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Triple Crossing Numbers of Graphs

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Abstract: We introduce the triple crossing number, a variation of the crossing number, of a graph, which is the minimal number of crossing points in all drawings of the graph with only triple crossings. It is defined to be zero for planar graphs, and to be infinite for non-planar graphs which do not admit a drawing with only triple crossings. In this paper, we determine the triple crossing numbers for all complete multipartite graphs which include all complete graphs.

Key words: crossing number, triple crossing number, complete multipartite graph

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1 Introduction

Let G be a graph. A drawing of G means a representation of the graph in the Euclidean plane or the 2-sphere, where vertices are points and edges are simple arcs joining their end-vertices. Since each edge is simple, no edge admits self crossings. Furthermore, we assume that the interiors of edges do not contain vertices, and that two edges do not intersect if they have a common vertex, and that two edges without common end-vertex intersect at most once, and if so, then they intersect transversally. These requirements are essential in this paper. A drawing is called a regular drawing (resp. semi-regular drawing) if it has only double (resp. triple) crossing points. From the requirements, we know that a graph has at least 6 vertices if it admits a semi-regular drawing with at least one triple crossing point.

The crossing number $\mathrm{cr}(G)$ of G is defined to be the minimal number of crossing points over all regular drawings of G. In particular, $\mathrm{cr}(G) = 0$ if G is planar. In this paper, we

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introduce a new variation of the crossing number. The triple crossing number $\mathrm{tcr}(G)$ is zero if G is planar, and ∞ if G does not admit a semi-regular drawing. Otherwise, $\mathrm{tcr}(G)$ is defined to be the minimal number of triple crossing points over all semi-regular drawings of G. In particular, $\mathrm{tcr}(G) = 0$ if and only if G is planar.

The triple crossing number can be regarded as a specialization of the degenerate crossing number introduced by Pach and Tóth[1]. In addition, for example, the Petersen graph is known to have the crossing number two (and thus non-planar), and hence has the triple crossing number one from Fig. 1.1. In general, we have the inequality $\mathrm{cr}(G) \leq 3\mathrm{tcr}(G)$ for these two notions, since we obtain a regular drawing from a semi-regular drawing by perturbing each triple crossing point into three double crossing points.

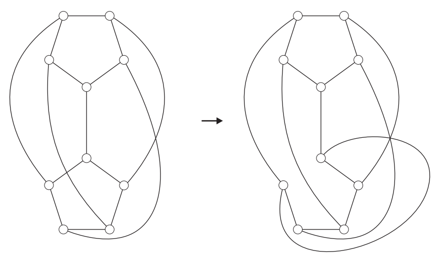


Fig. 1.1 The Petersen graph

In this paper, we determine the triple crossing numbers for all complete multipartite graphs. A complete multipartite graph is a graph whose vertex set can be partitioned into at least two, mutually disjoint non-empty sets, called the partite sets, so that two vertices u and v are adjacent if and only if u and v belong to different sets of the partition. If the partite sets are of sizes $n_1, \cdots, n_t$ ($n_i \geq 1$), then the graph is denoted by $K_{n_1,\cdots,n_t}$. We always assume that $n_i \geq n_j$ if $i < j$. In particular, if $n_i = 1$ for each i, then the graph $K_{1,\cdots,1}$ is the complete graph K_t with t vertices.

Here is how this paper is organized. After we describe basic lemmas, used in the paper repeatedly, in Section 2, we show that the triple crossing number of a complete t-partite graph is ∞ if $t \geq 5$ in Section 3. In the successive sections, we work on the cases when $t \leq 4$. Here we mention that the hardest part is the case where $t = 2$, in particular, long, but elementary, geometric arguments are needed to show that $K_{5,4}$, $K_{4,4}$, $K_{5,3}$ and $K_{n,3}$ with $n \geq 7$ do not admit a semi-regular drawing. This is treated in Sections 4, 5 and 6. After concluding the case where $t = 2$ in Section 7, the cases where $t = 4$ and $t = 3$ are established in Sections 8 and 9, respectively. Section 10 contains some remarks on our requirements for drawings and a generalization of triple crossing number.

2 Basic Lemmas

Basic terms of graph theory can be found in textbooks such as [2]–[3].

Lemma 2.1 *The complete bipartite graph $K_{3,3}$ and the complete graph K_5 with five vertices are non-planar. Also, a graph is non-planar if it contains $K_{3,3}$ or K_5 as a subgraph.*

For a graph with p vertices and q edges, let $d = 3p - q - 6$.

Lemma 2.2 *Let G be a plane graph with $p(\geq 3)$ vertices and q edges. Then G has a vertex of degree less than 6 and the inequality $d \geq 0$ holds. Furthermore, $d = 0$ if and only if each region of G is 3-sided.*

For the proofs of these lemmas, see [2].

Lemma 2.3 *Let G be a graph with $p(\geq 3)$ vertices and q edges. If G admits a semi-regular drawing, then $d \geq 0$. Thus, if $d < 0$, then $\mathrm{tcr}(G) = \infty$.*

Proof. Let D be a semi-regular drawing of G, and let k be the number of triple crossing points in D. If a new vertex is added to each triple crossing point, then we obtain a (simple) plane graph G'. Since G' has $p + k$ vertices and $q + 3k$ edges, we have $q + 3k \leq 3(p + k) - 6$ by Lemma 2.2, from which we have the conclusion.

Lemma 2.4 *Let G be a connected plane graph with $p(\geq 3)$ vertices, q edges and r faces.*

(1) *If $d = 1$, then one face is 4-sided, and the others are 3-sided.*
(2) *If $d = 2$, then either*
(a) *one face is 5-sided, and the others are 3-sided; or*
(b) *two faces are 4-sided, and the others are 3-sided.*
(3) *If $d = 3$, then either*
(a) *one face is 6-sided, and the others are 3-sided; or*
(b) *one face is 5-sided, another face is 4-sided, and the others are 3-sided; or*
(c) *three faces are 4-sided, and the others are 3-sided.*

Proof. By Euler's formula (see [2]), $p - q + r = 2$. Let r_i denote the number of i-sided faces of G. Then

$$3r_3 + 4r_4 + 5r_5 + 6r_6 + \sum_{i\geq 7} ir_i = 2q. \tag{2.1}$$

Thus

$$7r - 4r_3 - 3r_4 - 2r_5 - r_6 \leq 2q.$$

Since $q = p + r - 2$ and $d = 2p - r - 4$, we have

$$4r - d \leq 4r_3 + 3r_4 + 2r_5 + r_6 \leq 4(r_3 + r_4 + r_5 + r_6) \leq 4r. \tag{2.2}$$

In particular, the difference between the second and third terms, which is $r_4 + 2r_5 + 3r_6$, is at most d. We remark that $d \equiv r \pmod 2$.

When $d \in \{1, 2, 3\}$, $4r$ is the only multiple of four within the interval $[4r - d, 4r]$. Since $4(r_3 + r_4 + r_5 + r_6)$ is a multiple of four, we see $4(r_3 + r_4 + r_5 + r_6) = 4r$, giving $r_3 + r_4 + r_5 + r_6 = r$. Furthermore, if $r_5 = r_6 = 0$, then (2.1) reduces to $3r_3 + 4r_4 = 2q$. Combining this with $r_3 + r_4 = r$ gives $r_4 = 2q - 3r = d$.

(1) Since $r_4 + 2r_5 + 3r_6 \leq d = 1$, we have $r_5 = r_6 = 0$. Then $r_4 = 1$, and thus $r_3 = r - 1$.

(2) Since $r_4 + 2r_5 + 3r_6 \leq 2$, we have $r_6 = 0$ and $r_5 \leq 1$. If $r_5 = 1$, then $r_4 = 0$, giving $r_3 = r - 1$. This is the conclusion (a). If $r_5 = 0$, then $r_4 = 2$, and thus $r_3 = r - 2$. This is the conclusion (b).

(3) Since $r_4 + 2r_5 + 3r_6 \leq 3$, we have $r_6 \leq 1$. If $r_6 = 1$, then $r_4 = r_5 = 0$, and thus $r_3 = r - 1$. This is the conclusion (a).

Suppose $r_6 = 0$. Since $r_4 + 2r_5 \leq 3$, we see $r_5 \leq 1$.

If $r_5 = 1$, then $r_4 \leq 1$. From (2.1), $3r_3 + 4r_4 = 2q - 5$. Combining this with $r_3 + r_4 = r - 1$ gives $r_4 = 2q - 3r - 2$. Since $r \equiv 1 \pmod 2$, we have $r_4 = 1$. Hence $r_4 = r_5 = 1$ and $r_3 = r - 2$. This is the conclusion (b).

Finally suppose $r_5 = 0$. Then $r_4 = 3$, and thus $r_3 = r - 3$. This is the conclusion (c).

3 Complete t-partite Graphs $(t \geq 5)$

Theorem 3.1 *If G is a complete t-partite graph with $t \geq 5$, then G does not admit a semi-regular drawing. Thus,* $\operatorname{tcr}(G) = \infty$.

Proof. Assume for a contradiction that G admits a semi-regular drawing D. Let $t \geq 7$. If a new vertex is added to each triple crossing point, then we have a plane graph G'. However, the original vertices have degree at least $t - 1 (\geq 6)$, and the new vertices have degree 6. This contradicts Lemma 2.2.

Let $G = K_{n_1, n_2, \cdots, n_6}$. Then G has $p = \sum\limits_i n_i$ vertices and $q = \sum\limits_{i<j} n_i n_j$ edges. So

$$\begin{aligned} q - 3p + 6 &= (n_1 + n_4 - 3)(n_2 + n_3 - 3) + n_1 n_4 + n_2 n_3 \\ &\quad + (n_5 + n_6)(n_1 + n_2 + n_3 + n_4 - 3) + n_5 n_6 - 3 \\ &\geq (2n_4 - 3)^2 + 2n_4^2 \\ &\geq 3. \end{aligned}$$

This contradicts Lemma 2.3.

Finally, let $G = K_{n_1, n_2, \cdots, n_5}$. As above,

$$\begin{aligned} q - 3p + 6 &= (n_1 + n_4 - 3)(n_2 + n_3 - 3) + n_1 n_4 + n_2 n_3 \\ &\quad + n_5(n_1 + n_2 + n_3 + n_4 - 3) - 3 \\ &\geq (2n_4 - 3)^2 + 2n_4^2 + n_5 - 3 \\ &\geq 1. \end{aligned}$$

This contradicts Lemma 2.3 again.

Corollary 3.1 *Let K_n be the complete graph with n vertices. Then*

$$\operatorname{tcr}(K_n) = \begin{cases} 0, & n \leq 4; \\ \infty, & \text{otherwise}. \end{cases}$$

4 $K_{5,4}$

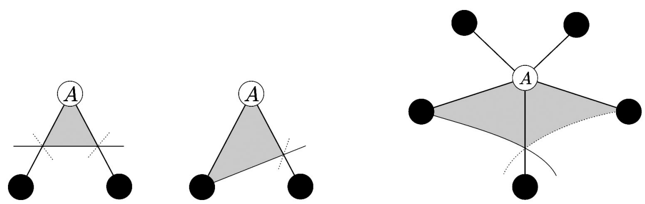
Throughout this section, we assume that $G = K_{5,4}$ admits a semi-regular drawing. We show that this is impossible.

Let $V_1 = \{x_1, x_2, x_3, x_4, x_5\}$ and $V_2 = \{A, B, C, D\}$ be the partite sets of G. For convenience, we refer to vertices of V_1 (resp. V_2) as black (resp. white) vertices. We denote the edge Ax_i by a_i for $1 \leq i \leq 5$. These are called A-lines. Similarly, define b_i, c_i, d_i, and call them B-, C-, D-lines, respectively. In particular, each black vertex is of degree four, and is incident with four distinct classes of lines.

We fix a semi-regular drawing of G hereafter, which is denoted by the same symbol G. Notice that any line in G connects a white vertex and a black vertex, and there may be triple crossing points on it. From our requirements for drawings, each triple crossing point of G arises from three distinct classes of lines. This fact will be referred to as property $(*)$ throughout the paper. Property $(*)$ is very useful and powerful. For example, if an A-line and a B-line intersect at a triple crossing point, then we can conclude that the remaining line through the triple crossing point is either a C- or D-line.

Let k be the number of triple crossing points. Add a new vertex to each triple crossing point. Then we have a plane graph G' with $9 + k$ vertices and $20 + 3k$ edges. Since $3(9 + k) - (20 + 3k) - 6 = 1$, the faces of the plane graph G' are all 3-sided, except a single 4-sided face by Lemma 2.4. For the semi-regular drawing G, a face means that of G', although it is an abuse of words. A 3-sided face is also called a triangle.

Take a look around vertex A. There are five faces of G, since A is not a cut vertex. We may assume that all five faces around vertex A are triangles without loss of generality, since the 4-sided face is incident with at most two white vertices. There are two types of triangles around vertex A as shown in Fig. 4.1. A type I triangle is incident with two triple crossing points, and a type II triangle is incident with a black vertex and a triple crossing point.



type I type II adjoint pair

Fig. 4.1 Two types of triangles at A and an adjoint pair of type II triangles

Notice that type II triangles appear in pairs. More precisely, this means that every type II triangle at A shares an A-line fully with another type II triangle. Such a pair of type II triangle is referred to as an adjoint pair of type II triangles. See Fig. 4.1. Hence the number of type II triangles around vertex A is either 0, 2 or 4. We will eliminate these three possibilities.

Lemma 4.1 *The number of type* II *triangles at vertex A is not four.*

Proof. Suppose that there are four type II triangles at A. Then we can assume that the local configuration at A is as shown in Fig. 4.2(1) by renaming B, C, D, if necessary. (Recall that each black vertex is incident with four distinct classes of lines.)

Then, by property $(*)$, the horizontal line is a B- or D-line, and the right lower line is a C- or D-line. See Fig. 4.2(2), where the symbol B/D, for example, indicates the class of the horizontal line. It does not mean that vertex B or D locates the left side of the horizontal line. Thus there are four cases as shown in Fig. 4.3.

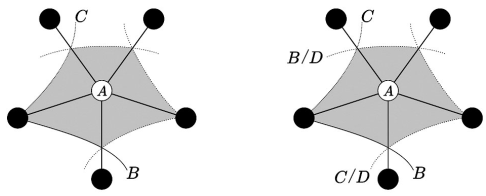


Fig. 4.2 Four type II triangles at A

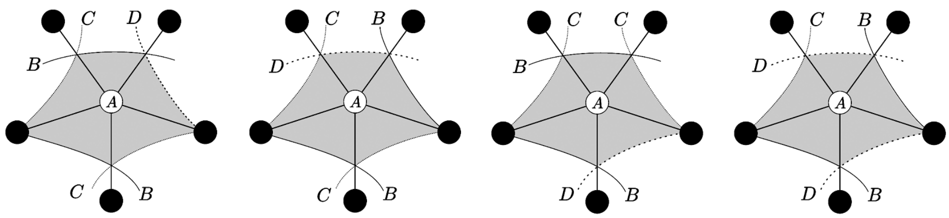


Fig. 4.3 Four cases where are four type II triangles

The class of the right upper line is determined by property (∗) and the fact that each black vertex is incident with four distinct classes of lines. For Fig. 4.3(4), the right upper line may be a C-line. But then, it can be reduced to (3) by renaming B and D and symmetry.

Claim 4.1 Fig. 4.3(1) is impossible.

Proof. Consider the B-line b_2. Let f_1, f_2 be the faces incident with b_2 and vertex x_2. See Fig. 4.4.

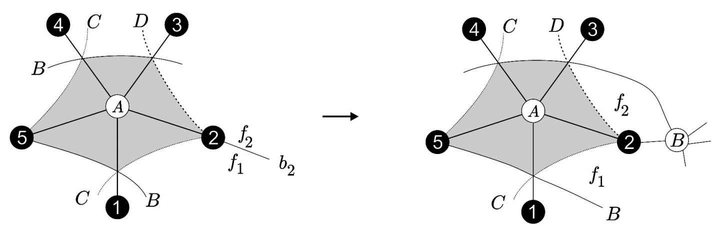


Fig. 4.4 The B-line b_2 and two faces f_1, f_2

If f_1 is 4-sided, then f_2 is 3-sided, because there is only one 4-sided face. Since two B-lines cannot intersect at a triple crossing point by property (∗), the horizontal B-line and b_2 meet at vertex B as shown in the second of Fig. 4.4 in order to make f_2 3-sided. However, f_1 cannot be 4-sided. Thus we can conclude that f_1 is 3-sided. Then vertex B is located around f_1 as shown in the first of Fig. 4.5 by the same reason as above.

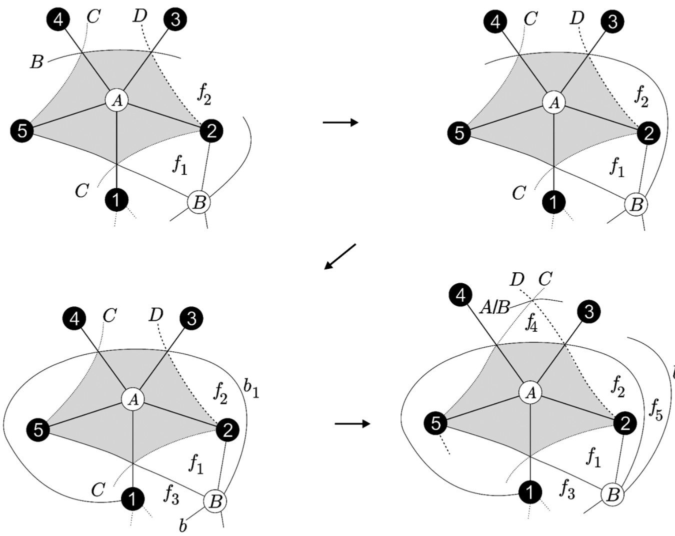


Fig. 4.5 f_1 is 3-sided

Also, then f_2 cannot be 4-sided, and thus 3-sided. Hence the B-line, intersecting two A-lines a_3 and a_4, turns out to be b_1. That is, it goes to black vertex x_1 as shown in the third of Fig. 4.5. (At this point, b_1 may contain triple crossing points on it after crossing a_4.) Consider the face f_3, which is adjacent to f_1 along the B-line b_5.

If f_3 is 3-sided, then the B-line b, which must be b_3 or b_4, intersects the A-line a_1. Then b can reach neither x_3 nor x_4, because b cannot cross b_1 or meet a_1 twice. Therefore, we found that f_3 is the only 4-sided face. Thus f_4 is 3-sided. We see that the line going through the upper triple crossing point of f_4 is either an A- or B-line by property $(*)$. Since f_5 is also 3-sided, the B-line b' goes to x_3, or crosses the A-line a_3. In any case, f_6 cannot be 3-sided as described in Fig. 4.6. This is a contradiction.

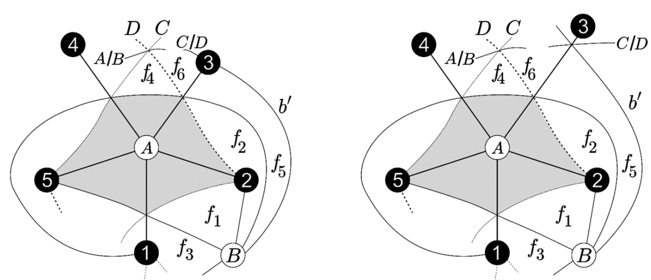


Fig. 4.6 b' goes to x_3 or crosses a_3

Claim 4.2 Fig. 4.3(2) is impossible.

Proof. First, since the horizontal D-line intersects a_3, a_4, c_5, it is either d_1 or d_2 from our requirements for drawings. By symmetry, we can assume that f_1 is 3-sided as in Fig. 4.7. Then vertex D is located around f_1. Then the horizontal D-line turns out to be d_1. However, neither f_2 nor f_3 is 3-sided, which contradicts that there is only one 4-sided face.

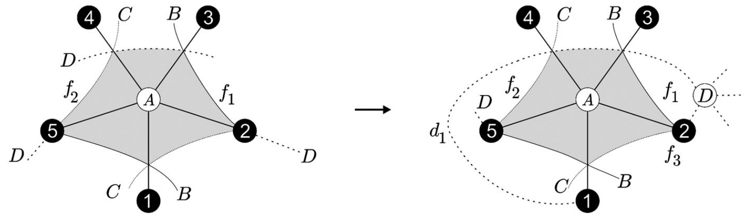


Fig. 4.7 f_1 is 3-sided

Claim 4.3 Fig. 4.3(3) is impossible.

Proof. In Fig. 4.8, either f_3 or f_4 is 4-sided. (For, if f_3 is not 4-sided, then it is 3-sided. Then vertex D is located there, which implies that f_4 is not 3-sided.) Thus both f_1 and f_2 are 3-sided. Then the B-line b_1 is determined. Since f_5 is 3-sided, the B-line b, which is b_3 or b_4, crosses the A-line a_1. Then b can reach neither x_3 nor x_4, a contradiction.

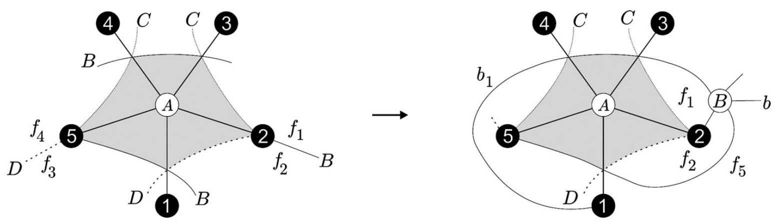


Fig. 4.8 Both f_1 and f_2 are 3-sided

Claim 4.4 Fig. 4.3(4) is impossible.

Proof. In Fig. 4.9, if f_1 is not 3-sided, then f_2 is 3-sided, and then vertex D is located there. But then, f_1 cannot be 4-sided. Hence f_1 is 3-sided. Similarly, so is f_2. Then the D-line d_1 is determined. If f_3 is 3-sided, then the D-line d, which is d_3 or d_4, crosses the A-line a_1. Then d cannot reach any black vertex as before. Hence f_3 is 4-sided. As in the proof of Claim 4.1, examining f_4, f_5, f_6 leads to a contradiction.

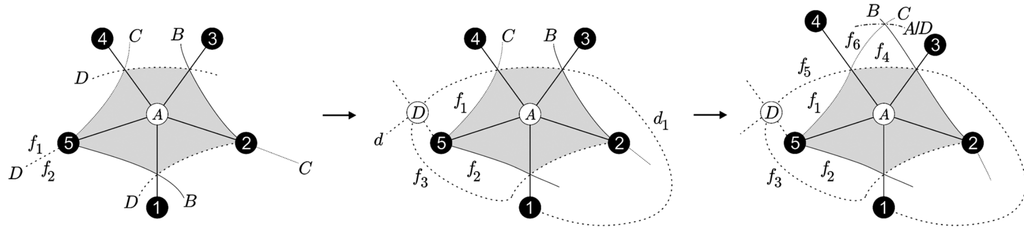


Fig. 4.9 Both f_1 and f_2 are 3-sided

This completes the proof of Lemma 4.1.

Lemma 4.2 *The number of type* II *triangles at vertex A is not two.*

Proof. Suppose that there are two type II triangles at A. Then we can assume that the local configuration at A is as shown in Fig. 4.10(1), up to renaming. By property $(*)$, the left upper line is a B- or D-line. Similarly, the right upper line is a C- or D-line. See Fig. 4.10(2). Then there are three cases, up to symmetry and relabeling of vertices, as shown in Fig. 4.11, where the class of the horizontal line is determined by property $(*)$.

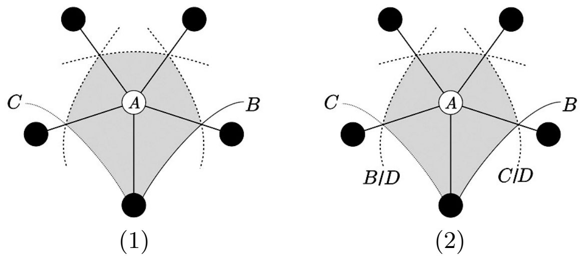


Fig. 4.10 Two type II triangles at A

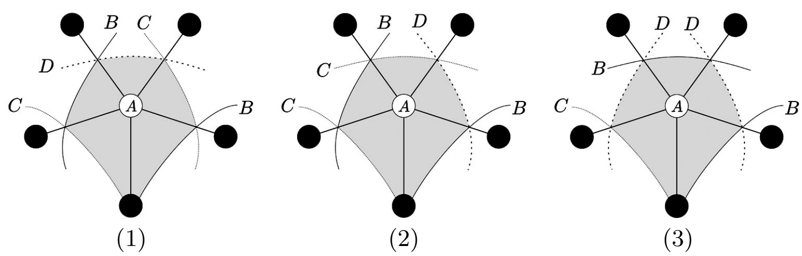


Fig. 4.11 Three cases where there are two type II triangles

Claim 4.5 Fig. 4.11(1) is impossible.

Proof. First, assume that f_1 is 4-sided in Fig. 4.12. Then the others are all 3-sided. Thus f_2 and f_3, and then f_4, f_5 are determined as in Fig. 4.12. (If an A-line goes through the left triple crossing point of f_2, then the face sharing a D-line with f_2 cannot be 3-sided. Similarly for f_3.)

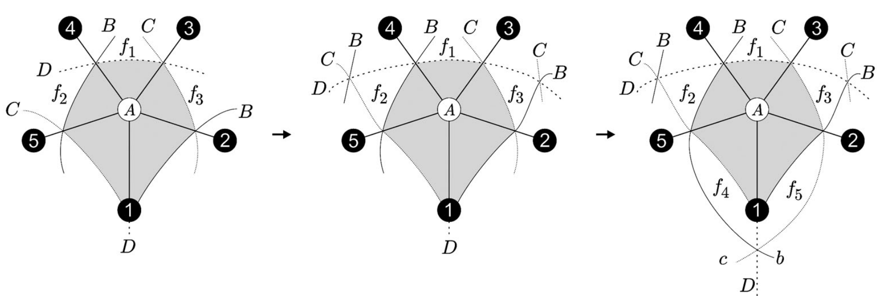


Fig. 4.12 Assume that f_1 is 4-sided

Consider the B-line b. It goes to x_2 or crosses the A-line a_2. Suppose that the former happens. Then $f_6, \cdots, f_9$ are determined as in Fig. 4.13. Moreover, the D-line d_5 is also determined.

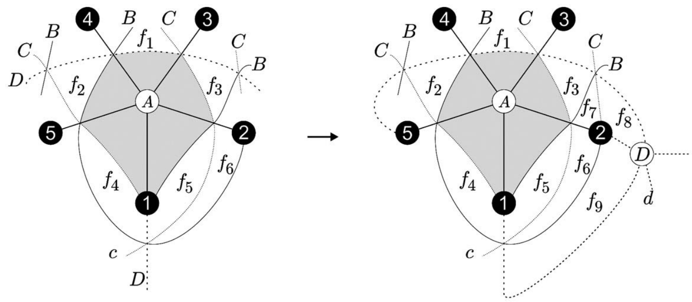


Fig. 4.13 The case where b goes to x_2

Then the C-line c cannot go to x_5, since it crosses d_5. Hence it crosses the A-line a_5. This forces the D-line d to cross the same a_5. Then it cannot reach any black vertex, a contradiction. Therefore, b crosses the A-line a_2. By the same reason, c crosses a_5.

Repeating the same argument, we obtain the configuration as shown in Fig. 4.14.

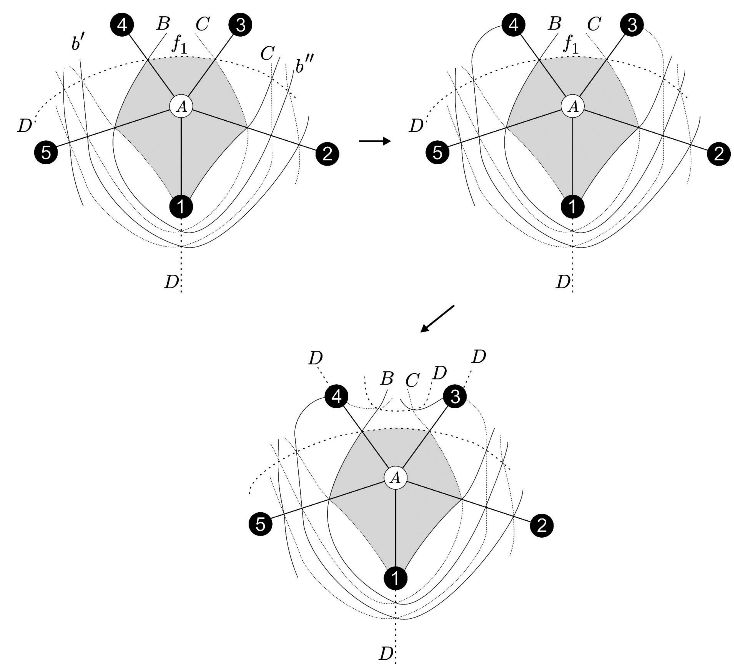


Fig. 4.14 A final contradiction when f_1 is 4-sided

If the B-line b' crosses the A-line a_4, then both b' and b'' go to x_3, a contradiction. Thus b' goes to x_4. Similarly, the C-line c_3 is determined. See the second of Fig. 4.14.

Then f_1 can be incident with neither vertex B nor C. For example, if f_1 is incident with B, then the left face of f_1 cannot be 3-sided. Hence f_1 is incident with two more triple crossing points. Then the upper horizontal line of f_1 is an A- or D-line by property $(*)$. From our requirements, it cannot be an A-line. Thus we have the third of Fig. 4.14, but then vertex D cannot be located.

Next, assume that f_1 is 3-sided. We see that a D-line goes through the upper triple crossing point of f_1 from our requirements and property $(*)$.

By symmetry, we can assume that f_2 is 3-sided. See Fig. 4.15.

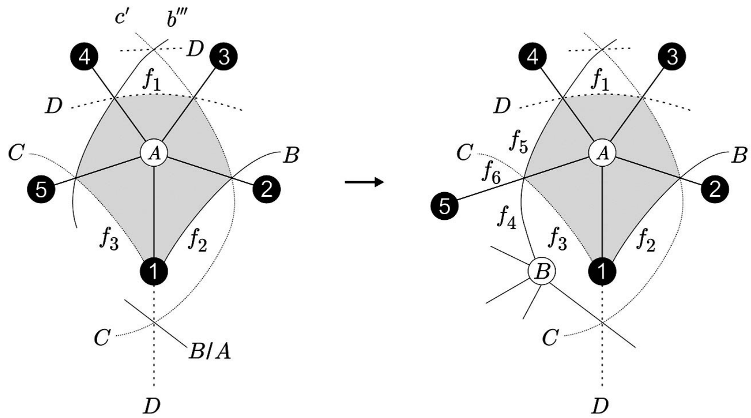


Fig. 4.15 f_1 is 3-sided

If f_3 is also 3-sided, then the B-line b''' and the C-line c' meet twice, a contradiction. Hence f_3 turns out to be 4-sided. Also, no A-line is adjacent to f_3, because each of the four A-lines $a_2, \cdots, a_5$ meets b''' or c'. Thus vertex B is located as in Fig. 4.15. Again, examining f_4, f_5, f_6 leads to a contradiction as in the proof of Claim 4.1.

Claim 4.6 Fig. 4.11(2) is impossible.

Proof. In Fig. 4.16, suppose that f_1 is 4-sided. Then f_2 is 3-sided, and so vertex D appears there. Then f_3 is not 3-sided, a contradiction. Hence f_1 is 3-sided.

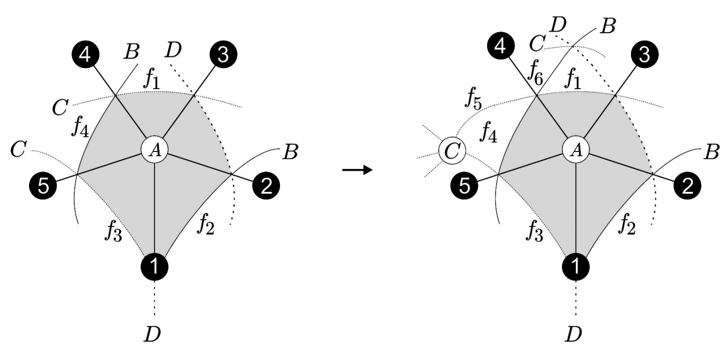


Fig. 4.16 f_1 is 3-sided

If f_2 is 3-sided, then f_3 is 4-sided as above. Otherwise, f_2 is 4-sided. In any case, f_4 is 3-sided, and vertex C appears. Also, f_5 and f_6 are 3-sided. But this is impossible as in the proof of Claim 4.1 again.

Claim 4.7 Fig. 4.11(3) is impossible.

Proof. In Fig. 4.17, at least two of f_1, f_2, f_3 are 3-sided. If f_1 and f_2 are 3-sided, then vertex D cannot be located correctly. Similarly for the case where f_1 and f_3 are 3-sided. Hence f_2 and f_3 are 3-sided. Then the D-line d goes to x_2 or x_3, and another D-line d' goes to x_4 or x_5. But this is impossible.

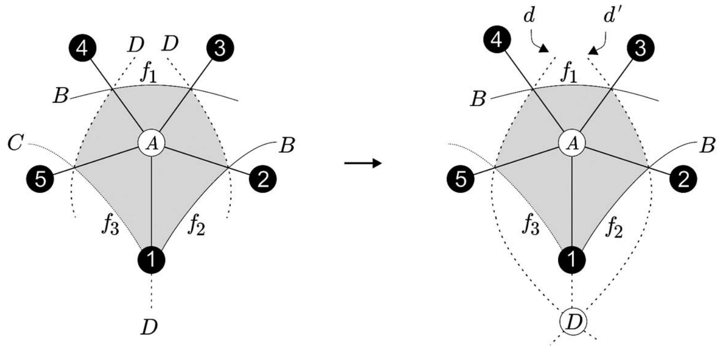


Fig. 4.17 f_2 and f_3 are 3-sided

This completes the proof of Lemma 4.2.

Lemma 4.3 *The number of type* II *triangles at vertex A is not zero.*

Proof. Assume that there are no type II triangles at A. Up to symmetry and relabeling of vertices, the local configuration at A can be assumed as in the first of Fig. 4.18.

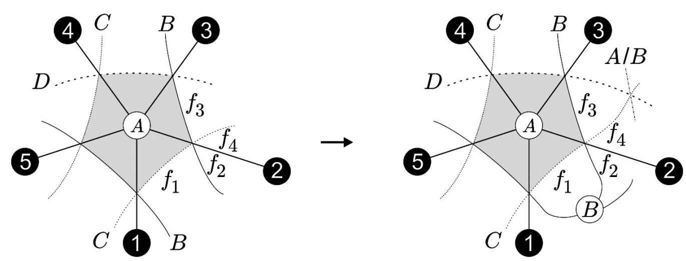


Fig. 4.18 Five type I triangles at A

By symmetry, we can assume that the right hand side does not contain a 4-sided face. More precisely, $f_1, \cdots, f_4$ are all 3-sided. Thus vertex B is located. Then examining f_2, f_3, f_4, as in the proof of Claim 4.1, leads to a contradiction. (In this case, f_3 can be incident with x_5. Then f_4 cannot be 3-sided likewise.)

Theorem 4.1 *$K_{5,4}$ does not admit a semi-regular drawing.*

Proof. This follows from Lemmas 4.1, 4.2 and 4.3.

5 $K_{4,4}$

Throughout this section, we assume that $G = K_{4,4}$ admits a semi-regular drawing. We will show that this is impossible.

Let $V_1 = \{x_1, x_2, x_3, x_4\}$ and $V_2 = \{A, B, C, D\}$ be the partite sets of G. As in Section 4, we refer to vertices of V_1 (resp. V_2) as black (resp. white) vertices, and use the same notions as A-lines, type I or II triangles, so on. Also, property $(*)$ holds from our requirements.

We fix a semi-regular drawing of G, which is denoted by G again. Let k be the number of triple crossing points. Add a new vertex to each triple crossing point. Then we have a plane graph G' with $8+k$ vertices and $16+3k$ edges. Since $3(8+k)-(16+3k)-6=2$, either

(1) one face of G' is 5-sided, and the others are 3-sided; or

(2) two faces of G' are 4-sided, and the others are 3-sided

by Lemma 2.4. As in Section 4, a face of G means that of G'.

5.1 Case (1)

We treat the case where one face of G is 5-sided, and the others are 3-sided. At most two white vertices appear in the 5-sided face. Hence we can assume that four faces at vertex A are all 3-sided. Thus the number of type II triangles at vertex A is either 0, 2 or 4. We will eliminate these three possibilities.

Lemma 5.1 *The number of type* II *triangles at vertex A is not four.*

Proof. Assume that there are four type II triangles at A. We may assume that the local configuration at A is as shown in Fig. 5.1(1), up to renaming. By property $(*)$, the right upper line is a C- or D-line, and the left upper line is a B- or D-line (see Fig. 5.1(2)). Up to symmetry and relabeling, there are two possibilities as shown in Fig. 5.2.

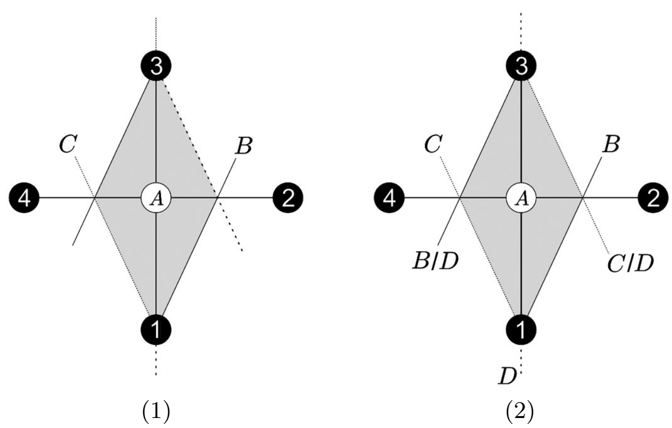


Fig. 5.1 Four type II triangles at A

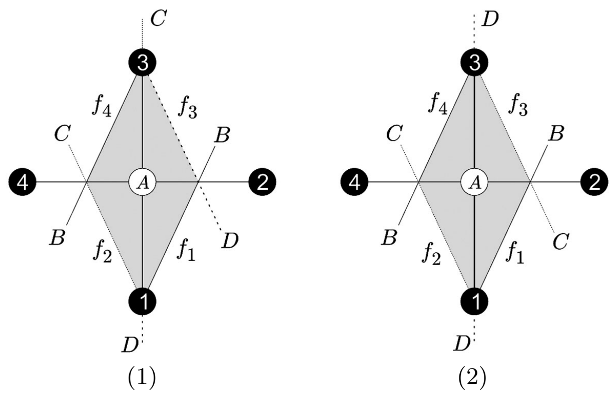


Fig. 5.2 Two cases where there are four type II triangles

Claim 5.1 Fig. 5.2(1) is impossible.

Proof. By symmetry, we can assume that both f_1 and f_2 are 3-sided. Since f_1 is 3-sided, vertex D is located. Then f_2 cannot be 3-sided, a contradiction.

Claim 5.2 Fig. 5.2(2) is impossible.

Proof. By symmetry, we can assume that f_1, f_2 are 3-sided again. Then the B-line b_3 meets the C-line c_3, a contradiction.

This complete the proof of Lemma 5.1.

Lemma 5.2 *The number of type* II *triangles at vertex A is not two.*

Proof. Assume that there are two type II triangles at A. As before, we may assume that the local configuration at A is as shown in Fig. 5.3(1). By property $(*)$, the right upper line is a C- or D-line, and the left upper line is a B- or D-line (see Fig. 5.3). Up to symmetry and relabeling, there are two possibilities as shown in Fig. 5.4.

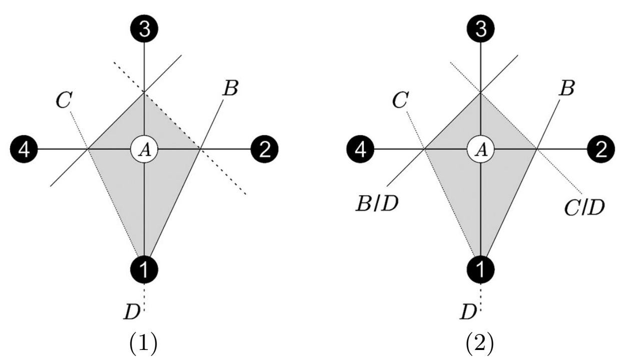


Fig. 5.3 Two type II triangles at A

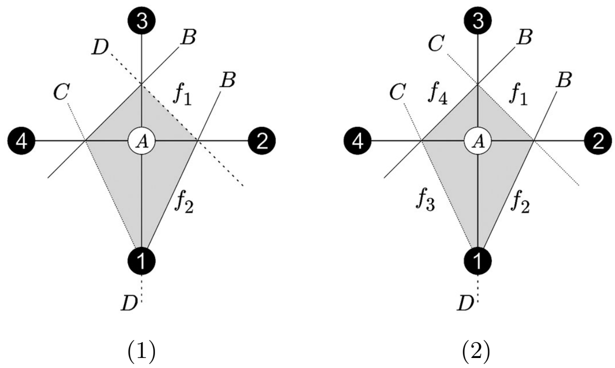


Fig. 5.4 Two cases where there are two type II triangles

Claim 5.3 Fig. 5.4(1) is impossible.

Proof. Assume that f_2 is not 3-sided. Then f_1 is 3-sided, and so vertex B is located there. Thus the B-line b_2 is determined. See the first of Fig. 5.5. Another B-line b crosses the A-line a_2, but then it cannot reach any black vertex.

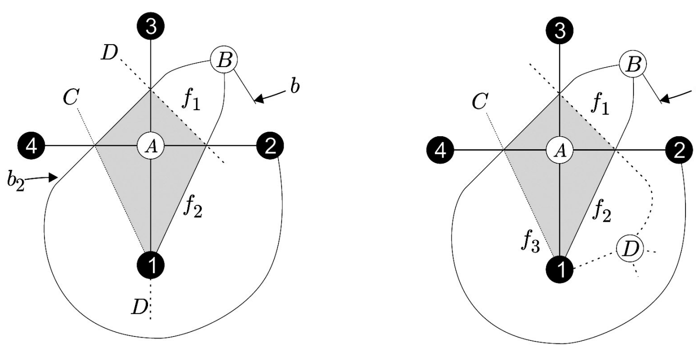


Fig. 5.5 b crosses a_2

Hence f_2 is 3-sided, so vertex D is located. Then f_3 cannot be 3-sided. Thus f_1 is 3-sided, so vertex B is located, and the B-line b_2 is determined again as in the second of Fig. 5.5. Examining b leads to a contradiction as above.

Claim 5.4 Fig. 5.4(2) is impossible.

Proof. By symmetry, we may assume that both f_1 and f_2 are 3-sided. Then vertex B is located. See the first of Fig. 5.6.

If f_3 is 3-sided, then the B-line b and the C-line c meet twice, a contradiction. Hence f_3 is not 3-sided, and b goes to x_2 as in Fig. 5.6. Then a line b' turns out to be a B-line. But this B-line cannot reach any black vertex, otherwise it crosses a_2 twice.

This completes the proof of Lemma 5.2.

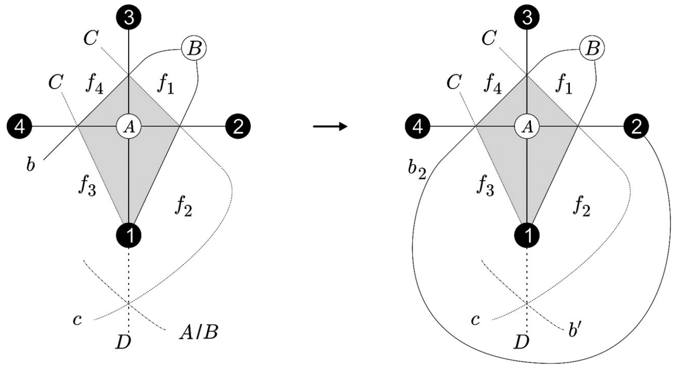


Fig. 5.6 Both f_1 and f_2 are 3-sided

Lemma 5.3 *The number of type* II *triangles at vertex* A *is not zero.*

Proof. Assume that there is no type II triangle at A. Up to symmetry and relabeling, there are two possibilities as shown in Fig. 5.7.

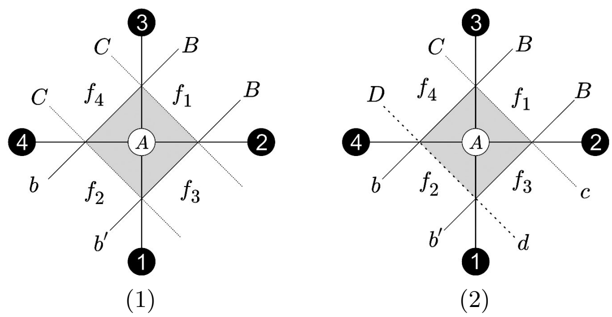


Fig. 5.7 Two cases where there are four type I triangles

In any case, we can assume that f_1 is 3-sided by symmetry. Then vertex B is located there. The B-line b goes to x_1 or x_2, and the B-line b' goes to x_3 or x_4. This is impossible.

Proposition 5.1 *Case* (1) *is impossible.*

Proof. This immediately follows from Lemmas 5.1, 5.2 and 5.3.

5.2 Case (2)

We treat the case where two faces of G are 4-sided, and the others are 3-sided. There are two subcases:

(2-1) All white vertices are incident with a 4-sided face.

(2-2) There is a white vertex which is not incident with a 4-sided face.

5.2.1 Subcase (2-1)

Around each 4-sided face, just two white vertices appear. Hence there is just one 4-sided face around each white vertex. Also, it implies that if a face is incident with two adjacent triple crossing points or an adjacent pair of a triple crossing point and a black vertex, then it must be 3-sided.

In this subcase, then the number of type II triangle at vertex A is 0, 1, 2 or 3.

Lemma 5.4 *The number of type* II *triangles at A is not three.*

Proof. Assume that there are thee type II triangle at A. Up to symmetry and renaming, the situation is as shown in Fig. 5.8(1), where f is 4-sided.

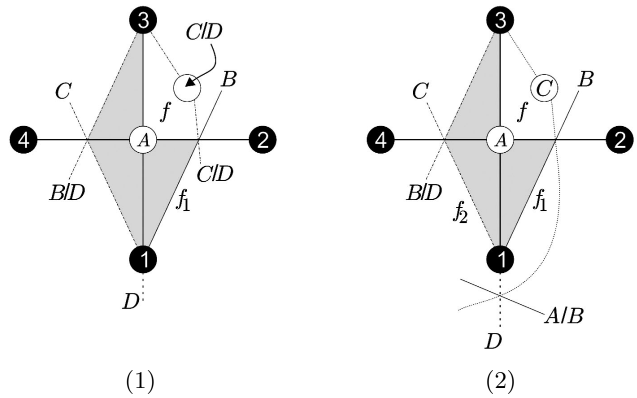


Fig. 5.8 Three type II triangles at A

As remarked above, f_1 is 3-sided. If f is incident with vertex D, then f_1 is incident with vertex D. This is impossible. Thus f is incident with vertex C. See Fig. 5.8(2). Since f_2 is also 3-sided, the left upper line of f_2 is a B-line. We have the configuration as in Fig. 5.9(1).

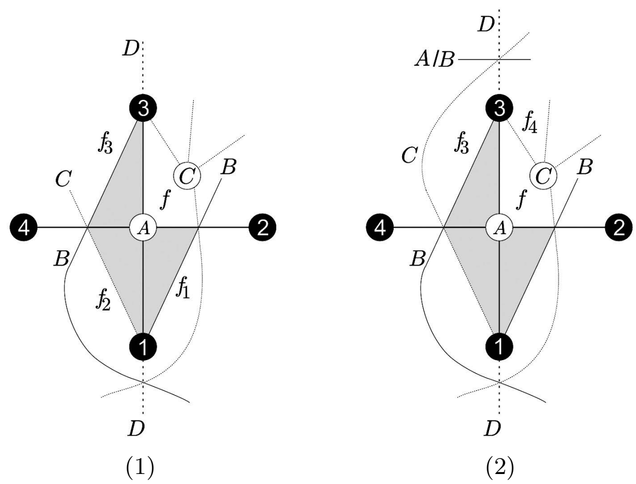


Fig. 5.9 Three type II triangles at A (continued)

Then f_3 is 3-sided, but this forces f_4 to be neither 3-sided nor 4-sided, a contradiction. See Fig. 5.9(2).

Lemma 5.5 *The number of type* II *triangles at A is not two.*

Proof. Assume that there are two type II triangles at A. Up to symmetry and renaming, there are two possibilities as shown in Fig. 5.10, where f is 4-sided. Notice that f_1 and f_2 are 3-sided.

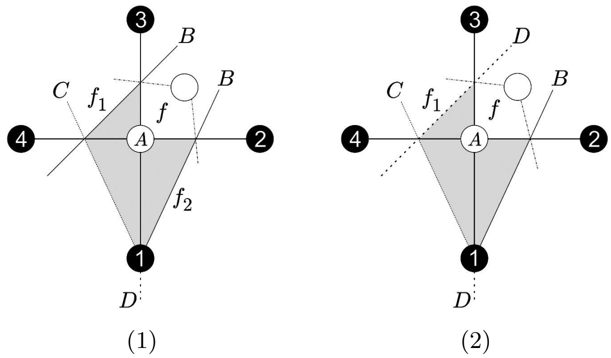


Fig. 5.10 Two cases where there are two type II triangles

For Fig. 5.10(1), either vertex C or D appears around f by property $(*)$. Since f_1 is 3-sided, the former is impossible. The latter is also impossible, because f_2 is 3-sided. For Fig. 5.10(2), vertex C appears around f by property $(*)$. Then f_1 gives a contradiction, again.

Lemma 5.6 *The number of type* II *triangles at A is not one.*

Proof. Suppose that there is one type II triangle at A. Up to symmetry and renaming, the configuration is as shown in Fig. 5.11, where f is 4-sided.

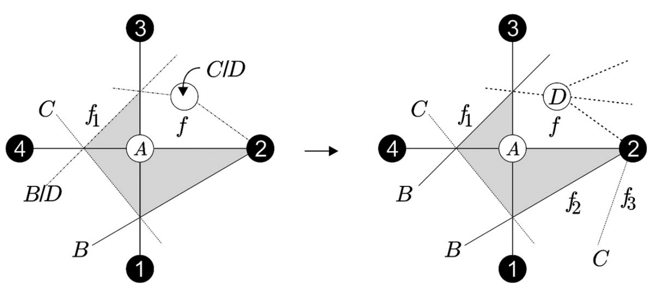


Fig. 5.11 One type II triangle at A

Notice that f_1 is 3-sided. If f is incident with vertex C, then f_1 is incident with C, an impossible. Hence f is incident with vertex D. Thus we have the configuration as in Fig. 5.11. The fact that f_2 is 3-sided forces f_3 to be 4-sided. Then f_3 is incident with vertices C and D, which contradicts the fact that D is incident with only one 4-sided face.

Lemma 5.7 *The number of type* II *triangles at A is not zero.*

Proof. Assume that there is no type II triangle at A. Up to symmetry and renaming, there are two possibilities as shown in Fig. 5.12, where f is 4-sided.

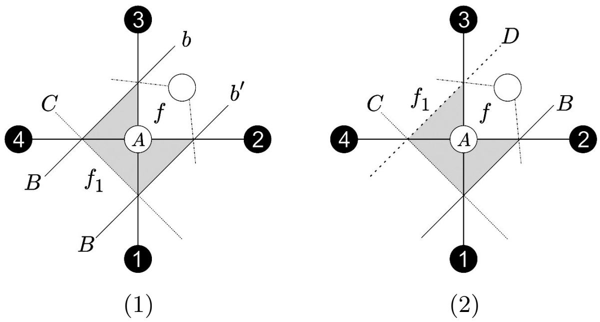


Fig. 5.12 Two cases where there is no type II triangle

For Fig. 5.12(1), f_1 is 3-sided, so vertex B is located there. Then b goes to x_1 or x_2, and b' goes to x_3 or x_4. This is impossible.

For Fig. 5.12(2), vertex C appears around f by property $(*)$. On the other hand, f_1 is 3-sided. This is impossible.

5.2.2 Subcase (2-2)

In this subcase, we may assume that vertex A is not incident with a 4-sided face without loss of generality. Thus the number of type II triangles at A is 0, 2 or 4.

Lemma 5.8 *The number of type* II *triangles at A is not four.*

Proof. Suppose that there are four type II triangles at A. Then there are two possibilities as in Fig. 5.2.

Claim 5.5 Fig. 5.2(1) is impossible.

Proof. Among the four faces $f_1, \cdots, f_4$, at least two are 3-sided. Furthermore, if f_1 (resp. f_3) is 3-sided, then f_2 (resp. f_4) is 4-sided, and vice versa. Up to symmetry, there are three possibilities:

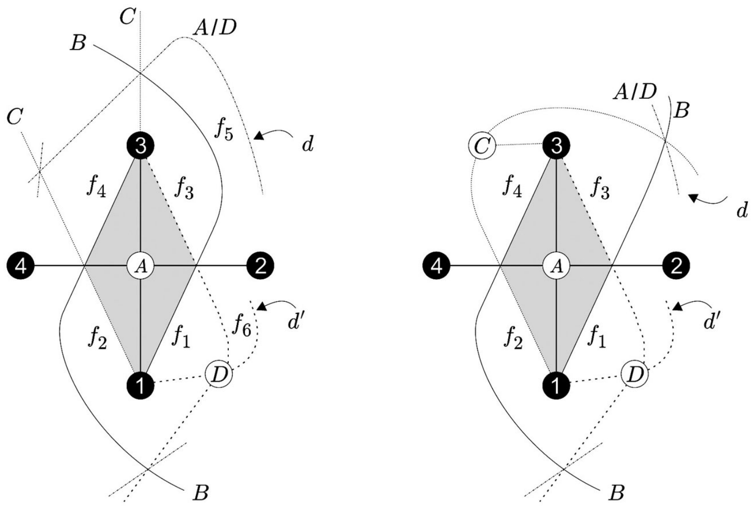
(a) f_1 and f_3 are 3-sided;

(b) f_1 and f_4 are 3-sided;

(c) f_2 and f_3 are 3-sided.

In case (a), f_2 and f_4 are 4-sided. Thus the others are all 3-sided. See Fig. 5.13(1). By examining f_5, the left upper line of f_4 is not an A-line. Since f_5 and f_6 are 3-sided, two D-lines d and d' go to x_2, or cross, a contradiction.

In case (b), f_2 and f_3 are 4-sided. By examining d and d' shown in Fig. 5.13(2), the same argument as (a) leads to a contradiction.



(1) (2)

Fig. 5.13 For (a) and (b)

In case (c), f_1 and f_4 are 4-sided. As above, we see that neither f_1 nor f_4 is incident with an A-line. See the first of Fig. 5.14. Thus we have two B-lines b and b' as shown there. Since f_5 and f_6 are 3-sided, vertex B is located as in the second of Fig. 5.14. But then f_7 cannot be 3-sided, a contradiction. Since f_5 and f_6 are 3-sided, vertex B is located as in the second of Fig. 5.14. But then f_7 cannot be 3-sided, a contradiction.

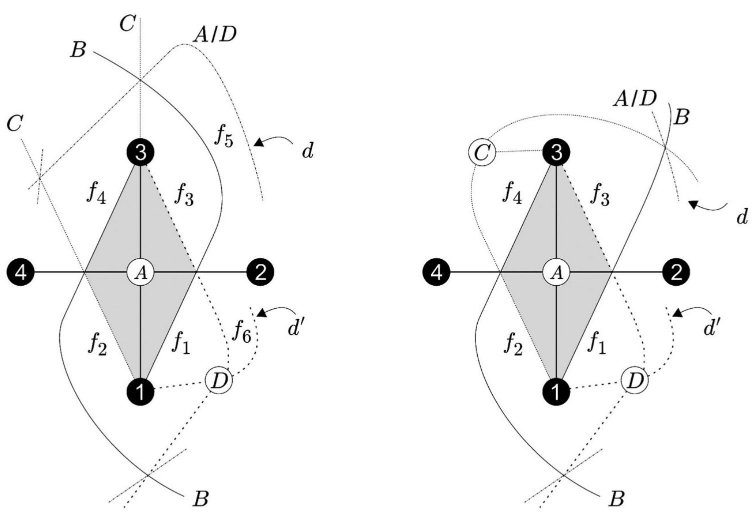


Fig. 5.14 For (c)

Claim 5.6 Fig. 5.2(2) is impossible.

Proof. If both f_1 and f_2 are 3-sided, then we have a contradiction as in the proof of Claim 5.2. Hence either of f_1 or f_2 is 4-sided. Similarly, either f_3 or f_4 is 4-sided. Then there are two possibilities, up to symmetry:

(d) f_1 and f_3 are 4-sided;

(e) f_1 and f_4 are 4-sided.

In case (d), f_2 and f_4 are 3-sided as in the first of Fig. 5.15.

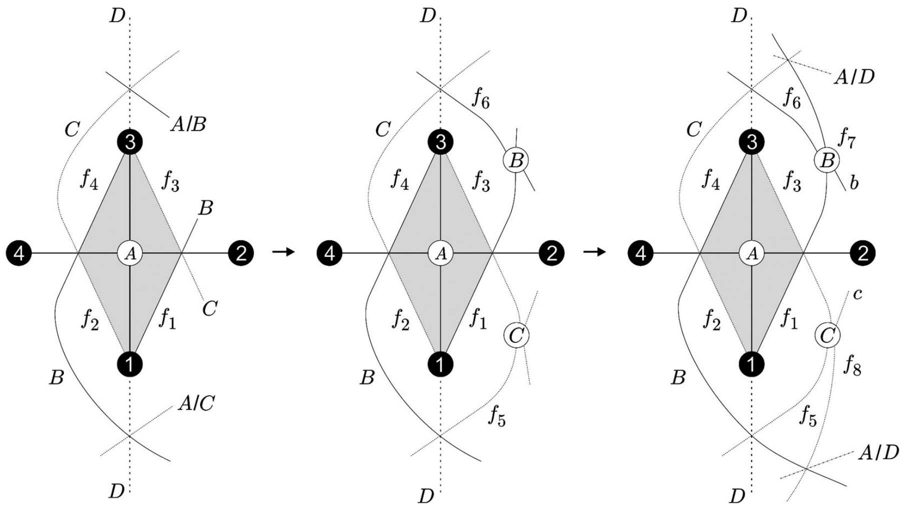


Fig. 5.15 For (d)

By property $(*)$, the right upper line ℓ of f_3 is an A- or B-line. If ℓ is an A-line, then it is either a_2 or a_4. But if ℓ is a_4, then ℓ meets c_1 twice, impossible. If ℓ is a_2, then ℓ meets b_1 twice, impossible. Thus ℓ is a B-line. By the same reason, the right lower line of f_1 is not an A-line, and so a C-line.

Thus vertices B and C are located as in the second of Fig. 5.15. After locating f_5 and f_6 as in the third of Fig. 5.15, consider the B-line b and the C-line c. If b crosses the A-line a_2, then so does c_1 through the same triple crossing point on a_2. Then f_7 cannot be 3-sided. Hence b, and then c, go to x_2. Then f_7 and f_8 cannot be 3-sided simultaneously.

In case (e), f_2 and f_3 are 3-sided. See Fig. 5.16.

By the same argument as (d), the right lower line of f_1 turns out to be a C-line. Similarly, the upper line of f_4 is a C-line. Since both f_1 and f_4 are 4-sided, vertex C is located as in the second of Fig. 5.16. Then f_5 is not 3-sided, a contradiction.

This completes the proof of Lemma 5.8.

Lemma 5.9 *The number of type* II *triangles at* A *is not two.*

Proof. Suppose that there are two type II triangles at A. Then the local configuration at A is Fig. 5.4(1) or (2).

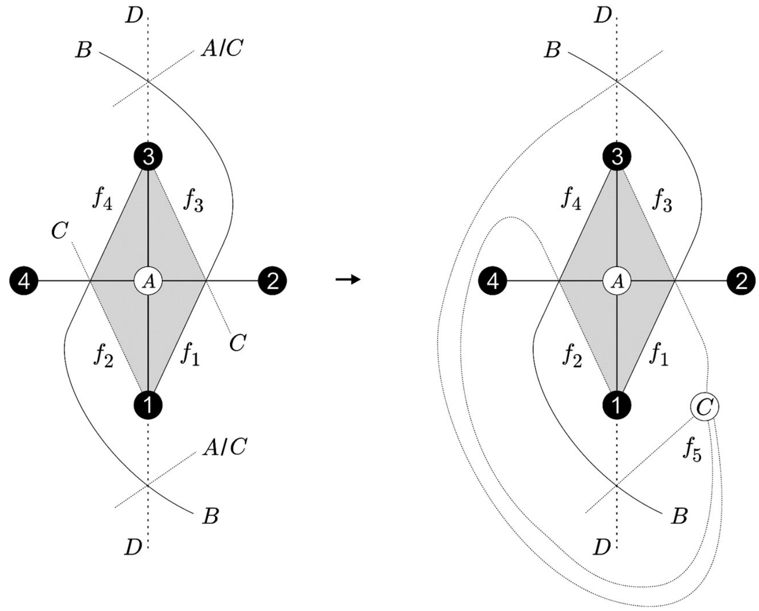


Fig. 5.16 For (e)

Claim 5.7 Fig. 5.4(1) is impossible.

Proof. We claim that f_1 is 4-sided. Assume that f_1 is 3-sided. Then vertex B is located, and the B-line b_2 is determined as in Fig. 5.17.

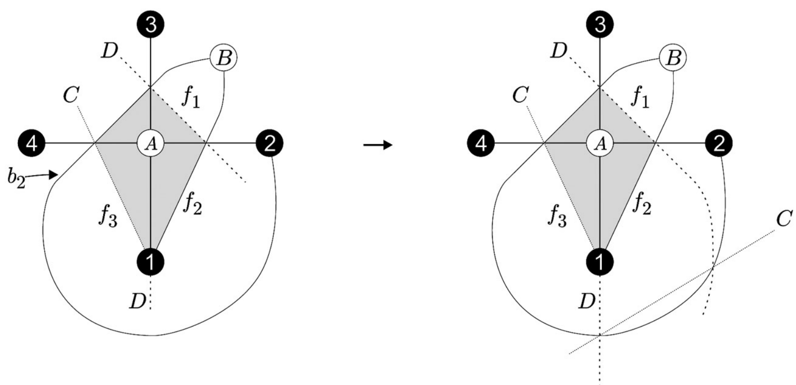


Fig. 5.17 The case where f_1 is 3-sided

Suppose further that f_2 is 4-sided. If vertex D is incident with f_2, then f_2 cannot be 4-sided. Hence f_2 is incident with two more triple crossing points. Then the fourth line of f_2 is an A-, B- or C-line by property $(*)$. However, the existence of b_2 implies that it is neither an A- nor B-line. Thus the right lower line of f_2 is a C-line. Then the second of Fig. 5.18 is the only possible configuration for f_2. But this is impossible, because two lines meet at most once.

Thus we see that f_2 is 3-sided, so vertex D is located as in Fig. 5.18. Then f_3 is 4-sided, and the D-lines d_3 and d_4, and thus d_2, are determined. But, f_4 can be neither 3-sided nor 4-sided, because the existence of b_2 disturbs an A-line and a B-line as above. We have thus shown that f_1 is 4-sided.

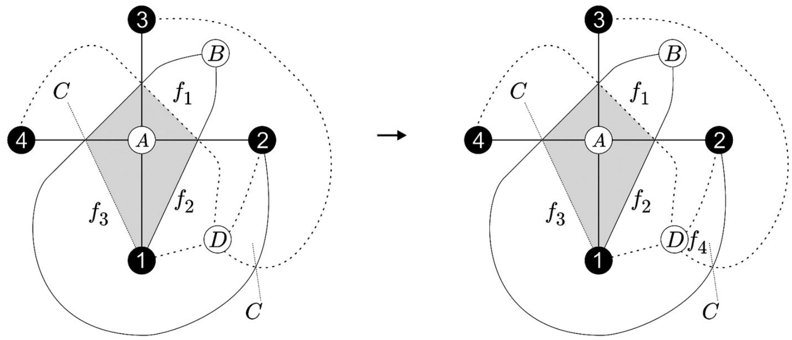


Fig. 5.18 The case where f_1 is 3-sided (continued)

Next, we claim that f_2 is 4-sided. Assume not. Then vertex D is located, and thus d_4 is determined. Also, f_3 is 4-sided. If f_3 is incident with x_2, then the D-line d_2 is determined, and thus d_3 cannot be drawn. Hence f_3 is incident with another triple crossing point as in the first of Fig. 5.19, where an A- or C-line goes through. If it is a C-line, then f_4 cannot be 3-sided. Hence it is an A-line, in particular, a_2. See the second of Fig. 5.19. Then d_2 cannot be drawn.

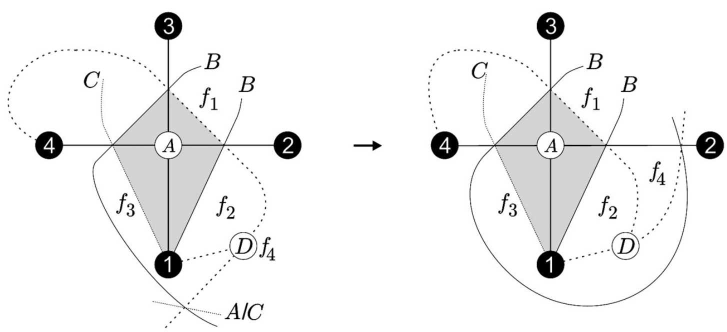


Fig. 5.19 The case where f_2 is 3-sided

Thus we have specified two 4-sided faces f_1 and f_2.

Now, f_3 is 3-sided. By property $(*)$, the right lower line of f_2 is an A- or C-line. See the first of Fig. 5.20. If it is an A-line, then it is a_2. But this is impossible, because the right upper line of f_2 already meets a_2. Thus the right lower line of f_2 is a C-line. Furthermore, if f_2 is incident with x_4, then the situation is drawn as in the second of Fig. 5.20. Then b_2

is determined, and thus vertex B is located. However, b_1 cannot be drawn.

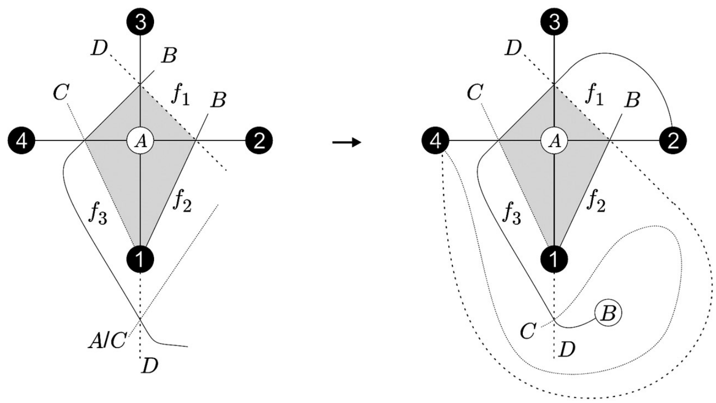


Fig. 5.20 f_1 and f_2 are 4-sided

Thus f_2 is incident with a triple crossing point at its right. See the first of Fig. 5.21. By property $(*)$, an A- or B-line goes through the triple crossing point. But it cannot be an A-line by examining the (3-sided) face right above f_2. After locating vertex B, the B-line b_2 is determined. Then b_1 cannot be drawn.

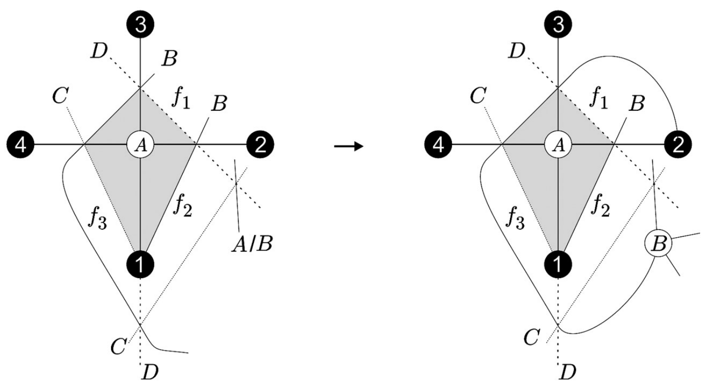


Fig. 5.21 f_1 and f_2 are 4-sided (continued)

Claim 5.8 Fig. 5.4(2) is impossible.

Proof. We claim that f_1 and f_4 are 4-sided. Assume that f_1 is 3-sided. Then vertex B is located, and then the B-line b_2 is determined. See the first of Fig. 5.22.

Suppose further that f_2 is 4-sided. Then f_2 is incident with either vertex C, or vertex D, or two more triple crossing points. If f_2 is incident with vertex C, then c_4 is determined, and then c_1 cannot be drawn. If f_2 is incident with vertex D, then an A- or B-line appears at the triple crossing point where a C-line meets a D-line. But this is impossible by the existence of b_2 as in the proof of Claim 5.7. If f_2 is incident with two more triple crossing points, then a similar argument to the proof of Claim 5.7 gives a contradiction again (see Fig. 5.22). Thus f_2 is 3-sided.

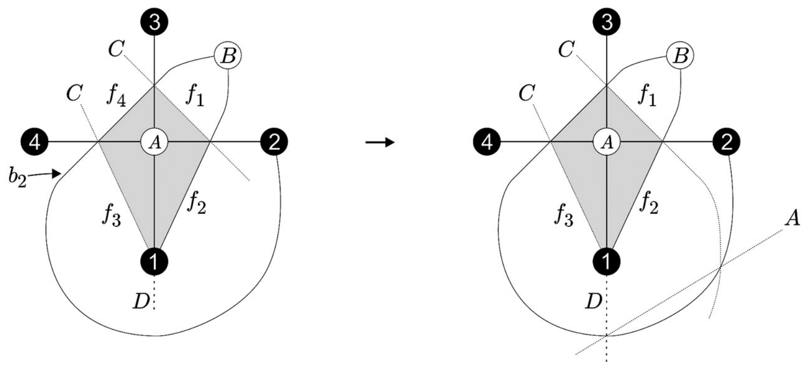


Fig. 5.22 The case where f_1 is 3-sided

Then the situation is as in the proof of Claim 5.4, leading to a contradiction. Thus f_1 is 4-sided. By the same argument, f_4 is 4-sided. Then f_2 and f_3 are 3-sided. But this is impossible, because there are a B-line and C-line meeting twice.

This completes the proof of Lemma 5.9.

Lemma 5.10 *The number of type* II *triangles at A is not zero.*

Proof. Suppose that there is no type II triangle at A. Then the local configuration at A is Fig. 5.7(1) or (2).

In Fig. 5.7(1), if f_1 or f_2 is 3-sided, then we have a contradiction as in the proof of Lemma 5.3. Thus both are 4-sided. By the same reason, f_3 and f_4 are 4-sided, a contradiction.

In Fig. 5.7(2), if f_1 or f_2 is 3-sided, then we have a contradiction as above. Hence f_1 and f_2 are 4-sided. Thus f_3 and f_4 are 3-sided. Then the C-line c meets the D-line d twice, a contradiction.

Proposition 5.2 *Case* (2) *is impossible.*

Proof. This follows from Lemmas 5.4–5.10.

Theorem 5.1 *$K_{4,4}$ does not admit a semi-regular drawing.*

Proof. This follows from Propositions 5.1 and 5.2.

6 $K_{n,3}$

Let $G = K_{n,3}$ with $n \geq 5$. In this section, we show that if $n \neq 6$ then G does not admit a semi-regular drawing. Hereafter, we assume that $n \geq 5$ and $n \neq 6$.

6.1 Exceptional faces

Let V_1 and $V_2 = \{A,\ B,\ C\}$ be the partite sets of G. As before, we refer to a vertex of V_1 (resp. V_2) as a black (resp. white) vertex. Any black vertex is incident with an A-line, B-line and C-line.

Suppose that G admits a semi-regular drawing. Fix such a drawing, denoted by G again. Property $(*)$ holds. That is, at each triple crossing point, an A-line, a B-line and a C-line meet. Let k be the number of triple crossing points. Add a new vertex to each triple crossing point. Then we have a plane graph G' with $n+3+k$ vertices and $3n+3k$ edges. Since $3(n+3+k)-(3n+3k)-6=3$, the faces of G' are 3-sided, except at most three faces, by Lemma 2.4. We refer to a non-triangular face as an exceptional face. More precisely, Lemma 2.4 claims that either

(1) G' has only one exceptional face, which is 6-sided; or

(2) G' has just two exceptional faces, which are 5-sided and 4-sided, respectively; or

(3) G' has just three exceptional faces, which are 4-sided.

As before, a face of G means that of G'. Let N be the number (counted with multiplicities) of white vertices which are incident with exceptional faces. Then $0 \le N \le 6$, because two white vertices are not adjacent in G'. Since a white vertex is not a cut-vertex of G', a white vertex cannot appear around one exceptional face twice.

Lemma 6.1 *Two exceptional faces are not incident with the same pair of white vertices.*

Proof. Suppose that two exceptional faces f and f' are incident with white vertices A and B, say. Then both f and f' are 4-sided, or one is 4-sided and the other 5-sided. See Fig. 6.1. Recall that any black vertex is incident with a C-line. Thus, in any case, we cannot place C-lines.

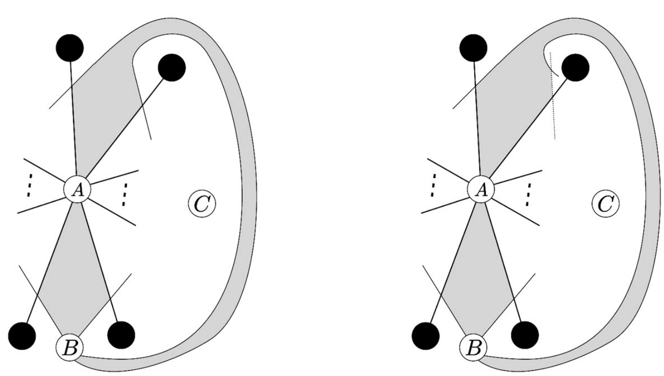


Fig. 6.1 Two exceptional faces with the same pair of white vertices

Recall that there are two types of triangles at a white vertex as shown in Fig. 4.1. Let $X = B$ or C. At vertex A, if a type I triangle is bounded by two A-lines and an X-line, then it is said to be of type I-X. See Fig. 6.2. Furthermore, a type I-X triangle is said to be good if the face sharing the X-line with the type I-X triangle is 3-sided. Otherwise, it is bad. In particular, a bad type I triangle is adjacent to an exceptional face, which is referred to as its associated exceptional face.

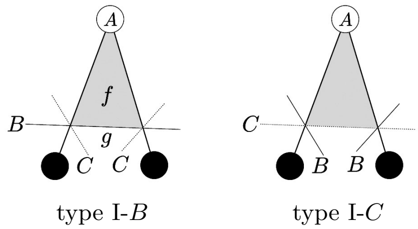


Fig. 6.2 A type I-B triangle and a type I-C triangle

Lemma 6.2 *Let $\{X,\ Y\} = \{B,\ C\}$. If there is a good type* I-X *triangle at vertex A, then there is neither an exceptional face incident with both A and Y, nor another good type* I-X *triangle at A. In particular, the number of good type* I *triangles is at most two.*

Proof. Let f_1 be a good type I-C triangle at A. Then the face f_2 sharing the C-line with f_1 is 3-sided, so vertex B is located there. Suppose that an exceptional face f is incident with vertex A and B. a similar situation to the proof of Lemma 6.1. Hence we cannot place C-lines. The existence of another good type I-C triangle is excluded by a similar argument. See Fig. 6.3. Here, we cannot place the C-lines going to the left upper black vertex and the left lower black vertex simultaneously.

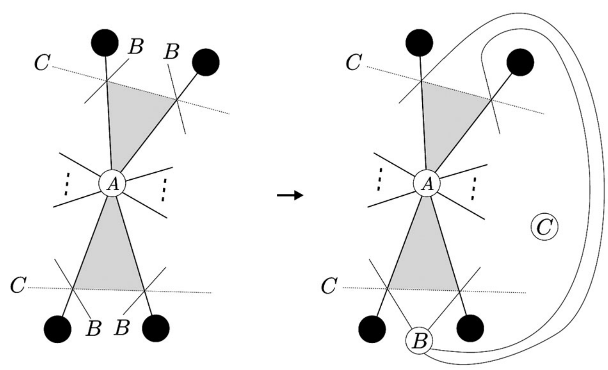


Fig. 6.3 Two good type I-C triangles at A

Lemma 6.3 *Suppose that there is a bad type* I *triangle f at vertex A. Let g be the associated exceptional face of f. If g is k-sided $(4 \le k \le 6)$, then g is incident with at most $k-4$ white vertices.*

Proof. We may assume that f is a bad type I-B triangle without loss of generality. As shown in the first of Fig. 6.2, the boundary of g contains a sequence of

a C-line, a triple crossing point, a B-line, a triple crossing point, a C-line.

The existence of this sequence forces g to admit at most $k-4$ white vertices.

Lemma 6.4 *Two bad type* I *triangles at vertex A cannot have the same associated exceptional face.*

Proof. Let f_1 and f_2 be bad type I triangles at A whose associated exceptional faces coincide. Let g be the common associated exceptional face. When g is 4- or 5-sided, the situation is as shown in Fig. 6.4, where labels B and C may be exchanged.

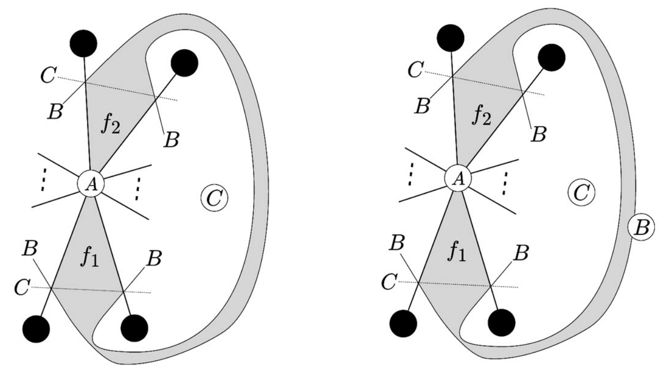


Fig. 6.4 Two bad type I triangles with the same associated 4- or 5-sided exceptional face

Then we cannot place C-lines as before. If g is 6-sided, then the situation is as shown in Fig. 6.5. Similarly, we cannot draw C-lines.

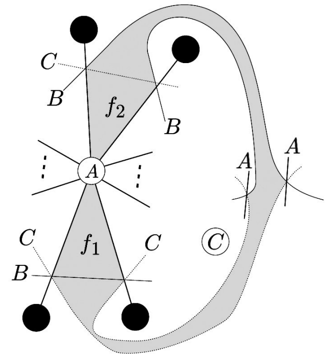


Fig. 6.5 Two bad type I triangles with the same associated 6-sided exceptional face

Lemma 6.5 *Suppose that G satisfies one of the following conditions*:

(1) *G has a* 5*-sided exceptional face incident with exactly one white vertex and a* 4*-sided exceptional face incident with at least one white vertex*;

(2) *G has three* 4*-sided exceptional faces, only one of which is incident with no white vertex.*

Then there is at most one bad type I *triangle at vertex A.*

Proof. Let f_i be a bad type I triangle at A, and let g_i be the associated exceptional face for $i = 1, 2$. By Lemma 6.3, we have $g_1 = g_2$. But this contradicts Lemma 6.4.

Lemma 6.6 *There is at most three bad type* I *triangles at vertex A.*

Proof. Suppose that there are four bad type I triangles at vertex A. Since G has at most three exceptional faces and each bad type I triangle is adjacent to an exceptional face, there exist two bad type I triangles whose associated exceptional faces coincide, contradicting Lemma 6.4.

Recall that an adjoint pair of type II triangles at vertex A is a pair of type II triangles sharing an A-line fully. See Fig. 6.6(1).

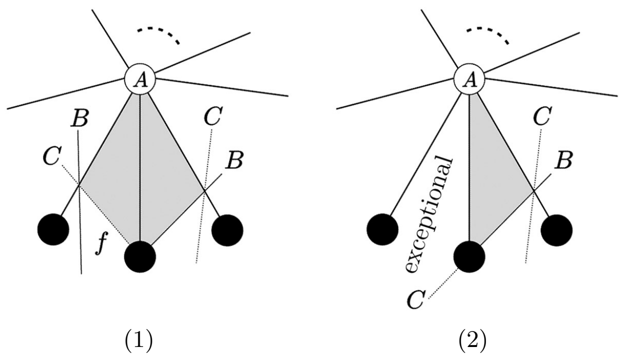


Fig. 6.6 Type II triangles at A

Lemma 6.7 *Suppose that G satisfies one of the following conditions*:

(1) *G has a single* 6*-sided exceptional face, which is incident with all white vertices*;

(2) *G has a* 5*-sided exceptional face incident with two white vertices and a* 4*-sided exceptional face incident with a white vertex*;

(3) *G has three* 4*-sided exceptional faces, each of which is incident with a white vertex. Then there is no adjoint pair of type* II *triangles at vertex A. Hence, if there is a type* II *triangle at A, then it shares an A-line fully with an exceptional face* (*Fig.* 6.6(2)).

Proof. Suppose that there is an adjoint pair of type II triangle at A. Then f, indicated in Fig. 6.6(1), is not 3-sided. However, f cannot be an exceptional face from the assumption, a contradiction.

Lemma 6.8 *There is at most one adjoint pairs of type* II *triangle at vertex A.*

Proof. As in the proof of Lemma 6.7, each adjoint pair of type II triangles yields an exceptional face. If two such pairs share the same exceptional face, then the exceptional face must be a 6-sided face without a white vertex. The situation is as shown in Fig. 6.7.

Then we cannot draw C-lines to the left two black vertices from vertex C.

Hence the number of adjoint pairs of type II triangles is no greater than the number of exceptional faces. When G has a 6-sided exceptional face, we have the conclusion.

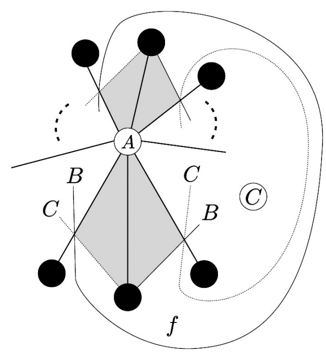


Fig. 6.7 Two adjoint pairs of type II triangles sharing the same exceptional face

Assume that G has at least two, then two or three, exceptional faces. Suppose that there are two adjoint pairs of type II triangle. Then these pairs correspond to distinct exceptional faces as above. In any case, there exists an adjoint pair of type II triangles which yields a 4-sided exceptional face f. Then the situation is as shown in Fig. 6.8.

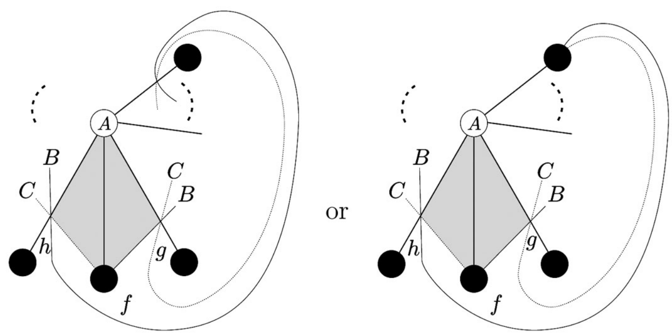


Fig. 6.8 An adjoint pair of type II triangles adjacent to a 4-sided face

This implies that both g and h are exceptional. Another adjoint pair of type II triangles yields one more exceptional face. Thus G would have 4 exceptional faces, a contradiction.

Lemma 6.9 *If n is odd, then each white vertex is incident with an exceptional face, hence $N \geq 3$.*

Proof. Assume that only triangles appear at a white vertex, A, say. Each triangle at A is incident with either a B-line or a C-line. Moreover, such triangles appear alternatively around A. Hence n must be even.

Lemma 6.10 *Suppose that vertex A is incident with only one exceptional face f. If f is 4-sided and incident with two white vertices, then n is even.*

Proof. We may assume that f is incident with A and B. Then each triangle adjacent to f at A is incident with a C-line. By the same reason as the proof of Lemma 6.9, n is even.

6.2 Reduction

Lemma 6.11 $N \neq 6$.

Proof. Let $N = 6$. This happens only when G has three 4-sided exceptional faces, each of which is incident with two white vertices. In particular, a face incident with two adjacent triple crossing points is 3-sided. By Lemma 6.1, there is one exceptional face for each pair of A, B, C. We examine the local configuration at vertex A.

By Lemmas 6.2 and 6.3, there is no type I triangle. By Lemma 6.7, each type II triangle is adjacent to an exceptional face.

If two exceptional faces at A share an A-line, then there are at most two type II triangles. This implies that $n \leq 4$, a contradiction. (We remark that this situation can happen when $n = 4$.) Otherwise, there are at most four type II triangles. In fact, both sides of a type II triangle cannot be exceptional faces, because there is only one exceptional (4-sided) face for each pair of A, B, C (see Fig. 6.6(2)). Hence there are exactly four type II triangles and two exceptional faces around A, giving $n = 6$, a contradiction.

Lemma 6.12 $N \neq 5$.

Proof. Assume $N = 5$. This happens only when G has three 4-sided exceptional faces f_1, f_2, f_3, two of which are incident with two white vertices, the other to one white vertex. By Lemma 6.1, we may assume that f_1 is incident with A and B, f_2 is incident with B and C, and f_3 is incident with B or C.

We examine the local configuration at A. By Lemma 6.3, there is no bad type I triangle. Assume that f_3 is incident with B. By Lemma 6.2, a good type I-C triangle is impossible, and at most one good type I-B triangle is possible. By Lemma 6.7, there are at most two type II triangles, which are adjacent to f_1. Hence we have $n \leq 4$, a contradiction.

The case where f_3 is incident with C is similar.

Lemma 6.13 $N \neq 4$.

Proof. Assume $N = 4$. Then G has at least two exceptional faces.

First, suppose that G has a 5-sided face f and a 4-sided face f'. Then both f and f' are incident with two white vertices. We may assume that f are incident with A, B, and f' to B, C. This case is handled by the same argument as in the second paragraph of the proof of Lemma 6.12.

Next, suppose that G has three 4-sided faces f_1, f_2, f_3. By Lemma 6.1, there are five possibilities for three exceptional faces as shown in Table 6.1, up to renaming. By Lemmas 6.9 and 6.10, $n \geq 8$ in any case.

Table 6.1 Five possibilities

	f_1	f_2	f_3
(a)	A, B	B, C	none
(b)	A, B	B	B
(c)	B, C	B	C
(d)	A, B	C	C
(e)	A, B	B	C

From (b) to (e), there is neither bad type I triangle (by Lemma 6.3) nor adjoint pair of type II triangle (by Lemma 6.7). Moreover, there are at most two good type I triangles (by Lemma 6.2) and at most two type II triangles (by Lemma 6.7). This gives $n \leq 5$, a contradiction.

Consider (a). By Lemma 6.2, there is no good type I-C triangle at A, and at most one good type I-B triangle is possible. By Lemma 6.5, there is at most one bad type I triangle. In total, there are at most two type I triangles. By Lemma 6.8, there is at most one adjoint pair of type II triangles, and further at most two type II triangles, which are adjacent to f_1, can be possible. Hence we have $n \leq 7$, a contradiction.

Lemma 6.14 $N \neq 3$.

Proof. Assume $N = 3$. We divide the proof into three cases, according to the set of exceptional faces of G'.

Case 1. G has a single 6-sided exceptional face.

Let f be the exceptional face. Then each white vertex is incident with f. By Lemmas 6.2 and 6.3, there is no type I triangle. By Lemma 6.7, there are at most two type II triangles adjacent to f. Hence $n \leq 3$, a contradiction.

Case 2. G has a 5-sided exceptional face and a 4-sided exceptional face.

Let f_1 and f_2 be the 5-sided, 4-sided exceptional faces, respectively. According to white vertices incident with them, there are four possibilities as in Table 6.2, up to renaming.

Table 6.2 Four possibilities and triangles at A

	f_1	f_2	good I-B	good I-C	bad I	adjoint II pair
(a)	B, C	B	≤ 1	≤ 1	$\times$	$\times$
(b)	A, B	C	≤ 1	$\times$	$\times$	$\times$
(c)	B	B, C	≤ 1	≤ 1	≤ 1	≤ 1
(d)	C	A, B	≤ 1	$\times$	≤ 1	≤ 1

By Lemmas 6.9 and 6.10, we have $n \geq 8$, except case (b).

(a) There is neither bad type I triangle (by Lemma 6.3) nor type II triangle (by Lemma 6.7). By Lemma 6.2, there are at most two good type I triangles. So, $n \leq 2$, a contradiction.

(b) By Lemma 6.7, there are at most two type II triangles, which are incident with f_1. There is neither bad type I triangle nor good type I-C triangle. Thus $n \leq 4$, a contradiction.

(c) There can be a good type I-X triangle at A for $X \in \{B, C\}$. Hence there are at most two good type I triangles by Lemma 6.2. By Lemma 6.5, there is at most one bad type I triangle. There is at most one adjoint pair of type II triangles by Lemma 6.8.

Claim 6.1 A bad type I triangle does not coexist with an adjoint pair of type II triangles.

Proof. Suppose that there is an adjoint pair of type II triangles. Then it is adjacent to f_1 (see Fig. 6.6(1)). If there is a bad type I triangle, then its associated exceptional face is also f_1 by Lemma 6.3. Hence f_1 is not incident with a white vertex, a contradiction.

In any case, we have $n \leq 4$, a contradiction.

(d) There is no good type I-C triangle. By Lemma 6.5, there is at most one bad type I triangle. Claim 6.1 holds again. There are at most two type II triangles, which are incident with f_2, by Lemma 6.8. In any case, we have $n \leq 6$, a contradiction.

Case 3. G has three 4-sided exceptional faces.

Let f_1, f_2, f_3 be the exceptional faces. There are five possibilities as in Table 6.3. Again, we have $n \geq 8$, except case (e), by Lemmas 6.9 and 6.10.

Table 6.3 Five possibilities and triangles at A

	f_1	f_2	f_3	good I	bad I	adjoint II pair	n
(a)	B, C	B	none	≤ 2	≤ 1	≤ 1	≤ 5
(b)	A, B	C	none	≤ 1	≤ 1	≤ 1	≤ 7
(c)	B	B	B	≤ 2	$\times$	$\times$	≤ 2
(d)	B	B	C	≤ 2	$\times$	$\times$	≤ 2
(e)	A	B	C	≤ 2	$\times$	$\times$	≤ 3

The number of good type I triangles is at most two, except (b), by Lemma 6.2. For (b), there is no good type I-C triangle by Lemma 6.2. For (c), (d) and (e), there is neither bad type I triangle (by Lemma 6.3) nor adjoint pair of type II triangles (by Lemma 6.7). Thus (c) and (d) are settled as in Table 6.3. For (a) and (b), there is at most one bad type I triangle (by Lemma 6.5) and at most one adjoint pair of type II triangles (by Lemma 6.8). Thus these cases are also settled as in Table 6.3.

The remaining case is (e). If there is no type II triangle, then we have $n \leq 3$, a contradiction. Otherwise, let g be a type II triangle. Then g is adjacent to the exceptional face f_1. We may assume that g is incident with a B-line. Let h be the face sharing this B-line with g. Then h is 3-sided, so vertex C is located there. This implies that f_1 is incident with A and C, a contradiction.

Lemma 6.15 $N \geq 3$.

Proof. Assume $N \leq 2$. We can assume that vertex A is not incident with an exceptional face. By Lemma 6.9, n is even, so $n \geq 8$. We estimate the number of triangles at A as before. By Lemmas 6.2 and 6.6, there are at most two good type I triangles and at most three bad type I triangles.

Since A is not incident with an exceptional face, type II triangles appear as adjoint pairs. By Lemma 6.8, there is at most one adjoint pair of type II triangles. Then we have $n \leq 7$, a contradiction.

Theorem 6.1 *Let $n \geq 5$ and $n \neq 6$. Then $K_{n,3}$ cannot admit a semi-regular drawing.*

Proof. By Lemma 6.15, $N \geq 3$. However, this is impossible by Lemmas 6.11–6.14.

7 Complete Bipartite Graphs

We have already shown that $K_{4,4}$, $K_{5,4}$, $K_{5,3}$ and $K_{n,3}$ with $n \geq 7$ do not admit a semi-regular drawing in Sections 4, 5 and 6.

Theorem 7.1 *Let $G = K_{n_1,n_2}$. If $n_2 \leq 2$, then $\mathrm{tcr}(G) = 0$. If $n_2 \geq 3$, then $\mathrm{tcr}(G) = \infty$ except $K_{3,3}$, $K_{4,3}$, $K_{6,3}$, $K_{6,4}$. Moreover, $\mathrm{tcr}(K_{3,3}) = \mathrm{tcr}(K_{4,3}) = 1$, $\mathrm{tcr}(K_{6,3}) = 2$ and $\mathrm{tcr}(K_{6,4}) = 4$.*

Proof. If $n_2 \leq 2$, then G is planar, and thus $\mathrm{tcr}(G) = 0$. The graph G has $p = n_1 + n_2$ vertices and $q = n_1 n_2$ edges. Then

$$q - 3p + 6 = (n_1 - 3)(n_2 - 3) - 3.$$

Hence if $n_2 \geq 5$, or $n_2 = 4$ and $n_1 \geq 7$, then

$$q - 3p + 6 > 0,$$

and thus $\mathrm{tcr}(G) = \infty$ by Lemma 2.3.

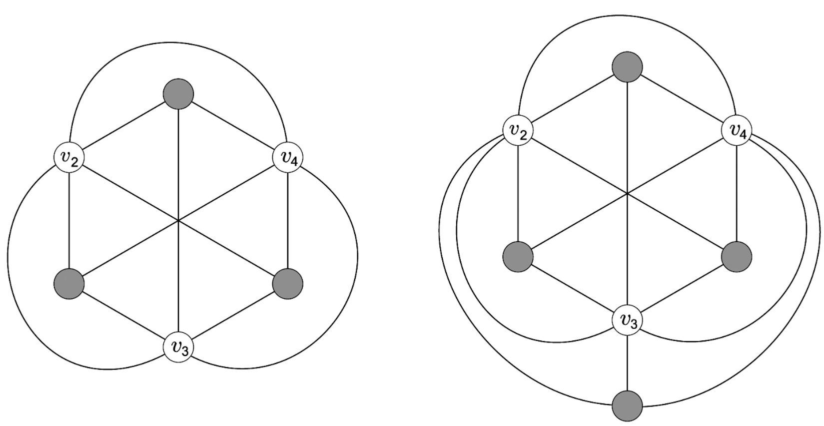


Fig. 7.1 $K_{3,1,1,1}$ and $K_{4,1,1,1}$

Fig. 7.1 (after removing three edges v_2v_3, v_3v_4, v_4v_2) shows that $\mathrm{tcr}(K_{3,3}) = \mathrm{tcr}(K_{4,3}) = 1$, since $K_{3,3}$ and $K_{4,3}$ are not planar from Lemma 2.1. Note that $\mathrm{tcr}(K_{6,3}) \geq 2$, since $\mathrm{cr}(K_{6,3}) = 6$ (see [4]) and $3\mathrm{tcr}(K_{6,3}) \geq \mathrm{cr}(K_{6,3})$. Thus we have that $\mathrm{tcr}(K_{6,3}) = 2$ from Fig. 7.2 (after removing three edges v_2v_3, v_3v_4, v_4v_2).

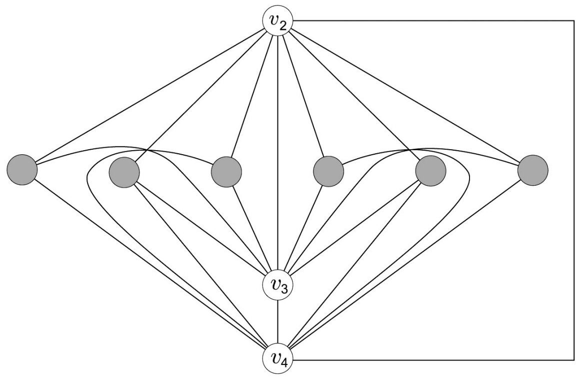


Fig. 7.2 $K_{6,1,1,1}$

Similarly, we have that $\mathrm{tcr}(K_{6,4}) = 4$ from the fact $\mathrm{cr}(K_{6,4}) = 12$ (see [4]) and Fig. 7.3.

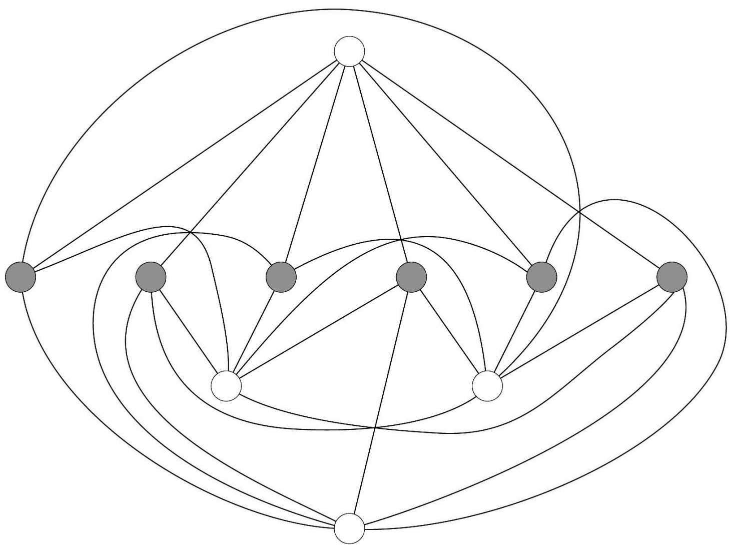


Fig. 7.3 $K_{6,4}$

Theorems 4.1, 5.1 and 6.1 show that $G = K_{n_1,n_2}$ has no semi-regular drawing for $(n_1, n_2) = (4, 4)$, $(5, 4)$, $(5, 3)$, $(n, 3)$ with $n \geq 7$.

8 Complete 4-partite Graphs

Theorem 8.1 *Let* $G = K_{n_1,n_2,n_3,n_4}$. *Then* $\mathrm{tcr}(G) = \infty$, *except* $K_{n_1,1,1,1}$ *with* $n_1 \in \{1,\ 2,\ 3,\ 4,\ 6\}$. *Also,*

$$\mathrm{tcr}(K_{n_1,1,1,1}) = \begin{cases} 0, & n_1 = 1, 2; \\ 1, & n_1 = 3, 4; \\ 2, & n_1 = 6. \end{cases}$$

Proof. The graph G has $p = \sum_i n_i$ vertices and $q = \sum_{i<j} n_i n_j$ edges. If $n_2 \geq 2$, then

$$\begin{aligned} q - 3p + 6 &= (n_1 - 1)(n_2 - 1) + (n_1 + n_2)(n_3 + n_4 - 2) + (n_3 - 3)(n_4 - 3) - 4 \\ &\geq 1 + 4(n_3 + n_4 - 2) + (n_3 - 3)(n_4 - 3) - 4 \\ &= (n_3 + 1)(n_4 + 1) - 3 \geq 1. \end{aligned}$$

Hence we have $\mathrm{tcr}(G) = \infty$ by Lemma 2.3.

Consider the case where $n_2 = n_3 = n_4 = 1$. If $n_1 = 1$ or 2, then G is planar, and thus $\mathrm{tcr}(G) = 0$. Suppose $n_1 \geq 3$. Then G is non-planar by Lemma 2.1, because G contains $K_{3,3}$ as a subgraph. Let V be the partite set of G with n_1 elements, and let v_2, v_3, v_4 be the other vertices of G. Notice that if G admits a semi-regular drawing, then no edge of the triangle $v_2v_3v_4$ contains a triple crossing point. By removing the three edges of the triangle from G, we obtain a semi-regular drawing of a complete bipartite graph $K_{n_1,3}$. However, this is impossible by Theorem 7.1, unless $n_1 = 3$, 4 or 6. Since $K_{3,1,1,1}$ and $K_{4,1,1,1}$ admit a semi-regular drawing with one triple crossing as shown in Fig. 7.1, they have triple crossing number one.

Finally, $K_{6,1,1,1}$ admits a semi-regular drawing with two triple crossings as shown in Fig. 7.2. Since $\mathrm{tcr}(K_{6,3}) = 2$ by Theorem 7.1, $\mathrm{tcr}(K_{6,1,1,1}) = 2$.

9 Complete Tripartite Graphs

Theorem 9.1 *Let $G = K_{n_1,n_2,n_3}$. Then we have the value of the triple crossing number of G as in Table* 9.1.

Table 9.1 $\mathrm{tcr}(K_{n_1,n_2,n_3})$

n_3	n_2	n_1	$\mathrm{tcr}(G)$
≥ 3			∞
2	≥ 3		∞
	2	≥ 3	∞
		2	0
1	≥ 4		∞
	3	≥ 4	∞
		3	1
	2	$\neq 2, 3, 4, 6$	∞
		6	2
		$3, 4$	1
		2	0
	1		0

Proof. The graph G has $p = \sum_i n_i$ vertices and $q = \sum_{i<j} n_i n_j$ edges. Then

$$\begin{aligned} q - 3p + 6 &= (n_1 + n_3 - 3)(n_2 + n_3 - 3) - n_3^2 + 3n_3 - 3 \\ &\geq (2n_3 - 3)^2 - n_3^2 + 3n_3 - 3 \\ &= 3(n_3 - 1)(n_3 - 2). \end{aligned}$$

If $n_3 \geq 3$, then

$$q - 3p + 6 > 0.$$

Thus we obtain $\mathrm{tcr}(G) = \infty$ by Lemma 2.3.

Let $n_3 = 2$. Then

$$q - 3p + 6 = (n_1 - 1)(n_2 - 1) - 1.$$

If $n_2 \geq 3$, then

$$q - 3p + 6 > 0.$$

If $n_2 = 2$, then

$$q - 3p + 6 > 0,$$

except when $n_1 = 2$. For these cases, $\mathrm{tcr}(G) = \infty$ by Lemma 2.3 again. Since $K_{2,2,2}$ is planar, $\mathrm{tcr}(K_{2,2,2}) = 0$.

Let $n_3 = 1$. We have

$$q - 3p + 6 = (n_1 - 2)(n_2 - 2) - 1.$$

If $n_2 \geq 4$, or if $n_2 = 3$ and $n_1 \geq 4$, then

$$q - 3p + 6 > 0.$$

For these cases, $\mathrm{tcr}(G) = \infty$. Since $K_{3,3,1}$ is not planar (it contains $K_{3,3}$ as a subgraph) and it admits a semi-regular drawing with one triple crossing as shown in Fig. 9.1, we have $\mathrm{tcr}(K_{3,3,1}) = 1$.

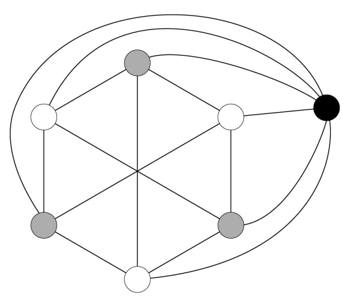


Fig. 9.1 $K_{3,3,1}$

The remaining cases are when $n_2 = 1, 2$. If $n_2 = 1$ or $n_1 = n_2 = 2$, G is planar. Therefore consider the case where $n_2 = 2$ and $n_1 \geq 3$. Let V_1 and V_2 be the partite sets of G with n_1 and n_2 elements, respectively, and let v_3 be the remaining vertex. Assume that G admits a semi-regular drawing. Notice that no edge connecting v_3 and a vertex of V_2 contains a triple crossing point, since $|V_2| = 2$. Thus we obtain a semi-regular drawing of a complete bipartite graph $K_{n_1,3}$ by removing two edges between v_3 and V_2. Then we have that $n_1 = 3$, 4 or 6 from Theorem 7.1. In either case, G is not planar, since G contains $K_{3,3}$ as a subgraph. Thus we have

$$\mathrm{tcr}(K_{3,2,1}) = \mathrm{tcr}(K_{4,2,1}) = 1$$

by Fig. 7.1 (after removing the edge v_2v_4). Since $\mathrm{tcr}(K_{6,3}) = 2$ by Theorem 7.1, we obtain

$$\mathrm{tcr}(K_{6,2,1}) = 2$$

by Fig. 7.2 (after removing the edge v_2v_4).

10 Comments

In this paper, we require that two edges intersect at most once, and two edges with a common end-vertex do not intersect. This is one natural standpoint in the study of the crossing number (see [5]–[6]), but this might be so strong that most complete multipartite graphs do not admit semi-regular drawings. If we relax it, then $K_{4,4}$, for example, admits a semi-regular drawing as shown in Fig. 10.1.

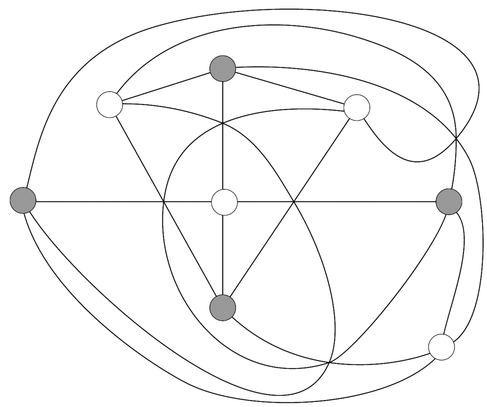


Fig. 10.1 Two edges intersect twice

In general, for $n \geq 4$, we can define the n-fold crossing number for a graph G to be the minimal number of n-fold crossing points over all drawings with only n-fold crossings. By a similar argument, we can show that Theorem 3.1 holds for the n-fold crossing number. Furthermore, if G is a non-planar complete t-partite graph with $t \geq 3$, then we can show that G does not admit a drawing with only n-fold crossings by similar arguments to those of Sections 2, 8 and 9. It might be possible to determine the values of this invariant for complete bipartite graphs.

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