nabla\triangle^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\}\\
&\quad\cdot\left\{\lim_{t\to\infty}(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\}^{-1}\\
&= \left\{\int_{\mathbb{R}^n}|\eta|^{4m+4}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\left\{\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}^{-1}\\
&= \frac{n+4m+2}{4\alpha},\\
R_{m,2} &= \left\{\lim_{t\to\infty}(1+t)^{2m+3+n/2}\int_{\mathbb{R}^n}|\triangle^{m+1}[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\}\\
&\quad\cdot\left\{\lim_{t\to\infty}(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\}^{-1}\\
&= \left\{\int_{\mathbb{R}^n}|\eta|^{4m+6}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\left\{\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}^{-1}\\
&= \frac{(n+4m+2)(n+4m+4)}{(4\alpha)^2}.
\end{aligned}$$

Remark 11 The existence of the integrals

$$\int_0^\infty\int_{\mathbb{R}^n}u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t$$

is not a problem, for all $i=1,2,3,\cdots,n$ and $j=1,2,3,\cdots,n$, because there holds the following decay estimate with sharp rate

$$(1+t)^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le C,$$

for all $t>0$, where $C>0$ is a positive constant, independent of $\mathbf{u}$ and $(\mathbf{x},t)$.

Remark 12 The exact limits are not completely explicit, because we cannot express the integral

$$\int_0^\infty\int_{\mathbb{R}^n}u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,$$

in terms of the integrals

$$\int_{\mathbb{R}^n}\phi_{ij}(\mathbf{x})\mathrm{d}\mathbf{x},\qquad\int_0^\infty\int_{\mathbb{R}^n}\psi_{ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t.$$

However, based on the results of Theorems 2 and 4, roughly speaking, we have

$$\int_0^\infty\int_{\mathbb{R}^n}u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\approx\int_0^\infty\int_{\mathbb{R}^n}v_i(\mathbf{x},t)v_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,$$

for all $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$, where $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ represents the global smooth solution to the linear heat equation. Similarly, we may use the following approximation

$$\int_0^\infty \int_{\mathbb{R}^n} u_i(\mathbf{x}, t) u_j(\mathbf{x}, t) \mathrm{d}\mathbf{x}\mathrm{d}t \approx \int_{\mathbb{R}^n} \phi_{ij}(\mathbf{x}) \mathrm{d}\mathbf{x} + \int_0^\infty \int_{\mathbb{R}^n} \psi_{ij}(\mathbf{x}, t) \mathrm{d}\mathbf{x}\mathrm{d}t,$$

for all $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$.

Remark 13 These exact limits are rigorously correct, no matter how many global weak solutions there exist and how many times singularity occur during any finite time interval (T_1, T_2), where $0 < T_1 < T_2 < \infty$.

Remark 14 A very interesting point is that the exact limits of any order derivatives of the global smooth solution depend only on the integrals

$$\alpha_{ij} = \int_{\mathbb{R}^n} \phi_{ij}(\mathbf{x}) \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n} \psi_{ij}(\mathbf{x}, t) \mathrm{d}\mathbf{x}\mathrm{d}t,$$
$$\lambda_{ij} = \int_{\mathbb{R}^n} \phi_{ij}(\mathbf{x}) \mathrm{d}\mathbf{x} + \int_0^\infty \int_{\mathbb{R}^n} \psi_{ij}(\mathbf{x}, t) \mathrm{d}\mathbf{x}\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}^n} u_i(\mathbf{x}, t) u_j(\mathbf{x}, t) \mathrm{d}\mathbf{x}\mathrm{d}t,$$

for all $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$. The exact limits do not depend on the integrals of the derivatives of these functions.

Remark 15 Numerical simulations for nonlinear equations in higher dimensional spaces, with small diffusion coefficient, have always been a challenging problem. The exact limits may be used to indirectly check the accuracy and stability of numerical schemes for the Navier-Stokes equations.

Remark 16 We have reasons and strong evidences to believe that for similar equations, very similar results hold.

3.3 Examples of initial function and external force for the n-dimensional incompressible Navier-Stokes equations

Note that the initial function and the external force are divergence free. Upon performing the Fourier transformation to the equations $\nabla \cdot \mathbf{u}_0 = 0$ and $\nabla \cdot \mathbf{f} = 0$, we get

$$\xi \cdot \widehat{\mathbf{u}}_0(\xi) = 0, \qquad \xi \cdot \widehat{\mathbf{f}}(\xi, t) = 0,$$

for all $\xi \in \mathbb{R}^n$. Let $\xi = \varepsilon \widehat{\mathbf{u}}_0(\mathbf{0})$ in $\xi \cdot \widehat{\mathbf{u}}_0(\xi) = 0$, and let $\xi = \delta \widehat{\mathbf{f}}(\mathbf{0}, t)$ in $\xi \cdot \widehat{\mathbf{f}}(\xi, t) = 0$, where $0 < \varepsilon \ll 1$ and $0 < \delta \ll 1$ are sufficiently small positive constants. Now

$$\widehat{\mathbf{u}}_0(\mathbf{0}) \cdot \widehat{\mathbf{u}}_0(\varepsilon \widehat{\mathbf{u}}_0(\mathbf{0})) = 0, \qquad \widehat{\mathbf{f}}(\mathbf{0}, t) \cdot \widehat{\mathbf{f}}(\delta \widehat{\mathbf{f}}(\mathbf{0}, t), t) = 0.$$

Then let $\varepsilon \to 0$ and $\delta \to 0$. Now it is easy to see that

$$\widehat{\mathbf{u}}_0(\mathbf{0}) = \int_{\mathbb{R}^n} \mathbf{u}_0(\mathbf{x}) \mathrm{d}\mathbf{x} = \mathbf{0}, \qquad \widehat{\mathbf{f}}(\mathbf{0}, t) = \int_{\mathbb{R}^n} \mathbf{f}(\mathbf{x}, t) \mathrm{d}\mathbf{x} = \mathbf{0},$$

for all $t > 0$.

For the n-dimensional incompressible Navier-Stokes equations, we made the assumptions that there exist real scalar functions ϕ_{ij} and ψ_{ij} in Subsection 1.4, such that

$$\mathbf{u}_0(\mathbf{x}) = \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\phi_{1j}(\mathbf{x}), \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\phi_{2j}(\mathbf{x}), \cdots, \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\phi_{nj}(\mathbf{x})\right),$$

$$\mathbf{f}(\mathbf{x},t) = \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\psi_{1j}(\mathbf{x},t), \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\psi_{2j}(\mathbf{x},t), \cdots, \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\psi_{nj}(\mathbf{x},t)\right).$$

The conditions $\nabla\cdot\mathbf{u}_0 = 0$ and $\nabla\cdot\mathbf{f} = 0$ are equivalent to the following

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\phi_{ij} = 0, \qquad \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\psi_{ij} = 0.$$

We provide the following examples for this kind of initial functions and external forces.

Example 1 Let $\alpha > 0$ be a positive constant. Define the functions $\phi_{ij} = \phi_{ij}(\mathbf{x})$ in the initial function by

$$\phi_{ij}(\mathbf{x}) = \exp\left(-\frac{|\mathbf{x}|^2}{4\alpha}\right), \quad \text{for all } (i,j) \neq (n,n),$$

$$\phi_{nn}(\mathbf{x}) = [1+\omega(\mathbf{x})]\exp\left(-\frac{|\mathbf{x}|^2}{4\alpha}\right).$$

Define the initial function

$$\mathbf{u}_0(\mathbf{x}) = \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\phi_{1j}(\mathbf{x}), \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\phi_{2j}(\mathbf{x}), \cdots, \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\phi_{nj}(\mathbf{x})\right).$$

Let $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$. Therefore,

$$\frac{\partial}{\partial x_i}\left[\exp\left(-\frac{|\mathbf{x}|^2}{4\alpha}\right)\right] = -\frac{x_i}{2\alpha}\exp\left(-\frac{|\mathbf{x}|^2}{4\alpha}\right),$$

$$\frac{\partial^2}{\partial x_i \partial x_j}\left[\exp\left(-\frac{|\mathbf{x}|^2}{4\alpha}\right)\right] = \frac{x_i x_j}{4\alpha^2}\exp\left(-\frac{|\mathbf{x}|^2}{4\alpha}\right) - \frac{\delta_{ij}}{2\alpha}\exp\left(-\frac{|\mathbf{x}|^2}{4\alpha}\right),$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_i \partial x_j}\left[\exp\left(-\frac{|\mathbf{x}|^2}{4\alpha}\right)\right] = \left[\left(\frac{1}{2\alpha}\sum_{i=1}^{n} x_i\right)^2 - \frac{n}{2\alpha}\right]\exp\left(-\frac{|\mathbf{x}|^2}{4\alpha}\right).$$

Moreover

$$\frac{\partial}{\partial x_n}\left[\omega(\mathbf{x})\exp\left(-\frac{|\mathbf{x}|^2}{4\alpha}\right)\right] = \frac{\partial}{\partial x_n}\omega(\mathbf{x})\exp\left(-\frac{|\mathbf{x}|^2}{4\alpha}\right) - \frac{x_n}{2\alpha}\omega(\mathbf{x})\exp\left(-\frac{|\mathbf{x}|^2}{4\alpha}\right),$$

$$\frac{\partial^2}{\partial x_n^2}\left[\omega(\mathbf{x})\exp\left(-\frac{|\mathbf{x}|^2}{4\alpha}\right)\right] = \left\{\frac{\partial^2}{\partial x_n^2}\omega(\mathbf{x}) - \frac{x_n}{\alpha}\frac{\partial}{\partial x_n}\omega(\mathbf{x}) + \frac{x_n^2}{4\alpha^2}\omega(\mathbf{x}) - \frac{1}{2\alpha}\omega(\mathbf{x})\right\}\exp\left(-\frac{|\mathbf{x}|^2}{4\alpha}\right).$$

Additionally, we have

$$\nabla \cdot \mathbf{u}_0(\mathbf{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\phi_{ij}(\mathbf{x})$$
$$= \left\{ \frac{\partial^2}{\partial x_n^2}\omega - \frac{x_n}{\alpha}\frac{\partial}{\partial x_n}\omega + \frac{x_n^2}{4\alpha^2}\omega - \frac{1}{2\alpha}\omega + \left(\frac{1}{2\alpha}\sum_{i=1}^{n} x_i\right)^2 - \frac{n}{2\alpha} \right\} \exp\left(-\frac{|\mathbf{x}|^2}{4\alpha}\right).$$

Therefore, if we choose the function $\omega = \omega(\mathbf{x})$ to be the unique solution to the initial value problems for the nonhomogeneous, second order, linear differential equation

$$\frac{\partial^2}{\partial x_n^2}\omega - \frac{x_n}{\alpha}\frac{\partial}{\partial x_n}\omega + \frac{x_n^2}{4\alpha^2}\omega - \frac{1}{2\alpha}\omega + \left(\frac{1}{2\alpha}\sum_{i=1}^{n} x_i\right)^2 - \frac{n}{2\alpha} = 0,$$
$$\omega(x_1, x_2, \cdots, x_{n-1}, 0) = 0, \qquad \frac{\partial}{\partial x_n}\omega(x_1, x_2, \cdots, x_{n-1}, 0) = 0,$$

then we see that the initial function $\mathbf{u}_0(\mathbf{x})$ is divergence free. In the initial value problems, $\mathbf{x} = (x_1, x_2, \cdots, x_{n-1}, x_n) \in \mathbb{R}^n$ and x_1, x_2, x_3, $\cdots$, x_{n-1} are treated as real parameters. Note that the existence and uniqueness of the global smooth solution $\omega = \omega(\mathbf{x})$ are guaranteed. There exists a positive constant $C > 0$, independent of $\mathbf{x}$, such that there holds the following estimate

$$|\omega(\mathbf{x})| \le C(1 + |\mathbf{x}|^4),$$

for all $\mathbf{x} \in \mathbb{R}^n$. Hence $[1 + \omega(\mathbf{x})] \exp\left(-\frac{|\mathbf{x}|^2}{4\alpha}\right) \in C^\infty(\mathbb{R}^n) \cap W^{\infty,1}(\mathbb{R}^n)$. Overall, $\phi_{ij} \in C^\infty(\mathbb{R}^n) \cap W^{\infty,1}(\mathbb{R}^n)$, for all positive integers $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$.

Example 2 Let $\alpha > 0$ be a positive constant and $t > 0$ be a positive parameter. Define the functions $\psi_{ij} = \psi_{ij}(\mathbf{x}, t)$ in the external force $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$ by

$$\psi_{ij}(\mathbf{x}, t) = \exp\left[-\frac{|\mathbf{x}|^2}{4\alpha(1+t)}\right], \text{ for all } (i, j) \neq (n, n),$$
$$\psi_{nn}(\mathbf{x}, t) = [1 + \omega(\mathbf{x}, t)] \exp\left[-\frac{|\mathbf{x}|^2}{4\alpha(1+t)}\right],$$

where $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$. Define the external force

$$\mathbf{f}(\mathbf{x}, t) = \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\psi_{1j}(\mathbf{x}, t), \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\psi_{2j}(\mathbf{x}, t), \cdots, \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\psi_{nj}(\mathbf{x}, t) \right).$$

Define $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$. Note that

$$\frac{\partial}{\partial x_i}\left\{\exp\left[-\frac{|\mathbf{x}|^2}{4\alpha(1+t)}\right]\right\} = -\frac{x_i}{2\alpha(1+t)}\exp\left[-\frac{|\mathbf{x}|^2}{4\alpha(1+t)}\right],$$
$$\frac{\partial^2}{\partial x_i \partial x_j}\left\{\exp\left[-\frac{|\mathbf{x}|^2}{4\alpha(1+t)}\right]\right\} = \left[\frac{x_i x_j}{4\alpha^2(1+t)^2} - \frac{\delta_{ij}}{2\alpha(1+t)}\right]\exp\left[-\frac{|\mathbf{x}|^2}{4\alpha(1+t)}\right],$$

$$\sum_{i=1}^n\sum_{j=1}^n \frac{\partial^2}{\partial x_i \partial x_j}\left\{\exp\left[-\frac{|\mathbf{x}|^2}{4\alpha(1+t)}\right]\right\} = \left[\frac{1}{4\alpha^2(1+t)^2}\left(\sum_{i=1}^n x_i\right)^2 - \frac{n}{2\alpha(1+t)}\right] \cdot \exp\left[-\frac{|\mathbf{x}|^2}{4\alpha(1+t)}\right].$$

Moreover

$$\begin{aligned}
&\frac{\partial}{\partial x_n}\left\{\omega(\mathbf{x},t)\exp\left[-\frac{|\mathbf{x}|^2}{4\alpha(1+t)}\right]\right\}\\
=&\left[\frac{\partial}{\partial x_n}\omega - \frac{x_n}{2\alpha(1+t)}\omega(\mathbf{x},t)\right]\exp\left[-\frac{|\mathbf{x}|^2}{4\alpha(1+t)}\right],\\
&\frac{\partial^2}{\partial x_n^2}\left\{\omega(\mathbf{x},t)\exp\left[-\frac{|\mathbf{x}|^2}{4\alpha(1+t)}\right]\right\}\\
=&\left[\frac{\partial^2}{\partial x_n^2}\omega - \frac{x_n}{\alpha(1+t)}\frac{\partial}{\partial x_n}\omega + \frac{x_n^2}{4\alpha^2(1+t)^2}\omega(\mathbf{x},t) - \frac{1}{2\alpha(1+t)}\omega\right]\exp\left[-\frac{|\mathbf{x}|^2}{4\alpha(1+t)}\right].
\end{aligned}$$

Additionally

$$\begin{aligned}
\nabla\cdot\mathbf{f}(\mathbf{x},t) &= \sum_{i=1}^n\sum_{j=1}^n \frac{\partial^2}{\partial x_i\partial x_j}\psi_{ij}\\
&= \sum_{i=1}^n\sum_{j=1}^n \frac{\partial^2}{\partial x_i\partial x_j}\left\{\exp\left[-\frac{|\mathbf{x}|^2}{4\alpha(1+t)}\right]\right\} + \frac{\partial^2}{\partial x_n^2}\left\{\omega(\mathbf{x})\exp\left[-\frac{|\mathbf{x}|^2}{4\alpha(1+t)}\right]\right\}\\
&= \left\{\frac{\partial^2}{\partial x_n^2}\omega - \frac{x_n}{\alpha(1+t)}\frac{\partial}{\partial x_n}\omega + \frac{x_n^2}{4\alpha^2(1+t)^2}\omega - \frac{1}{2\alpha(1+t)}\omega\right.\\
&\quad\left. + \frac{1}{4\alpha^2(1+t)^2}\left(\sum_{i=1}^n x_i\right)^2 - \frac{n}{2\alpha(1+t)}\right\}\exp\left[-\frac{|\mathbf{x}|^2}{4\alpha(1+t)}\right].
\end{aligned}$$

Therefore, if we choose the function $\omega = \omega(\mathbf{x},t)$ to be the unique solution to the initial value problems for the nonhomogeneous, second order, linear differential equation

$$\frac{\partial^2}{\partial x_n^2}\omega - \frac{x_n}{\alpha(1+t)}\frac{\partial}{\partial x_n}\omega + \frac{x_n^2}{4\alpha^2(1+t)^2}\omega - \frac{1}{2\alpha(1+t)}\omega + \frac{1}{4\alpha^2(1+t)^2}\left(\sum_{i=1}^n x_i\right)^2 - \frac{n}{2\alpha(1+t)} = 0,$$

$$\omega(x_1,x_2,\cdots,x_{n-1},0,t)=0, \quad \frac{\partial}{\partial x_n}\omega(x_1,x_2,\cdots,x_{n-1},0,t)=0,$$

then we see that the external force is divergence free. Overall

$$\psi_{ij}\in C^\infty(\mathbb{R}^n\times\mathbb{R}^+)\cap L^1(\mathbb{R}^n\times\mathbb{R}^+).$$

Now the functions ϕ_{ij}, $\mathbf{u}_0$, ψ_{ij} and $\mathbf{f}$ in the above examples satisfy all assumptions made in (A2). This kind of initial functions and external forces form a vector subspace of $L^1(\mathbb{R}^n)\cap H^2(\mathbb{R}^n)$ and $L^1(\mathbb{R}^n\times\mathbb{R}^+)\cap L^1(\mathbb{R}^+,L^2(\mathbb{R}^n))$, respectively.

3.4 Open problems

Consider the Cauchy problem for the n-dimensional incompressible Navier-Stokes equations

$$\frac{\partial}{\partial t}\mathbf{u}-\alpha\triangle\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=\mathbf{f}(\mathbf{x},t),\quad \nabla\cdot\mathbf{u}=0,\nabla\cdot\mathbf{f}=0,$$
$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}),\quad \nabla\cdot\mathbf{u}_0=0.$$

Given any large initial function $\mathbf{u}_0\in L^1(\mathbb{R}^n)\cap H^{2m+1}(\mathbb{R}^n)$ and given any large external force $\mathbf{f}\in L^1(\mathbb{R}^n\times\mathbb{R}^+)\cap L^1(\mathbb{R}^+,L^2(\mathbb{R}^n))\cap L^2(\mathbb{R}^+,H^{2m}(\mathbb{R}^n))$, the existence and uniqueness of the global smooth solution $\mathbf{u}\in L^\infty(\mathbb{R}^+,H^{2m+1}(\mathbb{R}^n))$, such that $\nabla\mathbf{u}\in L^2(\mathbb{R}^+,H^{2m+1}(\mathbb{R}^n))$, have not been accomplished yet, where $m\geq 1$ is a positive integer.

How to find the exact values of the integrals

$$\int_0^\infty\int_{\mathbb{R}^n}u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,$$

in terms of the diffusion coefficient α, the dimension n and the integrals

$$\int_{\mathbb{R}^n}\phi_{ij}(\mathbf{x})\mathrm{d}\mathbf{x},\qquad \int_0^\infty\int_{\mathbb{R}^n}\psi_{ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t?$$

3.5 Some technical lemmas

(The Gagliardo-Nirenberg's interpolation inequality) Let $1\leq p\leq\infty$, $1\leq q\leq\infty$ and $1\leq r\leq\infty$. Let $m\geq 1$, $n\geq 1$, $k\geq 0$ be integers, such that $k<m$. There exists a positive constant $\alpha\in(k/m,1)$, determined by

$$\frac{n}{p}-k=\alpha\Big(\frac{n}{r}-m\Big)+(1-\alpha)\frac{n}{q}.$$

There exists a positive constant $C=C(m,n,k,p,q,r)>0$, such that there holds the following estimate

$$\begin{aligned}&\left\{\sum_{\beta_1+\beta_2+\beta_3+\cdots+\beta_n=k}\int_{\mathbb{R}^n}\left|\frac{\partial^{\beta_1+\beta_2+\beta_3+\cdots+\beta_n}\mathbf{u}}{\partial x_1^{\beta_1}\partial x_2^{\beta_2}\partial x_3^{\beta_3}\cdots\partial x_n^{\beta_n}}\right|^p\mathrm{d}\mathbf{x}\right\}^{1/p}\\ \leq &C\left\{\sum_{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n=m}\int_{\mathbb{R}^n}\left|\frac{\partial^{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n}\mathbf{u}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}\cdots\partial x_n^{\alpha_n}}\right|^r\mathrm{d}\mathbf{x}\right\}^{\alpha/r}\left\{\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x})|^q\mathrm{d}\mathbf{x}\right\}^{(1-\alpha)/q},\end{aligned}$$

for all $u\in W^{m,r}(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)$.

References

[1] Luis A. Caffarelli, Robert V. Kohn and Louis Nirenberg, Partial regularity of suitable weak solutions of the Navier-Stokes equations, *Communications on Pure and Applied Mathematics*, **35**(1982),771-831.

[2] H. Fujita and T. Kato, On the Navier-Stokes initial value problem I, *Archive for Rational Mechanics and Analysis*, **16**(1964),269-315.

[3] John G. Heywood, The Navier-Stokes equations: on the existence, regularity and decay of solutions, *Indiana University Mathematics Journal*, **29**(1980),639-681.

[4] Tosio Kato, Strong L^λ-solutions of the Navier-Stokes equations in $\mathbb{R}^m$, with applications to weak solutions, *Math Z*, **187**(1984),471-480.

[5] Zhen Lei and Fanghua Lin, Global mild solutions of Navier-Stokes equations, *Communications in Pure and Applied Mathematics*, **64**(2011),1297-1304.

[6] Zhen Lei, Fanghua Lin and Yi Zhou, Structure of helicity and global solution of incompressible Navier-Stokes equations, *Archive for Rational Mechanics and Analysis*, **218**(2015),1417-1430.

[7] Jean Leray, Sur le mouvement d'un liquide visquex emplissant l'espace, *Acta Math*, **63**(1934),193-248.

[8] Fanghua Lin, A new proof of the Caffarelli-Kohn-Nirenberg theorem, *Communications in Pure and Applied Mathematics*, **51**(1998),241-257.

[9] Marcel Oliver and Edriss S. Titi, Remark on the rate of decay of higher order derivatives for solutions to the Navier-Stokes equations in $\mathbb{R}^n$, *Journal of Functional Analysis*, **172**(2000),1-18.

[10] James Serrin, On the interior regularity of weak solutions of the Navier-Stokes equations, *Archive for Rational Mechanics and Analysis*, **9**(1962),187-195.

[11] Tetsuro Miyakawa and Hermann Sohr, On energy inequality, smoothness and large time behavior in L^2 for weak solutions of the Navier-Stokes equations in exterior domains, *Mathematische Zeitschrift*, **199**(1988),455-478.

[12] Weimin Peng and Yi Zhou, Global large solutions to incompressible Navier-Stokes equations with gravity, *Mathematical Methods in Applied Sciences*, **38**(2015),590-597.

[13] Maria E. Schonbek and Michael Wiegner, On the decay of higher-order norms of the solutions of Navier-Stokes equations, *Proceedings of the Royal Society of Edinburgh*, **126A**(1996),677-685.

[14] Roger Temam, Navier-Stokes equations. Theory and Numerical Analysis, American Mathematical Society, Providence, Rhode Island. 1984.

[15] Roger Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Applied Mathematical Sciences, Volume 68. Spring-Verlag, New York, First Edition, 1988.

[16] Jindrich Necas, Michael Ruzicka and Vladimir Sverak, On Leray's self-similar solutions of the Navier-Stokes equations, *Acta Math*, **176**(1996),283-294.

[17] Linghai Zhang, Sharp rate of decay of solutions to 2-dimensional Navier-Stokes equations, *Communications in Partial Differential Equations*, **20**(1995),119-127.

[18] Linghai Zhang, Decay estimates with sharp rates of global solutions of nonlinear systems of fluid dynamics equations, *Discrete and Continuous Dynamical Systems, Series S*, **9**(2016),2181-2200.

(*edited by Mengxin He*)