Communications
in
Mathematical
Research
32(1)(2016), 88–96

Normality Criteria of Meromorphic Functions

Wang Qiong[1], Yuan Wen-jun[2], Chen Wei[3]
and Tian Hong-gen[1,*]

(*1. School of Mathematics Science, Xinjiang Normal University, Urumqi, 830054*)

(*2. School of Mathematics and Information Sciences, Guangzhou University, Guangzhou, 510006*)

(*3. School of Mathematics Sciences, Shandong University, Jinan, 250000*)

Communicated by Ji You-qing

Abstract: In this paper, we consider normality criteria for a family of meromorphic functions concerning shared values. Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, m, n, k and d be four positive integers satisfying $m \geq n+2$ and $d \geq \dfrac{k+1}{m-n-1}$, and $a(\neq 0)$, b be two finite constants. Suppose that every $f \in \mathcal{F}$ has all its zeros and poles of multiplicity at least k and d, respectively. If $(f^n)^{(k)} - af^m$ and $(g^n)^{(k)} - ag^m$ share the value b for every pair of functions $(f,\ g)$ of $\mathcal{F}$, then $\mathcal{F}$ is normal in D. Our results improve the related theorems of Schwick (Schwick W. Normality criteria for families of meromorphic function. *J. Anal. Math.*, 1989, **52**: 241–289), Li and Gu (Li Y T, Gu Y X. On normal families of meromorphic functions. *J. Math. Anal. Appl.*, 2009, **354**: 421–425).

Key words: meromorphic function, shared value, normal criterion

2010 MR subject classification: 30D30, 30D45

Document code: A

Article ID: 1674-5647(2016)01-0088-09

DOI: 10.13447/j.1674-5647.2016.01.07

1 Introduction and Main Results

Let $\mathbf{C}$ be the set of complex numbers, D be a domain in $\mathbf{C}$, which means that D is a connected nonempty open subset of $\mathbf{C}$. Let $\mathcal{F}$ be a family of meromorphic functions defined in D. For $\{f,\ g\} \subset \mathcal{F}$, $\{a,\ b\} \subset \mathbb{P}^1 = \mathbf{C} \cup \{\infty\}$, we write $f = a \Rightarrow g = b$ ($f = a \Leftrightarrow g = b$) if $f^{-1}(a) \subset g^{-1}(b)$ ($f^{-1}(a) = g^{-1}(b)$), and say that f and g share a ignoring multiplicities

Received date: Jan. 6. 2015.

Foundation item: The NSF (11461070, 11271090) of China, the NSF (S2012010010121, 2015A030313346) of Guangdong Province, and the Graduate Research, and Innovation Projects (XJGRI2015106) of Xinjiang Province.

* **Corresponding author**.

E-mail address: wq1298592600@163.com (Wang Q), tianhg@xjnu.edu.cn (Tian H G).

(IM, for short) if $f^{-1}(a) = g^{-1}(a)$ (see [1]). Here, the family $\mathcal{F}$ is said to be normal in D if any sequence of $\mathcal{F}$ must contain a subsequence that locally uniformly spherically converges to a meromorphic function or ∞ in D (see [2]).

In 1989, Schwick[3] proved a normality criterion:

Theorem 1.1 *Let k, $n(\geq k+3)$ be two positive integers, and $\mathcal{F}$ be a family of meromorphic functions defined in a domain D. If $(f^n)^{(k)} \neq 1$ for every function $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

In 1998, Wang and Fang[4] proved:

Theorem 1.2 *Let k, $n(\geq k+1)$ be two positive integers, and f be a transcendental meromorphic function. Then $(f^n)^{(k)}$ assumes every finite non-zero value infinitely often.*

For families of meromorphic functions, the connection between normality and shared values has been studied frequently.

By the ideas of shared values, Li and Gu[5] proved the following results:

Theorem 1.3 *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, k, $n(\geq k+2)$ be two positive integers, and $a \neq 0$ be a finite complex number. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share a in D for every pair of functions f, $g \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

In 2011, Liu and Li[6] studied Theorem 1.3, in which the value a was replaced by the fix-point z, and got the following result:

Theorem 1.4 *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, k, $n(\geq k+1)$ be two positive integers. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z in D for every pair of functions f, $g \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

Lately, some theorems in this area appear. Hu and Meng[7], Jiang and Gao[8] studied the functions of the form $f(f^{(k)})^n$. Ding *et al.*[9] studied the functions of the form $f^m(f^{(k)})^n$ and Sun[10] studied the form $P(f)(f^{(k)})^m$.

Naturally, we pose the following question:

Question Whether the form $(f^n)^{(k)} - af^m$ in above Theorems can have similar results?

In this paper, we prove the following theorems and deal with this question.

Theorem 1.5 *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, m, n, k be three positive integers satisfying $m \geq n+k+3$, and $a(\neq 0)$, b be two finite complex constants. If $(f^n)^{(k)} - af^m \neq b$ for every functions f of $\mathcal{F}$, then $\mathcal{F}$ is normal in D.*

Whether the condition $m \geq n+k+3$ in Theorem 1.5 can be improved? We get the following results:

Theorem 1.6 *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, m, n, $k(\geq 2)$ and d be four positive integers satisfying $m \geq n+k+1$ and $d \geq 2$, and $a(\neq 0)$, b be two finite complex constants. Suppose that every $f \in \mathcal{F}$ has all its poles of multiplicity at least d and $(f^n)^{(k)} - af^m \neq b$, then $\mathcal{F}$ is normal in D.*

By the ideas of shared values, we can get the following results:

Theorem 1.7 *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, m, n, k and d be four positive integers satisfying $m \geq n+2$ and $d \geq \dfrac{k+1}{m-n-1}$, and $a(\neq 0)$, b be two finite constants. Suppose that every $f \in \mathcal{F}$ has all its zeros and poles of multiplicity at least k and d, respectively. If $(f^n)^{(k)} - af^m$ and $(g^n)^{(k)} - ag^m$ share the value b IM for every pair of functions $(f,\ g)$ of $\mathcal{F}$, then $\mathcal{F}$ is normal in D.*

2 Some Lemmas

In order to improve our theorems, we require the following Lemmas.

Lemma 2.1[11] *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc Δ such that all zeros of functions in $\mathcal{F}$ have multiplicity $\geq p$, and all poles of functions in $\mathcal{F}$ have multiplicity $\geq q$. Let α be a real number satisfying $-q < \alpha < p$. Then $\mathcal{F}$ is not normal in any neighbourhood of $z_0 \in \Delta$ if and only if there exist*

(a) *points $z_j \in \Delta$, $z_j \to z_0$;*

(b) *functions $f_j \in \mathcal{F}$, and*

(c) *positive numbers $\rho_j \to 0$,*

such that $g_j(\xi) = \rho_j^{-\alpha} f_j(z_j + \rho_j \xi) \to g(\xi)$ spherically uniformly on compact subsets of **C**, *where $g(\xi)$ is a nonconstant meromorphic function satisfying that all zeros of g have multiplicity $\geq p$ and all poles of functions in $\mathcal{F}$ have multiplicity $\geq q$ and order at most 2.*

Lemma 2.2 *Let $f(z)$ be meromorphic functions such that $(f^n)^{(k)}(z) \not\equiv 0$, $a(\neq 0)$ be a finite constant, and m, n, k and d be four positive integers satisfying $m \geq n+k+1$. Then*

$$(m-n)T(r,f) \leq (k+1)\bar{N}(r,\ f) + \bar{N}\Big(r,\ \frac{1}{f}\Big) + \bar{N}\Big(r,\ \frac{1}{(f^n)^{(k)} - af^m}\Big) + S(r,\ f).$$

Proof. Set

$$\Psi(z) = \frac{(f^n)^{(k)}(z)}{af^m(z)}. \tag{2.1}$$

Since $(f^n)^{(k)}(z) \not\equiv 0$, we know that $\Psi(z) \not\equiv 0$.

By (2.1), we have

$$\frac{af^m(z)}{f^n(z)} = \frac{(f^n)^{(k)}(z)}{f^n(z)\Psi(z)}. \tag{2.2}$$

Thus, we get

$$\begin{aligned}(m-n)m(r,\ f) &= m(r,\ f^{m-n})\\ &\le m(r,\ af^{m-n}) + \log^+\frac{1}{|a|}\\ &\le m\Big(r,\ \frac{(f^n)^{(k)}}{f^n\,\Psi}\Big) + \log^+\frac{1}{|a|}\\ &\le m\Big(r,\ \frac{1}{\Psi}\Big) + m\Big(r,\ \frac{(f^n)^{(k)}}{f^n}\Big) + \log^+\frac{1}{|a|},\end{aligned}$$

which implies that

$$(m-n)m(r,\ f) \le m\Big(r,\ \frac{1}{\Psi}\Big) + S(r,\ f). \tag{2.3}$$

We see that a zero Ψ is attained at pole of f and zeros of $(f^n)^{(k)}$ which is not zero of f, and a pole of f must be zero of Ψ by the condition $m \ge n+k+1$. The pole of f cannot be zero of $\Psi - 1$. Hence, if we denote $\bar{N}_0(r)$ by the counting function of zeros of both Ψ and $(f^n)^{(k)}$, we see that

$$\bar{N}\Big(r,\ \frac{1}{\Psi}\Big) = \bar{N}(r,\ f) + \bar{N}_0(r), \tag{2.4}$$

$$\bar{N}(r,\ \Psi) \le \bar{N}\Big(r,\ \frac{1}{f}\Big), \tag{2.5}$$

$$\bar{N}\Big(r,\ \frac{1}{\Psi-1}\Big) = \bar{N}\Big(r,\ \frac{1}{(f^n)^{(k)} - af^n}\Big) + \bar{N}_0(r). \tag{2.6}$$

On the other hand, we have

$$\begin{aligned}mN(r,\ f) &= N(r,\ af^m)\\ &= N\Big(r,\ \frac{(f^n)^{(k)}}{\Psi}\Big)\\ &\le N(r,\ (f^n)^{(k)}) + N\Big(r,\ \frac{1}{\Psi}\Big) - \bar{N}_0(r)\\ &\le nN(r,\ f) + k\bar{N}(r,\ f) + N\Big(r,\ \frac{1}{\Psi}\Big) - \bar{N}_0(r).\end{aligned}$$

So we have

$$(m-n)N(r,\ f) \le k\bar{N}(r,\ f) + N\Big(r,\ \frac{1}{\Psi}\Big) - \bar{N}_0(r). \tag{2.7}$$

Therefore, by (2.3)–(2.7) and Nevanlinna's first and second fundamental theorems, we have

$$\begin{aligned}(m-n)T(r,f) &\le T\Big(r,\ \frac{1}{\Psi}\Big) + k\bar{N}(r,\ f) - \bar{N}_0(r) + S(r,\ f)\\ &\le \bar{N}(r,\ \Psi) + \bar{N}\Big(r,\ \frac{1}{\Psi}\Big) + \bar{N}\Big(r,\ \frac{1}{\Psi-1}\Big) + k\bar{N}(r,\ f) - \bar{N}_0(r) + S(r,\ f)\\ &\le \bar{N}\Big(r,\ \frac{1}{f}\Big) + (k+1)\bar{N}(r,\ f) + \bar{N}\Big(r,\ \frac{1}{(f^n)^{(k)} - af^m}\Big) + S(r,\ f).\end{aligned}$$

Then, we have the inequality

$$(m-n)T(r,f) \le (k+1)\bar{N}(r,f) + \bar{N}\Big(r,\ \frac{1}{f}\Big) + \bar{N}\Big(r,\ \frac{1}{(f^n)^{(k)} - af^m}\Big) + S(r,\ f). \tag{2.8}$$

Lemma 2.3 *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, m, n, k and d be four positive integers satisfying $m \ge n+2$ and $d \ge \dfrac{k+1}{m-n-1}$, and $a(\ne 0)$,* *b be two finite complex constants. Suppose that every $f \in \mathcal{F}$ has all its zeros and poles of multiplicity at least k and d, respectively, then*

$$T(r, f) \le \frac{1}{k}\bar{N}\Big(r,\ \frac{1}{f}\Big) + \bar{N}\Big(r,\ \frac{1}{(f^n)^{(k)} - af^m}\Big) + S(r,\ f).$$

Proof. By the argument as Lemma 2.2, since the condition that all zeros and poles of f are multiplicities at least k and d, respectively, we get

$$\bar{N}(r,\ f) \le \frac{1}{d}N(r,\ f) \le \frac{1}{d}T(r,\ f) \le \frac{m-n-1}{k+1}T(r,\ f), \tag{2.9}$$

$$\bar{N}\Big(r,\ \frac{1}{f}\Big) \le \frac{1}{k}N\Big(r,\ \frac{1}{f}\Big) \le \frac{1}{k}T(r,\ f). \tag{2.10}$$

Hence, by (2.9), (2.10) and the inequality (2.8), we get

$$T(r, f) \le \frac{1}{k}\bar{N}\Big(r,\ \frac{1}{f}\Big) + \bar{N}\Big(r,\ \frac{1}{(f^n)^{(k)} - af^m}\Big) + S(r,\ f).$$

3 Proof of Theorems

Proof of Theorem 1.5 Suppose that $\mathcal{F}$ is not normal at z_0. Then by Lemma 2.1, there exist $f_j \in \mathcal{F}$, $z_j \to z_0$ and $\rho_j \to 0^+$ such that

$$g_j(\xi) = \rho_j^{\frac{k}{m-n}} f_j(z_j + \rho_j\xi) \to g(\xi)$$

spherically uniformly on compact subsets of $\mathbf{C}$, where $g(\xi)$ is a nonconstant meromorphic function on $\mathbf{C}$. We have

$$\begin{aligned}
&(g_j^n)^{(k)}(\xi) - ag_j^m(\xi) - \rho_j^{\frac{km}{m-n}} b \\
&= \rho_j^{\frac{km}{m-n}} (f_j^n)^{(k)}(z_j + \rho_j\xi) - a\rho_j^{\frac{km}{m-n}} f_j^m(z_j + \rho_j\xi) - \rho_j^{\frac{km}{m-n}} b \\
&= \rho_j^{\frac{km}{m-n}} ((f_j^n)^{(k)}(z_j + \rho_j\xi) - af_j^m(z_j + \rho_j\xi) - b) \\
&\to (g^n)^{(k)}(\xi) - ag^m(\xi)
\end{aligned}$$

spherically uniformly on compact subsets of $\mathbf{C}$ outside poles of g. By hypothesis, $(f^n)^{(k)} - af^m \ne b$ for every functions f of $\mathcal{F}$. Applying Hurwitz theorem, we obtain that

$$(g^n)^{(k)} - ag^m \equiv 0$$

or

$$(g^n)^{(k)} - ag^m \ne 0.$$

If $(g^n)^{(k)} - ag^m \equiv 0$, then g has not poles. By the logarithmic derivative lemma, we get

$$(m-n)m(r,\ g) = S(r,\ g).$$

Hence

$$T(r,\ g) = S(r,\ g),$$

and this contradicts with g is nonconstant meromorphic function. Thus,

$$(g^n)^{(k)} - ag^m \ne 0.$$

Then $(g^n)^{(k)} \not\equiv 0$.

Indeed, if $(g^n)^{(k)} \equiv 0$, then g^n is polynomial with degree at most $k-1$, which is a contradiction with $(g^n)^{(k)} - ag^m \neq 0$. Applying Lemma 2.2 with meromorphic function g, we get

$$(m-n)T(r,\ g) \leq (k+1)\bar{N}(r,\ g) + \bar{N}\Big(r,\ \frac{1}{g}\Big) + \bar{N}\Big(r,\ \frac{1}{(g^n)^{(k)} - ag^m}\Big) + S(r,\ f). \tag{3.1}$$

This implies

$$(m-n-k-2)T(r,\ g) \leq S(r,\ f).$$

By $m \geq n+k+3$, we conclude that g is constant function. This is a contradiction. Hence, $\mathcal{F}$ is normal in D.

Proof of Theorem 1.6 By the argument as Theorem 1.5, we can assume that $\mathcal{F}$ is not normal at z_0. Then by Lemma 2.1, there exist $f_j \in \mathcal{F}$, $z_j \to z_0$ and $\rho_j \to 0^+$ such that

$$g_j(\xi) = \rho_j^{\frac{k}{m-n}} f_j(z_j + \rho_j \xi) \to g(\xi)$$

spherically uniformly on compact subsets of $\mathbf{C}$, where $g(\xi)$ is a nonconstant meromorphic function on $\mathbf{C}$ and all its poles has multiplicity at least 2. Hence, from the inequality (3.1), we get

$$(m-n)T(r,\ g) \leq \frac{(k+1)}{2}\bar{N}(r,\ g) + \bar{N}\Big(r,\ \frac{1}{g}\Big) + \bar{N}\Big(r,\ \frac{1}{(g^n)^{(k)} - ag^m}\Big) + S(r,\ f).$$

By hypothesis,

$$m \geq n+k+1 > n + \frac{(k+1)}{2} + 1,$$

we get that g is a constant function. This is a impossible. Hence, $\mathcal{F}$ is normal in D.

Proof of Theorem 1.7 Suppose that $\mathcal{F}$ is not normal at z_0. Then by Lemma 2.1, there exist $f_j \in \mathcal{F}$, $z_j \to z_0$ and $\rho_j \to 0^+$ such that

$$g_j(\xi) = \rho_j^{\frac{k}{m-n}} f_j(z_j + \rho_j \xi) \to g(\xi)$$

spherically uniformly on compact subsets of $\mathbf{C}$, where $g(\xi)$ is a nonconstant meromorphic function on $\mathbf{C}$ and whose zeros and poles has multiplicity at least k and d, respectively. Moreover, the order of g is at most 2. We have

$$\begin{aligned}
&(g_j^n)^{(k)}(\xi) - ag_j^m(\xi) \\
&= \rho_j^{\frac{km}{m-n}} (f_j^n)^{(k)}(z_j + \rho_j\xi) - a\rho_j^{\frac{km}{m-n}} f_j^m(z_j + \rho_j\xi) - \rho_j^{\frac{km}{m-n}} b \\
&= \rho_j^{\frac{km}{m-n}} ((f_j^n)^{(k)}(z_j + \rho_j\xi) - af_j^m(z_j + \rho_j\xi) - b) \\
&\to (g^n)^{(k)}(\xi) - ag^m(\xi)
\end{aligned}$$

spherically uniformly on compact subsets of $\mathbf{C}$ outside poles of g. Hence, we apply Hurwitz theorem and obtain that

$$(g^n)^{(k)} - ag^m \equiv 0$$

or

$$(g^n)^{(k)} - ag^m \neq 0.$$

If $(g^n)^{(k)} - ag^m \equiv 0$, since all poles of g have multiplicity at least d, we have

$$\begin{aligned} mT(r,g) &= T(r,\ g^m) \\ &= T(r,\ (g^n)^{(k)}) + O(1) \\ &= m(r,\ (g^n)^{(k)}) + N(r,\ (g^n)^{(k)}) + O(1) \\ &\le nm(r,\ g) + nN(r,\ g) + k\bar{N}(r,\ g) + S(r,\ g) \\ &\le nT(r,\ g) + \frac{k(m-n-1)}{k+1}T(r,\ g) + S(r,\ g) \\ &< (m-1)T(r,\ g) + S(r,\ g). \end{aligned}$$

Therefore, g is a constant, a contradiction. So

$$(g^n)^{(k)} - ag^m \not\equiv 0.$$

By Lemma 2.3, we have

$$\begin{aligned} T(r,g) &\le \frac{1}{k}\bar{N}\Big(r,\ \frac{1}{g}\Big) + \bar{N}\Big(r,\ \frac{1}{(g^n)^{(k)} - ag^m}\Big) + S(r,\ f) \\ &\le \frac{1}{k}T\Big(r,\ \frac{1}{g}\Big) + \bar{N}\Big(r,\ \frac{1}{(g^n)^{(k)} - ag^m}\Big) + S(r,\ f). \end{aligned}$$

Then

$$T(r,g) \le \Big(1 + \frac{1}{k-1}\Big)\bar{N}\Big(r, \frac{1}{(g^n)^{(k)} - ag^m}\Big) + S(r,f). \tag{3.2}$$

If $(g^n)^{(k)} - ag^m \neq 0$, then (3.2) gives that g is a constant. Hence, $(g^n)^{(k)} - ag^m$ is a meromorphic function and has at least one zero.

Next, we prove that $(g^n)^{(k)} - ag^m$ has just a unique zero.

Suppose to the contrary, let $\xi_0,\ \xi_0^*$ be two distinct zeros of $(g^n)^{(k)}(\xi) - ag^m(\xi)$, and choose $\delta > 0$ small enough such that

$$D(\xi_0,\ \delta) \cap D(\xi_0^*,\ \delta) = \emptyset,$$

where

$$D(\xi_0,\ \delta) = \{\xi : |\xi - \xi_0| < \delta\}, \qquad D(\xi_0^*,\ \delta) = \{\xi : |\xi - \xi_0^*| < \delta\}.$$

By Hurwitz theorem, there exists a sequence of points $\xi_j \in D(\xi_0,\ \delta)$ and $\xi_j^* \in D(\xi_0^*,\ \delta)$ such that for large enough j,

$$\begin{aligned} (f_j^n)^{(k)}(z_j + \rho_j\xi_j) + af_j^m(z_j + \rho_j\xi_j) - b = 0, \\ (f_j^n)^{(k)}(z_j + \rho_j\xi_j^*) + af_j^m(z_j + \rho_j\xi_j^*) - b = 0. \end{aligned}$$

By the assumption that for each pair of functions $f, g \in \mathcal{F}$, $(f^n)^{(k)} - af^m$ and $(g^n)^{(k)} - ag^m$ share b in D, we know that for any positive integer m,

$$\begin{aligned} (f_m^n)^{(k)}(z_j + \rho_j\xi_j) + af_m^m(z_j + \rho_j\xi_j) - b = 0, \\ (f_m^n)^{(k)}(z_j + \rho_j\xi_j^*) + af_m^m(z_j + \rho_j\xi_j^*) - b = 0. \end{aligned}$$

Fix m, take $j \to \infty$ and note $z_j + \rho_j\xi_j \to 0$, $z_j + \rho_j\xi_j^* \to 0$, we get

$$(f_m^n)^{(k)}(0) + af_m^m(0) - b = 0.$$

Since $(f^n)_m^{(k)} + af_m^m - b$ has no accumulation point, one has

$$z_j + \rho_j\xi_j = 0, \qquad z_j + \rho_j\xi_j^* = 0.$$

Hence,

$$\xi_j = \frac{z_j}{\rho_j}, \qquad \xi_j^* = \frac{z_j}{\rho_j}.$$

This contradicts with $\xi_j \in D(\xi_0,\ \delta)$, $\xi_j^* \in D(\xi_0^*,\ \delta)$ and $D(\xi_0,\ \delta)\bigcap D(\xi_0^*,\ \delta) = \emptyset$. So $(f^n)^{(k)} + af^m$ has just a unique zero, which can be denoted by ξ_0.

Noting that g has zeros and poles of multiplicities at least k and d respectively, then (3.2) deduces that g is a rational function with degree at most 2.

If g is a polynomial and noting that $\deg g \le 2$ and the multiplicities of zeros are at least k, we distinguish two cases.

Case 1 $\deg g = 1$.

We can write $g = A(\xi - \xi_1)$, where A is a nonzero constant. So

$$(g^n)' + ag^m = (\xi - \xi_1)^{n-1}[nA^n - aA^m(\xi - \xi_1)^{(m-n-1)}].$$

Obviously, $(g^n)' + ag^m$ has at least two distinct zeros, a contradiction.

Case 2 $\deg g = 2$.

We distinguish two cases again.

Case 2.1 $k = 1$.

We can write $g = A(\xi - \xi_1)(\xi - \xi_2)$, where A is a nonzero constant. Then

$$\begin{aligned}&(g^n)' + ag^m\\ &= (\xi - \xi_1)^{n-1}(\xi - \xi_2)^{n-1}[A^n n2(\xi - \xi_1 - \xi_2) - aA^m(\xi - \xi_1)^{(m-n+1)}(\xi - \xi_1)^{(m-n+1)}].\end{aligned}$$

Obviously, $(g^n)' + ag^m$ has at least three distinct zeros, a contradiction.

Case 2.2 $k = 2$.

We can write $g = A(\xi - \xi_1)^2$, where A is a nonzero constant. Then

$$(g^n)'' + ag^m = (\xi - \xi_1)^{2n-2}[2n(2n-1)A^n - aA^m(\xi - \xi_1)^{(2m-2n+2)}].$$

Obviously, $(g^n)'' + ag^m$ has at least two distinct zeros, a contradiction.

Suppose that g is a rational function with $\deg g \le 2$ and noting the multiplicities of poles are at least $d \ge \dfrac{k+1}{m-n-1}$, we also distinguish two subcases.

Subcase 1 $\deg g = 1$.

We can write

$$g = \frac{A\xi + B}{C\xi + D},$$

where A, C are nonzero constants and $AD + BC \ne 0$. Then

$$\begin{aligned}(g^n)' + ag^m &= \frac{n(A\xi + B)^{n-1}(AD - CB)}{(C\xi + D)^{n+1}} - \frac{a(A\xi + B)^m}{(C\xi + D)^m}\\ &= \frac{(A\xi + B)^{n-1}[n(C\xi + D)^{m-n-1}(AD - CB) - a(A\xi + B)^{m-n-1}]}{(C\xi + D)^m}.\end{aligned}$$

Noting $m \ge n + 2$, $(g^n)' + ag^m$ has at least two distinct zeros, a contradiction.

Subcase 2 $\deg g = 2$.

We distinguish three cases.

Subcase 2.1 $g = 0$, and g has only one zero.

In this case we have

$$g = \frac{A(\xi - \xi_1)^2}{A_1\xi^2 + B\xi + C},$$

where A, A_1 are two nonzero constants. We conclude that $k = 2$. It follows that

$$d \ge \frac{k+1}{m-n-1} \ge 3,$$

a contradiction.

Subcase 2.2 $g = 0$, and g has two distinct zeros.

In this case we have
$$g = \frac{A(\xi - \xi_1)(\xi - \xi_2)}{A_1\xi^2 + B\xi + C},$$
where A, A_1 are two nonzero constants. We conclude that $k = 1$ and
$$d \geq \frac{k+1}{m-n-1} \geq 2.$$
Furthermore,
$$g = \frac{A(\xi - \xi_1)(\xi - \xi_2)}{(\xi - \xi_3)},$$
where A is a nonzero constant. So
$$\begin{aligned}&(g^n)' + ag^m \\ = &\frac{(\xi - \xi_1)^{n-1}(\xi - \xi_1)^{n-1}[g_1(\xi)(\xi - \xi_3)^{2m-2b+2} - aA^m(\xi - \xi_1)^{2m-2n-1}(\xi - \xi_1)^{2m-2n-1}]}{(\xi - \xi_3)^{2m}},\end{aligned}$$
where $\deg g_1 < 2$. Obviously, $(g^n)' + ag^m$ has at least three distinct zeros, a contradiction.

Subcase 2.3 $g \neq 0$.

From Lemma 2.3, we get
$$T(r,\ f) \leq \bar{N}\Big(r,\ \frac{1}{(f^n)^{(k)} - af^m}\Big) + S(r,\ f),$$
which gives that g is a constant. This is a contradiction.

The proof is completed.

References

[1] Schiff J. Normal Families. Berlin: Springer-Verlag, 1993.

[2] Hayman W K. Meromorphic Functions. Oxford: Clarendon Press, 1964.

[3] Schwick W. Normality criteria for families of meromorphic function. *J. Anal. Math.*, 1989, **52**: 241–289.

[4] Wang Y F, Fang M L. Picard values and normal families of meromorphic functions with multiple zeros. *Acta. Math. Sin.*, 1998, **14**(1): 17–26.

[5] Li Y T, Gu Y X. On normal families of meromorphic functions. *J. Math. Anal. Appl.*, 2009, **354**: 421–425.

[6] Liu Z H, Li Y C. Normal families of meromorphic functions concerning shared fixed-points. *Int. Math. Forum.*, 2011, **6**(31): 1507–1511.

[7] Hu P C, Meng D W. Normality criteria of meromorphic functions with multiple zeros. *J. Math. Anal. Appl.*, 2009, **357**: 323–329.

[8] Jiang Y B, Gao Z S. Normal families of meromorphic functions sharing a holomorphic fuction and the converese of function Bloch principle. *Acta. Math. Sci.*, 2012, **32B**(4): 1503–1512.

[9] Ding J J, Ding L W, Yuan W J. Normal families of meromorphic functions concerning shared values. *Complex Var. Elliptic Equ.*, 2013, **58**(1): 113–121.

[10] Sun C X. Normal families and shared values of meromorphic functions. *Ann. Math.*, 2013, **34A**(2): 205–210.

[11] Zalcman L. Normal families: new perspectives. *Bull. Am. Math. Soc.*, 1998, **35**: 215–230.