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L-octo-algebras

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Abstract: L-octo-algebra with 8 operations as the Lie algebraic analogue of octo-algebra such that the sum of 8 operations is a Lie algebra is discussed. Any octo-algebra is an L-octo-algebra. The relationships among L-octo-algebras, L-quadri-algebras, L-dendriform algebras, pre-Lie algebras and Lie algebras are given. The close relationships between L-octo-algebras and some interesting structures like Rota-Baxter operators, classical Yang-Baxter equations and some bilinear forms satisfying certain conditions are given also.

Key words: L-octo-algebra, L-quadri-algebra, bimodule

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1 Introduction

Octo-algebras are a remarkable class of Loday algebras (see [1]). Loday algebras which include dendriform trialgebras (see [2]–[3]), NS-algebras (see [4]), octo-algebras (see [5]), ennea-algebras (see [6]) and dendriform-Nijenhuis algebras (see [4]) were first introduced by Loday[7] in 1995 with motivation from algebraic K-theory. At first, they introduced due to their own interesting motivations, then they developed as independent algebraic systems. Loday algebras are closely related to the study of CYBE, $\mathcal{O}$-operator, operads and so on (see [8]–[10]).

In this paper, we introduce the notion of L-octo-algebra and discuss the relationships among Lie algebra, L-dendriform algebra, L-quadri-algebras and L-octo-algebras. This paper is organized as follows: In Section 2, we recall some basic facts on pre-Lie algebras, L-dendriform algebras and L-quadri-algebras; The definition of L-octo-algebras and the associated L-quadri-algebras, L-dendriform algebras and pre-Lie algebras on L-octo-algebras are given in Section 3; We give the bimodules on L-quadri-algebras and the bimodule of the

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associated L-quadri-algebras, L-dendriform algebras, pre-Lie algebras on L-octo-algebras and the construction of L-octo-algebras on L-quadri-algebras by $\mathcal{O}$-operators on L-quadri-algebras, 2-cocycle on L-quadri-algebra in Sections 4 and 5, respectively; Finally, we give the bilinear forms on L-octo-algebras and the LO-equation on L-octo-algebras in Section 6. Throughout this paper, all algebras are finite dimensional and over a field F of characteristic zero.

2 Pre-Lie Algebras, L-dendriform Algebras and L-quadri-algebras

Proposition 2.1[11]–[12] *Let $(A, \circ)$ be a pre-Lie algebra. Then the commutator*

$$[x, y] = x \circ y - y \circ x, \qquad x, y \in A \tag{2.1}$$

defines a Lie algebra $\mathfrak{g}(A)$, which is called the sub-adjacent Lie algebra of A.

Proposition 2.2[11] *Let $(A, \triangleright, \triangleleft)$ be an L-dendriform algebra. If we define*

$$x \bullet y = x \triangleright y + x \triangleleft y, \qquad x, y \in A, \tag{2.2}$$

$$x \circ y = x \triangleright y - y \triangleleft x, \qquad x, y \in A, \tag{2.3}$$

then $(A, \bullet)$ and $(A, \circ)$ are pre-Lie algebras, which are called the associated horizontal and vertical pre-Lie algebras.

Proposition 2.3[12] *Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be an L-quadri-algebra.*

(1) *$(A, \succ, \prec)$ and $(A, \vee, \wedge)$ are dendriform algebras. They are called the associated vertical and depth L-dendriform algebra of $(A, \searrow, \nearrow, \nwarrow, \swarrow)$;*

(2) *If we define*

$$x \triangleright y = x \searrow y - y \nwarrow x, \quad x \triangleleft y = x \nearrow y - y \swarrow x, \qquad x, y \in A, \tag{2.4}$$

then $(A, \triangleright, \triangleleft)$ is a dendriform algebra, which is called the associated horizontal L-dendriform algebra of $(A, \searrow, \nearrow, \nwarrow, \swarrow)$.

3 L-octo-algebras

Definition 3.1[12] *Let A be a vector space with eight bilinear products denoted by $\searrow_1$, $\searrow_2$, $\nearrow_1$, $\nearrow_2$, $\nwarrow_1$, $\nwarrow_2$, $\swarrow_1 \swarrow_2$: $A \otimes A \to A$. $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$ is called an L-octo-algebra if for any $x, y, z \in A$,*

$$x \searrow_2 (y \searrow_2 z) - (x *_{12} y) \searrow_2 z = y \searrow_2 (x \searrow_2 z) - (y *_{12} x) \searrow_2 z,$$

$$x \searrow_2 (y \nearrow_2 z) - (x \vee_{12} y) \nearrow_2 z = y \nearrow_2 (x \succ_2 z) - (y \wedge_{12} x) \nearrow_2 z,$$

$$x \searrow_2 (y \nearrow_1 z) - (x \vee_2 y) \nearrow_1 z = y \nearrow_1 (x \succ_{12} z) - (y \wedge_1 x) \nearrow_1 z,$$

$$x \nearrow_2 (y \succ_1 z) - (x \wedge_2 y) \nearrow_1 z = y \searrow_1 (x \nearrow_{12} z) - (y \vee_1 x) \nearrow_1 z,$$

$$x \searrow_2 (y \searrow_1 z) - (x *_2 y) \searrow_1 z = y \searrow_1 (x \searrow_{12} z) - (y *_1 x) \searrow_1 z,$$

$$x \searrow_2 (y \nwarrow_1 z) - (x \searrow_2 y) \nwarrow_1 z = y \nwarrow_1 (x *_{12} z) - (y \nwarrow_1 x) \nwarrow_1 z,$$

$$x \nearrow_2 (y \prec_1 z) - (x \nearrow_2 y) \nwarrow_1 z = y \swarrow_1 (x \wedge_{12} z) - (y \swarrow_1 x) \nwarrow_1 z,$$

$$x \nearrow_1 (y \prec_{12} z) - (x \nearrow_1 y) \nwarrow_1 z = y \swarrow_2 (x \wedge_1 z) - (y \swarrow_2 x) \nwarrow_1 z,$$

$$x \searrow_1 (y \swarrow_{12} z) - (x \succ_1 y) \swarrow_1 z = y \swarrow_2 (x \vee_1 z) - (y \prec_2 x) \swarrow_1 z,$$

$$x \searrow_1 (y \nwarrow_{12} z) - (x \searrow_1 y) \nwarrow_1 z = y \nwarrow_2 (x *_1 z) - (y \nwarrow_2 x) \nwarrow_1 z,$$

$$x \searrow_2 (y \swarrow_1 z) - (x \succ_2 y) \swarrow_1 z = y \swarrow_1 (x \vee_{12} z) - (y \prec_1 x) \swarrow_1 z,$$

$$x \searrow_2 (y \swarrow_2 z) - (x \succ_{12} y) \swarrow_2 z = y \swarrow_2 (x \vee_2 z) - (y \prec_{12} x) \swarrow_2 z,$$

$$x \nearrow_2 (y \prec_2 z) - (x \nearrow_{12} y) \nwarrow_2 z = y \swarrow_2 (x \wedge_2 z) - (y \swarrow_{12} x) \nwarrow_2 z,$$

$$x \searrow_2 (y \nwarrow_2 z) - (x \searrow_{12} y) \nwarrow_2 z = y \nwarrow_2 (x *_2 z) - (y \nwarrow_{12} x) \nwarrow_2 z,$$

where

$$x \vee_i y = x \searrow_i y + x \swarrow_i y, \quad x \wedge_i y = x \nearrow_i y + x \nwarrow_i y, \qquad i = 1, 2,$$

$$x \succ_i y = x \searrow_i y + x \nearrow_i y, \quad x \prec_i y = x \nwarrow_i y + x \swarrow_i y, \qquad i = 1, 2,$$

$$x \searrow_{12} y = x \searrow_1 y + x \searrow_2 y, \qquad x \nearrow_{12} y = x \nearrow_1 y + x \nearrow_2 y,$$

$$x \nwarrow_{12} y = x \nwarrow_1 y + x \nwarrow_2 y, \qquad x \swarrow_{12} y = x \swarrow_1 y + x \swarrow_2 y,$$

$$x \vee_{12} y = x \vee_1 y + x \vee_2 y, \qquad x \wedge_{12} y = x \wedge_1 y + x \wedge_2 y,$$

$$x \succ_{12} y = x \succ_1 y + x \succ_2 y, \qquad x \prec_{12} y = x \prec_1 y + x \prec_2 y,$$

$$x *_1 y = x \searrow_1 y + x \nearrow_1 y + x \nwarrow_1 y + x \swarrow_1 y,$$

$$x *_2 y = x \searrow_2 y + x \nearrow_2 y + x \nwarrow_2 y + x \swarrow_2 y,$$

$$x *_{12} y = x *_1 y + x *_2 y.$$

Proposition 3.1[12] *Let* $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$ *be an L-octo-algebra. Then* $(A, \vee_2, \wedge_2, \wedge_1, \vee_1)$, $(A, \succ_2, \succ_1, \prec_1, \prec_2)$, $(A, \searrow_{12}, \nearrow_{12}, \nwarrow_{12}, \swarrow_{12})$ *are L-quadri-algebras. If we define*

$$x \triangleright_1^2 y = x \searrow_2 y - y \nwarrow_1 x, \quad x \triangleleft_1^2 y = x \nearrow_2 y - y \swarrow_1 x,$$

$$x \triangleright_2^1 y = x \searrow_1 y - y \nwarrow_2 x, \quad x \triangleleft_2^1 y = x \nearrow_1 y - y \swarrow_2 x, \qquad x, y \in A, \tag{3.1}$$

then $(A, \triangleright_1^2, \triangleleft_1^2, \triangleleft_2^1, \triangleright_2^1)$ *is an L-quadri-algebra.*

Proposition 3.2 *For an L-octo-algebra* $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$, *define*

$$\begin{array}{ll} x \triangleright_1 y = x *_2 y, & x \triangleleft_1 y = x *_1 y, \\ x \triangleright_2 y = x \vee_{12} y, & x \triangleleft_2 y = x \wedge_{12} y, \\ x \triangleright_3 y = x \vee_2 y - y \wedge_1 x, & x \triangleleft_3 y = x \wedge_2 y - y \vee_1 x, \\ x \triangleright_4 y = x \succ_{12} y, & x \triangleleft_4 y = x \prec_{12} y, \\ x \triangleright_5 y = x \succ_2 y - y \prec_1 x, & x \triangleleft_5 y = x \succ_1 y - y \prec_2 x, \\ x \triangleright_6 y = x \searrow_{12} y - y \nwarrow_{12} x, & x \triangleleft_6 y = x \nearrow_{12} y - y \swarrow_{12} x. \end{array}$$

Then $(A, \triangleright_i, \triangleleft_i)$ $(i = 1, 2, 3, 4, 5, 6)$ *are L-dendriform algebras.* $(A, \triangleright_1, \triangleleft_1)$ *is the associated vertical L-dendriform algebra of* $(A, \vee_2, \wedge_2, \wedge_1, \vee_1)$ *and the associated depth L-dendriform algebra of* $(A, \succ_2, \succ_1, \prec_1, \prec_2)$; $(A, \triangleright_2, \triangleleft_2)$ *is the associated depth L-dendriform algebra of*

$(A, \vee_2, \wedge_2, \wedge_1, \vee_1)$ and the associated depth L-dendriform algebra of $(A, \searrow_{12}, \nearrow_{12}, \nwarrow_{12}, \swarrow_{12})$; $(A, \rhd_3, \lhd_3)$ is the associated horizontal L-dendriform algebra of $(A, \vee_2, \wedge_2, \wedge_1, \vee_1)$ and the associated horizontal L-dendriform algebra of $(A, \rhd_1^2, \lhd_1^2, \lhd_2^1, \rhd_2^1)$; $(A, \rhd_4, \lhd_4)$ is the associated vertical L-dendriform algebra of $(A, \succ_2, \succ_1, \prec_1, \prec_2)$ and the associated vertical L-dendriform algebra of $(A, \searrow_{12}, \nearrow_{12}, \nwarrow_{12}, \swarrow_{12})$; $(A, \rhd_5, \lhd_5)$ is the associated horizontal L-dendriform algebra of $(A, \succ_2, \succ_1, \prec_1, \prec_2)$ and the associated vertical L-dendriform algebra of $(A, \rhd_1^2, \lhd_1^2, \lhd_2^1, \rhd_2^1)$; $(A, \rhd_6, \lhd_6)$ is the associated horizontal L-dendriform algebra of $(A, \searrow_{12}, \nearrow_{12}, \nwarrow_{12}, \swarrow_{12})$ and the associated depth L-dendriform algebra of $(A, \rhd_1^2, \lhd_1^2, \lhd_2^1, \rhd_2^1)$.

Proof. It follows straightly from Propositions 2.3 and 3.1.

Proposition 3.3 *Let $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$ be an L-octo-algebra.*

(1) *If we define $x \circ_1 y = x *_{12} y$, then $(A, \circ_1)$ is a pre-Lie algebra. It is the horizontal pre-Lie algebra of the L-dendriform algebra $(A, \rhd_1, \lhd_1)$, the horizontal pre-Lie algebra of the L-dendriform algebra $(A, \rhd_2, \lhd_2)$ and the horizontal pre-Lie of $(A, \rhd_4, \lhd_4)$ of the L-octo-algebra $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$;*

(2) *If we define $x \circ_2 y = x *_2 y - y *_1 x$, then $(A, \circ_2)$ is a pre-Lie algebra. It is the vertical pre-Lie algebra of the L-dendriform algebra $(A, \rhd_1, \lhd_1)$, the horizontal pre-Lie algebra of the L-dendriform algebra $(A, \rhd_3, \lhd_3)$ and the vertical pre-Lie algebra of the L-dendriform algebra $(A, \rhd_5, \lhd_5)$ of the L-octo-algebra $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$;*

(3) *If we define $x \circ_3 y = x \vee_{12} y - y \wedge_{12} x$, then $(A, \circ_3)$ is a pre-Lie algebra. It is the vertical pre-Lie algebra of the L-dendriform algebra $(A, \rhd_2, \lhd_2)$, the vertical pre-Lie algebra of the L-dendriform algebra $(A, \rhd_3, \lhd_3)$ and the vertical pre-Lie algebra of the L-dendriform algebra $(A, \rhd_6, \lhd_6)$ of the L-octo-algebra $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$;*

(4) *If we define $x \circ_4 y = x \succ_{12} y - y \prec_{12} x$, then $(A, \circ_4)$ is a pre-Lie algebra. It is the vertical pre-Lie algebra of the L-dendriform algebra $(A, \rhd_4, \lhd_4)$, the horizontal pre-Lie algebra of the L-dendriform algebra $(A, \rhd_5, \lhd_5)$ and the horizontal pre-Lie algebra of the L-dendriform algebra $(A, \rhd_6, \lhd_6)$ of the L-octo-algebra $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$;*

(5) *Define $[x, y] = x *_{12} y - y *_{12} x$. Then $(A, [\cdot, \cdot])$ is a Lie algebra on the L-octo-algebra $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$. It is also the Lie algebra of the L-quadri-algebra $(A, \vee_2, \wedge_2, \wedge_1, \vee_1)$, $(A, \succ_2, \succ_1, \prec_1, \prec_2)$, $(A, \searrow_{12}, \nearrow_{12}, \nwarrow_{12}, \swarrow_{12})$ and $(A, \rhd_1^2, \lhd_1^2, \lhd_2^1, \rhd_2^1)$.*

Proof. It follows straightly from Propositions 2.1, 2.2, 3.1 and 3.2.

4 Bimodules of L-quadri-algebras

Definition 4.1 *Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be an L-quadri-algebra and V be a vector space. Let $l_\searrow, r_\searrow, l_\nearrow, r_\nearrow, l_\nwarrow, r_\nwarrow, l_\swarrow, r_\swarrow : A \to gl(V)$ be eight linear maps. $(l_\searrow, r_\searrow, l_\nearrow, r_\nearrow, l_\nwarrow, r_\nwarrow, l_\swarrow, r_\swarrow, V)$ is called a bimodule of $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ if the following fourteen equations hold (for any $x, y \in A$)*

$$[l_\searrow(x), l_\searrow(y)] = l_\searrow(x * y) - l_\searrow(y * x), \tag{4.1}$$

$$r_{\searrow}(x \searrow y) - r_{\searrow}(y)r_*(x) = l_{\searrow}(x)r_{\searrow}(y) - r_{\searrow}(y)l_*(x), \tag{4.2}$$

$$r_{\searrow}(y \nearrow x) - r_{\nearrow}(x)r_{\vee}(y) = l_{\nearrow}(y)r_{\succ}(x) - r_{\nearrow}(x)l_{\wedge}(y), \tag{4.3}$$

$$r_{\nearrow}(x \succ y) - r_{\nearrow}(y)r_{\wedge}(x) = l_{\searrow}(x)r_{\nearrow}(y) - r_{\nearrow}(y)l_{\vee}(x), \tag{4.4}$$

$$l_{\searrow}(x)l_{\nearrow}(y) - l_{\nearrow}(y)l_{\searrow}(x) = l_{\nearrow}(x \vee y) - l_{\nearrow}(y \wedge x) + l_{\nearrow}(y)l_{\nearrow}(x), \tag{4.5}$$

$$r_{\searrow}(x \nwarrow y) - r_{\nwarrow}(y)r_{\searrow}(x) = l_{\nwarrow}(x)r_*(y) - r_{\nwarrow}(y)l_{\nwarrow}(x), \tag{4.6}$$

$$r_{\nwarrow}(x * y) - r_{\nwarrow}(y)r_{\nwarrow}(x) = [l_{\searrow}(x),\ r_{\nwarrow}(y)], \tag{4.7}$$

$$l_{\searrow}(x)l_{\nwarrow}(y) - l_{\nwarrow}(x \searrow y) = l_{\nwarrow}(y)l_*(x) - l_{\nwarrow}(y \nwarrow x), \tag{4.8}$$

$$r_{\nearrow}(x \prec y) - r_{\nwarrow}(y)r_{\nearrow}(x) = l_{\swarrow}(x)r_{\nearrow}(y) + l_{\swarrow}(x)r_{\nwarrow}(y) - r_{\nwarrow}(y)l_{\swarrow}(x), \tag{4.9}$$

$$l_{\nearrow}(x)r_{\prec}(y) - r_{\nwarrow}(y)l_{\nearrow}(x) = r_{\swarrow}(x \wedge y) - r_{\nwarrow}(y)r_{\swarrow}(x), \tag{4.10}$$

$$l_{\nearrow}(x)l_{\prec}(y) - l_{\nwarrow}(x \nearrow y) = l_{\swarrow}(y)l_{\wedge}(x) - l_{\nwarrow}(y \swarrow x), \tag{4.11}$$

$$r_{\searrow}(x \swarrow y) - r_{\swarrow}(y)r_{\succ}(x) = l_{\swarrow}(x)r_{\vee}(y) - r_{\swarrow}(y)l_{\prec}(x), \tag{4.12}$$

$$l_{\searrow}(x)r_{\swarrow}(y) - r_{\swarrow}(y)l_{\succ}(x) = -r_{\swarrow}(y)r_{\prec}(x) + r_{\swarrow}(x \vee y), \tag{4.13}$$

$$l_{\searrow}(x)l_{\swarrow}(y) - l_{\swarrow}(x \succ y) = l_{\swarrow}(y)l_{\vee}(x) - l_{\swarrow}(y \prec x), \tag{4.14}$$

where

$$\begin{aligned}
&r_*(x) = r_{\searrow}(x) + r_{\nearrow}(x) + r_{\nwarrow}(x) + r_{\swarrow}(x), \qquad l_*(x) = l_{\searrow}(x) + l_{\nearrow}(x) + l_{\nwarrow}(x) + l_{\swarrow}(x),\\
&r_{\succ}(x) = r_{\searrow}(x) + r_{\nearrow}(x), \qquad l_{\succ}(x) = l_{\searrow}(x) + l_{\nearrow}(x), \qquad r_{\prec}(x) = r_{\nwarrow}(x) + r_{\swarrow}(x),\\
&l_{\prec}(x) = l_{\nwarrow}(x) + l_{\swarrow}(x), \qquad r_{\vee}(x) = r_{\searrow}(x) + r_{\swarrow}(x), \qquad l_{\vee}(x) = l_{\searrow}(x) + l_{\swarrow}(x),\\
&r_{\wedge}(x) = r_{\nearrow}(x) + r_{\nwarrow}(x), \qquad l_{\wedge}(x) = l_{\nearrow}(x) + l_{\nwarrow}(x).
\end{aligned}$$

In fact, according to the definition of the bimodule of an L-quadri-algebra, we can check straightly that $(l_{\searrow}, r_{\searrow}, l_{\nearrow}, r_{\nearrow}, l_{\nwarrow}, r_{\nwarrow}, l_{\swarrow}, r_{\swarrow}, V)$ is a bimodule of an L-quadri-algebra $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ if and only if the direct sum $A \oplus V$ of the underlying vector spaces of A and V is turned into an L-quadri-algebra by defining multiplications in $A \oplus V$ by

$$\begin{aligned}
(x+u) \searrow (y+v) &= x \searrow y + l_{\searrow}(x)v + r_{\searrow}(y)u,\\
(x+u) \nearrow (y+v) &= x \nearrow y + l_{\nearrow}(x)v + r_{\nearrow}(y)u,\\
(x+u) \nwarrow (y+v) &= x \nwarrow y + l_{\nwarrow}(x)v + r_{\nwarrow}(y)u,\\
(x+u) \swarrow (y+v) &= x \swarrow y + l_{\swarrow}(x)v + r_{\swarrow}(y)u,
\end{aligned}$$

where $x, y \in A$, $u, v \in V$. We denote it by $A \ltimes_{l_{\searrow}, r_{\searrow}, l_{\nearrow}, r_{\nearrow}, l_{\nwarrow}, r_{\nwarrow}, l_{\swarrow}, r_{\swarrow}} V$.

Proposition 4.1 *Let* $(l_{\searrow}, r_{\searrow}, l_{\nearrow}, r_{\nearrow}, l_{\nwarrow}, r_{\nwarrow}, l_{\swarrow}, r_{\swarrow}, V)$ *be a bimodule of an L-quadri-algebra* $(A, \searrow, \nearrow, \nwarrow, \swarrow)$. *Then* $(l^*_{\searrow} + l^*_{\nearrow} + l^*_{\nwarrow} + l^*_{\swarrow} - r^*_{\searrow} - r^*_{\nearrow} - r^*_{\nwarrow} - r^*_{\swarrow}, -r^*_{\searrow}, r^*_{\searrow} + r^*_{\swarrow} - l^*_{\nearrow} - l^*_{\nwarrow}, r^*_{\searrow} + r^*_{\nearrow}, l^*_{\nwarrow} - r^*_{\searrow}, -r^*_{\searrow} - r^*_{\nearrow} - r^*_{\nwarrow} - r^*_{\swarrow}, r^*_{\searrow} + r^*_{\nearrow} - l^*_{\nwarrow} - l^*_{\swarrow}, r^*_{\searrow} + r^*_{\swarrow}, V^*)$ *is a bimodule of* $(A, \searrow, \nearrow, \nwarrow, \swarrow)$.

Proof. It can be checked straightly by the definition of the bimodule of L-quadri-algebra.

Proposition 4.2 *Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be an L-quadri-algebra. Then*

(1) $(L_\searrow, R_\searrow, L_\nearrow, R_\nearrow, L_\nwarrow, R_\nwarrow, L_\swarrow, R_\swarrow, A)$ *is a bimodule of* $(A, \searrow, \nearrow, \nwarrow, \swarrow)$;

(2) $(L^*_\searrow + L^*_\nearrow + L^*_\nwarrow + L^*_\swarrow - R^*_\searrow - R^*_\nearrow - R^*_\nwarrow - R^*_\swarrow,\ -R^*_\searrow,\ R^*_\searrow + R^*_\swarrow - L^*_\nearrow - L^*_\nwarrow,\ R^*_\searrow + R^*_\nearrow,\ L^*_\nwarrow - R^*_\searrow,\ -R^*_\searrow - R^*_\nearrow - R^*_\nwarrow - R^*_\swarrow,\ R^*_\searrow + R^*_\nearrow - L^*_\nwarrow - L^*_\swarrow,\ R^*_\searrow + R^*_\swarrow,\ A^*)$ *is a bimodule of* $(A, \searrow, \nearrow, \nwarrow, \swarrow)$.

Proof. We can check it by the Definition 4.1 and Proposition 4.1.

Proposition 4.3 *Let A be a vector space with eight bilinear products denoted by $\searrow_1$, $\searrow_2$, $\nearrow_1$, $\nearrow_2$, $\nwarrow_1$, $\nwarrow_2$, $\swarrow_1$, $\swarrow_2$: $A \otimes A \to A$. Then*

(1) $(\searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2, A)$ *is an L-octo-algebra if and only if* $(A, \vee_2, \wedge_2, \wedge_1, \vee_1)$ *is an L-quadri-algebra and* $(L_{\searrow_2}, R_{\swarrow_2}, L_{\nearrow_2}, R_{\nwarrow_2}, L_{\nearrow_1}, R_{\nwarrow_1}, L_{\searrow_1}, R_{\swarrow_1}, A)$ *is a bimodule*;

(2) $(\searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2, A)$ *is an L-octo-algebra if and only if* $(A, \succ_2, \succ_1, \prec_1, \prec_2)$ *is an L-quadri-algebra and* $(L_{\searrow_2}, R_{\nearrow_2}, L_{\searrow_1}, R_{\nearrow_1}, L_{\swarrow_1}, R_{\nwarrow_1}, L_{\swarrow_2}, R_{\nwarrow_2}, A)$ *is a bimodule*;

(3) $(\searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2, A)$ *is an L-octo-algebra if and only if* $(A, \searrow_{12}, \nearrow_{12}, \nwarrow_{12}, \swarrow_{12})$ *is an L-quadri-algebra and* $(L_{\searrow_2}, R_{\searrow_1}, L_{\nearrow_2}, R_{\nearrow_1}, L_{\nwarrow_2}, R_{\nwarrow_1}, L_{\swarrow_2}, R_{\swarrow_1}, A)$ *is a bimodule.*

Proof. It follows from Definitions 4.1 and 3.1.

Proposition 4.4 *Let $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$ be an L-octo-algebra and $(A, \vee_2, \wedge_2, \wedge_1, \vee_1)$, $(A, \succ_2, \succ_1, \prec_1, \prec_2)$, $(A, \searrow_{12}, \nearrow_{12}, \nwarrow_{12}, \swarrow_{12})$ be the associated L-quadri-algebra. Then*

(1) $(L^*_{\searrow_2}+L^*_{\nearrow_2}+L^*_{\nearrow_1}+L^*_{\searrow_1}-R^*_{\swarrow_2}-R^*_{\nwarrow_2}-R^*_{\nwarrow_1}-R^*_{\swarrow_1},\ -R^*_{\swarrow_2},\ R^*_{\swarrow_2}+R^*_{\swarrow_1}-L^*_{\nearrow_2}-L^*_{\nearrow_1},\ R^*_{\swarrow_2}+R^*_{\nwarrow_2},\ L^*_{\nearrow_1}-R^*_{\swarrow_2},\ -R^*_{\swarrow_2}-R^*_{\nwarrow_2}-R^*_{\nwarrow_1}-R^*_{\swarrow_1},\ R^*_{\swarrow_2}+R^*_{\nwarrow_2}-L^*_{\nearrow_1}-L^*_{\searrow_1},\ R^*_{\swarrow_2}+R^*_{\swarrow_1},\ A^*)$ *is a bimodule of* $(A, \vee_2, \wedge_2, \wedge_1, \vee_1)$;

(2) $(L^*_{\searrow_2}+L^*_{\searrow_1}+L^*_{\swarrow_2}+L^*_{\swarrow_1}-R^*_{\nearrow_2}-R^*_{\nearrow_1}-R^*_{\nwarrow_2}-R^*_{\nwarrow_1},\ -R^*_{\nearrow_2},\ R^*_{\nearrow_2}+R^*_{\nwarrow_2}-L^*_{\searrow_1}-L^*_{\swarrow_1},\ R^*_{\nearrow_2}+R^*_{\nearrow_1},\ L^*_{\swarrow_1}-R^*_{\nearrow_2},\ -R^*_{\nearrow_2}-R^*_{\nwarrow_2}-R^*_{\nearrow_1}-R^*_{\nwarrow_1},\ R^*_{\nearrow_2}+R^*_{\nearrow_1}-L^*_{\swarrow_1}-L^*_{\swarrow_2},\ R^*_{\nearrow_2}+R^*_{\nwarrow_2},\ A^*)$ *is a bimodule of* $(A, \succ_2, \succ_1, \prec_1, \prec_2)$;

(3) $(L^*_{\nwarrow_2}+L^*_{\swarrow_2}+L^*_{\searrow_2}+L^*_{\nearrow_2}-R^*_{\searrow_1}-R^*_{\nearrow_1}-R^*_{\nwarrow_1}-R^*_{\swarrow_1},\ -R^*_{\searrow_1},\ R^*_{\searrow_1}+R^*_{\swarrow_1}-L^*_{\nearrow_2}-L^*_{\nwarrow_2},\ R^*_{\searrow_1}+R^*_{\nearrow_1},\ L^*_{\nwarrow_2}-R^*_{\searrow_1},\ -R^*_{\searrow_1}-R^*_{\nearrow_1}-R^*_{\nwarrow_1}-R^*_{\swarrow_1},\ R^*_{\searrow_1}+R^*_{\nearrow_1}-L^*_{\swarrow_2}-L^*_{\nwarrow_2},\ R^*_{\searrow_1}+R^*_{\swarrow_1},\ A^*)$ *is a bimodule of* $(A, \searrow_{12}, \nearrow_{12}, \nwarrow_{12}, \swarrow_{12})$.

Proof. It follows from Propositions 4.1 and 4.3.

Proposition 4.5 *Let $(l_\searrow, r_\searrow, l_\nearrow, r_\nearrow, l_\nwarrow, r_\nwarrow, l_\swarrow, r_\swarrow, V)$ be the bimodule of an L-quadri-algebra $(A, \searrow, \nearrow, \nwarrow, \swarrow)$. Then*

(1) $(l_\searrow, r_\nearrow, l_\swarrow, r_\nwarrow, V)$ *is a bimodule of the associated vertical L-dendriform algebra* $(A, \succ, \prec)$ *of* $(A, \searrow, \nearrow, \nwarrow, \swarrow)$;

(2) $(l_\searrow, r_\swarrow, l_\nearrow, r_\nwarrow, V)$ *is a bimodule of the associated depth L-dendriform algebra* $(A, \vee, \wedge)$ *of* $(A, \searrow, \nearrow, \nwarrow, \swarrow)$;

(3) *$(l_{\searrow}, -l_{\nwarrow}, l_{\nearrow}, -l_{\swarrow}, V)$ is a bimodule of the associated depth L-dendriform algebra $(A, \rhd, \lhd)$ of $(A, \searrow, \nearrow, \nwarrow, \swarrow)$.*

Proof. This conclusion can be proved straightly by Definition 4.1.

Corollary 4.1 *Let $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$ be an L-octo-algebra. Then*

(1) *$(L_{\searrow_2}, R_{\nwarrow_2}, L_{\searrow_1}, R_{\nwarrow_1}, A)$ is a bimodule of the associated L-dendriform algebra $(A, \rhd_1, \lhd_1)$;*

(2) *$(L_{\searrow_2}, R_{\swarrow_1}, L_{\nearrow_2}, R_{\nwarrow_1}, A)$ is a bimodule of the associated L-dendriform algebra $(A, \rhd_2, \lhd_2)$;*

(3) *$(L_{\searrow_2}, -L_{\nearrow_1}, L_{\nearrow_2}, -L_{\searrow_1}, A)$ is a bimodule of the associated L-dendriform algebra $(A, \rhd_3, \lhd_3)$;*

(4) *$(L_{\searrow_2}, R_{\nearrow_1}, L_{\swarrow_2}, R_{\nwarrow_1}, A)$ is a bimodule of the associated L-dendriform algebra $(A, \rhd_4, \lhd_4)$;*

(5) *$(L_{\searrow_2}, -L_{\swarrow_1}, L_{\searrow_1}, -L_{\swarrow_2}, A)$ is a bimodule of the associated L-dendriform algebra $(A, \rhd_5, \lhd_5)$;*

(6) *$(L_{\searrow_2}, -L_{\nwarrow_2}, L_{\nearrow_2}, -L_{\swarrow_2}, A)$ is a bimodule of the associated L-dendriform algebra $(A, \rhd_6, \lhd_6)$.*

Proof. It follows from Definition 3.1.

Corollary 4.2 *Let $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$ be an L-octo-algebra. Then*

(1) *$(L_{\searrow_2}, R_{\nwarrow_1}, A)$ is a bimodule of the associated pre-Lie algebra $(A, \circ_1)$;*

(2) *$(L_{\searrow_2}, -L_{\searrow_1}, A)$ is a bimodule of the associated pre-Lie algebra $(A, \circ_2)$;*

(3) *$(L_{\searrow_2}, -L_{\nearrow_2}, A)$ is a bimodule of the associated pre-Lie algebra $(A, \circ_3)$;*

(4) *$(L_{\searrow_2}, -L_{\swarrow_2}, A)$ is a bimodule of the associated pre-Lie algebra $(A, \circ_4)$;*

(5) *$(L_{\searrow_2}, A)$ is a representation of the associated Lie algebra $(A, [\,\cdot\,, \cdot\,])$.*

Proof. It follows from Definition 3.1.

5 Construction of L-octo-algebras

Definition 5.1 *Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be an L-quadri-algebra, $(l_{\searrow}, r_{\searrow}, l_{\nearrow}, r_{\nearrow}, l_{\nwarrow}, r_{\nwarrow}, l_{\swarrow}, r_{\swarrow}, V)$ be a bimodule of $(A, \searrow, \nearrow, \nwarrow, \swarrow)$. A linear map $T : V \to A$ is called an $\mathcal{O}$-operator of $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ associated to $(l_{\searrow}, r_{\searrow}, l_{\nearrow}, r_{\nearrow}, l_{\nwarrow}, r_{\nwarrow}, l_{\swarrow}, r_{\swarrow}, V)$ if T satisfies*

$$T(u) \searrow T(v) = T(l_{\searrow}(T(u))v + r_{\searrow}(T(v))u),$$

$$T(u) \nearrow T(v) = T(l_{\nearrow}(T(u))v + r_{\nearrow}(T(v))u),$$

$$T(u) \nwarrow T(v) = T(l_{\nwarrow}(T(u))v + r_{\nwarrow}(T(v))u),$$

$$T(u) \swarrow T(v) = T(l_{\swarrow}(T(u))v + r_{\swarrow}(T(v))u), \qquad u, v \in V.$$

In particular, an $\mathcal{O}$-operator of an L-quadri-algebra $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ associated to $(L_{\searrow}, R_{\searrow}, L_{\nearrow}, R_{\nearrow}, L_{\nwarrow}, R_{\nwarrow}, L_{\swarrow}, R_{\swarrow}, A)$ is called a Rota-Baxter operator of weight zero on $(A, \searrow, \nearrow, \nwarrow, \swarrow)$.

Proposition 5.1 *Let* $(l_{\searrow}, r_{\searrow}, l_{\nearrow}, r_{\nearrow}, l_{\nwarrow}, r_{\nwarrow}, l_{\swarrow}, r_{\swarrow}, V)$ *be a bimodule of an L-quadri-algebra* $(A, \searrow, \nearrow, \nwarrow, \swarrow)$. *If* T *is an* $\mathcal{O}$*-operator of* $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ *associated to* $(l_{\searrow}, r_{\searrow}, l_{\nearrow}, r_{\nearrow}, l_{\nwarrow}, r_{\nwarrow}, l_{\swarrow}, r_{\swarrow}, V)$, *then there exists an L-octo-algebra structure on* V *defined by*

$$\begin{aligned}
&u \searrow_1 v = -r_{\nwarrow}(T(u))v, && u \searrow_2 v = l_{\searrow}(T(u))v, && u \nearrow_1 v = -r_{\swarrow}(T(u))v,\\
&u \nearrow_2 v = l_{\nearrow}(T(u))v, && u \nwarrow_1 v = -r_{\searrow}(T(u))v, && u \nwarrow_2 v = l_{\nwarrow}(T(u))v,\\
&u \swarrow_1 v = -r_{\nearrow}(T(u))v, && u \swarrow_2 v = l_{\swarrow}(T(u))v, && u, v \in V.
\end{aligned} \tag{5.1}$$

Therefore, there exists an L-quadri-algebra structure on V *defined by* (3.1) *as the associated L-quadri-algebra of* $(V, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$ *and* T *is the homomorphism of L-quadri-algebras. Furthermore,* $T(V) = \{T(v) \mid v \in V\} \subseteq A$ *is an L-quadri subalgebra of* $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ *and there is an induced L-octo-algebra structure on* A *given by*

$$\begin{aligned}
&T(u) \searrow_1 T(v) = T(u \searrow_1 v), && T(u) \searrow_2 T(v) = T(u \searrow_2 v),\\
&T(u) \nearrow_1 T(v) = T(u \nearrow_1 v), && T(u) \nearrow_2 T(v) = T(u \nearrow_2 v),\\
&T(u) \nwarrow_1 T(v) = T(u \nwarrow_1 v), && T(u) \nwarrow_2 T(v) = T(u \nwarrow_2 v),\\
&T(u) \swarrow_1 T(v) = T(u \swarrow_1 v), && T(u) \swarrow_2 T(v) = T(u \swarrow_2 v), \qquad u, v \in V.
\end{aligned} \tag{5.2}$$

Moreover, the corresponding associated L-quadri-algebra structure on $T(V)$ *is just the L-quadri-algebra structure of* $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ *and* T *is an homomorphism of L-quadri-algebras.*

Proof. For any $u, v, w \in V$, we have

$$u \searrow_2 (v \searrow_2 w) - v \searrow_2 (u \searrow_2 w) = l_{\searrow}(T(u))l_{\searrow}(T(v))w - l_{\searrow}(T(v))l_{\searrow}(T(u))w.$$

On the other hand,

$$\begin{aligned}
&(u *_{12} v) \searrow_2 w - (v *_{12} u) \searrow_2 w\\
&= l_{\searrow}(T[(l_{\searrow}(T(u))v + r_{\searrow}(T(v))u)] + T[(l_{\nearrow}(T(u))v + r_{\nearrow}(T(v))u)]\\
&\quad + T[(l_{\nwarrow}(T(u))v + r_{\nwarrow}(T(v))u)] + T[(l_{\swarrow}(T(u))v + r_{\swarrow}(T(v))u)]\\
&\quad - T[(l_{\searrow}(T(v))u + r_{\searrow}(T(u))v)] - T[(l_{\nearrow}(T(v))u + r_{\nearrow}(T(u))v)]\\
&\quad - T[(l_{\nwarrow}(T(v))u + r_{\nwarrow}(T(u))v)] - T[(l_{\swarrow}(T(v))u + r_{\swarrow}(T(u))v)])w\\
&= l_{\searrow}(T(u) * T(v) - T(v) * T(u))w.
\end{aligned}$$

Similarly we can prove the other identities in the definition of L-octo-algebra. The rest of this proposition follows immediately.

Theorem 5.1 *Let* $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ *be an L-quadri-algebra. Then there exists an L-octo-algebra structure on* A *such that* $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ *is the associated L-octo-algebra if and only if there exists an invertible* $\mathcal{O}$*-operator of* $(A, \searrow, \nearrow, \nwarrow, \swarrow)$.

Proof. Let T be an invertible $\mathcal{O}$-operator of $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ associated to the bimodule $(l_{\searrow}, r_{\searrow}, l_{\nearrow}, r_{\nearrow}, l_{\nwarrow}, r_{\nwarrow}, l_{\swarrow}, r_{\swarrow}, V)$. By Proposition 5.1, there exists an L-octo-algebra structure on V given by (5.1). Therefore, we define an L-octo-algebra structure on A defined by (5.2) such that T is an isomorphism of L-octo-algebras, that is

$$x \searrow_1 y = -T(r_{\nwarrow}(x)T^{-1}(y)), \qquad x \searrow_2 y = T(l_{\searrow}(x)T^{-1}(y)),$$

$$x \nearrow_1 y = -T(r_{\swarrow}(x)T^{-1}(y)), \qquad x \nearrow_2 y = T(l_{\nearrow}(x)T^{-1}(y)),$$

$$x \nwarrow_1 y = -T(r_{\searrow}(x)T^{-1}(y)), \qquad x \nwarrow_2 y = T(l_{\nwarrow}(x)T^{-1}(y)),$$

$$x \swarrow_1 y = -T(r_{\nearrow}(x)T^{-1}(y)), \qquad x \swarrow_2 y = T(l_{\swarrow}(x)T^{-1}(y)).$$

Moreover, the associated L-quadri-algebra defined by (3.1) is $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ since

$$x \searrow y = T(T^{-1}(x)) \searrow T(T^{-1}(y)) = T(l_{\searrow}(x)T^{-1}(y) + r_{\searrow}(y)T^{-1}(x)) = x \searrow_2 y - y \nwarrow_1 x,$$

$$x \nearrow y = T(T^{-1}(x)) \nearrow T(T^{-1}(y)) = T(l_{\nearrow}(x)T^{-1}(y) + r_{\nearrow}(y)T^{-1}(x)) = x \nearrow_2 y - y \swarrow_1 x,$$

$$x \nwarrow y = T(T^{-1}(x)) \nwarrow T(T^{-1}(y)) = T(l_{\nwarrow}(x)T^{-1}(y) + r_{\nwarrow}(y)T^{-1}(x)) = x \nwarrow_2 y - y \searrow_1 x,$$

$$x \swarrow y = T(T^{-1}(x)) \swarrow T(T^{-1}(y)) = T(l_{\swarrow}(x)T^{-1}(y) + r_{\swarrow}(y)T^{-1}(x)) = x \swarrow_2 y - y \nearrow_1 x.$$

Conversely, let $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$ be an L-octo-algebra and $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be the associated L-quadri-algebra $(A, \rhd_1^2, \lhd_1^2, \lhd_2^1, \rhd_2^1)$. Then the identity map $\mathrm{id} : A \to A$ is an invertible $\mathcal{O}$-operator of $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ associated to the bimodule $(L_{\searrow_2}, -L_{\nwarrow_1}, L_{\nearrow_2}, -L_{\swarrow_1}, L_{\searrow_2}, -L_{\nwarrow_1}, L_{\swarrow_2}, -L_{\swarrow_1}, A)$.

Proposition 5.2 *Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be an L-quadri-algebra and $r \in A \otimes A$ be symmetric. Then r is an $\mathcal{O}$-operator of $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ associate to $(L^*_{\searrow} + L^*_{\nearrow} + L^*_{\nwarrow} + L^*_{\swarrow} - R^*_{\searrow} - R^*_{\nearrow} - R^*_{\nwarrow} - R^*_{\swarrow}, -R^*_{\searrow}, R^*_{\searrow} + R^*_{\swarrow} - L^*_{\nearrow} - L^*_{\nwarrow}, R^*_{\searrow} + R^*_{\nearrow}, L^*_{\nwarrow} - R^*_{\searrow}, -R^*_{\searrow} - R^*_{\nearrow} - R^*_{\nwarrow} - R^*_{\swarrow}, R^*_{\searrow} + R^*_{\nearrow} - L^*_{\nwarrow} - L^*_{\swarrow}, R^*_{\searrow} + R^*_{\swarrow}, A^*)$ if and only if r satisfies the following equations*:

$$r_{13} \rhd r_{23} = -(r_{12} * r_{23}) + r_{12} \nwarrow r_{13}, \tag{5.3}$$

$$r_{13} \lhd r_{23} = (r_{12} \wedge r_{23}) - r_{12} \prec r_{13}. \tag{5.4}$$

Moreover, if r is invertible, then r satisfies (5.3)–(5.4) *if and only if the nondegenerate bilinear form $\mathcal{B}$ induced by r satisfies*

$$\mathcal{B}(x \swarrow y, z) = -\mathcal{B}(y, z \bullet x) - \mathcal{B}(x, z \vee y), \tag{5.5}$$

$$\mathcal{B}(x \nearrow y, z) = \mathcal{B}(y, x \wedge z - z \vee x) - \mathcal{B}(x, z \succ y). \tag{5.6}$$

Proof. Let $F_r : A^* \to A$ be the $\mathcal{O}$-operator defined by r. Then

$$r_{13} \searrow r_{23} = r_{13} \searrow r_{12} - (r_{12} \vee r_{23} + r_{12} \wedge r_{23}) + (r_{23} \vee r_{12} + r_{23} \wedge r_{12}).$$

Similarly, by the definition of $\mathcal{O}$-operator, we can get

$$r_{13} \nearrow r_{23} = -r_{13} \succ r_{12} + r_{12} \wedge r_{23} - r_{23} \vee r_{12},$$

$$r_{13} \nwarrow r_{23} = r_{13} * r_{12} - r_{12} \nwarrow r_{23} + r_{23} \searrow r_{12},$$

$$r_{13} \swarrow r_{23} = -r_{13} \vee r_{12} + r_{12} \prec r_{23} - r_{23} \succ r_{12}.$$

Let σ be any element in the permutation group Σ_3 acting on $\{1, 2, 3\}$. Then σ induces a linear map from $A \otimes A \otimes A$ to $A \otimes A \otimes A$ by

$$\sigma(x_1 \otimes x_2 \otimes x_3) = x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes x_{\sigma(3)}.$$

By the action of Σ and combining these equations we can get the conclusion.

Definition 5.2 *Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be an L-quadri-algebra and $r \in A \otimes A$ be symmetric.* (5.3)–(5.4) *is called LQ-equation in $(A, \searrow, \nearrow, \nwarrow, \swarrow)$. On the other hand, a symmetric bilinear form $\mathcal{B}$ on A satisfying* (5.5)–(5.6) *is called a 2-cocycle of $(A, \searrow, \nearrow, \nwarrow, \swarrow)$.*

Proposition 5.3 *Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be an L-quadri-algebra with a nondegenerate 2-cocycle $\mathcal{B}$. Then there exists an L-octo-algebra structure $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$ on A defined by*

$$\begin{aligned}
&\mathcal{B}(x \searrow_1 y, z) = -\mathcal{B}(y, z * x), && \mathcal{B}(x \searrow_2 y, z) = \mathcal{B}(y, z * x - x * z),\\
&\mathcal{B}(x \nearrow_1 y, z) = \mathcal{B}(y, z \vee x), && \mathcal{B}(x \nearrow_2 y, z) = \mathcal{B}(y, x \wedge z - z \vee x),\\
&\mathcal{B}(x \nwarrow_1 y, z) = -\mathcal{B}(y, z \searrow x), && \mathcal{B}(x \nwarrow_2 y, z) = \mathcal{B}(y, z \searrow x - x \nwarrow z),\\
&\mathcal{B}(x \swarrow_1 y, z) = \mathcal{B}(y, z \succ x), && \mathcal{B}(x \swarrow_2 y, z) = \mathcal{B}(y, x \prec z - z \succ x).
\end{aligned} \tag{5.7}$$

Proof. By Proposition 5.2, the invertible linear map $T : A^* \to A$ is an invertible $\mathcal{O}$-operator associated to the bimodule $(L^*_{\searrow} + L^*_{\nearrow} + L^*_{\nwarrow} + L^*_{\swarrow} - R^*_{\searrow} - R^*_{\nearrow} - R^*_{\nwarrow} - R^*_{\swarrow}, -R^*_{\searrow}, R^*_{\searrow} + R^*_{\swarrow} - L^*_{\nearrow} - L^*_{\nwarrow}, R^*_{\searrow} + R^*_{\nearrow}, L^*_{\nwarrow} - R^*_{\searrow}, -R^*_{\searrow} - R^*_{\nearrow} - R^*_{\nwarrow} - R^*_{\swarrow}, R^*_{\searrow} + R^*_{\nearrow} - L^*_{\nwarrow} - L^*_{\swarrow}, R^*_{\searrow} + R^*_{\swarrow}, A^*)$. As a result, there is an L-octo-algebra structure on A defined by

$$\begin{aligned}
\mathcal{B}(x \searrow_1 y, z) &= \mathcal{B}(T((R^*_{\searrow} + R^*_{\nearrow} + R^*_{\nwarrow} + R^*_{\swarrow})(x)T^{-1}(y)), z)\\
&= \langle (R^*_{\searrow} + R^*_{\nearrow} + R^*_{\nwarrow} + R^*_{\swarrow})(x)T^{-1}(y), z\rangle\\
&= -\langle T^{-1}(y), z * x\rangle\\
&= -\mathcal{B}(y, z * x).
\end{aligned}$$

Similarly, by a direct computation we can get the other identities in (5.7).

Corollary 5.1 *Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be an L-quadri-algebra with a nondegenerate 2-cocycle $\mathcal{B}$. Then the corresponding L-quadri-algebra $(A, \vee_2, \wedge_2, \wedge_1, \vee_1)$, $(A, \succ_2, \succ_1, \prec_1, \prec_2)$, $(A, \searrow_{12}, \nearrow_{12}, \nwarrow_{12}, \swarrow_{12})$ of the L-octo-algebra $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$ is given by*

$$\begin{aligned}
&\mathcal{B}(x \vee_2 y, z) = \mathcal{B}(y, z \prec x - x \succ z), && \mathcal{B}(x \wedge_2 y, z) = \mathcal{B}(y, x \nearrow z - z \swarrow x),\\
&\mathcal{B}(x \wedge_1 y, z) = \mathcal{B}(y, z \swarrow x), && \mathcal{B}(x \vee_1 y, z) = -\mathcal{B}(y, z \prec x),\\
&\mathcal{B}(x \succ_2 y, z) = \mathcal{B}(y, z \wedge x - x \vee z), && \mathcal{B}(x \succ_1 y, z) = -\mathcal{B}(y, z \wedge x),\\
&\mathcal{B}(x \prec_1 y, z) = \mathcal{B}(y, z \nearrow x), && \mathcal{B}(x \prec_2 y, z) = \mathcal{B}(y, x \swarrow z - z \nearrow x),\\
&\mathcal{B}(x \searrow_{12} y, z) = -\mathcal{B}(y, x * z), && \mathcal{B}(x \nearrow_{12} y, z) = \mathcal{B}(y, x \wedge z),\\
&\mathcal{B}(x \nwarrow_{12} y, z) = -\mathcal{B}(y, x \nwarrow z), && \mathcal{B}(x \swarrow_{12} y, z) = \mathcal{B}(y, x \prec z).
\end{aligned} \tag{5.8}$$

Proposition 5.4 *Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be an L-quadri-algebra and $(l_{\searrow}, r_{\searrow}, l_{\nearrow}, r_{\nearrow}, l_{\nwarrow}, r_{\nwarrow}, l_{\swarrow}, r_{\swarrow}, V)$ be a bimodule. Let $(l^*_{\searrow} + l^*_{\nearrow} + l^*_{\nwarrow} + l^*_{\swarrow} - r^*_{\searrow} - r^*_{\nearrow} - r^*_{\nwarrow} - r^*_{\swarrow}, -r^*_{\searrow}, r^*_{\searrow} + r^*_{\swarrow} - l^*_{\nearrow} - l^*_{\nwarrow}, r^*_{\searrow} + r^*_{\nearrow}, l^*_{\nwarrow} - r^*_{\searrow}, -r^*_{\searrow} - r^*_{\nearrow} - r^*_{\nwarrow} - r^*_{\swarrow}, r^*_{\searrow} + r^*_{\nearrow} - l^*_{\nwarrow} - l^*_{\swarrow}, r^*_{\searrow} + r^*_{\swarrow}, V^*)$ be the bimodule of A. Let $T : V \to A$ be a linear map which can be identified as an element in the vector space*

$(A\oplus V)\otimes(A\oplus V)$. *Then* $r = T-\sigma(T)$ *is a symmetric solution of LQ-equation in the L-quadri-algebra* $A\ltimes_{(l^*_{\searrow}+l^*_{\nearrow}+l^*_{\nwarrow}+l^*_{\swarrow}-r^*_{\searrow}-r^*_{\nearrow}-r^*_{\nwarrow}-r^*_{\swarrow},-r^*_{\searrow},r^*_{\searrow}+r^*_{\swarrow}-l^*_{\nearrow}-l^*_{\nwarrow},r^*_{\searrow}+r^*_{\nearrow},l^*_{\nwarrow}-r^*_{\searrow},-r^*_{\searrow}-r^*_{\nearrow}-r^*_{\nwarrow}-r^*_{\swarrow},r^*_{\searrow}+r^*_{\nearrow}-l^*_{\nwarrow}-l^*_{\swarrow},r^*_{\searrow}+r^*_{\swarrow})}V^*$ *if and only if* T *is an* $\mathcal{O}$*-operator of* $(A,\ \searrow,\ \nearrow,\ \nwarrow,\ \swarrow)$ *associate to* $(l^*_{\searrow}+l^*_{\nearrow}+l^*_{\nwarrow}+l^*_{\swarrow}-r^*_{\searrow}-r^*_{\nearrow}-r^*_{\nwarrow}-r^*_{\swarrow},\ -r^*_{\searrow}, r^*_{\searrow}+r^*_{\swarrow}-l^*_{\nearrow}-l^*_{\nwarrow},\ r^*_{\searrow}+r^*_{\nearrow},\ l^*_{\nwarrow}-r^*_{\searrow},\ -r^*_{\searrow}-r^*_{\nearrow}-r^*_{\nwarrow}-r^*_{\swarrow},\ r^*_{\searrow}+r^*_{\nearrow}-l^*_{\nwarrow}-l^*_{\swarrow},\ r^*_{\searrow}+r^*_{\swarrow},\ V^*)$.

Proof. It follows straightly by a direct computation.

Corollary 5.2 *Let* $(A,\ \searrow_1,\ \searrow_2,\ \nearrow_1,\ \nearrow_2,\ \nwarrow_1,\ \nwarrow_2,\ \swarrow_1,\ \swarrow_2)$ *be an L-octo-algebra and* $(A,\ \vee_2,\ \wedge_2,\ \wedge_1,\ \vee_1)$, $(A,\ \succ_2,\ \succ_1,\ \prec_1,\ \prec_2)$, $(A,\ \searrow_{12},\ \nearrow_{12},\ \nwarrow_{12},\ \swarrow_{12})$ *be the associated L-quadri-algebra. Let* $\{e_1, e_2, \cdots, e_n\}$ *be a basis of* A *and* $\{e_1^*,\ e_2^*,\ \cdots,\ e_n^*\}$ *be a basis of* A^*. *Then*

$$r=\sum_{i=1}^{n}(e_i\otimes e_i^*-e_i^*\otimes e_i)$$

is a skew symmetric solution of LD-equation in the L-quadri-algebras

$$A\ltimes_{(L^*_{\succ_{12}}-R^*_{\prec_{12}},-R^*_{\swarrow_2},R^*_{\swarrow_{12}}-L^*_{\nearrow_{12}},R^*_{\prec_2},L^*_{\nearrow_1}-R^*_{\swarrow_2},-R^*_{\prec_{12}},R^*_{\prec_2}-L^*_{\succ_1},R^*_{\swarrow_{12}})}A^*,$$

$$A\ltimes_{(L^*_{\vee_{12}}-R^*_{\wedge_{12}},-R^*_{\nearrow_2},R^*_{\wedge_2}-L^*_{\vee_1},R^*_{\nearrow_{12}},L^*_{\swarrow_1}-R^*_{\nearrow_2},-R^*_{\wedge_{12}},R^*_{\nearrow_{12}}-L^*_{\swarrow_{12}},R^*_{\wedge_2})}A^*,$$

$$A\ltimes_{(L^*_{*_2}-R^*_{*_1},-R^*_{\searrow_1},R^*_{\vee_1}-L^*_{\wedge_2},R^*_{\succ_1},L^*_{\nwarrow_2}-R^*_{\searrow_1},-R^*_{*_1},R^*_{\succ_1}-L^*_{\prec_2},R^*_{\vee_1})}A^*,$$

respectively. Moreover, there is a natural 2*-cocycle* $\mathcal{B}$ *of this L-quadri-algebras induced by* r *which is given by*

$$\mathcal{B}(x+a^*,\ y+b^*)=-\langle a^*,\ y\rangle+\langle x,\ b^*\rangle,\qquad x,y\in A,\ a^*,b^*\in A^*.$$

Proof. This is because id is an $\mathcal{O}$-operator of $(A,\ \vee_2,\ \wedge_2,\ \wedge_1,\ \vee_1)$, $(A,\ \succ_2,\ \succ_1,\ \prec_1,\ \prec_2)$, $(A,\ \searrow_{12},\ \nearrow_{12},\ \nwarrow_{12},\ \swarrow_{12})$.

6 Bilinear Forms on L-octo-algebras and LO-equation

Theorem 6.1 *Let* $(A,\ \searrow_1,\ \searrow_2,\ \nearrow_1,\ \nearrow_2,\ \nwarrow_1,\ \nwarrow_2,\ \swarrow_1,\ \swarrow_2)$ *be an L-octo-algebra and* $r\in A\otimes A$ *be skew-symmetric. Let* $(A,\ \vee_2,\ \wedge_2,\ \wedge_1,\ \vee_1)$, $(A,\ \succ_2,\ \succ_1,\ \prec_1,\ \prec_2)$, $(A,\ \searrow_{12},\ \nearrow_{12},\ \nwarrow_{12},\ \swarrow_{12})$ *be the associated L-quadri-algebra. Then the following statements are equivalent*:

(1) r *is the* $\mathcal{O}$*-operator of* $(A,\ \vee_2,\ \wedge_2,\ \wedge_1,\ \vee_1)$ *associated to the bimodule*
$(L^*_{\succ_{12}}-R^*_{\prec_{12}},\ -R^*_{\swarrow_2},\ R^*_{\swarrow_{12}}-L^*_{\nearrow_{12}},\ R^*_{\prec_2},\ L^*_{\nearrow_1}-R^*_{\swarrow_2},\ -R^*_{\prec_{12}},\ R^*_{\prec_2}-L^*_{\succ_1},\ R^*_{\swarrow_{12}},\ A^*)$;

(2) r *is the* $\mathcal{O}$*-operator of* $(A,\ \succ_2,\ \succ_1,\ \prec_1,\ \prec_2)$ *associated to the bimodule*
$(L^*_{\vee_{12}}-R^*_{\wedge_{12}},\ -R^*_{\nearrow_2},\ R^*_{\wedge_2}-L^*_{\vee_1},\ R^*_{\nearrow_{12}},\ L^*_{\swarrow_1}-R^*_{\nearrow_2},\ -R^*_{\wedge_{12}},\ R^*_{\nearrow_{12}}-L^*_{\swarrow_{12}},\ R^*_{\wedge_2},\ A^*)$;

(3) r *is the* $\mathcal{O}$*-operator of* $(A,\ \searrow_{12},\ \nearrow_{12},\ \nwarrow_{12},\ \swarrow_{12})$ *associated to the bimodule*
$(L^*_{*_2}-R^*_{*_1},\ -R^*_{\searrow_1},\ R^*_{\vee_1}-L^*_{\wedge_2},\ R^*_{\succ_1},\ L^*_{\nwarrow_2}-R^*_{\searrow_1},\ -R^*_{*_1},\ R^*_{\succ_1}-L^*_{\prec_2},\ R^*_{\vee_1},\ A^*)$;

(4) r *satisfies*

$$r_{13}\vee_2 r_{23}=-r_{12}\succ_{12} r_{23}+r_{23}\prec_{12} r_{12}-r_{13}\swarrow_2 r_{12},\tag{6.1}$$

$$r_{13}\wedge_2 r_{23}=r_{12}\nearrow_{12} r_{23}-r_{23}\swarrow_{12} r_{12}+r_{13}\prec_2 r_{12},\tag{6.2}$$

$$r_{13}\wedge_1 r_{23}=-r_{12}\nearrow_1 r_{23}+r_{23}\swarrow_2 r_{12}-r_{13}\prec_{12} r_{12},\tag{6.3}$$

$$r_{13} \vee_1 r_{23} = r_{12} \succ_1 r_{23} - r_{23} \prec_2 r_{12} + r_{13} \swarrow_{12} r_{12}. \tag{6.4}$$

Proof. We just prove the equivalence between (1) and (4). Let $\{e_1, e_2, \cdots, e_n\}$ be a basis of A and $\{e_1^*, e_2^*, \cdots, e_n^*\}$ be the dual basis. Suppose that

$$r(e_i^*) = \sum_{j=1}^{n} a_{ij} e_j,$$

and

$$\begin{aligned} &e_i \searrow_1 e_j = \sum_{k=1}^{n} a_{ij}^k e_k, \quad && e_i \searrow_2 e_j = \sum_{k=1}^{n} b_{ij}^k e_k, \quad && e_i \nearrow_1 e_j = \sum_{k=1}^{n} c_{ij}^k e_k, \\ &e_i \nearrow_2 e_j = \sum_{k=1}^{n} d_{ij}^k e_k, && e_i \nwarrow_1 e_j = \sum_{k=1}^{n} m_{ij}^k e_k, && e_i \nwarrow_2 e_j = \sum_{k=1}^{n} n_{ij}^k e_k, \\ &e_i \swarrow_1 e_j = \sum_{k=1}^{n} s_{ij}^k e_k, && e_i \swarrow_2 e_j = \sum_{k=1}^{n} t_{ij}^k e_k. \end{aligned}$$

Since r is an $\mathcal{O}$-operator, we have

$$\begin{aligned} &r(e_i^*) \vee_2 r(e_j^*) \\ = {}& r[(L^*_{\searrow_2} + L^*_{\nearrow_2} + L^*_{\searrow_1} + L^*_{\nearrow_1} - R^*_{\nwarrow_2} - R^*_{\swarrow_2} - R^*_{\nwarrow_1} - R^*_{\swarrow_1})(r(e_i^*))e_j^* - R^*_{\swarrow_2}(r(e_j^*))e_i^*]. \end{aligned}$$

So we can get

$$\begin{aligned} \sum_{k,l=1}^{n} a_{ik} a_{jl} (b_{kl}^t + t_{kl}^t) = & - \sum_{k,l=1}^{n} a_{ik} a_{lt} (b_{kl}^j + d_{kl}^j + a_{kl}^j + c_{kl}^j) \\ & + \sum_{k,l=1}^{n} a_{ik} a_{lt} (m_{lk}^j + n_{lk}^j + s_{lk}^j + t_{lk}^j) + \sum_{k,l=1}^{n} a_{jk} a_{lt} t_{lk}^i. \end{aligned}$$

This is the coefficient of $e_i \otimes e_j \otimes e_t$ in

$$r_{13} \vee_2 r_{23} = -r_{12} \succ_{12} r_{23} + r_{23} \prec_{12} r_{12} - r_{13} \swarrow_2 r_{12}.$$

Similarly, we can compute the other identities in (4). Hence this conclusion holds.

Theorem 6.2 *Let $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$ be an L-octo-algebra and $r \in A \otimes A$ be skew-symmetric. Suppose that r is invertible. Then r satisfies* (6.1)–(6.4) *if and only if the nondegenerate bilinear form $\mathcal{B}$ induced by r satisfies*

$$\mathcal{B}(z \vee_2 x, y) = \mathcal{B}(z \succ_{12} y, x) - \mathcal{B}(y \prec_{12} z, x) - \mathcal{B}(y \swarrow_2 x, z), \tag{6.5}$$

$$\mathcal{B}(z \wedge_2 x, y) = -\mathcal{B}(z \nearrow_{12} y, x) + \mathcal{B}(y \swarrow_{12} z, x) + \mathcal{B}(y \prec_2 x, z), \tag{6.6}$$

$$\mathcal{B}(z \wedge_1 x, y) = \mathcal{B}(z \nearrow_1 y, x) - \mathcal{B}(y \swarrow_2 z, x) - \mathcal{B}(y \prec_{12} x, z), \tag{6.7}$$

$$\mathcal{B}(z \vee_1 x, y) = -\mathcal{B}(z \succ_1 y, x) + \mathcal{B}(y \prec_2 z, x) + \mathcal{B}(y \swarrow_{12} x, z). \tag{6.8}$$

Proof. Suppose that $r = \sum\limits_i a_i \otimes b_i$. Then $r(v^*) = \sum\limits_i v^*(a_i) b_i$. Since r is nondegenerate, for any $x, y, z \in A$, there exists $u^*, v^*, w^* \in A^*$ such that $x = r(u^*)$, $y = r(v^*)$, $z = r(w^*)$, so

$$\begin{aligned} &\langle r_{13} \swarrow_2 r_{12}, \ w^* \otimes u^* \otimes v^* \rangle \\ = {}& \left\langle w^* \otimes u^* \otimes v^*, \ \Big(\sum_i a_i \otimes 1 \otimes b_i\Big) \swarrow_2 \Big(\sum_j a_j \otimes b_j \otimes 1\Big) \right\rangle \end{aligned}$$

$$\begin{aligned}
&= \sum_{i,j} \langle w^* \otimes u^* \otimes v^*,\ (a_i \swarrow_2 a_j) \otimes b_j \otimes b_i \rangle \\
&= \sum_{i,j} w^*(a_i \swarrow_2 a_j) v^*(b_i) u^*(b_j) \\
&= \langle r(v^*) \swarrow_2 r(u^*),\ w^* \rangle \\
&= -\mathcal{B}(y \swarrow_2 x,\ z).
\end{aligned}$$

Similarly, we can compute $\langle r_{13} \vee_2 r_{23},\ w^* \otimes u^* \otimes v^* \rangle$, $\langle r_{12} \succ_{12} r_{23},\ w^* \otimes u^* \otimes v^* \rangle$, $\langle r_{23} \prec_{12} r_{12},\ w^* \otimes u^* \otimes v^* \rangle$. As a result, we have

$$\mathcal{B}(z \vee_2 x,\ y) = \mathcal{B}(z \succ_{12} y,\ x) - \mathcal{B}(y \prec_{12} z,\ x) - \mathcal{B}(y \swarrow_2 x,\ z).$$

Similarly, we can get the other identities in (6.5)–(6.8).

Definition 6.1 *Let* $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$ *be an L-octo-algebra and* $r \in A \otimes A$ *be skew-symmetric.* (6.1)–(6.4) *are called LO-equations* (*a set of equations*) *in* $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$. *On the other hand, a skew symmetric bilinear form* $\mathcal{B}$ *on* A *is called a* 2*-cocycle if* $\mathcal{B}$ *satisfies* (6.5)–(6.8).

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