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Ore Extensions over Weakly 2-primal Rings

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Abstract: A weakly 2-primal ring is a common generalization of a semicommutative ring, a 2-primal ring and a locally 2-primal ring. In this paper, we investigate Ore extensions over weakly 2-primal rings. Let α be an endomorphism and δ an α-derivation of a ring R. We prove that (1) If R is an (α, δ)-compatible and weakly 2-primal ring, then $R[x;\ \alpha, \delta]$ is weakly semicommutative; (2) If R is (α, δ)-compatible, then R is weakly 2-primal if and only if $R[x;\ \alpha, \delta]$ is weakly 2-primal.

Key words: (α, δ)-compatible ring, weakly 2-primal ring, weakly semicommutative ring, nil-semicommutative ring, Ore extension

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1 Introduction

Throughout this paper, R denotes an associative ring with identity, α is an endomorphism of R and δ is an α-derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$ for $a, b \in R$. We denote by $R[x;\ \alpha, \delta]$ the Ore extension whose elements are the polynomials over R, the addition is defined as usual, and the multiplication subject to the reaction $xr = \alpha(r)x + \delta(r)$ for any $r \in R$. Particularly, if $\delta = 0_R$, we denote by $R[x; \alpha]$ the skew polynomial ring; if $\alpha = 1_R$, we denote by $R[x; \delta]$ the differential polynomial ring. For a ring R, we denote by $\mathrm{nil}(R)$ the set of all nilpotent elements of R, $\mathrm{Nil}_*(R)$ its lower nil-radical, $\mathrm{Nil}^*(R)$ its upper nil-radical and L-rad(R) its Levitzki radical. For a nonempty subset M of a ring R, the symbol $\langle M\rangle$ denotes the subring (may not with 1) generated by M.

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Recall that a ring R is called reduced if it has no nonzero nilpotent elements; R is symmetric if $abc = 0$ implies $acb = 0$ for all $a, b, c \in R$; R is semicommutative if $ab = 0$ implies $aRb = 0$ for all $a, b \in R$. In [1], semicommutative property is called the insertion-of-factors-property, or IFP. There are many papers to study semicommutative rings and their generalization (see [2]–[5]). Liu and Zhao ([6], Lemma 3.1) has proved that if R is a semicommutative ring, then $\mathrm{nil}(R)$ is an ideal of R. Liang *et al.*[5] called a ring R to be weakly semicommutative if $ab = 0$ implies $aRb \subseteq \mathrm{nil}(R)$ for any $a, b \in R$. This notion is a proper generalization of semicommutative rings by Example 2.2 in [5]. According to Chen[2], a ring R is called nil-semicommutative if $ab \in \mathrm{nil}(R)$ implies $aRb \subseteq \mathrm{nil}(R)$ for any $a, b \in R$. A nil-semicommutative ring is weakly semicommutative, but the converse is not true by Example 2.2 in [2]. Recall that a ring R is 2-primal if $\mathrm{nil}(R) = \mathrm{Nil}_*(R)$. Hong *et al.*[7] called a ring R to be locally 2-primal if each finite subset generates a 2-primal ring, and have shown that if R is a nil ring then R is locally 2-primal if and only if R is a Levitzki radical ring. Chen and Cui[3] called a ring R to be weakly 2-primal if the set of nilpotent elements in R coincides with its Levitzki radical, that is, $\mathrm{nil}(R)$=L-rad(R). Due to Marks[8], a ring R is called NI if $\mathrm{nil}(R) = \mathrm{Nil}^*(R)$. It is obvious that a ring R is NI if and only if $\mathrm{nil}(R)$ forms an ideal, if and only if $R/\mathrm{Nil}^*(R)$ is reduced. Hwang *et al.*[9] considered basic structure and some extensions of NI rings, and Proposition 2.1 in [3] has presented their some characterizations. The following implications hold:

$$\text{Reduced} \Rightarrow \text{Symmetric} \Rightarrow \text{Semicommutative} \Rightarrow \text{2-primal} \Rightarrow \text{Locally 2-primal}$$
$$\Rightarrow \text{Weakly 2-primal} \Rightarrow NI\text{-ring} \Rightarrow \text{Weakly semicommutative.}$$

In general, each of these implications is irreversible (see [3], [7]).

According to Annin[10], for an endomorphism α and an α-derivation δ, a ring R is said to be α-compatible if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$. Moreover, R is called to be δ-compatible if for each $a, b \in R$, $ab = 0 \Rightarrow a\delta(b) = 0$. If R is both α-compatible and δ-compatible, R is called (α, δ)-compatible. Liang *et al.*[5] have proved that if R is α-compatible semicommutative, then $R[x;\ \alpha]$ is weakly semicommutative. Chen and Cui[3] have shown that if R is weakly 2-primal and α-compatible, then $R[x;\ \alpha]$ is weakly 2-primal and hence weakly semicommutative. In this paper, we extend respectively the above results to more general cases, the Ore extensions over weakly 2-primal rings, and generalize recent some related work on polynomial rings and skew polynomial rings. In particular, we show that if R is an (α, δ)-compatible and weakly 2-primal ring, then $R[x;\ \alpha, \delta]$ is a weakly semicommutative ring; if R is (α, δ)-compatible, then R is weakly 2-primal if and only if $R[x;\ \alpha, \delta]$ is weakly 2-primal. At the same time, we also extend a main result proved by Chen[2] to the Ore extensions $R[x;\ \alpha, \delta]$ over weakly 2-primal ring, and obtain that if R is an (α, δ)-compatible and weakly 2-primal ring, then $R[x;\ \alpha, \delta]$ is a nil-semicommutative ring.

In the following, for integers i, j with $0 \le i \le j$, $f_i^j \in \mathrm{End}(R, +)$ denotes the map which is the sum of all possible words in α, δ built with i letters α and $j - i$ letters δ. For instance, $f_2^4 = \alpha^2\delta^2 + \delta^2\alpha^2 + \delta\alpha^2\delta + \alpha\delta^2\alpha + \alpha\delta\alpha\delta + \delta\alpha\delta\alpha$. In particular, $f_0^0 = 1$, $f_i^i = \alpha^i$, $f_0^i = \delta^i$, $f_{j-1}^j = \alpha^{j-1}\delta + \alpha^{j-2}\delta\alpha + \cdots + \delta\alpha^{j-1}$. For every $f_i^j \in \mathrm{End}(R, +)$ with $0 \le i \le j$, it has C_j^i

monomials in α, δ built with i letters α and $j-i$ letters δ. As is known to all that for any integer n and $r \in R$, we have $x^n r = \sum\limits_{i=0}^{n} f_i^n(r)x^i$ in the ring $R[x;\ \alpha, \delta]$.

2 Weakly Semicommutative Property of $R[x; \alpha, \delta]$

In this section, we discuss the weakly semicommutative property and nil-semicommutative property of Ore extensions $R[x;\ \alpha, \delta]$ over weakly 2-primal rings. In general, one may suspect that if R is (α, δ)-compatible, then R is weakly semicommutative (resp., nil-semicommutative) if and only if $R[x;\ \alpha, \delta]$ is weakly semicommutative (resp., nil-semicommutative). Since any subring of a weakly semicommutative (resp., nil-semicommutative) ring is also a weakly semicommutative (resp., nil-semicommutative) ring, it is clear that if $R[x; \alpha, \delta]$ is weakly semicommutative (resp., nil-semicommutative), then R is weakly semicommutative (resp., nil-semicommutative). Unfortunately, the converse is negative. Chen ([2], Theorem 2.6) has proved that there exists a nil-semicommutative ring R over which the polynomial ring $R[x]$ is not nil-semicommutative. Example 2.1 in the following shows that there exists a weakly semicommutative ring R over which the polynomial ring $R[x]$ is not weakly semicommutative.

Example 2.1[4] Let Z_2 be the field of integers modulo 2 and $S = Z_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle$ be the free algebra in noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over Z_2. Let $A = Z_2[a_0, a_1, a_2, b_0, b_1, b_2, c]$ be the subalgebra in S, of polynomials with zero constant terms. Note that A is a ring without identity and consider an ideal of Z_2+A, say I, generated by a_0b_0, $a_0b_1 + a_1b_0$, $a_0b_2 + a_1b_1 + a_2b_0$, $a_1b_2 + a_2b_1$, a_2b_2, a_0rb_0, $(a_0 + a_1 + a_2)r(b_0 + b_1 + b_2)$ with $r \in A$ and $r_1r_2r_3r_4$ with r_1, r_2, r_3, $r_4 \in A$. Then, clearly, $A^4 \in I$. Let $T = (Z_2 + A)/I$. Then T is semicommutative by Example 2 in [5]. Thus $R = T[x]$ is weakly semicommutative by Corollary 3.1 in [5]. Next we prove that $R[y]$ is not weakly semicommutative. Notice that $(a_0 + a_1x + a_2x^2)(b_0 + b_1x + b_2x^2) \in I[x]$, then

$(a_0 + (a_0 + a_1x)y + (a_0 + a_1x + a_2x^2)y^2)(b_0 + (b_0 + b_1x)y + (b_0 + b_1x + b_2x^2)y^2) \in I[x][y]$,

but

$(a_0 + (a_0 + a_1x)y + (a_0 + a_1x + a_2x^2)y^2)c(b_0 + (b_0 + b_1x)y + (b_0 + b_1x + b_2x^2)y^2) \notin I[x][y]$

since $a_0cb_1 + a_1cb_0 \notin I$. Therefore, $T[x]$ is not weakly semicommutative.

To prove the main results of this section, we need the following lemma and several propositions.

Lemma 2.1[11] *Let R be an (α, δ)-compatible ring. Then*

(1) *If $ab = 0$, then $a\alpha^n(b) = \alpha^n(a)b = 0$ for all positive integers n;*

(2) *If $\alpha^k(a)b = 0$ for some positive integer k, then $ab = 0$;*

(3) *If $ab = 0$, then $\alpha^n(a)\delta^m(b) = \delta^m(a)\alpha^n(b) = 0$ for all positive integers m, n.*

Proposition 2.1 *Let R be an (α, δ)-compatible ring. Then*

(1) *If $ab = 0$, then $af_i^j(b) = 0$ for all $0 \le i \le j$ and $a, b \in R$;*

(2) *For $a, b \in R$ and any positive integer m, $ab \in \mathrm{nil}(R)$ if and only if $a\alpha^m(b) \in \mathrm{nil}(R)$.*

Proof. (1) If $ab = 0$, then $a\alpha^i(b) = a\delta^j(b) = 0$ for all $i \geq 0$ and $j \geq 0$ by Lemma 2.1. Hence $af_i^j(b) = 0$ for all $0 \leq i \leq j$.

(2) It is an immediate consequence of Lemma 3.1 in [5] and Lemma 2.8 in [12].

Proposition 2.2 *Let R be an (α, δ)-compatible ring. Then*

(1) *If $abc = 0$, then $a\delta(b)c = 0$ for any $a, b, c \in R$;*

(2) *If $abc = 0$, then $af_i^j(b)c = 0$ for all $0 \leq i \leq j$ and $a, b, c \in R$;*

(3) *If $ab \in \mathrm{nil}(R)$, then $a\delta(b) \in \mathrm{nil}(R)$ for any $a, b \in R$.*

Proof. (1) If $abc = 0$, we have $\alpha(ab)\delta(c) = 0$, $\alpha(a)\alpha(b)\delta(c) = 0$ and $a\alpha(b)\delta(c) = 0$. On the other hand, we also have $a\delta(bc) = 0$, $a(\delta(b)c + \alpha(b)\delta(c)) = 0$ and $a\delta(b)c + a\alpha(b)\delta(c) = 0$. So $a\delta(b)c = 0$.

(2) If $abc = 0$, we have $a\alpha(bc) = 0$, $a\alpha(b)\alpha(c) = 0$ and $a\alpha(b)c = 0$. It follows that $a\alpha^m(b)c = 0$ and $a\delta^n\alpha^m(b)c = 0$ for any positive integers m, n. Meanwhile, we can obtain that $a\delta(b)c = 0$ by (1), which implies that $a\delta^j(b)c = 0$ and $a\alpha^i\delta^j(b)c = 0$. Therefore, we have $af_i^j(b)c = 0$ for all $0 \leq i \leq j$.

(3) Since $ab \in \mathrm{nil}(R)$, there exists some positive integer k such that $(ab)^k = 0$. In the following computations, we use freely (1):

$$\begin{aligned}
&(ab)^k = ab(ab \cdots ab) = 0 \\
\Rightarrow\ & a\delta(b)(ab \cdots ab) = (a\delta(b)a)b(ab \cdots ab) = 0 \\
\Rightarrow\ & (a\delta(b)a)\delta(b)(ab \cdots ab) = 0 \\
\Rightarrow\ & \cdots \\
\Rightarrow\ & (a\delta(b))^{k-1}ab1 = 0 \\
\Rightarrow\ & (a\delta(b))^k = 0.
\end{aligned}$$

This implies that $a\delta(b) \in \mathrm{nil}(R)$.

Proposition 2.3 *If R is an (α, δ)-compatible NI ring, then $ab \in \mathrm{nil}(R)$ implies $af_i^j(b) \in \mathrm{nil}(R)$ for all $0 \leq i \leq j$ and $a, b \in R$.*

Proof. If $ab \in \mathrm{nil}(R)$, then we have $a\alpha^i(b)$, $a\delta^j(b) \in \mathrm{nil}(R)$ for all $i \geq 0$ and $j \geq 0$ by Propositions 2.1 and 2.2. This implies $a\delta^j\alpha^i(b)$, $a\alpha^i\delta^j(b) \in \mathrm{nil}(R)$. Since R is NI, we have $af_i^j(b) \in \mathrm{nil}(R)$ for all $0 \leq i \leq j$.

Proposition 2.4 *Let R be an (α, δ)-compatible NI ring, and $f(x) = \sum\limits_{i=0}^{m} a_i x^i$, $g(x) = \sum\limits_{j=0}^{n} b_j x^j \in R[x;\ \alpha, \delta]$. Then $f(x)g(x) = 0$ implies $a_i b_j \in \mathrm{nil}(R)$ for each i, j.*

Proof. Suppose $f(x) = \sum\limits_{i=0}^{m} a_i x^i$, $g(x) = \sum\limits_{j=0}^{n} b_j x^j \in R[x;\ \alpha, \delta]$ such that $f(x)g(x) = 0$. Then we have

$$f(x)g(x) = \Big(\sum_{i=0}^{m} a_i x^i\Big)\Big(\sum_{j=0}^{n} b_j x^j\Big)$$

$$
\begin{aligned}
&= \left(\sum_{i=0}^{m} a_i x^i\right) b_0 + \left(\sum_{i=0}^{m} a_i x^i\right) b_1 x + \cdots + \left(\sum_{i=0}^{m} a_i x^i\right) b_n x^n \\
&= \sum_{i=0}^{m} a_i f_0^i(b_0) + \left(\sum_{i=1}^{m} a_i f_1^i(b_0)\right) x + \cdots + \left(\sum_{i=s}^{m} a_i f_s^i(b_0)\right) x^s + \cdots + a_m \alpha^m(b_0) x^m \\
&\quad + \left(\sum_{i=0}^{m} a_i f_0^i(b_1) + \cdots + \left(\sum_{i=s}^{m} a_i f_s^i(b_1)\right) x^s + \cdots + a_m \alpha^m(b_1) x^m\right) x + \cdots \\
&\quad + \left(\sum_{i=0}^{m} a_i f_0^i(b_n) + \left(\sum_{i=1}^{m} a_i f_1^i(b_n)\right) x + \cdots + a_m \alpha^m(b_n) x^m\right) x^n \\
&= \sum_{i=0}^{m} a_i f_0^i(b_0) + \left(\sum_{i=1}^{m} a_i f_1^i(b_0) + \sum_{i=0}^{m} a_i f_0^i(b_1)\right) x + \cdots \\
&\quad + \left(\sum_{s+t=k} \left(\sum_{i=s}^{m} a_i f_s^i(b_t)\right)\right) x^k + \cdots + a_m \alpha^m(b_n) x^{m+n} \\
&= 0.
\end{aligned}
$$

It follows that

$$\varDelta_{m+n} = a_m \alpha^m(b_n) = 0, \tag{2.1}$$

$$\varDelta_{m+n-1} = a_m \alpha^m(b_{n-1}) + a_{m-1} \alpha^{m-1}(b_n) + a_m f_{m-1}^m(b_n) = 0, \tag{2.2}$$

$$\varDelta_{m+n-2} = a_m \alpha^m(b_{n-2}) + \sum_{i=m-1}^{m} f_{m-1}^i(b_{n-1}) + \sum_{i=m-2}^{m} f_{m-2}^i(b_n) = 0, \tag{2.3}$$

$$\vdots$$

$$\varDelta_k = \sum_{s+t=k} \left(\sum_{i=s}^{m} a_i f_s^i(b_t)\right) = 0. \tag{2.4}$$

From (2.1), we have $a_m b_n = 0$ since R is (α, δ)-compatible. Thus, by Proposition 2.1, $a_m f_s^t(b_n) = 0$ for all $0 \le s \le t$. From (2.2), we have

$$\varDelta'_{m+n-1} = a_m \alpha^m(b_{n-1}) + a_{m-1} \alpha^{m-1}(b_n) = 0. \tag{2.5}$$

If we multiply (2.5) on the left side by b_n, then we obtain

$$b_n a_m \alpha^m(b_{n-1}) + b_n a_{m-1} \alpha^{m-1}(b_n) = 0.$$

Since $a_m b_n = 0$, we have $b_n a_m \in \mathrm{nil}(R)$. So

$$b_n a_{m-1} \alpha^{m-1}(b_n) = -b_n a_m \alpha^m(b_{n-1}) \in \mathrm{nil}(R),$$

because the $\mathrm{nil}(R)$ of an NI ring R is an ideal. Thus, $b_n a_{m-1} b_n \in \mathrm{nil}(R)$ by Proposition 2.1, and hence $b_n a_{m-1} \in \mathrm{nil}(R)$, $a_{m-1} b_n \in \mathrm{nil}(R)$ and $a_{m-1} \alpha^{m-1}(b_n) \in \mathrm{nil}(R)$. It follows that $a_m \alpha^m(b_{n-1}) \in \mathrm{nil}(R)$ and so $a_m b_{n-1} \in \mathrm{nil}(R)$ by Proposition 2.1. Therefore, $a_m b_{n-1},\ a_{m-1} b_n \in \mathrm{nil}(R)$. By Proposition 2.3 and (2.3),

$$
\begin{aligned}
\varDelta_{m+n-2} &= a_m \alpha^m(b_{n-2}) + a_{m-1} \alpha^{m-1}(b_{n-1}) + a_m f_{m-1}^m(b_{n-1}) + a_{m-2} \alpha^{m-2}(b_n) \\
&\quad + a_{m-1} f_{m-2}^{m-1}(b_n) + a_m f_{m-2}^m(b_n) \\
&= 0,
\end{aligned}
$$

we have

$$\varDelta'_{m+n-2} = a_m \alpha^m(b_{n-2}) + a_{m-1} \alpha^{m-1}(b_{n-1}) + a_{m-2} \alpha^{m-2}(b_n) \in \mathrm{nil}(R). \tag{2.6}$$

If we multiply (2.6) on the left side by b_n, b_{n-1}, b_{n-2}, respectively, then we obtain $a_{m-2}b_n \in \mathrm{nil}(R)$, $a_{m-1}b_{n-1} \in \mathrm{nil}(R)$ and $a_m b_{n-2} \in \mathrm{nil}(R)$ in turn.

Continuing this procedure yields that $a_i b_j \in \mathrm{nil}(R)$ for all i, j.

The index of nilpotency of a nilpotent element x in a ring R is the least positive integer n such that $x^n = 0$. The index of nilpotency of a subset I of R is the supremum of the indices of nilpotency of all nilpotent elements in I. If such a supremum is finite, then I is said to be of bounded index of nilpotency.

Proposition 2.5 *Let R be (α,δ)-compatible and $f(x) = \sum\limits_{i=0}^{n} a_i x^i \in R[x;\ \alpha,\delta]$. Then*

(1) *If R is an NI ring, then $f(x) \in \mathrm{nil}(R[x;\ \alpha,\delta])$ implies $a_i \in \mathrm{nil}(R)$ for all $0 \le i \le n$;*

(2) *If R is a weakly 2-primal ring, then $a_i \in \mathrm{nil}(R)$ for all $0 \le i \le n$ implies $f(x) \in \mathrm{nil}(R[x;\ \alpha,\delta])$;*

(3) *If $\mathrm{Nil}^*(R)$ is nilpotent, then $a_i \in \mathrm{nil}(R)$ for $0 \le i \le n$ implies $f(x) \in \mathrm{nil}(R[x;\ \alpha,\delta])$;*

(4) *If R is of bounded index of nilpotency, then $a_i \in \mathrm{nil}(R)$ for all $0 \le i \le n$ implies $f(x) \in \mathrm{nil}(R[x;\ \alpha,\delta])$.*

Proof. (1) Let $f(x) = \sum\limits_{i=0}^{n} a_i x^i \in \mathrm{nil}(R[x;\ \alpha,\delta])$. Then there exists a positive integer k such that

$$\begin{aligned} f(x)^k &= (a_0 + a_1 x + \cdots + a_n x^n)^k \\ &= \text{lower order terms} + a_n\alpha^n(a_n)\alpha^{2n}(a_n)\cdots\alpha^{(k-1)n}(a_n)x^{nk} \\ &= 0. \end{aligned}$$

Hence

$$\begin{aligned} & a_n\alpha^n(a_n)\alpha^{2n}(a_n)\cdots\alpha^{(k-1)n}(a_n) = 0 \\ \Rightarrow\ & a_n\alpha^n((a_n)\alpha^n(a_n)\cdots\alpha^{(k-2)n}(a_n)) = 0 \\ \Rightarrow\ & a_n^2\alpha^n(a_n)\cdots\alpha^{(k-3)n}(a_n)\alpha^{(k-2)n}(a_n) = 0 \\ \Rightarrow\ & a_n^3\alpha^n(a_n)\cdots\alpha^{(k-3)n}(a_n) = 0 \\ \Rightarrow\ & \cdots \\ \Rightarrow\ & a_n^k = 0 \\ \Rightarrow\ & a_n \in \mathrm{nil}(R). \end{aligned}$$

So by Proposition 2.3, $a_n = 1 \cdot a_n \in \mathrm{nil}(R)$ implies $1 \cdot f_s^t(a_n) = f_s^t(a_n) \in \mathrm{nil}(R)$ for all $0 \le s \le t$. Let $Q = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$. Then

$$\begin{aligned} 0 &= (Q + a_n x^n)^k \\ &= (Q + a_n x^n)(Q + a_n x^n)\cdots(Q + a_n x^n) \\ &= (Q^2 + Q\cdot a_n x^n + a_n x^n \cdot Q + a_n x^n \cdot a_n x^n)(Q + a_n x^n)\cdots(Q + a_n x^n) \\ &= \cdots \\ &= Q^k + \varDelta, \end{aligned}$$

where $\varDelta \in R[x;\ \alpha,\delta]$. Notice that the coefficients of $\varDelta$ can be written as sums of monomials in a_i and $f_u^v(a_j)$, where $a_i, a_j \in \{a_0, a_1, \cdots, a_n\}$ and $0 \le u \le v$ are positive integers, and each monomial has a_n or $f_s^t(a_n)$. Since $\mathrm{nil}(R)$ is an ideal of R, we obtain that each monomial is in $\mathrm{nil}(R)$, and then $\Delta \in \mathrm{nil}(R)[x;\ \alpha,\delta]$. Thus

$$\begin{aligned}&(a_0 + a_1x + \cdots + a_{n-1}x^{n-1})^k\\ &= \text{lower order terms} + a_{n-1}\alpha^{n-1}(a_{n-1})\cdots\alpha^{(n-1)(k-1)}(a_{n-1})x^{(n-1)k} \in \mathrm{nil}(R)[x;\ \alpha,\delta].\end{aligned}$$

Hence, by Proposition 2.3,

$$\begin{aligned}&a_{n-1}\alpha^{n-1}(a_{n-1})\cdots\alpha^{(n-1)(k-1)}(a_{n-1}) \in \mathrm{nil}(R)\\ \Rightarrow\ & a_{n-1}\alpha^{n-1}(a_{n-1}\alpha^{n-1}(a_{n-1})\cdots\alpha^{(n-1)(k-2)}(a_{n-1})) \in \mathrm{nil}(R)\\ \Rightarrow\ & a_{n-1}^2\alpha^{n-1}(a_{n-1})\cdots\alpha^{(n-1)(k-2)}(a_{n-1}) \in \mathrm{nil}(R)\\ \Rightarrow\ & a_{n-1}^3\alpha^{n-1}(a_{n-1})\cdots\alpha^{(n-1)(k-3)}(a_{n-1}) \in \mathrm{nil}(R)\\ \Rightarrow\ & \cdots\\ \Rightarrow\ & a_{n-1}^{k-1} \in \mathrm{nil}(R)\\ \Rightarrow\ & a_{n-1} \in \mathrm{nil}(R).\end{aligned}$$

By using induction on n, we have $a_i \in \mathrm{nil}(R)$ for all $0 \le i \le n$.

(2) Consider the finite subset $\{a_0, a_1, \cdots, a_n\}$. Since R is weakly 2-primal and hence $\mathrm{nil}(R)$=L-rad(R), $\langle a_0, a_1, \cdots, a_n\rangle$ is nilpotent subring of R. So there exists a positive integer k such that any product of k elements $a_{i1}a_{i2}\cdots a_{ik}$ from $\{a_0, a_1, \cdots, a_n\}$ is zero. Note that the coefficients of $f(x)^{k+1} = \left(\sum\limits_{i=0}^{n} a_ix^i\right)^{k+1}$ in $R[x;\ \alpha,\delta]$ can be written as sums of monomials of length $k+1$ in a_i and $f_u^v(a_j)$, where $a_i, a_j \in \{a_0, a_1, \cdots, a_n\}$ and $0 \le u \le v$ are positive integers. For each monomial $a_{i_1}f_{s_2}^{t_2}(a_{i_2})\cdots f_{s_{k+1}}^{t_{k+1}}(a_{i_{k+1}})$, where $a_{i_1}, a_{i_2}, \cdots, a_{i_{k+1}} \in \{a_0, a_1, \cdots, a_n\}$ and $t_j,\ s_j\ (t_j \ge s_j,\ 2 \le j \le k+1)$ are nonnegative integers, we obtain $a_{i_1}f_{s_2}^{t_2}(a_{i_2})\cdots\ f_{s_{k+1}}^{t_{k+1}}(a_{i_{k+1}}) = 0$ by Propositions 2.1 and 2.2. Therefore, we have $f(x)^{k+1} = 0$ and so $f(x) \in \mathrm{nil}(R[x;\ \alpha,\delta])$.

(3) In this case, $\mathrm{nil}(R) = \mathrm{Nil}^*(R)$=L-rad$(R)$, the proof is similar to that of (2).

(4) By Proposition 22.2 in [13], in this case, R is locally nilpotent, and hence $\mathrm{nil}(R) = \mathrm{Nil}^*(R)$=L-rad$(R) = R$.

Corollary 2.1 *Let R be a weakly 2-primal ring. If R is (α,δ)-compatible, then*

$$\mathrm{nil}(R[x;\ \alpha,\delta]) = \mathrm{nil}(R)[x;\ \alpha,\delta]).$$

Corollary 2.2 *Let R be (α,δ)-compatible. Then*

(1) *If R is weakly 2-primal, then $R[x;\ \alpha,\delta]$ is NI;*

(2) *If $\mathrm{Nil}^*(R)$ is nilpotent, then $R[x;\ \alpha,\delta]$ is NI;*

(3) *If R is of bounded index of nilpotency, then $R[x;\ \alpha,\delta]$ is NI.*

Proof. Since $\mathrm{nil}(R)[x;\ \alpha,\delta]$ is an ideal of $R[x;\ \alpha,\delta]$), we have

$$\mathrm{nil}(R[x;\ \alpha,\delta]) = \mathrm{nil}(R)[x;\ \alpha,\delta]) = \mathrm{Nil}^*(R)[x;\ \alpha,\delta]$$

by Proposition 2.5.

Corollary 2.3 *Let R be a weakly 2-primal ring. Then* $\mathrm{nil}(R[x]) = \mathrm{nil}(R)[x]$.

Proposition 2.6 *Let R be an (α,δ)-compatible weakly 2-primal ring. Then, for $f(x)=\sum\limits_{i=0}^{m}a_ix^i$, $g(x)=\sum\limits_{j=0}^{n}b_jx^j$, $h(x)=\sum\limits_{k=0}^{p}c_kx^k\in R[x;\ \alpha,\delta]$ and $c\in R$, we have*

(1) $fg\in\mathrm{nil}(R[x;\ \alpha,\delta])\Leftrightarrow a_ib_j\in\mathrm{nil}(R)$ *for all* $0\le i\le m$, $0\le j\le n$;

(2) $fgc\in\mathrm{nil}(R[x;\ \alpha,\delta])\Leftrightarrow a_ib_jc\in\mathrm{nil}(R)$ *for all* $0\le i\le m$, $0\le j\le n$;

(3) $fgh\in\mathrm{nil}(R[x;\ \alpha,\delta])\Leftrightarrow a_ib_jc_k\in\mathrm{nil}(R)$ *for all* $0\le i\le m$, $0\le j\le n$ *and* $0\le k\le p$.

Proof. (1) $\Rightarrow$. Suppose $f(x)=\sum\limits_{i=0}^{m}a_ix^i$, $g(x)=\sum\limits_{j=0}^{n}b_jx^j\in R[x;\ \alpha,\delta]$ such that $fg\in\mathrm{nil}(R[x;\ \alpha,\delta])$. Then

$$\begin{aligned}f(x)g(x)&=\Big(\sum_{i=0}^{m}a_ix^i\Big)\Big(\sum_{j=0}^{n}b_jx^j\Big)\\&=\sum_{i=0}^{m}a_if_0^i(b_0)+\Big(\sum_{i=1}^{m}a_if_1^i(b_0)+\sum_{i=0}^{m}a_if_0^i(b_1)\Big)x+\cdots\\&\quad+\Big(\sum_{s+t=k}\Big(\sum_{i=s}^{m}a_if_s^i(b_t)\Big)\Big)x^k+\cdots\\&\quad+a_m\alpha^m(b_n)x^{m+n}\in\mathrm{nil}(R[x;\ \alpha,\delta]).\end{aligned}$$

Thus, by Proposition 2.5, we have that

$$\varOmega_{m+n}=a_m\alpha^m(b_n)\in\mathrm{nil}(R),\tag{2.7}$$

$$\varOmega_{m+n-1}=a_m\alpha^m(b_{n-1})+a_{m-1}\alpha^{m-1}(b_n)+a_mf_{m-1}^m(b_n)\in\mathrm{nil}(R),\tag{2.8}$$

$$\varOmega_{m+n-2}=a_m\alpha^m(b_{n-2})+\sum_{i=m-1}^{m}f_{m-1}^i(b_{n-1})+\sum_{i=m-2}^{m}f_{m-2}^i(b_n)\in\mathrm{nil}(R),\tag{2.9}$$

$$\vdots$$

$$\varOmega_k=\sum_{s+t=k}\Big(\sum_{i=s}^{m}a_if_s^i(b_t)\Big)\in\mathrm{nil}(R).\tag{2.10}$$

From Proposition 2.1 and (2.7), we have $a_mb_n\in\mathrm{nil}(R)$. Next we show that $a_ib_n\in\mathrm{nil}(R)$ for all $0\le i\le m$. If we multiply (2.8) on the left side by b_n, then $b_na_{m-1}\alpha^{m-1}(b_n)\in\mathrm{nil}(R)$ since $\mathrm{nil}(R)$ is an ideal of R. Thus, by Proposition 2.1, $b_na_{m-1}b_n\in\mathrm{nil}(R)$, and so $b_na_{m-1}\in\mathrm{nil}(R)$, $a_{m-1}b_n\in\mathrm{nil}(R)$. Multiplying (2.9) on the left side by b_n, since $\mathrm{nil}(R)$ is an ideal of R, we obtain

$$\begin{aligned}b_na_{m-2}\alpha^{m-2}(b_n)&=b_n\varOmega_{m+n-2}-b_na_m\alpha^m(b_{n-2})-b_na_{m-1}\alpha^{m-1}(b_{n-1})-b_na_mf_{m-1}^m(b_{n-1})\\&\quad-b_na_{m-1}f_{m-2}^{m-1}(b_n)-b_na_mf_{m-2}^m(b_n)\in\mathrm{nil}(R).\end{aligned}$$

Thus $b_na_{m-2}\in\mathrm{nil}(R)$ and $a_{m-2}b_n\in\mathrm{nil}(R)$. Continuing this procedure yields that $a_ib_n\in\mathrm{nil}(R)$ for all $0\le i\le m$, and so $a_if_s^t(b_n)\in\mathrm{nil}(R)$ for any $0\le s\le t$ and $0\le i\le m$ by Proposition 2.3. Thus it is easy to verify that

$$\Big(\sum_{i=0}^{m}a_ix^i\Big)\Big(\sum_{j=0}^{n-1}b_jx^j\Big)\in\mathrm{nil}(R[x;\ \alpha,\delta]).$$

Applying the preceding method repeatedly, we obtain that $a_ib_j\in\mathrm{nil}(R)$ for all $0\le i\le m$ and $0\le j\le n$.

$\Leftarrow$. Let $a_ib_j \in \mathrm{nil}(R)$ for all i, j. Then $a_if_s^i \in \mathrm{nil}(R)$ for all i, j and all positive integers $0 \le s \le i$ by Proposition 2.3. Thus

$$\sum_{s+t=k}\left(\sum_{i=s}^{m} a_if_s^i(b_t)\right) \in \mathrm{nil}(R), \qquad k = 0, 1, 2, \cdots, m+n.$$

Hence, by Proposition 2.5,

$$fg = \sum_{k=0}^{m}\left(\sum_{s+t=k}\left(\sum_{i=s}^{m} a_if_s^i(b_t)\right)\right)x^k \in \mathrm{nil}(R[x;\ \alpha, \delta]).$$

(2) $\Rightarrow$.

$$\begin{aligned} g(x)c &= \left(\sum_{j=0}^{n} b_jx^j\right)c \\ &= \sum_{j=0}^{n} b_jf_0^j(c) + \left(\sum_{j=1}^{n} b_jf_1^j(c)\right)x + \cdots + \left(\sum_{j=s}^{n} b_jf_s^j(c)\right)x^s + \cdots + b_n\alpha^n(c)x^n \\ &= \Delta_0 + \Delta_1x + \cdots + \Delta_sx^s + \cdots + \Delta_nx^n, \end{aligned}$$

where $\Delta_s = \sum\limits_{j=s}^{n} b_jf_s^j(c)$, $0 \le s \le n$. By (1), we have

$$a_i\Delta_s = a_i\left(\sum_{j=s}^{n} b_jf_s^j(c)\right) \in \mathrm{nil}(R), \qquad 0 \le i \le m,\ 0 \le s \le n.$$

For $s = n$, we have

$$a_i\Delta_n = a_ib_n\alpha^n(c) \in \mathrm{nil}(R), \qquad 0 \le i \le m.$$

Then, by Proposition 2.1, $a_ib_nc \in \mathrm{nil}(R)$ for all $0 \le i \le m$.

For $s = n-1$, we have

$$a_i\Delta_{n-1} = a_ib_{n-1}\alpha^{n-1}(c) + a_ib_nf_{n-1}^n(c) \in \mathrm{nil}(R), \qquad 0 \le i \le m.$$

Since $a_ib_nc \in \mathrm{nil}(R)$, by Proposition 2.3, we have $a_ib_nf_{n-1}^n(c) \in \mathrm{nil}(R)$. Hence

$$a_ib_{n-1}\alpha^{n-1}(c) = a_i\Delta_{n-1} - a_ib_nf_{n-1}^n(c) \in \mathrm{nil}(R),$$

and so $a_ib_nc \in \mathrm{nil}(R)$ for all $0 \le i \le m$.

Now suppose that k is a positive integer such that $a_ib_jc \in \mathrm{nil}(R)$ for all $0 \le i \le m$ when $j > k$. We show that $a_ib_kc \in \mathrm{nil}(R)$ for all $0 \le i \le m$.

If $s = k$, for all $0 \le i \le m$, we have

$$a_i\Delta_k = a_ib_k\alpha^k(c) + a_ib_{k+1}f_k^{k+1}(c) + \cdots + a_ib_nf_k^n(c) \in \mathrm{nil}(R).$$

Since $a_ib_jc \in \mathrm{nil}(R)$ for $0 \le i \le m$ and $k < j \le n$, by Proposition 2.3, we have

$$a_ib_jf_k^j(c) \in \mathrm{nil}(R), \qquad 0 \le i \le m,\ k < j \le n.$$

It follows that $a_ib_k\alpha^k(c) \in \mathrm{nil}(R)$, and hence $a_ib_kc \in \mathrm{nil}(R)$ for all $0 \le i \le m$. By induction, we obtain that $a_ib_jc \in \mathrm{nil}(R)$ for all $0 \le i \le m$ and $0 \le j \le n$.

$\Leftarrow$. Suppose that $a_ib_jc \in \mathrm{nil}(R)$ for all $0 \le i \le m$ and $0 \le j \le n$. Then $a_ib_jf_s^j(c) \in \mathrm{nil}(R)$, and so $a_i\sum\limits_{j=s}^{n}(b_jf_s^j(c)) \in \mathrm{nil}(R)$ for all $0 \le i \le m$ and $0 \le j \le n$. By (1), we obtain $fgc \in \mathrm{nil}(R[x;\ \alpha, \delta])$.

(3)

$$fg = \sum_{l=0}^{m+n}\left(\sum_{s+t=l}\left(\sum_{i=s}^{m} a_if_s^i(b_t)\right)\right)x^l = \sum_{l=0}^{m+n}\Delta_lx^l.$$

$\Rightarrow$. First we show that $fgh \in \mathrm{nil}(R[x;\ \alpha,\delta])$ implies $fgc_k \in \mathrm{nil}(R[x;\ \alpha,\delta])$ for all $0 \le k \le p$. For any $0 \le k \le p$, since $fgh \in \mathrm{nil}(R[x;\ \alpha,\delta])$, by (1), we have

$$\Delta_l c_k = \sum_{s+t=l} \Big(\sum_{i=s}^{m} a_i f_s^i(b_t) \Big) c_k \in \mathrm{nil}(R), \qquad 0 \le l \le m+n,$$

and so $fgc_k \in \mathrm{nil}(R[x;\ \alpha,\delta])$ with $k \in 0,1,\cdots,p$. Now (2) implies that $a_i b_j c_k \in \mathrm{nil}(R)$ for all $0 \le i \le m$, $0 \le j \le n$ and $0 \le k \le p$.

$\Leftarrow$. Suppose that $a_i b_j c_k \in \mathrm{nil}(R)$ for all $0 \le i \le m$, $0 \le j \le n$ and $0 \le k \le p$. Then we have $fgc_k \in \mathrm{nil}(R[x;\ \alpha,\delta])$ for all $0 \le k \le p$, and so $\Delta_l c_k \in \mathrm{nil}(R)$ for all $0 \le l \le m+n$ and $0 \le k \le p$ by (2). Therefore, (1) implies $fgh \in \mathrm{nil}(R[x;\ \alpha,\delta])$.

Theorem 2.1 *Let R be a weakly 2-primal ring. If R is (α,δ)-compatible, then $R[x;\ \alpha,\delta]$ is weakly semicommutative.*

Proof. Assume $f(x) = \sum\limits_{i=0}^{m} a_i x^i$, $g(x) = \sum\limits_{j=0}^{n} b_j x^j \in R[x;\ \alpha,\delta]$ such that $f(x)g(x) = 0$. By Proposition 2.4, we have $a_i b_j \in \mathrm{nil}(R)$ for all i, j, and hence $b_j a_i \in \mathrm{nil}(R)$. Since $\mathrm{nil}(R)$ is an ideal of weakly 2-primal ring R, we know $rb_j a_i \in \mathrm{nil}(R)$, and hence $a_i r b_j \in \mathrm{nil}(R)$ for each $r \in R$. It follows that $fhg \in \mathrm{nil}(R[x;\ \alpha,\delta])$ by Proposition 2.6 for any $h(x) = \sum\limits_{k=0}^{p} c_k x^k \in R[x;\ \alpha,\delta]$.

Corollary 2.4 *If R is a weakly 2-primal ring, then $R[x]$ is weakly semicommutative.*

Corollary 2.5 *If R is an α-compatible and weakly 2-primal ring, then $R[x;\ \alpha]$ is a weakly semicommutative ring.*

Corollary 2.6 *If R is a δ-compatible and weakly 2-primal ring, then $R[x;\ \delta]$ is a weakly semicommutative ring.*

Chen ([2], Theorem 2.6) has shown that there exists a nil-semicommutative ring R over which the polynomial ring $R[x]$ is not nil-semicommutative. Nevertheless, we obtain that if R is semicommutative, then $R[x]$ is nil-semicommutative. For the more general case, we have the following theorem.

Theorem 2.2 *Let R be a weakly 2-primal ring. If R is (α,δ)-compatible, then $R[x;\ \alpha,\delta]$ is nil-semicommutative.*

Proof. Assume $f(x) = \sum\limits_{i=0}^{m} a_i x^i$, $g(x) = \sum\limits_{j=0}^{n} b_j x^j \in R[x;\ \alpha,\delta]$ such that $f(x)g(x) = 0$. By Proposition 2.4 we have $a_i b_j \in \mathrm{nil}(R)$ for all i, j. Since $\mathrm{nil}(R)$ is an ideal of weakly 2-primal ring R, $rb_j a_i \in \mathrm{nil}(R)$, and so $a_i r b_j \in \mathrm{nil}(R)$ for each $r \in R$. It follows that $fhg \in \mathrm{nil}(R[x;\ \alpha,\delta])$ for any $h(x) = \sum\limits_{k=0}^{p} c_k x^k \in R[x;\ \alpha,\delta]$ by Proposition 2.6.

Corollary 2.7 *If R is a weakly 2-primal ring, then $R[x]$ is nil-semicommutative.*

Corollary 2.8[2] *Let R be a weakly 2-primal ring. If R is an α-compatible ring, then $R[x;\ \alpha]$ is nil-semicommutative.*

Corollary 2.9 *Let R be a weakly 2-primal ring. If R is a δ-compatible ring, then $R[x;\ \delta]$ is nil-semicommutative.*

3 Weakly 2-primal Property of $R[x;\ \alpha,\delta]$

In this section, we consider the relationship between the properties of being the weakly 2-primal of a ring R and that of the Ore extension $R[x;\ \alpha,\delta]$.

Lemma 3.1[3] *Let R be an (α,δ)-compatible ring. If $k_1,\ k_2,\ \cdots,\ k_n$ are arbitrary nonnegative integers and $a_1,\ a_2,\ \cdots,\ a_n$ are arbitrary elements in R, then*

$$a_1a_2\cdots a_n=0 \quad \Leftrightarrow \quad \alpha^{k_1}(a_1)\alpha^{k_2}(a_2)\cdots\alpha^{k_n}(a_n)=0.$$

Proposition 3.1 *Let R be an (α,δ)-compatible ring. Then*

(1) *If $a_1a_2\cdots a_n=0$, then $\delta^{k_1}(a_1)\delta^{k_2}(a_2)\cdots\delta^{k_n}(a_n)=0$, where $k_1,\ k_2,\ \cdots,k_n$ are arbitrary nonnegative integers and $a_1,\ a_2,\ \cdots,\ a_n$ are arbitrary elements in R;*

(2) *If $a_1a_2\cdots a_n=0$, then $f_{s_1}^{t_1}(a_1)f_{s_2}^{t_2}(a_2)\cdots f_{s_n}^{t_n}(a_n)=0$ for all $a_i\in R$ and $0\leq s_i\leq t_i$, $i=1,2,\cdots,n$.*

Proof. (1) Let $abc=0$ for all $a,b,c\in R$. We have $a\delta(b)c=0$ by Proposition 2.2. According to Lemma 2.1 and δ-compatibility, we have $\delta(a)\delta(b)\delta(c)=0$. Thus $a_1a_2\cdots a_n=0$ implies $\delta^{k_1}(a_1)\delta^{k_2}(a_2)\cdots\delta^{k_n}(a_n)=0$.

(2) It is an immediate consequence of (1) and Lemma 3.1.

Theorem 3.1 *Let R be (α,δ)-compatible. Then R is weakly 2-primal if and only if $R[x;\ \alpha,\delta]$ is weakly 2-primal.*

Proof. By Proposition 3.1 in [3], each subring of weakly 2-primal rings is weakly 2-primal. So we just to prove the necessity.

Since R is weakly 2-primal, L-rad(R)=nil(R), and so $R/\mathrm{nil}(R)$ is reduced. The endomorphism α of R induces an endomorphism $\bar{\alpha}$ of $R/\mathrm{nil}(R)$ via $\bar{\alpha}(a+\mathrm{nil}(R))=\alpha(a)+\mathrm{nil}(R)$ since $\alpha(\mathrm{nil}(R))\subseteq\mathrm{nil}(R)$. And the α-derivation δ of R also induces an $\bar{\alpha}$-derivation $\bar{\delta}$ of $R/\mathrm{nil}(R)$ via $\bar{\delta}(a+\mathrm{nil}(R))=\delta(a)+\mathrm{nil}(R)$ since $\delta(\mathrm{nil}(R))\subseteq\mathrm{nil}(R)$.

We claim that $R/\mathrm{nil}(R)$ is $(\bar{\alpha},\bar{\delta})$-compatible. In fact, for any $a,b\in R$, if $\bar{a}\bar{b}=\bar{0}$ in $R/\mathrm{nil}(R)$, then $ab\in\mathrm{nil}(R)$. This implies $a\alpha(b)\in\mathrm{nil}(R)$ by Proposition 2.1. Hence $\bar{a}\bar{\alpha}(\bar{b})=\bar{0}$. If $\bar{a}\bar{\alpha}(\bar{b})=\bar{0}$ in $R/\mathrm{nil}(R)$, then $a\alpha(b)\in\mathrm{nil}(R)$. This implies $ab\in\mathrm{nil}(R)$ by Proposition 2.1. Hence $\bar{a}\bar{b}=\bar{0}$. Thus $R/\mathrm{nil}(R)$ is $\bar{\alpha}$-compatible. On the other hand, if $\bar{a}\bar{b}=\bar{0}$ in $R/\mathrm{nil}(R)$, then $ab\in\mathrm{nil}(R)$. This implies $a\delta(b)\in\mathrm{nil}(R)$ by Proposition 2.2. Hence $\bar{a}\bar{\delta}(\bar{b})=\bar{0}$. Thus $R/\mathrm{nil}(R)$ is $\bar{\delta}$-compatible. So $R/\mathrm{nil}(R)$ is $(\bar{\alpha},\bar{\delta})$-compatible.

We need to prove that $\mathrm{nil}(R[x;\ \alpha,\delta]) = \text{L-rad}(R[x;\ \alpha,\delta])$. It is enough to show that $\mathrm{nil}(R[x;\ \alpha,\delta]) \subseteq \text{L-rad}(R[x;\ \alpha,\delta])$ since the reverse inclusion is obvious. It is a routine task to check that there exists an onto ring homomorphism

$\beta : R[x;\ \alpha,\delta] \to R/\mathrm{nil}(R)[x;\ \bar{\alpha},\bar{\delta}]$ with $\beta(a_0 + a_1x + \cdots + a_nx^n) = \bar{a}_0 + \bar{a}_1x + \cdots + \bar{a}_nx^n$, where $\bar{a}_i = a + \mathrm{nil}(R)$, $0 \leqslant i \leqslant n$, and the meaning of $\bar{\alpha}$, $\bar{\delta}$ is the same as in the first paragraph.

First we show that $\mathrm{nil}(R[x;\ \alpha,\delta]) \subseteq \mathrm{nil}(R)[x;\ \alpha,\delta] = \text{L-rad}(R)[x;\ \alpha,\delta]$. Suppose that $f(x) = \sum\limits_{i=0}^{n} a_ix^i$ is nilpotent with the nilpotent index k in $R[x;\alpha,\delta]$. Then in $R/\mathrm{nil}(R)[x;\bar{\alpha},\bar{\delta}]$, $\bar{f}(x) = \sum\limits_{i=0}^{n} \bar{a}_ix^i$ satisfies $\bar{f}(x)^k = \bar{0}$. Because $R/\mathrm{nil}(R)$ is reduced and $(\bar{\alpha},\bar{\delta})$-compatible, we can obtain $\bar{a}_i \in \mathrm{nil}(R/\mathrm{nil}(R))$ for all $0 \le i \le n$ by Proposition 2.5. Hence $a_i \in \mathrm{nil}(R)$ for all $0 \le i \le n$. Thus $\mathrm{nil}(R[x;\ \alpha,\delta]) \subseteq \mathrm{nil}(R)[x;\ \alpha,\delta] = \text{L-rad}(R)[x;\ \alpha,\delta]$.

Next we prove that $\text{L-rad}(R)[x;\alpha,\delta]$ is locally nilpotent. Suppose that

$$f_1(x), f_2(x), \cdots, f_k(x) \in \text{L-rad}(R)[x;\alpha,\delta].$$

We prove that the finitely generated subring (without 1) $W = \langle f_1(x), f_2(x), \cdots, f_k(x)\rangle$ of $\text{L-rad}(R)[x;\ \alpha,\delta]$ is nilpotent. Write $f_i(x) = a_{i0}+a_{i1}x+\cdots+a_{in}x^n$, where a_{ij} is in $\text{L-rad}(R)$ for all $i = 1,2,\cdots,k$; $j = 0,1,2,\cdots,n$. Let $M = \{a_{i0}, a_{i1}, \cdots, a_{in} \mid i = 1,2,\cdots,k\}$. Then M is a finite subset of $\text{L-rad}(R)$. So the subring $\langle M\rangle$ (without 1) generated by M is nilpotent. There exists a positive integer p such that $\langle M\rangle^p = 0$. Hence for any $b_1, b_2, \cdots, b_p \in \langle M\rangle$, we have $b_1b_2\cdots b_p = 0$. Now we prove that $W^p = 0$. In fact, for any $g_1(x), g_2(x), \cdots, g_p(x) \in W$, we may write $g_j(x) = b_{j0} + b_{j1}x + \cdots + b_{jm}x^m$, $j = 1,2,\cdots,p$. It is easy to see that $b_{jt} \in M$ for all j and $t = 0,1,2,\cdots,m$. Note that

$$\begin{aligned}
g_1(x)g_2(x) &= \Big(\sum_{i=0}^{m} b_{1i}x^i\Big)\Big(\sum_{j=0}^{m} b_{2j}x^j\Big)\\
&= \Big(\sum_{i=0}^{m} b_{1i}x^i\Big)b_{20} + \Big(\sum_{i=0}^{m} b_{1i}x^i\Big)b_{21}x + \cdots + \Big(\sum_{i=0}^{m} b_{1i}x^i\Big)b_{2m}x^m\\
&= \sum_{i=0}^{m} b_{1i}f_0^i(b_{20}) + \cdots + \Big(\sum_{i=s}^{m} b_{1i}f_s^i(b_{20})\Big)x^s + \cdots + b_{1m}\alpha^m(b_{20})x^m\\
&\quad + \Big(\sum_{i=0}^{m} b_{1i}f_0^i(b_{21}) + \Big(\sum_{i=1}^{m} b_{1i}f_1^i(b_{21})\Big)x + \cdots + b_{1m}\alpha^m(b_{21})x^m\Big)x\\
&\quad + \cdots + \Big(\sum_{i=0}^{m} b_{1i}f_0^i(b_{2m}) + \Big(\sum_{i=1}^{m} b_{1i}f_1^i(b_{2m})\Big)x + \cdots + b_{1m}\alpha^m(b_{2m})x^m\Big)x^m\\
&= \sum_{i=0}^{m} b_{1i}f_0^i(b_{20}) + \Big(\sum_{i=1}^{m} b_{1i}f_1^i(b_{20}) + \sum_{i=0}^{m} b_{1i}f_0^i(b_{21})\Big)x\\
&\quad + \cdots + \Big(\sum_{s+t=k}\Big(\sum_{i=s}^{m} b_{1i}f_s^i(b_{2t})\Big)\Big)x^k + \cdots + b_{1m}\alpha^m(b_{2m})x^{2m}.
\end{aligned}$$

It is easy to check that the coefficients of $g_1(x)g_2(x)\cdots g_p(x)$ can be written as sums of monomials of length p in b_{ji} and $f_u^v(b_{jt})$, where $b_{ji}, b_{jt} \in \{b_{j_0}, b_{j1}, \cdots, b_{jm} \mid j = 1,2,\cdots,p\}$

and $0 \le u \le v$ are positive integers. Consider each monomial $b_{1i_1} f_{s_2}^{t_2}(b_{2i_2}) \cdots f_{s_p}^{t_p}(b_{pi_p})$, where $b_{1i_1}, b_{2i_2}, \cdots, b_{pi_p} \in \{b_{j_0}, b_{j1}, \cdots, b_{jm} \mid j = 1, 2, \cdots, p\}$ and t_j, s_j $(0 \le s_j \le t_j,\ 1 \le j \le p-1)$ are nonnegative integers. Since $b_{1i_1}, b_{2i_2}, \cdots, b_{pi_p} \in M$, we have $b_{1i_1} b_{2i_2} \cdots b_{pi_p} = 0$. Hence $b_{1i_1} f_{s_2}^{t_2}(b_{2i_2}) \cdots f_{s_p}^{t_p}(b_{pi_p}) = 0$ by Proposition 3.1. It follows that $g_1(x)g_2(x) \cdots g_p(x) = 0$, and so L-rad$(R)[x;\ \alpha, \delta]$ is locally nilpotent. Since nil$(R) =$ L-rad(R) is an ideal of R and $\alpha(\text{nil}(R)) \subseteq \text{nil}(R)$ and $\delta(\text{nil}(R)) \subseteq \text{nil}(R)$, L-rad$(R)[x;\ \alpha, \delta]$ is an ideal of $R[x;\ \alpha, \delta]$. Noting that L-rad$(R)[x;\ \alpha, \delta]$ is locally nilpotent, we have L-rad$(R)[x;\ \alpha, \delta] \subseteq$ L-rad$(R[x;\ \alpha, \delta])$. From the above argument, we have

$$\text{nil}(R[x;\ \alpha, \delta]) \subseteq \text{nil}(R)[x;\ \alpha, \delta] = \text{L-rad}(R)[x;\ \alpha, \delta] \subseteq \text{L-rad}(R[x;\ \alpha, \delta]).$$

Corollary 3.1 *Let R be a weakly 2-primal ring. If R is (α, δ)-compatible, then $R[x;\ \alpha, \delta]$ is NI and weakly semicommutative.*

Corollary 3.2[3] *Let R be α-compatible. Then R is weakly 2-primal if and only if $R[x;\ \alpha]$ is weakly 2-primal.*

Corollary 3.3 *Let R be δ-compatible. Then R is weakly 2-primal if and only if $R[x;\ \delta]$ is weakly 2-primal.*

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