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A COMPACT FINITE DIFFERENCE SCHEME FOR THE FOURTH-ORDER TIME MULTI-TERM FRACTIONAL SUB-DIFFUSION EQUATIONS WITH THE FIRST DIRICHLET BOUNDARY CONDITIONS

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Abstract. In this paper, a finite difference scheme is established for solving the fourth-order time multi-term fractional sub-diffusion equations with the first Dirichlet boundary conditions. Using the method of order reduction, the original problem is equivalent to a lower-order system. Then the system is considered at some particular points, and the first Dirichlet boundary conditions are also specially handled, so that the global convergence of the presented difference scheme reaches $O(\tau^2 + h^4)$, with τ and h the temporal and spatial step size, respectively. The energy method is used to give the theoretical analysis on the stability and convergence of the difference scheme, where some novel techniques have been applied due to the non-local property of fractional operators and the numerical treatment of the first Dirichlet boundary conditions. Numerical experiments further validate the theoretical results.

Key words. Multi-term, fractional sub-diffusion equations, the first Dirichlet boundary conditions, stability, convergence.

1. Introduction

With the development of science and technology, fractional differential equations are widely used in scientific research and engineering applications. Many phenomena in the fields of astronomy [1], finance [2], medicine [3], physics [4], etc. can use fractional differential equations to build models. Therefore, the theoretical researches and applications of fractional differential equations have become one of the hot issues of recent concern, which has the widespread good prospects for development. Since the solutions to many fractional differential equations cannot be accurately obtained or the form of the solution is relatively complicated, the numerical results are particularly important.

When the first-order or second-order time derivatives in the classical diffusion wave equation are replaced by fractional derivatives, the fractional diffusion wave equations are obtained. In recent years, many scholars have done a lot of researches on the second-order time fractional diffusion equations. Sun and Wu [5] analyzed the truncation errors of the $L1$ numerical approximation formula by using linear interpolation for the Caputo fractional derivative and then constructed a fully discrete difference scheme for the fractional wave equations by introducing new variables to convert the original system of equations into a lower-order system. The stability and convergence of the difference scheme were proved by the energy method. Based on the previous content, the numerical results in the case of the slow diffusion system were also briefly discussed. Du, Cao and Sun [6] further proposed the high order difference method for the fractional wave equations to improve the convergence order in space to the fourth-order. Gao and Sun [7] proposed a compact difference scheme for time fractional diffusion equations, where the stability

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and unconditional convergence of the scheme were shown by defining a new inner product. By selecting $\sigma = 1 - \frac{\alpha}{2}$, Alikhanov [8] obtained the $L2 - 1_\sigma$ formula to approximate the values of Caputo derivatives at some particular points and proved that the truncation error of this formula is $O(\tau^{3-\alpha})$, with α the order of the fractional derivative. Based on this formula, the finite difference scheme for the time fractional diffusion equation was established with the convergence accuracy of order two in both time and space. Vong and Lyu [9] proposed a finite difference scheme for a time-fractional Burgers-type equation, where the highlight of the scheme was that there is no need to use iterative methods to find the approximate solutions, and the unconditional stability together with convergence were proved.

For some physical phenomena, it is often not enough to describe these phenomena by the second-order spatial derivative term, hence the fourth-order derivative term in space need be introduced. By using the finite sine transform technique, Agrawal [10] converted a fractional differential equation from a space domain to a wave number domain and obtained the solutions to fourth-order fractional diffusion-wave equations by the method of inverse Laplace and inverse finite sine transforms. Hu and Zhang [11] applied the extrapolation technique to establish a compact difference scheme for solving the fourth-order fractional diffusion wave equations, and in Ref. [12], using the method of order reduction, an implicit compact difference scheme for the fourth-order fractional diffusion-wave equations was obtained. Wei and He [13] introduced a fully discrete local discontinuous Galerkin finite element method based on a finite difference discretization in time and local discontinuous Galerkin method in space for fourth-order time fractional equations and proved its unconditional stability and convergence. Yao and Wang [14] established a finite difference scheme with global convergence order $O(\tau^2 + h^4)$ for fourth-order fractional diffusion equations with Neumann boundary conditions by the special handling of the Neumann boundary condition. Liu et al. [15] proposed a finite element algorithm for solving nonlinear time fractional diffusion equations with the fourth-order derivative term.

The fractional diffusion wave equation plays an important role in the field of anomalous diffusion, especially the case with the time multi-term fractional derivatives. It's often called the multi-term fractional diffusion-wave equation. Jiang et al. [16] used the method of separation of variables to present the analytical solutions to the multi-term time-fractional diffusion-wave equation and the multi-term time-fractional diffusion equation. Liu et al. [17] investigated two implicit numerical methods to simulate the two-term mobile/immobile time fractional diffusion equation and the two-term time fractional diffusion equation, where the predictor-corrector method to solve the multi-term time fractional diffusion equations was proposed and the strict theoretical analysis was provided. Ren and Sun [18] obtained the difference scheme for solving one-dimensional and two-dimensional multi-term time fractional diffusion-wave equations by using the $L1$ approximation for the multi-term time Caputo fractional derivatives. Gao, Alikhanov and Sun [19] considered the interpolation approximation of the multi-term fractional derivatives at some special points and established a numerical algorithm for solving time multi-term fractional diffusion equations. Wei [20] established a fully discrete scheme using local discontinuous Galerkin method in space and classical $L1$ approximation in time and proved the stability and convergence of the resultant scheme. By extending the domain of the fractional Laplacian to a Banach space and using the multivariate Mittag-Leffler function, Sin, Ri and Kim [21] obtained the analytical solutions to the multi-term fractional diffusion equation. Reutskiy [22] introduced the backward substitution method for solving fractional partial differential equations. The method is based on the Fourier series expansion along the spatial coordinate that transforms the original equation into a sequence of multi-term fractional ordinary differential equations. Zaky [23] handled the multi-term time fractional diffusion equations by using a Legendre spectral tau method.

It can be found that there are limited works dealing with the fourth-order sub-diffusion equations with the first Dirichlet boundary conditions. Vong and Wang [24] derived a compact difference scheme to solve the problem, and the stability as well as the convergence were proved. Ji, Sun and Hao [25] presented a different way to numerically solve the same problem, where the method of order reduction was used and the special treatment of the first Dirichlet boundary conditions was introduced. It's noted that both Refs. [24] and [25] handle the problem with the single-term time fractional derivative, which was approximated by the $L1$ formula, and the convergence order of the resultant schemes in time was less than two. Different from the previous works, for the the fourth-order time multi-term fractional sub-diffusion equations with the first Dirichlet boundary conditions, in the present work, we are devoted to find the higher-order numerical solutions for the problem by the special handling of boundary conditions and higher-order approximation for the multi-term time fractional derivatives. The main advantages of the current work include:

- The multi-term time-fractional derivatives are discretized at some special points based on the interpolation approximation developed in Ref. [19], instead of the $L1$ approximation used in Refs. [24] and [25]. Then the global second-order convergence of the algorithm in time can be achieved.
- The first Dirichlet boundary conditions are handled skillfully to match the global fourth-order accuracy of the proposed scheme in space by defining a new and simple average operator, which is also different from some ways exsiting in the previous works.
- The error estimation on the developed scheme is proceeded successfully by the energy method. One can find that the essential difference between error estimation of single-term and multi-term time-fractional parabolic equations lies in Lemma 4.3, which plays a key role in the current proof. The difficulty in the proof of this lemma is to show the truth of $(2\sigma - 1)\hat{c}_0^{(n)} - \sigma\hat{c}_1^{(n)} > 0$, which has been illustrated using some novel techniques in Ref. [19]. In addition, the difficulty caused by the numerical treatment of the first Dirichlet boundary conditions in the analysis has been overcome with the help of ε-inequality and Lemma 4.1.

The outline of this paper is as follows: Section 2 is devoted to some necessary preliminaries before the construction of the difference scheme. In Section 3, a compact finite difference scheme is derived for the fourth-order time multi-term fractional sub-diffusion equations with the first Dirichlet boundary conditions. The stability and convergence of the scheme are rigorously proved by the energy method in Section 4. In Section 5, some numerical examples are provided to further validate our theoretical results. A brief conclusion ends this work finally.

2. Preliminaries

In the present work, we consider the following fourth-order time multi-term fractional sub-diffusion equations with the first Dirichlet boundary conditions:

$$\sum_{r=0}^{m}\lambda_r\, {}_0^C D_t^{\alpha_r}u(x,t)+\frac{\partial^4 u(x,t)}{\partial x^4}+qu(x,t)=f(x,t),\quad x\in(0,L),\ t\in(0,T], \tag{1}$$

$$u(0,t)=g_1(t),\quad u(L,t)=g_2(t),\quad t\in(0,T], \tag{2}$$

$$\frac{\partial u(0,t)}{\partial x}=\gamma_1(t),\quad \frac{\partial u(L,t)}{\partial x}=\gamma_2(t),\quad t\in(0,T], \tag{3}$$

$$u(x,0)=\phi(x),\quad x\in[0,L], \tag{4}$$

where $g_1(0)=\phi(0), g_2(0)=\phi(L), \phi'(0)=\gamma_1(0), \phi'(L)=\gamma_2(0), 0\le\alpha_m<\alpha_{m-1}<\cdots<\alpha_1<\alpha_0\le 1$ and at least one of $\alpha_i's$ belongs to $(0,1)$, q is a positive constant, the functions $f(x,t), g_i(t), \gamma_i(t)(i=1,2)$ and $\phi(x)$ are all given, ${}_0^C D_t^{\alpha}u(x,t)$ is the α-th order time fractional Caputo derivative of $u(x,t)$ defined by

$${}_0^C D_t^{\alpha}u(x,t)=\begin{cases} u(x,t)-u(x,0), & \alpha=0,\\ \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t \frac{u_s(x,s)}{(t-s)^{\alpha}}\mathrm{d}s, & 0<\alpha<1,\\ u_t(x,t), & \alpha=1.\end{cases}$$

For the numerical approach, the mesh partition is essential. For two positive integers M and N, let $h=L/M, \tau=T/N, x_i=ih(0\le i\le M), t_n=n\tau(0\le n\le N), \Omega_h=\{x_i|0\le i\le M\}, \Omega_\tau=\{t_n|0\le n\le N\}$, then the computational domain $[0,L]\times[0,T]$ is covered by $\Omega_h\times\Omega_\tau$.

We commence with the following lemmas which will be used in the subsequent discussions.

Lemma 2.1. *Denote $\theta(s)=(1-s)^3[10-3(1-s)^2]$ and $\xi(s)=(1-s)^3[5-3(1-s)^2]$. (I)If function $g\in C^6[x_0,x_1]$, then we have*

$$\begin{aligned}&\Big[\frac{2}{3}g''(x_0)+\frac{1}{3}g''(x_1)\Big]-\frac{2}{h}\Big[\frac{g(x_1)-g(x_0)}{h}-g'(x_0)\Big]\\ =&\frac{h^2}{12}g^{(4)}(x_0)+\frac{7h^3}{180}g^{(5)}(x_0)+\frac{h^4}{180}\int_0^1\theta(s)g^{(6)}(x_0+sh)\mathrm{d}s.\end{aligned}\tag{5}$$

(II)If function $g\in C^6[x_{M-1},x_M]$, then we have

$$\begin{aligned}&\Big[\frac{1}{3}g''(x_{M-1})+\frac{2}{3}g''(x_M)\Big]-\frac{2}{h}\Big[g'(x_M)-\frac{g(x_M)-g(x_{M-1})}{h}\Big]\\ =&\frac{h^2}{12}g^{(4)}(x_M)-\frac{7h^3}{180}g^{(5)}(x_M)+\frac{h^4}{180}\int_0^1\theta(s)g^{(6)}(x_M-sh)\mathrm{d}s.\end{aligned}\tag{6}$$

(III) [14] If function $g\in C^6[x_{i-1},x_{i+1}]$, then we have

$$\begin{aligned}\frac{1}{12}[g''(x_{i-1})+10g''(x_i)+g''(x_{i+1})]=&\frac{1}{h^2}[g(x_{i-1})-2g(x_i)+g(x_{i+1})]\\ &+\frac{h^4}{360}\int_0^1\xi(s)[g^{(6)}(x_i-sh)+g^{(6)}(x_i+sh)]\mathrm{d}s.\end{aligned}\tag{7}$$

Proof. By the formula of Taylor expansion with integral remainder

$$g(x_0+h)=\sum_{l=0}^{k}\frac{h^l}{l!}g^{(l)}(x_0)+\frac{h^{k+1}}{k!}\int_0^1 g^{(k+1)}(x_0+sh)(1-s)^k\mathrm{d}s,$$

we have

$$\begin{aligned} g(x_1) =& g(x_0) + g'(x_0)h + \frac{h^2}{2}g''(x_0) + \frac{h^3}{6}g'''(x_0) + \frac{h^4}{24}g^{(4)}(x_0) \\ &+ \frac{h^5}{120}g^{(5)}(x_0) + \frac{h^6}{120}\int_0^1 (1-s)^5 g^{(6)}(x_0+sh)\mathrm{d}s. \end{aligned}$$

Hence,

$$\begin{aligned} &\frac{2}{h}\Big[\frac{g(x_1)-g(x_0)}{h} - g'(x_0)\Big] \\ =& g''(x_0) + \frac{h}{3}g'''(x_0) + \frac{h^2}{12}g^{(4)}(x_0) + \frac{h^3}{60}g^{(5)}(x_0) \\ &+ \frac{h^4}{60}\int_0^1 (1-s)^5 g^{(6)}(x_0+sh)\mathrm{d}s. \end{aligned} \tag{8}$$

In addition, it follows from

$$\begin{aligned} g''(x_1) =& g''(x_0) + hg'''(x_0) + \frac{h^2}{2}g^{(4)}(x_0) + \frac{h^3}{6}g^{(5)}(x_0) \\ &+ \frac{h^4}{6}\int_0^1 (1-s)^3 g^{(6)}(x_0+sh)\mathrm{d}s \end{aligned}$$

that

$$\begin{aligned} \frac{2}{3}g''(x_0) + \frac{1}{3}g''(x_1) =& g''(x_0) + \frac{h}{3}g'''(x_0) + \frac{h^2}{6}g^{(4)}(x_0) \\ &+ \frac{h^3}{18}g^{(5)}(x_0) + \frac{h^4}{18}\int_0^1 (1-s)^3 g^{(6)}(x_0+sh)\mathrm{d}s. \end{aligned} \tag{9}$$

Subtraction of (8) from (9) will yield (5). In a similar way, (6) can be proved.

Remark 1: In Ref. [26], the simple form of (5) and (6) has been given. Here, a detailed result is illustrated in order to handle the first Dirichlet boundary conditions in the next part.

Lemma 2.2. *[14] If function $u \in C^3[t_{n-1}, t_n]$, σ is a constant and $0 < \sigma < 1$, it holds that*

$$u(t_{n-1+\sigma}) = \sigma u(t_n) + (1-\sigma)u(t_{n-1}) + O(\tau^2).$$

For simplicity, denote $u^{\sigma_n} = \sigma u(t_n) + (1-\sigma)u(t_{n-1}), \quad 1 \le n \le N$. In addition, for $\alpha \in [0,1]$, denote

$$a_0^{(\alpha)} = \sigma^{1-\alpha}, \quad a_l^{(\alpha)} = (l+\sigma)^{1-\alpha} - (l-1+\sigma)^{1-\alpha}, \quad l \geqslant 1,$$

$$b_l^{(\alpha)} = \frac{1}{2-\alpha}[(l+\sigma)^{2-\alpha} - (l-1+\sigma)^{2-\alpha}] - \frac{1}{2}[(l+\sigma)^{1-\alpha} + (l-1+\sigma)^{1-\alpha}], \ l \geqslant 1.$$

When $n = 1$,

$$C_0^{(n,\alpha)} = a_0^{(\alpha)};$$

When $n \geqslant 2$,

$$C_k^{(n,\alpha)} = \begin{cases} a_0^{(\alpha)} + b_1^{(\alpha)}, \quad k = 0, \\ a_k^{(\alpha)} + b_{k+1}^{(\alpha)} - b_k^{(\alpha)}, \quad 1 \leqslant k \leqslant n-2, \\ a_k^{(\alpha)} - b_k^{(\alpha)}, \quad k = n-1. \end{cases}$$

Lemma 2.3. *[19] Suppose function $u(x,\cdot) \in C^3[0,T]$. Then it holds*

$$\sum_{r=0}^{m} \lambda_r \, {}_0^C D_t^{\alpha_r} u(x_i, t_{n-1+\sigma}) = \sum_{r=0}^{m} \lambda_r \frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)} \sum_{k=0}^{n-1} C_k^{(n,\alpha_r)} [u(x_i, t_{n-k}) - u(x_i, t_{n-k-1})] + O(\tau^{3-\alpha_0}),$$

where $0 \le \alpha_m < \alpha_{m-1} < \cdots < \alpha_0 \le 1$, σ is the root of the nonlinear equation

$$\sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(3-\alpha_r)} \sigma^{1-\alpha_r} [\sigma - (1 - \frac{\alpha_r}{2})] \tau^{2-\alpha_r} = 0.$$

3. The derivation of the compact finite difference scheme

For any mesh function $u = (u_0, u_1, \ldots, u_M)$ defined on Ω_h, introduce the following notations:

$$\delta_x u_{i-\frac{1}{2}} = \frac{1}{h}(u_i - u_{i-1}),\ 1 \le i \le M; \quad \delta_x^2 u_i = \frac{1}{h}(\delta_x u_{i+\frac{1}{2}} - \delta_x u_{i-\frac{1}{2}}),\ 1 \le i \le M-1$$

and the average operator

$$(\mathcal{H}u)_i = \begin{cases} \frac{2}{3}u_0 + \frac{1}{3}u_1, & i = 0, \\ \frac{1}{12}(u_{i-1} + 10u_i + u_{i+1}), & 1 \le i \le M-1, \\ \frac{1}{3}u_{M-1} + \frac{2}{3}u_M, & i = M. \end{cases}$$

Let $v(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}$. Then Eqs. (1)-(4) are equivalent to

$$\sum_{r=0}^{m} \lambda_r \, {}_0^C D_t^{\alpha_r} u(x,t) + \frac{\partial^2 v(x,t)}{\partial x^2} + qu(x,t) = f(x,t), \quad x \in (0,L),\ t \in (0,T], \tag{10}$$

$$v(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in (0,L), \quad t \in [0,T], \tag{11}$$

$$u(0,t) = g_1(t), \quad u(L,t) = g_2(t),\ t \in (0,T], \tag{12}$$

$$\frac{\partial u(0,t)}{\partial x} = \gamma_1(t), \quad \frac{\partial u(L,t)}{\partial x} = \gamma_2(t), \quad t \in (0,T], \tag{13}$$

$$u(x,0) = \phi(x), \quad x \in [0,L]. \tag{14}$$

Define the grid functions

$$U_i^n = u(x_i, t_n), \quad V_i^n = v(x_i, t_n), \quad 0 \le i \le M, 0 \le n \le N;$$
$$f_i^{n-1+\sigma} = f(x_i, t_{n-1+\sigma}), \quad 0 \le i \le M, 1 \le n \le N.$$

Suppose the exact solution $u \in C^{(8,3)}([0,L] \times [0,T])$. Considering Eqs. (10), (11) at the point $(x_i, t_{n-1+\sigma})$, we have

$$\sum_{r=0}^{m} \lambda_r \, {}_0^C D_t^{\alpha_r} u(x_i, t_{n-1+\sigma}) + \frac{\partial^2 v(x_i, t_{n-1+\sigma})}{\partial x^2} + qu(x_i, t_{n-1+\sigma})$$
$$= f(x_i, t_{n-1+\sigma}), \quad 0 \le i \le M,\ 1 \le n \le N,$$
$$v(x_i, t_{n-1+\sigma}) = \frac{\partial^2 u(x_i, t_{n-1+\sigma})}{\partial x^2}, \quad 0 \le i \le M,\ 1 \le n \le N.$$

Performing the average operator $\mathcal{H}$ on both hand sides of the above two equations yields

$$\begin{aligned}&\mathcal{H}\sum_{r=0}^{m}\lambda_r\,{}_0^CD_t^{\alpha_r}u(x_i,t_{n-1+\sigma})+\mathcal{H}\frac{\partial^2 v(x_i,t_{n-1+\sigma})}{\partial x^2}+q\mathcal{H}u(x_i,t_{n-1+\sigma})\\ &=\mathcal{H}f(x_i,t_{n-1+\sigma}),\quad 1\le i\le M-1,\ 1\le n\le N,\end{aligned}\tag{15}$$

$$\mathcal{H}v(x_i,t_{n-1+\sigma})=\mathcal{H}\frac{\partial^2 u(x_i,t_{n-1+\sigma})}{\partial x^2},\quad 0\le i\le M,\ 1\le n\le N.\tag{16}$$

According to Lemmas 2.1-2.3, we obtain

$$\begin{aligned}&\mathcal{H}\sum_{r=0}^{m}\lambda_r\frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)}\sum_{k=0}^{n-1}C_k^{(n,\alpha_r)}(U_i^{n-k}-U_i^{n-k-1})+\delta_x^2V_i^{\sigma_n}+q\mathcal{H}U_i^{\sigma_n}\\ &=\mathcal{H}f_i^{n-1+\sigma}+R_i^{\sigma_n},\quad 1\le i\le M-1,\ 1\le n\le N,\end{aligned}\tag{17}$$

$$\mathcal{H}V_i^{\sigma_n}=\delta_x^2U_i^{\sigma_n}+S_i^{\sigma_n},\quad 1\le i\le M-1,\ 1\le n\le N,\tag{18}$$

where there is a positive constant c_1 such that

$$|R_i^{\sigma_n}|\le c_1(\tau^2+h^4),\quad |S_i^{\sigma_n}|\le c_1(\tau^2+h^4),\quad 1\le i\le M-1, 1\le n\le N.\tag{19}$$

Next, the case of Eq. (16) with $i=0$ and $i=M$ will be specially discussed, respectively. Letting $x\to 0^+$ in Eq. (1), one can obtain by using the boundary condition (2) that

$$\frac{\partial^4 u(0,t)}{\partial x^4}=f(0,t)-\sum_{r=0}^{m}\lambda_r\,{}_0^CD_t^{\alpha_r}g_1(t)-qg_1(t),\quad 1\le n\le N.\tag{20}$$

Meanwhile, differentiating the both hand sides of Eq. (1) with respect to x once and letting $x\to 0^+$, using the boundary condition (3), we obtain

$$\frac{\partial^5 u(0,t)}{\partial x^5}=f_x(0,t)-\sum_{r=0}^{m}\lambda_r\,{}_0^CD_t^{\alpha_r}\gamma_1(t)-q\gamma_1(t),\quad 1\le n\le N.\tag{21}$$

Similarly, letting $x\to L^-$, we can get two equalities similar to the above two ones.

When $i=0$, Eq. (16) is

$$\mathcal{H}v(0,t_{n-1+\sigma})=\mathcal{H}\frac{\partial^2 u(0,t_{n-1+\sigma})}{\partial x^2},\quad 1\le n\le N.$$

Using Lemma 2.1, Lemma 2.2 and Eqs. (20), (21), we can obtain

$$\begin{aligned}\mathcal{H}V_0^{\sigma_n}=&\frac{2}{h}[\delta_xU_{\frac12}^{\sigma_n}-\gamma_1(t_{n-1+\sigma})]+\frac{h^2}{12}\Big[f(0,t_{n-1+\sigma})-\sum_{r=0}^{m}\lambda_r\,{}_0^CD_t^{\alpha_r}g_1(t_{n-1+\sigma})\\ &-qg_1(t_{n-1+\sigma})\Big]+\frac{7h^3}{180}\Big[f_x(0,t_{n-1+\sigma})-\sum_{r=0}^{m}\lambda_r\,{}_0^CD_t^{\alpha_r}\gamma_1(t_{n-1+\sigma})\\ &-q\gamma_1(t_{n-1+\sigma})\Big]+S_0^{\sigma_n}\\ =&\frac{2}{h}\delta_xU_{\frac12}^{\sigma_n}+p(t_{n-1+\sigma})+S_0^{\sigma_n},\quad 1\le n\le N,\end{aligned}\tag{22}$$

where

$$\begin{aligned}p(t)=&-\frac{2}{h}\gamma_1(t)+\frac{h^2}{12}\Big[f(0,t)-\sum_{r=0}^{m}\lambda_r\,{}_0^CD_t^{\alpha_r}g_1(t)-qg_1(t)\Big]\\ &+\frac{7h^3}{180}\Big[f_x(0,t)-\sum_{r=0}^{m}\lambda_r\,{}_0^CD_t^{\alpha_r}\gamma_1(t)-q\gamma_1(t)\Big]\end{aligned}$$

and there exists a positive constant c_2 such that

$$|S_0^{\sigma_n}| \le c_2(\tau^2 + h^4), \quad 1 \le n \le N. \tag{23}$$

Similarly, we can get

$$\begin{aligned}\mathcal{H}V_M^{\sigma_n} =& \frac{2}{h}[\gamma_2(t_{n-1+\sigma}) - \delta_x U_{M-\frac{1}{2}}^{\sigma_n}] \\ &+ \frac{h^2}{12}\Big[f(x_M, t_{n-1+\sigma}) - \sum_{r=0}^{m} \lambda_r \, {}_0^C D_t^{\alpha_r} g_2(t_{n-1+\sigma}) - q g_2(t_{n-1+\sigma})\Big] \\ &- \frac{7h^3}{180}\Big[f_x(x_M, t_{n-1+\sigma}) - \sum_{r=0}^{m} \lambda_r \, {}_0^C D_t^{\alpha_r} \gamma_2(t_{n-1+\sigma}) - q\gamma_2(t_{n-1+\sigma})\Big] + S_M^{\sigma_n} \\ =& -\frac{2}{h}\delta_x U_{M-\frac{1}{2}}^{\sigma_n} + q(t_{n-1+\sigma}) + S_M^{\sigma_n}, \quad 1 \le n \le N, \end{aligned} \tag{24}$$

where

$$\begin{aligned} q(t) =& \frac{2}{h}\gamma_2(t) + \frac{h^2}{12}\Big[f(x_M, t) - \sum_{r=0}^{m} \lambda_r \, {}_0^C D_t^{\alpha_r} g_2(t) - q g_2(t)\Big] \\ &- \frac{7h^3}{180}\Big[f_x(x_M, t) - \sum_{r=0}^{m} \lambda_r \, {}_0^C D_t^{\alpha_r} \gamma_2(t) - q\gamma_2(t)\Big] \end{aligned}$$

and there exists a constant c_3 such that

$$|S_M^{\sigma_n}| \le c_3(\tau^2 + h^4), \quad 1 \le n \le N. \tag{25}$$

Noticing the initial-boundary conditions (12) and (14), one has

$$U_0^n = g_1(t_n), \quad U_M^n = g_2(t_n), \quad 1 \le n \le N, \tag{26}$$

$$U_i^0 = \phi(x_i), \quad 0 \le i \le M. \tag{27}$$

Omitting the small terms $R_i^{\sigma_n}, S_i^{\sigma_n}$ in Eqs. (17), (18), (22), (24) and replacing the exact solution $\{U_i^n, V_i^n | 0 \le i \le M, 0 \le n \le N\}$ with the numerical one $\{u_i^n, v_i^n | 0 \le i \le M, 0 \le n \le N\}$, for Eqs. (10)-(14), we construct the following difference scheme:

$$\begin{aligned}&\mathcal{H}\sum_{r=0}^{m} \lambda_r \frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)} \sum_{k=0}^{n-1} C_k^{(n,\alpha_r)}(u_i^{n-k} - u_i^{n-k-1}) \\ &+ \delta_x^2 v_i^{\sigma_n} + q\mathcal{H}u_i^{\sigma_n} = \mathcal{H}f_i^{n-1+\sigma}, \quad 1 \le i \le M-1, \ 1 \le n \le N,\end{aligned} \tag{28}$$

$$\mathcal{H}v_i^{\sigma_n} = \delta_x^2 u_i^{\sigma_n}, \quad 1 \le i \le M-1, \ 1 \le n \le N, \tag{29}$$

$$\mathcal{H}v_0^{\sigma_n} = \frac{2}{h}\delta_x u_{\frac{1}{2}}^{\sigma_n} + p(t_{n-1+\sigma}), \quad 1 \le n \le N, \tag{30}$$

$$\mathcal{H}v_M^{\sigma_n} = -\frac{2}{h}\delta_x u_{M-\frac{1}{2}}^{\sigma_n} + q(t_{n-1+\sigma}), \quad 1 \le n \le N, \tag{31}$$

$$u_i^0 = \phi(x_i), \quad 0 \le i \le M, \tag{32}$$

$$u_0^n = g_1(t_n), \quad u_M^n = g_2(t_n), \quad 1 \le n \le N. \tag{33}$$

Remark 2: For the fractional ODEs, one can collocate the equation at $t = t_{n-1+\sigma}$ directly to yield the $(3-\alpha_0)$-th order convergence in view of Lemma 2.3. However, for the time-fractional PDEs, the spatial partial derivatives at $t = t_{n-1+\sigma}$ need be numerically evaluated together and we use Lemma 2.2 to handle it. That is, a weighted average of spatial partial derivatives at $t = t_n$ and $t = t_{n-1}$ is used to approximate the value of spatial partial derivatives at $t = t_{n-1+\sigma}$, so that the global convergence rate of difference scheme in time can only achieve two. The idea to treat the spatial partial derivatives is similar to that of Crank-Nicolson scheme for standard parabolic equation. For the time-fractional PDEs, the ideas can be found in some existing works, such as Refs. [8, 9, 14, 19] and so on.

Next, an equivalence result of the difference scheme (28)-(33) can be obtained.

Theorem 3.1. *The difference scheme* (28)-(33) *is equivalent to*

$$\begin{aligned}
&\frac{19}{36}\Big[\mathcal{H}\sum_{r=0}^{m}\lambda_r\frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)}\sum_{k=0}^{n-1}C_k^{(n,\alpha_r)}(u_1^{n-k}-u_1^{n-k-1})+q\mathcal{H}u_1^{\sigma_n}\Big]\\
&+\frac{1}{18}\Big[\mathcal{H}\sum_{r=0}^{m}\lambda_r\frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)}\sum_{k=0}^{n-1}C_k^{(n,\alpha_r)}(u_2^{n-k}-u_2^{n-k-1})+q\mathcal{H}u_2^{\sigma_n}\Big]\\
&+\frac{2}{h^3}\delta_x u_{\frac12}^{\sigma_n}-\frac{5}{3h^2}\delta_x^2u_1^{\sigma_n}+\frac{2}{3h^2}\delta_x^2u_2^{\sigma_n}\\
&=\frac{19}{36}\mathcal{H}f_1^{n-1+\sigma}+\frac{1}{18}\mathcal{H}f_2^{n-1+\sigma}-\frac{1}{h^2}p(t_{n-1+\sigma}),\quad 1\le n\le N,
\end{aligned}\tag{34}$$

$$\begin{aligned}
&\mathcal{H}^2\sum_{r=0}^{m}\lambda_r\frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)}\sum_{k=0}^{n-1}C_k^{(n,\alpha_r)}(u_i^{n-k}-u_i^{n-k-1})+\delta_x^4u_i^{\sigma_n}+q\mathcal{H}^2u_i^{\sigma_n}\\
&=\mathcal{H}^2f_i^{n-1+\sigma},\quad 2\le i\le M-2,\ 1\le n\le N,
\end{aligned}\tag{35}$$

$$\begin{aligned}
&\frac{19}{36}\Big[\mathcal{H}\sum_{r=0}^{m}\lambda_r\frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)}\sum_{k=0}^{n-1}C_k^{(n,\alpha_r)}(u_{M-1}^{n-k}-u_{M-1}^{n-k-1})+q\mathcal{H}u_{M-1}^{\sigma_n}\Big]\\
&+\frac{1}{18}\Big[\mathcal{H}\sum_{r=0}^{m}\lambda_r\frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)}\sum_{k=0}^{n-1}C_k^{(n,\alpha_r)}(u_{M-2}^{n-k}-u_{M-2}^{n-k-1})+q\mathcal{H}u_{M-2}^{\sigma_n}\Big]\\
&-\frac{2}{h^3}\delta_x u_{M-\frac12}^{\sigma_n}-\frac{5}{3h^2}\delta_x^2u_{M-1}^{\sigma_n}+\frac{2}{3h^2}\delta_x^2u_{M-2}^{\sigma_n}\\
&=\frac{19}{36}\mathcal{H}f_{M-1}^{n-1+\sigma}+\frac{1}{18}\mathcal{H}f_{M-2}^{n-1+\sigma}-\frac{1}{h^2}q(t_{n-1+\sigma}),\quad 1\le n\le N,
\end{aligned}\tag{36}$$

$$u_i^0=\phi(x_i),\quad 0\le i\le M,\tag{37}$$

$$u_0^n=g_1(t_n),\quad u_M^n=g_2(t_n),\quad 1\le n\le N\tag{38}$$

and

$$\mathcal{H}v_i^{\sigma_n}=\delta_x^2u_i^{\sigma_n},\quad 1\le i\le M-1,\ 1\le n\le N,\tag{39}$$

$$\mathcal{H}v_0^{\sigma_n}=\frac{2}{h}\delta_x u_{\frac12}^{\sigma_n}+p(t_{n-1+\sigma}),\quad 1\le n\le N,\tag{40}$$

$$\mathcal{H}v_M^{\sigma_n}=-\frac{2}{h}\delta_x u_{M-\frac12}^{\sigma_n}+q(t_{n-1+\sigma}),\quad 1\le n\le N.\tag{41}$$

Proof. Performing the average operator $\mathcal{H}$ and the operator δ_x^2 on both hand sides of (28) and (29), respectively, we have

$$\begin{aligned}
&\mathcal{H}^2\sum_{r=0}^{m}\lambda_r\frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)}\sum_{k=0}^{n-1}C_k^{(n,\alpha_r)}(u_i^{n-k}-u_i^{n-k-1})+\mathcal{H}\delta_x^2v_i^{\sigma_n}+q\mathcal{H}^2u_i^{\sigma_n}\\
&=\mathcal{H}^2f_i^{n-1+\sigma},\quad 2\le i\le M-2,\quad 1\le n\le N,
\end{aligned}\tag{42}$$

$$\mathcal{H}\delta_x^2v_i^{\sigma_n}=\delta_x^4u_i^{\sigma_n},\quad 2\le i\le M-2,\quad 1\le n\le N.\tag{43}$$

Substituting (43) into (42), we obtain

$$\mathcal{H}^2\sum_{r=0}^{m}\lambda_r\frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)}\sum_{k=0}^{n-1}C_k^{(n,\alpha_r)}(u_i^{n-k}-u_i^{n-k-1})+\delta_x^4u_i^{\sigma_n}+q\mathcal{H}^2u_i^{\sigma_n}$$
$$=\mathcal{H}^2f_i^{n-1+\sigma},\quad 2\le i\le M-2,\ 1\le n\le N, \tag{44}$$

which is exactly (35).

For $i=0$, rewriting the left hand of (30) gives

$$\mathcal{H}v_0^{\sigma_n}=h^2(b_1\delta_x^2v_1^{\sigma_n}+b_2\delta_x^2v_2^{\sigma_n})+b_3\mathcal{H}v_1^{\sigma_n}+b_4\mathcal{H}v_2^{\sigma_n},\quad 1\le n\le N. \tag{45}$$

Comparing the coefficients on both hand sides, we get the following system of linear equations

$$\begin{cases} b_1+\dfrac{b_3}{12}=\dfrac{2}{3},\\ -2b_1+b_2+\dfrac{5}{6}b_3+\dfrac{1}{12}b_4=\dfrac{1}{3},\\ b_1-2b_2+\dfrac{b_3}{12}+\dfrac{5}{6}b_4=0,\\ b_2+\dfrac{b_4}{12}=0,\end{cases}$$

which implies $b_1=\frac{19}{36}, b_2=\frac{1}{18}, b_3=\frac{5}{3}$ and $b_4=-\frac{2}{3}$. Applying the results of (28) and (29) with $i=1,2$ into (45), noticing (30) and (45) yields

$$\frac{2}{h}\delta_xu_{\frac12}^{\sigma_n}+p(t_{n-1+\sigma})=h^2\Big\{\frac{19}{36}\Big[-q\mathcal{H}u_1^{\sigma_n}+\mathcal{H}f_1^{n-1+\sigma}$$
$$-\mathcal{H}\sum_{r=0}^{m}\lambda_r\frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)}\sum_{k=0}^{n-1}C_k^{(n,\alpha_r)}(u_1^{n-k}-u_1^{n-k-1})\Big]$$
$$+\frac{1}{18}\Big[-\mathcal{H}\sum_{r=0}^{m}\lambda_r\frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)}\sum_{k=0}^{n-1}C_k^{(n,\alpha_r)}(u_2^{n-k}-u_2^{n-k-1})$$
$$-q\mathcal{H}u_2^{\sigma_n}+\mathcal{H}f_2^{n-1+\sigma}\Big]\Big\}+\frac{5}{3}\delta_x^2u_1^{\sigma_n}-\frac{2}{3}\delta_x^2u_2^{\sigma_n},\quad 1\le n\le N. \tag{46}$$

In a similar way, we get

$$-\frac{2}{h}\delta_xu_{M-\frac12}^{\sigma_n}+q(t_{n-1+\sigma})=h^2\Big\{\frac{19}{36}\Big[-q\mathcal{H}u_{M-1}^{\sigma_n}+\mathcal{H}f_{M-1}^{n-1+\sigma}$$
$$-\mathcal{H}\sum_{r=0}^{m}\lambda_r\frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)}\sum_{k=0}^{n-1}C_k^{(n,\alpha_r)}(u_{M-1}^{n-k}-u_{M-1}^{n-k-1})\Big]$$
$$+\frac{1}{18}\Big[-\mathcal{H}\sum_{r=0}^{m}\lambda_r\frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)}\sum_{k=0}^{n-1}C_k^{(n,\alpha_r)}(u_{M-2}^{n-k}-u_{M-2}^{n-k-1})$$
$$-q\mathcal{H}u_{M-2}^{\sigma_n}+\mathcal{H}f_{M-2}^{n-1+\sigma}\Big]\Big\}+\frac{5}{3}\delta_x^2u_{M-1}^{\sigma_n}-\frac{2}{3}\delta_x^2u_{M-2}^{\sigma_n},\quad 1\le n\le N. \tag{47}$$

Multiplying (46) and (47) by $\frac{1}{h^2}$, respectively, and rearranging the terms, we can acquire (34) and (36). The proof ends.

Based on Theorem 3.1, we can calculate the numerical solution $\{u_i^n | 0 \le i \le M, 0 \le n \le N\}$ directly from the difference scheme (34)-(38) for the problem (1)-(4), whereas, the theoretical analysis which follows will still start from the difference scheme (28)-(33), which is more convenient for the analysis.

4. Stability and convergence analysis of the finite difference scheme

In this section, the stability and convergence of the compact difference scheme (28)-(33) will be studied. To this end, we introduce some lemmas, which play a vital role in the subsequent analysis.

Lemma 4.1. *Let v be a grid function defined on Ω_h, then it holds that*

$$h\sum_{i=1}^{M-1}(\mathcal{H}v_i)^2 \ge \frac{5}{12}h\sum_{i=1}^{M-1}(v_i)^2 - \frac{5}{72}h(v_0^2+v_M^2). \tag{48}$$

Proof. A direct calculation shows

$$\begin{aligned}
& h\sum_{i=1}^{M-1}(\mathcal{H}v_i)^2 \\
= & h\sum_{i=1}^{M-1}\Big(\frac{1}{12}v_{i-1}+\frac{10}{12}v_i+\frac{1}{12}v_{i+1}\Big)^2 \\
= & \frac{1}{144}h\sum_{i=1}^{M-1}(v_{i-1}^2+20v_{i-1}v_i+2v_{i-1}v_{i+1}+100v_i^2+20v_iv_{i+1}+v_{i+1}^2) \\
\ge & \frac{1}{144}h\sum_{i=1}^{M-1}[v_{i-1}^2-10(v_{i-1}^2+v_i^2)-(v_{i-1}^2+v_{i+1}^2) \\
& +100v_i^2-10(v_i^2+v_{i+1}^2)+v_{i+1}^2] \\
= & \frac{1}{144}h\sum_{i=1}^{M-1}(-10v_{i-1}^2+80v_i^2-10v_{i+1}^2) \\
= & \frac{1}{144}\Big(-10h\sum_{i=0}^{M-2}v_i^2+80h\sum_{i=1}^{M-1}v_i^2-10h\sum_{i=2}^{M}v_i^2\Big) \\
= & \frac{1}{144}\Big(-10h\sum_{i=1}^{M-1}v_i^2-10hv_0^2+10hv_{M-1}^2+80h\sum_{i=1}^{M-1}v_i^2 \\
& -10h\sum_{i=1}^{M-1}v_i^2+10hv_1^2-10hv_M^2\Big) \\
\ge & \frac{5}{12}h\sum_{i=1}^{M-1}(v_i)^2-\frac{5}{72}h(v_0^2+v_M^2),
\end{aligned}$$

where the inequality $2ab \ge -(a^2+b^2)$ has been used in the third step.

Lemma 4.2. *For any grid function u and v defined on Ω_h, if $u_0 = u_M = 0$, we have*

$$h\sum_{i=1}^{M-1}\delta_x^2 v_i \cdot \mathcal{H}u_i - h\sum_{i=1}^{M-1}\delta_x^2 u_i \cdot \mathcal{H}v_i = \big(\delta_x u_{\frac{1}{2}}\big)v_0 - \big(\delta_x u_{M-\frac{1}{2}}\big)v_M. \tag{49}$$

Proof. Noticing that $\mathcal{H}u_i = \big(I + \frac{h^2}{12}\delta_x^2\big)u_i$ for $1 \le i \le M-1$ and $u_0 = u_M = 0$, we have

$$\begin{aligned}
& h\sum_{i=1}^{M-1}\delta_x^2 v_i \cdot \mathcal{H}u_i - h\sum_{i=1}^{M-1}\delta_x^2 u_i \cdot \mathcal{H}v_i \\
&= h\sum_{i=1}^{M-1}\delta_x^2 v_i \cdot \Big(I + \frac{h^2}{12}\delta_x^2\Big)u_i - h\sum_{i=1}^{M-1}\delta_x^2 u_i \cdot \Big(I + \frac{h^2}{12}\delta_x^2\Big)v_i \\
&= h\sum_{i=1}^{M-1}(\delta_x^2 v_i)u_i - h\sum_{i=1}^{M-1}(\delta_x^2 u_i)v_i \\
&= \big(\delta_x u_{\frac{1}{2}}\big)v_0 - \big(\delta_x u_{M-\frac{1}{2}}\big)v_M,
\end{aligned}$$

where the summation formula by parts has been applied in the last step above. This completes the proof.

Lemma 4.3. *[19] Denote* $\hat{c}_k^{(n)} = \sum_{r=0}^{m}\lambda_r\frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)}C_k^{(n,\alpha_r)}$ *and suppose that* u *is a grid function defined on* Ω_h*, then*

$$\begin{aligned}
& h\sum_{i=1}^{M-1}\sum_{k=0}^{n-1}\hat{c}_k^{(n)}(u_i^{n-k} - u_i^{n-k-1})u_i^{\sigma_n} \\
\geq & \frac{1}{2}\sum_{k=0}^{n-1}\hat{c}_k^{(n)}\Big[h\sum_{i=1}^{M-1}(u_i^{n-k})^2 - h\sum_{i=1}^{M-1}(u_i^{n-k-1})^2\Big], \quad 1 \le n \le N. \qquad (50)
\end{aligned}$$

With these preparations, the following theorem can be obtained.

Theorem 4.1. *(A prior estimate) Suppose that* $\{u_i^n | 0 \le i \le M, 0 \le n \le N\}$ *and* $\{v_i^{\sigma_n} | 0 \le i \le M, 1 \le n \le N\}$ *be the solution of the following difference scheme*

$$\begin{aligned}
& \mathcal{H}\sum_{r=0}^{m}\lambda_r\frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)}\sum_{k=0}^{n-1}C_k^{(n,\alpha_r)}(u_i^{n-k} - u_i^{n-k-1}) + \delta_x^2 v_i^{\sigma_n} + q\mathcal{H}u_i^{\sigma_n} \\
& = P_i^{n-1+\sigma}, \quad 1 \le i \le M-1,\ 1 \le n \le N, \qquad (51) \\
& \mathcal{H}v_i^{\sigma_n} = \delta_x^2 u_i^{\sigma_n} + Q_i^{n-1+\sigma}, \quad 1 \le i \le M-1,\ 1 \le n \le N, \qquad (52) \\
& \mathcal{H}v_0^{\sigma_n} = \frac{2}{h}\delta_x u_{\frac{1}{2}}^{\sigma_n} + Q_0^{n-1+\sigma}, \quad 1 \le n \le N, \qquad (53) \\
& \mathcal{H}v_M^{\sigma_n} = -\frac{2}{h}\delta_x u_{M-\frac{1}{2}}^{\sigma_n} + Q_M^{n-1+\sigma}, \quad 1 \le n \le N, \qquad (54) \\
& u_i^0 = \omega_i, \quad 0 \le i \le M, \qquad (55) \\
& u_0^n = 0, \quad u_M^n = 0, \quad 1 \le n \le N. \qquad (56)
\end{aligned}$$

Then we have

$$\begin{aligned}
h\sum_{i=1}^{M-1}(u_i^n)^2 \le & \frac{12}{5}\Bigg\{\frac{2}{\sum_{r=0}^{m}\frac{\lambda_r}{T^{\alpha_r}\Gamma(1-\alpha_r)}}\max_{1\le l\le n}\Bigg[\frac{1}{2q}h\sum_{i=1}^{M-1}(P_i^{l-1+\sigma})^2 \\
& + 2h\sum_{i=1}^{M-1}(Q_i^{l-1+\sigma})^2 + h(Q_0^{l-1+\sigma})^2 + h(Q_M^{l-1+\sigma})^2\Bigg] \\
& + h\sum_{i=1}^{M-1}(\mathcal{H}u_i^0)^2\Bigg\}, \quad 1 \le n \le N. \qquad (57)
\end{aligned}$$

Proof. Multiplying the both hand sides of (51) and (52) by $h\mathcal{H}u_i^{\sigma_n}, h\mathcal{H}v_i^{\sigma_n}$, respectively, and summing up for i from 1 to $M-1$, then adding the results, we get

$$\begin{aligned}
&h\sum_{i=1}^{M-1}\sum_{k=0}^{n-1}\hat{c}_k^{(n)}(\mathcal{H}u_i^{n-k}-\mathcal{H}u_i^{n-k-1})\mathcal{H}u_i^{\sigma_n}+h\sum_{i=1}^{M-1}\delta_x^2 v_i^{\sigma_n}\cdot\mathcal{H}u_i^{\sigma_n}\\
&+qh\sum_{i=1}^{M-1}(\mathcal{H}u_i^{\sigma_n})^2+h\sum_{i=1}^{M-1}(\mathcal{H}v_i^{\sigma_n})^2=h\sum_{i=1}^{M-1}P_i^{n-1+\sigma}\cdot\mathcal{H}u_i^{\sigma_n}\\
&+h\sum_{i=1}^{M-1}\delta_x^2 u_i^{\sigma_n}\cdot\mathcal{H}v_i^{\sigma_n}+h\sum_{i=1}^{M-1}Q_i^{n-1+\sigma}\cdot\mathcal{H}v_i^{\sigma_n},\quad 1\le n\le N.
\end{aligned}\tag{58}$$

Using the ε-inequality ($ab\le\varepsilon a^2+\frac{1}{4\varepsilon}b^2$), for $1\le n\le N$, we obtain

$$h\sum_{i=1}^{M-1}P_i^{n-1+\sigma}\cdot\mathcal{H}u_i^{\sigma_n}\le qh\sum_{i=1}^{M-1}(\mathcal{H}u_i^{\sigma_n})^2+\frac{1}{4q}h\sum_{i=1}^{M-1}(P_i^{n-1+\sigma})^2,\tag{59}$$

$$h\sum_{i=1}^{M-1}Q_i^{n-1+\sigma}\cdot\mathcal{H}v_i^{\sigma_n}\le\frac{h}{4}\sum_{i=1}^{M-1}(\mathcal{H}v_i^{\sigma_n})^2+h\sum_{i=1}^{M-1}(Q_i^{n-1+\sigma})^2.\tag{60}$$

Noticing Lemmas 4.1-4.3 and substituting (59), (60) into (58), we get

$$\begin{aligned}
&\frac{1}{2}\sum_{k=0}^{n-1}\hat{c}_k^{(n)}\left[h\sum_{i=1}^{M-1}(\mathcal{H}u_i^{n-k})^2-h\sum_{i=1}^{M-1}(\mathcal{H}u_i^{n-k-1})^2\right]+\frac{5}{16}h\sum_{i=1}^{M-1}(v_i^{\sigma_n})^2\\
\le&-(\delta_x u_{\frac12}^{\sigma_n})v_0^{\sigma_n}+(\delta_x u_{M-\frac12}^{\sigma_n})v_M^{\sigma_n}+\frac{5h}{96}(v_0^{\sigma_n})^2+\frac{5h}{96}(v_M^{\sigma_n})^2\\
&+\frac{1}{4q}h\sum_{i=1}^{M-1}(P_i^{n-1+\sigma})^2+h\sum_{i=1}^{M-1}(Q_i^{n-1+\sigma})^2,\quad 1\le n\le N.
\end{aligned}\tag{61}$$

From Eqs. (53) and (54), using the ε-inequality, one can know that

$$\begin{aligned}
&-(\delta_x u_{\frac12}^{\sigma_n})v_0^{\sigma_n}+(\delta_x u_{M-\frac12}^{\sigma_n})v_M^{\sigma_n}+\frac{5h}{96}(v_0^{\sigma_n})^2+\frac{5h}{96}(v_M^{\sigma_n})^2\\
=&-\frac{h}{2}(\mathcal{H}v_0^{\sigma_n}-Q_0^{n-1+\sigma})v_0^{\sigma_n}-\frac{h}{2}(\mathcal{H}v_M^{\sigma_n}-Q_M^{n-1+\sigma})v_M^{\sigma_n}\\
&+\frac{5h}{96}(v_0^{\sigma_n})^2+\frac{5h}{96}(v_M^{\sigma_n})^2\\
=&-\frac{h}{2}\Big(\frac{2}{3}v_0^{\sigma_n}+\frac{1}{3}v_1^{\sigma_n}-Q_0^{n-1+\sigma}\Big)v_0^{\sigma_n}-\frac{h}{2}\Big(\frac{2}{3}v_M^{\sigma_n}+\frac{1}{3}v_{M-1}^{\sigma_n}\\
&-Q_M^{n-1+\sigma}\Big)v_M^{\sigma_n}+\frac{5h}{96}(v_0^{\sigma_n})^2+\frac{5h}{96}(v_M^{\sigma_n})^2\\
\le&-\frac{h}{3}(v_0^{\sigma_n})^2+\frac{h}{6}\Big[\frac{1}{4}(v_0^{\sigma_n})^2+(v_1^{\sigma_n})^2\Big]+\frac{h}{2}\Big[\frac{1}{4}(v_0^{\sigma_n})^2+(Q_0^{n-1+\sigma})^2\Big]\\
&-\frac{h}{3}(v_M^{\sigma_n})^2+\frac{h}{6}\Big[\frac{1}{4}(v_M^{\sigma_n})^2+(v_{M-1}^{\sigma_n})^2\Big]\\
&+\frac{h}{2}\Big[\frac{1}{4}(v_M^{\sigma_n})^2+(Q_M^{n-1+\sigma})^2\Big]+\frac{5h}{96}(v_0^{\sigma_n})^2+\frac{5h}{96}(v_M^{\sigma_n})^2\\
=&-\frac{11h}{96}(v_0^{\sigma_n})^2+\frac{h}{6}(v_1^{\sigma_n})^2-\frac{11h}{96}(v_M^{\sigma_n})^2+\frac{h}{6}(v_{M-1}^{\sigma_n})^2\\
&+\frac{h}{2}(Q_0^{n-1+\sigma})^2+\frac{h}{2}(Q_M^{n-1+\sigma})^2
\end{aligned}$$

$$\leq \frac{h}{6}[(v_1^{\sigma_n})^2 + (v_{M-1}^{\sigma_n})^2] + \frac{h}{2}[(Q_0^{n-1+\sigma})^2 + (Q_M^{n-1+\sigma})^2], \quad 1 \leq n \leq N. \tag{62}$$

Noticing

$$\frac{h}{6}[(v_1^{\sigma_n})^2 + (v_{M-1}^{\sigma_n})^2] \leq \frac{5}{16} h \sum_{i=1}^{M-1} (v_i^{\sigma_n})^2,$$

the substitution of (62) into (61) produces

$$\begin{aligned}
&\frac{1}{2}\sum_{k=0}^{n-1} \hat{c}_k^{(n)} \left[h \sum_{i=1}^{M-1} (\mathcal{H}u_i^{n-k})^2 - h \sum_{i=1}^{M-1} (\mathcal{H}u_i^{n-k-1})^2 \right] \\
\leq& \frac{1}{4q} h \sum_{i=1}^{M-1} (P_i^{n-1+\sigma})^2 + h \sum_{i=1}^{M-1} (Q_i^{n-1+\sigma})^2 \\
&+ \frac{h}{2}[(Q_0^{n-1+\sigma})^2 + (Q_M^{n-1+\sigma})^2], \quad 1 \leq n \leq N,
\end{aligned}$$

that is

$$\begin{aligned}
&\sum_{k=0}^{n-1} \hat{c}_k^{(n)} \left[h \sum_{i=1}^{M-1} (\mathcal{H}u_i^{n-k})^2 - h \sum_{i=1}^{M-1} (\mathcal{H}u_i^{n-k-1})^2 \right] \\
\leq& \frac{1}{2q} h \sum_{i=1}^{M-1} (P_i^{n-1+\sigma})^2 + 2h \sum_{i=1}^{M-1} (Q_i^{n-1+\sigma})^2 \\
&+ h[(Q_0^{n-1+\sigma})^2 + (Q_M^{n-1+\sigma})^2], \quad 1 \leq n \leq N,
\end{aligned}$$

or

$$\begin{aligned}
\hat{c}_0^{(n)} h \sum_{i=1}^{M-1} (\mathcal{H}u_i^n)^2 \leq& \sum_{k=1}^{n-1} (\hat{c}_{k-1}^{(n)} - \hat{c}_k^{(n)}) h \sum_{i=1}^{M-1} (\mathcal{H}u_i^{n-k})^2 + \hat{c}_{n-1}^{(n)} h \sum_{i=1}^{M-1} (\mathcal{H}u_i^0)^2 \\
&+ \frac{1}{2q} h \sum_{i=1}^{M-1} (P_i^{n-1+\sigma})^2 + 2h \sum_{i=1}^{M-1} (Q_i^{n-1+\sigma})^2 \\
&+ h[(Q_0^{n-1+\sigma})^2 + (Q_M^{n-1+\sigma})^2], \quad 1 \leq n \leq N.
\end{aligned}$$

Noticing [19]

$$\hat{c}_{n-1}^{(n)} \geq \sum_{r=0}^{m} \lambda_r \frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)} \frac{1-\alpha_r}{2} (n-1+\sigma)^{-\alpha_r} \geq \frac{1}{2} \sum_{r=0}^{m} \frac{\lambda_r}{T^{\alpha_r}\Gamma(1-\alpha_r)},$$

further one can get

$$\begin{aligned}
\hat{c}_0^{(n)} h \sum_{i=1}^{M-1} (\mathcal{H}u_i^n)^2 \leq& \sum_{k=1}^{n-1} (\hat{c}_{k-1}^{(n)} - \hat{c}_k^{(n)}) h \sum_{i=1}^{M-1} (\mathcal{H}u_i^{n-k})^2 + \hat{c}_{n-1}^{(n)} \Bigg\{ h \sum_{i=1}^{M-1} (\mathcal{H}u_i^0)^2 \\
&+ \frac{2}{\sum_{r=0}^{m} \frac{\lambda_r}{T^{\alpha_r}\Gamma(1-\alpha_r)}} \left[\frac{1}{2q} h \sum_{i=1}^{M-1} (P_i^{n-1+\sigma})^2 + h(Q_0^{n-1+\sigma})^2 \right. \\
&\left. + 2h \sum_{i=1}^{M-1} (Q_i^{n-1+\sigma})^2 + h(Q_M^{n-1+\sigma})^2 \right] \Bigg\}, \quad 1 \leq n \leq N.
\end{aligned}$$

The induction method applied to the above inequality will lead to

$$h \sum_{i=1}^{M-1} (\mathcal{H}u_i^n)^2 \leq \frac{2}{\sum_{r=0}^{m} \frac{\lambda_r}{T^{\alpha_r}\Gamma(1-\alpha_r)}} \max_{1 \leq l \leq n} \left[\frac{1}{2q} h \sum_{i=1}^{M-1} (P_i^{l-1+\sigma})^2 \right.$$

$$+ 2h\sum_{i=1}^{M-1}(Q_i^{l-1+\sigma})^2 + h(Q_0^{l-1+\sigma})^2 + h(Q_M^{l-1+\sigma})^2\Big]$$
$$+ h\sum_{i=1}^{M-1}(\mathcal{H}u_i^0)^2, \quad 1 \le n \le N.$$

Noticing (56) and Lemma 4.1, further one can reach the desired inequality (57). The proof ends.

From Theorem 3.1 and Theorem 4.1, one can read off the unconditional stability of the difference scheme (34)-(38) with respect to both the initial value and the source term $f(x,t)$.

Theorem 4.2. *(Stability) The difference scheme* (34)-(38) *is unconditionally stable with respect to the right hand term f and the initial value u^0.*

Theorem 4.3. *(Convergence) Suppose that $\{u_i^n, v_i^n \mid 0 \le i \le M, 0 \le n \le N\}$ is the solution of the difference scheme* (28)-(33). *Let*

$$e_i^n = U_i^n - u_i^n, \quad \epsilon_i^n = V_i^n - v_i^n, \quad 0 \le i \le M, \quad 0 \le n \le N,$$

then there exists a positive constant c, independent of h and τ, such that

$$\sqrt{h\sum_{i=1}^{M-1}(e_i^n)^2} \le c(\tau^2 + h^4), \quad 1 \le n \le N,$$

where

$$c^2 = \frac{24L}{5\sum_{r=0}^{m}\frac{\lambda_r}{T^{\alpha_r}\Gamma(1-\alpha_r)}}\left[\Big(\frac{1}{2q}+2\Big)c_1^2 + c_2^2 + c_3^2\right].$$

Proof. Subtracting Eqs. (28)-(33) from (17)-(18), (22), (24) and (26)-(27), respectively, we have the error system

$$\mathcal{H}\sum_{r=0}^{m}\lambda_r\frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)}\sum_{k=0}^{n-1}C_k^{(n,\alpha_r)}(e_i^{n-k} - e_i^{n-k-1})$$
$$+\delta_x^2\epsilon_i^{\sigma_n} + q\mathcal{H}e_i^{\sigma_n} = R_i^{\sigma_n}, \quad 1 \le i \le M-1, \ 1 \le n \le N, \tag{63}$$
$$\mathcal{H}\epsilon_i^{\sigma_n} = \delta_x^2 e_i^{\sigma_n} + S_i^{\sigma_n}, \quad 1 \le i \le M-1, \ 1 \le n \le N, \tag{64}$$
$$\mathcal{H}\epsilon_0^{\sigma_n} = \frac{2}{h}\delta_x e_{\frac{1}{2}}^{\sigma_n} + S_0^{\sigma_n}, \quad 1 \le n \le N, \tag{65}$$
$$\mathcal{H}\epsilon_M^{\sigma_n} = -\frac{2}{h}\delta_x e_{M-\frac{1}{2}}^{\sigma_n} + S_M^{\sigma_n}, \quad 1 \le n \le N, \tag{66}$$
$$e_i^0 = 0, \quad 0 \le i \le M, \tag{67}$$
$$e_0^n = 0, \quad e_M^n = 0, \quad 1 \le n \le N. \tag{68}$$

The application of Theorem 4.1 into (63)-(68) produces

$$h\sum_{i=1}^{M-1}(e_i^n)^2 \le \frac{12}{5}\Bigg\{\frac{2}{\sum_{r=0}^{m}\frac{\lambda_r}{T^{\alpha_r}\Gamma(1-\alpha_r)}}\max_{1\le l\le n}\Big[\frac{1}{2q}h\sum_{i=1}^{M-1}(R_i^{\sigma_l})^2 + h(S_0^{\sigma_l})^2$$
$$+ 2h\sum_{i=1}^{M-1}(S_i^{\sigma_l})^2 + h(S_M^{\sigma_l})^2\Big] + h\sum_{i=1}^{M-1}(\mathcal{H}e_i^0)^2\Bigg\}, \quad 1 \le n \le N.$$

Noticing (19), (23) and (25), together with (67)-(68), further it follows

$$h\sum_{i=1}^{M-1}(e_i^n)^2 \le \frac{24}{5\sum_{r=0}^{m}\frac{\lambda_r}{T^{\alpha_r}\Gamma(1-\alpha_r)}}\left[\Big(\frac{L}{2q}+2L\Big)c_1^2(\tau^2+h^4)^2\right.$$

$$
\begin{aligned}
&+ (c_2^2 + c_3^2)h(\tau^2 + h^4)^2\Big] \\
\le &\frac{24L}{5\sum_{r=0}^{m} \frac{\lambda_r}{T^{\alpha_r}\Gamma(1-\alpha_r)}} \Big[\Big(\frac{1}{2q} + 2\Big)c_1^2 + c_2^2 + c_3^2\Big](\tau^2 + h^4)^2, \\
&\qquad\qquad\qquad\qquad\qquad\qquad 1 \le n \le N.
\end{aligned}
$$

The proof ends.

TABLE 1. Numerical errors and convergence orders of the difference scheme (34)-(38) in time for solving Example 5.1 ($M = 100$).

		$(\lambda_0,\lambda_1,\lambda_2)=(3,2,1)$		$(\lambda_0,\lambda_1,\lambda_2)=(1,2,3)$	
$(\alpha_0,\alpha_1,\alpha_2)$	τ	err(h,τ)	order$_t$	err(h,τ)	order$_t$
	1/10	7.021464e − 3	1.96	6.076584e − 3	1.95
	1/20	1.807379e − 3	1.98	1.568653e − 3	1.97
$(\frac{1}{3},\frac{1}{4},\frac{1}{5})$	1/40	4.593740e − 4	1.99	3.996679e − 4	1.98
	1/80	1.160392e − 4	1.99	1.011902e − 4	1.99
	1/160	2.922078e − 5	—	2.554141e − 5	—
	1/10	1.202742e − 2	1.97	1.068150e − 2	1.95
	1/20	3.071287e − 3	1.98	2.762612e − 3	1.97
$(\frac{2}{3},\frac{1}{2},\frac{1}{3})$	1/40	7.774683e − 4	1.99	7.070557e − 4	1.97
	1/80	1.959674e − 4	1.99	1.799903e − 4	1.98
	1/160	4.928183e − 5	—	4.567631e − 5	—
	1/10	1.460960e − 2	1.99	1.404951e − 2	1.96
	1/20	3.673673e − 3	2.00	3.612040e − 3	1.98
$(1,\frac{1}{2},0)$	1/40	9.202870e − 4	2.00	9.139085e − 4	1.99
	1/80	2.302377e − 4	2.00	2.296745e − 4	2.00
	1/160	5.757304e − 5	—	5.755096e − 5	—

5. Numerical examples

In this section, we are devoted to use some numerical examples to further validate our previous theoretical results. Denote

$$
\mathrm{err}(h,\tau) = \max_{\substack{0\leqslant i\leqslant M\\ 0\leqslant n\leqslant N}} |u(x_i,t_n) - u_i^n|, \quad \mathrm{order}_t = \log_2 \frac{\mathrm{err}(h,\tau)}{\mathrm{err}(h,\tau/2)},
$$
$$
\mathrm{order}_x = \log_2 \frac{\mathrm{err}(h,\tau)}{\mathrm{err}(h/2,\tau)}.
$$

Example 5.1. *In* (1)-(4)*, take* $L = 1$*,* $T = 1$*,* $m = 2$*,* $q = 1$*,* $\phi(x) = 0$*,* $f(x,t) = [24\sum_{r=0}^{m} \lambda_r t^{4-\alpha_r}/\Gamma(5-\alpha_r) + \pi^4 t^4 + t^4]\sin \pi x$*,* $g_1(t) = 0$*,* $g_2(t) = 0$*,* $\gamma_1(t) = \pi t^4$*,* $\gamma_2(t) = -\pi t^4$*.*

The exact solution of this example is $u(x,t) = t^4 \sin \pi x$.

Taking different values of λ_r and $\alpha_r (r = 0, 1, \ldots, m)$, we calculate the maximum errors and convergence orders using the difference scheme (34)-(38). Firstly, when the spatial step size h is fixed to be sufficiently small and the temporal step size τ varies from $\frac{1}{10}$ to $\frac{1}{160}$, the computational results will be listed in Table 1, from which one can see that the convergence order in time of the difference scheme (34)-(38) we proposed reaches the second-order accuracy, which is in agreement with our theoretical results.

TABLE 2. Numerical errors and convergence orders of the difference scheme (34)-(38) in space for solving Example 5.1 ($N = 10000$).

$(\alpha_0, \alpha_1, \alpha_2)$	h	$(\lambda_0, \lambda_1, \lambda_2) = (3, 2, 1)$ err(h, τ)	order$_x$	$(\lambda_0, \lambda_1, \lambda_2) = (1, 2, 3)$ err(h, τ)	order$_x$
$(\frac{1}{3}, \frac{1}{4}, \frac{1}{5})$	1/4	6.061004e − 4	3.90	6.067914e − 4	3.90
	1/8	4.058035e − 5	3.98	4.062686e − 5	3.98
	1/16	2.579793e − 6	4.06	2.583560e − 6	4.05
	1/32	1.548619e − 7	—	1.558968e − 7	—
$(\frac{2}{3}, \frac{1}{2}, \frac{1}{3})$	1/4	6.008189e − 4	3.90	6.031158e − 4	3.90
	1/8	4.022607e − 5	3.98	4.037848e − 5	3.98
	1/16	2.552319e − 6	4.11	2.562481e − 6	4.10
	1/32	1.482862e − 7	—	1.493519e − 7	—
$(1, \frac{1}{2}, 0)$	1/4	5.965199e − 4	3.90	6.036876e − 4	3.90
	1/8	3.993985e − 5	3.98	4.041412e − 5	3.98
	1/16	2.532271e − 6	4.12	2.562545e − 6	4.12
	1/32	1.452571e − 7	—	1.471400e − 7	—

Similarly, Table 2 lists the maximum errors and convergence orders under different values of h, from which, we can easily get the similar conclusion as shown above, that is, the convergence order of the difference scheme (34)-(38) in space matches the theoretical results we proved previously.

TABLE 3. Numerical errors and convergence orders of the difference scheme (34)-(38) in time for solving Example 5.2 ($M = 100$).

$(\alpha_0, \alpha_1, \alpha_2)$	τ	$(\lambda_0, \lambda_1, \lambda_2) = (3, 2, 1)$ err(h, τ)	order$_t$	$(\lambda_0, \lambda_1, \lambda_2) = (1, 2, 3)$ err(h, τ)	order$_t$
$(\frac{1}{3}, \frac{1}{4}, \frac{1}{5})$	1/10	7.021464e − 3	1.96	6.076584e − 3	1.95
	1/20	1.807379e − 3	1.98	1.568653e − 3	1.97
	1/40	4.593742e − 4	1.99	3.996681e − 4	1.98
	1/80	1.160392e − 4	1.99	1.011901e − 4	1.99
	1/160	2.922070e − 5	—	2.554131e − 5	—
$(\frac{2}{3}, \frac{1}{2}, \frac{1}{3})$	1/10	1.202742e − 2	1.97	1.068150e − 2	1.95
	1/20	3.071287e − 3	1.98	2.762612e − 3	1.97
	1/40	7.774684e − 4	1.99	7.070557e − 4	1.97
	1/80	1.959674e − 4	1.99	1.799902e − 4	1.98
	1/160	4.928188e − 5	—	4.567628e − 5	—
$(1, \frac{1}{2}, 0)$	1/10	1.460960e − 2	1.99	1.404951e − 2	1.96
	1/20	3.673673e − 3	2.00	3.612040e − 3	1.98
	1/40	9.202869e − 4	2.00	9.139085e − 4	1.99
	1/80	2.302375e − 4	2.00	2.296744e − 4	2.00
	1/160	5.757295e − 5	—	5.755088e − 5	—

Example 5.2. *In* (1)-(4), *take* $L = 1$, $T = 1$, $m = 2$, $\phi(x) = \cos x$, $q = 1$, $f(x,t) = [24 \sum_{r=0}^{m} \lambda_r t^{4-\alpha_r}/\Gamma(5-\alpha_r) + \pi^4 t^4 + t^4] \sin \pi x + 2\cos x$, $g_1(t) = 1$, $g_2(t) = \cos 1$, $\gamma_1(t) = \pi t^4$, $\gamma_2(t) = -\pi t^4 - \sin 1$.

The exact solution for this example is $u(x,t) = t^4 \sin \pi x + \cos x$.

Table 3 lists the maximum errors and the temporal convergence orders when the spatial step size $h = 1/100$ and τ is taken as 1/10, 1/20, 1/40, 1/80 and 1/160,

Table 4. Numerical errors and convergence orders of the difference scheme (34)-(38) in space for solving Example 5.2 ($N = 10000$).

		$(\lambda_0, \lambda_1, \lambda_2) = (3, 2, 1)$		$(\lambda_0, \lambda_1, \lambda_2) = (1, 2, 3)$	
$(\alpha_0, \alpha_1, \alpha_2)$	h	err(h, τ)	order$_x$	err(h, τ)	order$_x$
$(\frac{1}{3}, \frac{1}{4}, \frac{1}{5})$	1/8	4.056457e − 5	3.98	4.061108e − 5	3.97
	1/16	2.579443e − 6	4.06	2.583210e − 6	4.05
	1/32	1.548593e − 7	—	1.558946e − 7	—
$(\frac{2}{3}, \frac{1}{2}, \frac{1}{3})$	1/8	4.021022e − 5	3.98	4.036266e − 5	3.98
	1/16	2.551968e − 6	4.11	2.562130e − 6	4.10
	1/32	1.482846e − 7	—	1.493498e − 7	—
$(1, \frac{1}{2}, 0)$	1/8	3.992396e − 5	3.98	4.039831e − 5	3.98
	1/16	2.531920e − 6	4.12	2.562195e − 6	4.12
	1/32	1.452536e − 7	—	1.471360e − 7	—

respectively. From this table, we can see that, as what is expected, the convergence order of the finite difference scheme (34)-(38) in time is two. Besides, in order to examine the convergence order in space, we choose a sufficiently small temporal step size $\tau = 1/10000$ and h is taken as 1/8, 1/16 and 1/32, respectively. Table 4 presents the computational results which are in accord with the theoretical ones we proved in last section. It further illustrates the reliability of the difference scheme (34)-(38).

From Examples 5.1 and 5.2, we can observe that whether the solution of the differential equation satisfies the zero boundary value or not, the finite difference scheme (34)-(38) we proposed can both reach the convergence order $O(\tau^2 + h^4)$ in maximum norm, if the solution has the enough regularity, which is stronger than our theoretical results in L_2 norm.

6. Conclusion

In this paper, a class of the fourth-order time multi-term fractional sub-diffusion equations with the first Dirichlet boundary conditions is numerically considered. Firstly, the original problem is equivalently converted into a lower-order system by introducing the intermediate function $v(x, t)$. Then the system is considered at some particular points and the first Dirichlet boundary conditions are specially handled by the techniques different from the previous works in [24] and [25]. A finite difference scheme with the global convergence order $O(\tau^2 + h^4)$ is successfully proposed. The stability and convergence of the proposed scheme were rigorously proved by the energy method. Numerical results further validate our theoretical analysis. Noticing that the finite difference scheme we proposed requires the storage at all previous time steps, when the temporal step size is sufficiently small, the computational work and the storage will be huge. In further research, a fast evaluation method to deal with the Caputo fractional derivative in the current problem will be investigated.

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