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Global Exponential Stability in Lagrange Sense for Delayed Memristive Neural Networks with Parameter Uncertainties

Liangchen Li[1], Rui Xu[2,†] and Jiazhe Lin[1]

Abstract This paper addresses the Lagrange stability of memristive neural networks with leakage delay and time-varying transmission delays as well as parameter uncertainties. Based on the theory of Filippov's solution, by using Lyapunov-Krasovskii functionals and the free-weighting matrix method, sufficient conditions in terms of linear matrix inequality (LMI) are given to ascertain the networks with different kinds of activation functions to be stable in Lagrange sense. Meanwhile the estimation of globally attractive sets are given. Finally, numerical simulations are carried out to illustrate the effectiveness of theoretical results.

Keywords Memristive neural networks, Lagrange stability, Leakage delay, Uncertain parameters.

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1. Introduction

In the past few years, there have been increasing research interests in analyzing the dynamical behaviors of neural networks due to their widespread applications in various areas, such as optimization, signal processing, pattern recognition and so on. In these applications, the stability of neural networks is a precondition to ensure the results to be reliable. Up to now, much work has been done in the filed of stability of neural networks [1–5]. Most of these researches are about Lyapunov stability of monostable neural networks with a unique equilibrium attracting all trajectories asymptotically. However, monostable neural networks have been found computationally restrictive in many applications and multistable neural networks may be more appropriate [6, 7]. For example, the neural networks are required to have multistable equilibria when designed for associative memory or pattern recognition, so that they can get different results with diverse inputs (or initial values). In these applications, the neural networks are no longer globally stable in Lyapunov sense and it's meaningful to analyse their stability in Lagrange sense. Lagrange stability is concerned with the boundedness and the attractivity of systems. It has been proved that no equilibrium, chaos attractor or periodic state exists outside the global attractive set in a Lagrange stable neural network [6, 8]. Moreover, the global stability in Lyapunov sense can be regarded as a special case of stability in

[†]the corresponding author. Email address: rayxu88@126.com(R. Xu)
[1]Institute of Applied Mathematics, Army Engineering University, Shijiazhuang, Hebei 050003, China
[2]Complex Systems Research Center, Shanxi University, Taiyuan, Shanxi 030006, China

Lagrange sense when the attractive set is an equilibrium. So far, some researches about Lagrange stability of neural networks can be found in [6, 8–14].

Since the first memristor was fabricated by Hewlett-Packard Laboratory [15,16], it has attracted much attention because of its special features. Electrical measurements showed that memristors can produce many important features of synapses, such as long term potentiation, long term depression, spike-timing-dependent plasticity, and short term potentiation. Also the distinctive ability to memorize the passed quantity of electric charge and the nonvolatile nature make memirstor a potential media for the next generation storage technology. Memristive neural networks are constructed by replacing the resistors with memristors in VLSI circuits of conventional neural networks. This new type of neural networks will provide the great potential for building a brain-like neural computer by implementing the synapses of biological brains [17]. In [18], Hu and Wang proposed a simplified mathematical model to characterize the pinched hysteretic feature of the memristor. A memristive neural networks model was given in this paper. The employing of memristors made the neural networks state-dependent switching. The state-dependent switching neural networks are discontinuous on the right-hand side, which implies they have more abundant dynamical behaviors and are more difficult to be investigated. Moreover, the analysis of dynamical properties for the memristive neural networks has been found useful to address a number of interesting engineering tasks [19].

As well known, delays are inherent features in many practical networks. Transmission delays in neural networks can be caused by the finite switching speed of the neuron amplifiers and the finite signal propagation speed. In [6, 8–10], this kind of delays were considered when the authors analysed the Lagarange stability of neural networks. In [20], Gopalsamy proposed a kind of delays called leakage delay. Leakage delays are introduced to describe that the decay process of neurons is not instantaneous and time is required to isolate the static state. They always have a great impact on the dynamical behavior of neural networks [19, 21]. In [11], the authors studied the Lagrange stability of complex-valued neural networks with leakage delay. However, few scholars considered parameter uncertainties in the study of Lagrange stability for neural networks. Parameter uncertainties may arise because of the variations in system parameters (temperature variation for example), modeling errors or some ignored factors. These uncertainties may cause the instability and poor performance [21]. So, it's important to study the dynamical behaviors of neural networks by taking the uncertainties into account. In [21] and [22], Xiao et. al and Song et. al investigated the passivity of conventional and memristive neural networks with parameter uncertainties, respectively. Comparing with researches on passivity of neural networks without parameter uncertainties in [23] and [24], their results are more general and reasonable.

To our best knowledge results on Lagrange stability of memristive neural networks with leakage delay and parameters uncertainties have not been reported in the literature. Compared with traditional neural networks, the dynamical properties of memristive neural networks are more complex and difficult to analyse. To obtain more general and applicable results, delays and parameter uncertainties should be considered in the analysis. These factors also complicate the dynamical behavior of memristive neural networks, especially the leakage delay. Meanwhile, the activation functions in neural networks are various, but most of them are Lipschitz continuous or bounded.

Motivated by discussion above, the objective of this paper is to study the Lagrange stability of memristive neural networks with leakage delay and time-varying transmission delays as well as parameters uncertainties. The analysis are argued in two situations: 1: the activation functions of the memristive neural networks satisfy Lipshcitz continuity condition. 2: the activation functions are bounded. To deal with the discontinuity caused by using memristor, we make qualitative analysis of a relevant differential inclusion under the framework of Filippov's solution [25]. By constructing suitable Lyapunov-Krasovskii functionals and using the free-weighting matrix method, sufficient conditions in terms of LMI are given to ascertain the network is stable in Lagrange sense. The criteria can be easily checked by Matlab LMI Toolbox. Meanwhile the according estimation of globally attractive sets are also given.

The rest of the paper is organized as follows. The model description, some necessary definitions and lemmas are presented in Section 2. In Section 3, the main theorems are derived. And then, numerical simulations are given to demonstrate the effectiveness of the results in Section 4. Finally, we make a summary in Section 5.

Notations: $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$. The superscripts A^T and A^{-1} stand for matrix transposition and matrix inverse of A, respectively. $*$ denotes the symmetric block in symmetric matrix. Throughout this paper, solutions of all the networks considered are in the Filippov's sense [25]. Let $C([-\rho,0],\mathbb{R}^n)$ be the Banach space of continuous functions $\psi: [-\rho,0]\to\mathbb{R}^n$ with the norm $\|\psi\|_c=\sup_{s\in[-\rho,0]}\|\psi(s)\|$. For a given constant $S>0$, C_S is defined as the subset $\{\psi\in C:\|\psi\|_c<S\}$. $D^+V(t)$ stands for the upper right Dini derivative of $V(t)$. $\lambda_{\min}(\cdot)$ stands for the minimum eigenvalue of a certain matrix.

2. Model description and Preliminaries

Based on the study of memristive neural networks modeling in [19,26] and researches about uncertain systems in [21, 22, 27], we consider the following memristive neural networks with leakage delay and time-varying transmission delays as well as parameter uncertainties,

$$\begin{aligned}\dot{x}(t)=&-(\hat{D}+\Delta D)x(t-\delta)+(\hat{A}+\Delta A)f(x(t))\\&+(\hat{B}+\Delta B)f(x(t-\tau(t)))+U(t),\end{aligned}\tag{2.1}$$

where $x(t)=(x_1(t),x_2(t),\cdots,x_n(t))^T\in\mathbb{R}^n$ is the state vector of the network at time t; n corresponds to the number of neurons; $f(x(t))=(f_1(x_1(t)),f_2(x_2(t)),\cdots,f_n(x_n(t)))^T$ are the activation functions. $U(t)=(u_1(t),u_2(t),\cdots,u_n(t))^T\in\mathbb{R}^n$ is a continuous external input, satisfying $|u_i(t)|\le u_i^*$, ($u_i^*=\max_{t\ge0}|u_i(t)|$). Denote $U=(u_1^*,u_2^*,\cdots,u_n^*)^T$. δ and $\tau(t)=(\tau_1(t),\tau_2(t),\cdots,\tau_n(t))^T$ stand for the leakage delay and time-varying transmission delays, respectively. $\tau_i(t)$ are continuous and satisfy $\max\{\tau_1(t),\tau_2(t),\cdots,\tau_n(t)\}<\tau$. $\Delta A,\Delta B$, and ΔD are time-varying parameter uncertainties, which are assumed to be of the form

$$\Delta A=H_1T_1(t)E_1,\quad \Delta B=H_2T_2(t)E_2,\quad \Delta D=H_3T_3(t)E_3,\tag{2.2}$$

where $H_1, H_2, H_3, E_1, E_2, E_3$ are known real constant matrices of appropriate dimensions, $T_1(t), T_2(t), T_3(t)$ are unknown time-varying matrices satisfying

$$T_1^T(t)T_1(t) \leq I, \quad T_2^T(t)T_2(t) \leq I, \quad T_3^T(t)T_3(t) \leq I. \tag{2.3}$$

Especially, $H_3, T_3(t), E_3$ are diagonal matrices.

$\hat{D} = \mathrm{diag}(\hat{d}_1(x_1(t)), \hat{d}_2(x_2(t)), \cdots, \hat{d}_n(x_n(t)))$ describes the rate with which each neuron will reset its potential to the resting state in isolation when disconnected from the networks and external inputs at time t. $\hat{A} = (\hat{a}_{ij}(x_i(t)))_{n\times n}$ and $\hat{B} = (\hat{b}_{ij}(x_i(t)))_{n\times n}$ are the connection weight matrix and the delay connection weight matrix, respectively. According to the feature of the memristor and the current-voltage characteristic, $\hat{d}_i(x_i(t)), \hat{a}_{ij}(x_i(t)), \hat{b}_{ij}(x_i(t))$ satisfy

$$\hat{d}_i(x_i(t)) = \begin{cases} d_i^*, & |x_i(t)| \leq \chi_i, \\ d_i^{**}, & |x_i(t)| > \chi_i, \end{cases} \quad \hat{a}_{ij}(x_j(t)) = \begin{cases} a_{ij}^*, & |x_i(t)| \leq \chi_i, \\ a_{ij}^{**}, & |x_i(t)| > \chi_i, \end{cases}$$
$$\hat{b}_{ij}(x_j(t)) = \begin{cases} b_{ij}^*, & |x_i(t)| \leq \chi_i, \\ b_{ij}^{**}, & |x_i(t)| > \chi_i, \end{cases}$$

in which $\chi_i > 0, d_i^* > 0, d_i^{**} > 0, a_i^*, a_i^{**}, b_i^*, b_i^{**}, i, j = 1, 2, \cdots, n$ are constants. Denote $d_{\max} = \max\{d_i^*, d_i^{**}, i = 1, 2, \cdots, n\}$. Obviously, for each i and j, $\hat{d}_i(x_i(t)), \hat{a}_{ij}(x_j(t))$ and $\hat{b}_{ij}(x_j(t))$ have two possible value. A certain state of x_i will determine the value of $2n+1$ of them, thus the combination number of the possible form of $\hat{D}, \hat{A}$ and $\hat{B}$ is 2^n. Order these 2^n cases in the following way:

$$(D_1, A_1, B_1), (D_2, A_2, B_2), \cdots, (D_{2^n}, A_{2^n}, B_{2^n}).$$

Then, at any fixed time $t \geq 0$, the form of $\hat{D}, \hat{A}$ and $\hat{B}$ must be one of the 2^n cases. For each case, we define the characteristic function as

$$\Psi_i(t) = \begin{cases} 1, & \hat{D} = D_i, \hat{A} = A_i, \hat{B} = B_i, \\ 0, & \text{otherwise,} \end{cases} \quad i = 1, 2, \cdots, 2^n. \tag{2.4}$$

We can easily conclude that $\sum_{i=1}^{2^n} \Psi_i(t) = 1$ and

$$\hat{D} = \sum_{i=1}^{2^n} \Psi_i(t) D_i, \hat{A} = \sum_{i=1}^{2^n} \Psi_i(t) A_i, \hat{B} = \sum_{i=1}^{2^n} \Psi_i(t) B_i. \tag{2.5}$$

Then network (2.1) can be rewritten as

$$\begin{aligned} \dot{x}(t) = &- (\sum_{i=1}^{2^n} \Psi_i(t) D_i + \Delta D) x(t-\delta) + (\sum_{i=1}^{2^n} \Psi_i(t) A_i + \Delta A) f(x(t)) \\ &+ (\sum_{i=1}^{2^n} \Psi_i(t) B_i + \Delta B) f(x(t-\tau(t))) + U(t) \\ = &\sum_{i=1}^{2^n} \Psi_i(t)(-(\hat{D} + \Delta D) x(t-\delta) + (\hat{A} + \Delta A) f(x(t)) \\ &+ (\hat{B} + \Delta B) f(x(t-\tau(t))) + U(t)). \end{aligned} \tag{2.6}$$

The initial condition of network (2.1) is given as

$$x_i(t) = \psi_i(t), \quad t \in [-\rho, 0], \rho = \max\{\delta, \tau\}.$$

For any initial condition $\psi(t) = [\psi_1(t), \psi_2(t), \cdots, \psi_n(t)]^T \in C$, the solution of network (2.1) that starts from the initial condition ψ will be denoted by $x(t, \psi)$. If there is no need to emphasize the initial condition, any solution of network (2.1) will also simply be denoted by $x(t)$.

For the activation functions, we make the following assumptions:

(A1): For $i = 1, 2, \cdots, n$, $f_i(0) = 0$ and there exist constants F_i^-, F_i^+ such that

$$F_i^- \le \frac{f_i(x_1) - f_i(x_2)}{x_1 - x_2} \le F_i^+,$$

for all $x_1 \neq x_2$. We denote

$$F_1 = \operatorname{diag}\left(F_1^- F_1^+, F_2^- F_2^+, \cdots, F_n^- F_n^+\right),$$
$$F_2 = \operatorname{diag}\left(\frac{F_1^- + F_1^+}{2}, \frac{F_2^- + F_2^+}{2}, \cdots, \frac{F_n^- + F_n^+}{2}\right).$$

Remark 2.1. The constants $F_i^-, F_i^+ (i = 1, 2, \cdots, n)$ are allowed to be positive, negative or zero. Hence, this assumption is weaker than the assumptions in [9,11,14].

(A2): For $i = 1, 2, \cdots, n$, $f_i(x)$ is bounded, and there exist positive constant h_i such that $|f_i(x)| < h_i$. Denote $H = (h_1, h_2, \cdots, h_n)^T$.

Definition 2.1. Network (2.1) is said to be uniformly stable in Lagrange sense (or uniformly bounded), if for any $S > 0$, there exists a constant $\kappa = \kappa(S) > 0$ such that $\|x(t, \psi)\| < \kappa$ for all $\psi \in C_S$ and $t \ge 0$.

Definition 2.2. If there exist a radially unbounded and positive definite function $V(x(t))$, a functional $\kappa \in C$, positive constants ℓ, α, such that for any solution $x(t) = x(t, \psi)$ of network (2.1), $V(x(t)) > \ell, t \ge 0$, implies

$$V(x(t)) - \ell \le \kappa(\psi) \exp(-\alpha t),$$

then network (2.1) is said to be globally exponentially attractive, and the compact set $\Omega := \{x \in \mathbb{R}^n, V(x) < \ell\}$ is called a globally exponentially attractive set of network (2.1).

Definition 2.3. Network (2.1) is called globally exponentially stable in Lagrange sense, if it is both uniformly stable in Lagrange sense and globally exponentially attractive.

To prove our results, the following lemmas are necessary.

Lemma 2.1 ([22])**.** *Let h be a positive constant, and $P \in \mathbb{R}^{n\times n}$ be a positive definite constant matrix, then*

$$\left(\int_{t-h}^{t} x(s)ds\right)^T P \int_{t-h}^{t} x(s)ds \le h \int_{t-h}^{t} x^T(s) P x(s) ds,$$

for $t \ge 0$ and any vector function $x(s) \in \mathbb{R}^n$.

Lemma 2.2 ([27]). *Let H, E and $T(t)$ be real matrices of appropriate dimensions with $T(t)$ satisfying $T^T(t)T(t) < I$. Then for any scalar $\varepsilon > 0$,*

$$HT(t)E + (HT(t)E)^T \leq \varepsilon^{-1}HH^T + \varepsilon E^T E.$$

Lemma 2.3 ([28]). *Let $a, b \in \mathbb{R}^n$ and Q be a positive definite matrix, then $2a^T b \leq a^T Q^{-1} a + b^T Q b$.*

Lemma 2.4 ([29]). *The LMI $Y = \begin{bmatrix} y_{11} & y_{12} \\ y_{12}^T & y_{22} \end{bmatrix} < 0$ with $y_{11}^T = y_{11}, y_{22}^T = y_{22}$ is equivalent to one of the following conditions:*

(i) $y_{22} < 0,\ y_{11} - y_{12}y_{22}^{-1}y_{12}^T < 0,$

(ii) $y_{11} < 0,\ y_{22} - {y_{12}}^T y_{11}^{-1} y_{12} < 0.$

Lemma 2.5 ([30]). *Let $V(x(t)) : \mathbb{R}^n \to \mathbb{R}_+$ be a positive definite and radially unbounded function, and suppose there exist two positive constants ϖ, π such that*

$$D^+V(x(t)) \leq -\varpi V(x(t)) + \pi, \quad t \geq t_0,$$

then, when $V(x(t)) \geq \pi/\varpi, t \geq t_0$ have

$$V(x(t)) - \frac{\pi}{\varpi} \leq \left(V(x(t_0)) - \frac{\pi}{\varpi}\right) e^{-\varpi(t-t_0)}.$$

3. Main results

Theorem 3.1. *Under assumption (A1), if there exist positive definite matrices $P_1, P_2, P_3, P_4, Q_1, Q_2, Q_3, Q_4$, positive definite diagonal matrices K_1, K_2, matrices M_1, M_2, M_3, M_4 and positive scalars $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6$ such that the following LMIs hold:*

$$\Lambda_i = \begin{bmatrix} \Xi_i & \Gamma_1 \\ \Gamma_1^T & \Theta_1 \end{bmatrix} < 0, i = 1, 2, \cdots, 2^n, \tag{3.1}$$

where

$$\Gamma_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & M_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_4 & 0 & 0 \\ 0 & M_2H_3 & 0 & M_2H_1 & 0 & M_2H_2 & 0 & 0 & 0 & M_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ M_1H_3 & 0 & M_1H_1 & 0 & M_1H_2 & 0 & 0 & 0 & M_1 & 0 \end{bmatrix},$$

$$\Theta_1 = diag(-\varepsilon_1 I, -\varepsilon_2 I, -\varepsilon_3 I, -\varepsilon_4 I, -\varepsilon_5 I, -\varepsilon_6 I, -\frac{Q_1}{\delta}, -\frac{Q_2}{\tau}, -Q_3, -Q_4),$$

$$\Xi_i = \begin{bmatrix} \Sigma_{11} & M_4 & \Sigma_{13} & \Sigma_{14} & F_2K_1 & 0 & \Sigma_{17} \\ * & \Sigma_{22} & 0 & 0 & 0 & F_2K_2 & 0 \\ * & * & \Sigma_{33} & -P_1D_i & M_2A_i & M_2B_i & \Sigma_{37} \\ * & * & * & \Sigma_{44} & 0 & 0 & -P_1 \\ * & * & * & * & \Sigma_{55} & 0 & M_1A_i \\ * & * & * & * & * & \Sigma_{66} & M_1B_i \\ * & * & * & * & * & * & \Sigma_{77} \end{bmatrix},$$

in which

$\Sigma_{11} = P_1 + P_4 - M_3 - M_3^T - F_1K_1 + (1+\delta e^{\delta})P_3 + \delta {D_i}^T P_2 D_i - P_1D_i - {D_i}^T P_1^T,$
$\Sigma_{22} = -F_1K_2 - M_4 - M_4^T, \Sigma_{13} = P_1D_i + M_3,$
$\Sigma_{33} = -P_3 - M_2D_i - {D_i}^T M_2^T + (\varepsilon_1 + \varepsilon_2)\, E_3E_3^T,$
$\Sigma_{14} = P_1D_i - P_1, \Sigma_{44} = P_1 - \frac{P_2}{\delta e^{\delta}}, \Sigma_{55} = -K_1 + (\varepsilon_3 + \varepsilon_4)\, E_1E_1^T,$
$\Sigma_{66} = -K_2 + (\varepsilon_5 + \varepsilon_6)\, E_2E_2^T, \Sigma_{17} = P_1 + P_4, \Sigma_{37} = -M_1D_i - M_2,$
$\Sigma_{77} = e^{\delta}\delta Q_1 + e^{\tau}\tau Q_2 - M_1 - M_1^T.$

Then (2.1) *is globally exponentially stable in Lagrange sense, and*

$$\Omega_1 = \left\{ x(t) \in \mathbb{R}^n, \|x(t)\| \le \sqrt{\frac{U^T(Q_3+Q_4)\,U}{\lambda_{\min}(P_1)}} e^{\delta d_{\max}} \right\} \cap \left\{ x(t) \in \mathbb{R}^n, \|x(t)\| \le \sqrt{\frac{U^T(Q_3+Q_4)\,U}{\lambda_{\min}(P_4)}} \right\}$$

is a globally exponentially attractive set of network (2.1)*.*

Proof. Employ the following Lyapunov-Krasovskii functional:

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t), \tag{3.2}$$

where

$$V_1(t) = \left(x(t) - \hat{D} \int_{t-\delta}^{t} x(s)ds \right)^T P_1 \left(x(t) - \hat{D} \int_{t-\delta}^{t} x(s)ds \right),$$
$$V_2(t) = \int_{-\delta}^{0} \int_{t+\theta}^{t} e^{s-t} \left(x(s)^{\mathrm{T}} \hat{D}^{\mathrm{T}} P_2 \hat{D} x(s) + e^{\delta} x(s)^{\mathrm{T}} P_3 x(s) + e^{\delta} \dot{x}^T(s) Q_1 \dot{x}(s) \right) \mathrm{d}s\mathrm{d}\theta,$$
$$V_3(t) = \int_{-\tau}^{0} \int_{t+\theta}^{t} e^{s-t+\tau} \dot{x}^T(s) Q_2 \dot{x}(s) \mathrm{d}s\mathrm{d}\theta,$$
$$V_4(t) = x^T(t) P_4 x(t) + \int_{t-\delta}^{t} x^T(s) P_3 x(s) \mathrm{d}s.$$

By computing the upper right Dini derivative of $V(t)$ along the solutions of network (2.1), we can derive that

$$D^+V(t) = D^+V_1(t) + D^+V_2(t) + D^+V_3(t) + D^+V_4(t), \tag{3.3}$$

where

$$\begin{aligned} D^+V_1(t) =& 2\left(x(t) - \hat{D}\int_{t-\delta}^{t} x(s)ds\right)^T P_1\left(\dot{x}(t) - \hat{D}x(t) + \hat{D}x(t-\delta)\right), \\ D^+V_2(t) =& -V_2(t) + \delta x^T(t)\hat{D}^T P_2\hat{D}x(t) + e^{\delta}\delta x(t)^{\mathrm{T}} P_3 x(t) \\ & + e^{\delta}\delta\dot{x}^T(t)Q_1\dot{x}(t) - \int_{t-\delta}^{t} e^{s-t}x^T(s)\hat{D}^T P_2\hat{D}x(s)\mathrm{d}s \\ & - \int_{t-\delta}^{t} e^{s-t+\delta}x^T(s)P_3x(s)\mathrm{d}s - \int_{t-\delta}^{t} e^{s-t+\delta}\dot{x}^T(s)Q_1\dot{x}(s)\mathrm{d}s, \\ D^+V_3(t) =& -V_3(t) + e^{\tau}\tau\dot{x}^T(t)Q_2\dot{x}(t) - \int_{t-\tau}^{t} e^{s-t+\tau}\dot{x}^T(s)Q_2\dot{x}(s)\mathrm{d}s, \\ D^+V_4(t) =& 2x^T(t)P_4\dot{x}(t) + x^T(t)P_3x(t) - x^T(t-\delta)P_3x(t-\delta). \end{aligned} \tag{3.4}$$

Obviously,

$$-\int_{t-\delta}^{t} e^{s-t+\delta}x^T(s)P_3x(s)\mathrm{d}s \le -\int_{t-\delta}^{t} x^T(s)P_3x(s)\mathrm{d}s, \tag{3.5}$$

$$-\int_{t-\delta}^{t} e^{s-t+\delta}\dot{x}^T(s)Q_1\dot{x}(s)\mathrm{d}s \le -\int_{t-\delta}^{t} \dot{x}^T(s)Q_1\dot{x}(s)\mathrm{d}s, \tag{3.6}$$

$$-\int_{t-\tau}^{t} e^{s-t+\tau}\dot{x}^T(s)Q_2\dot{x}(s)\mathrm{d}s \le -\int_{t-\tau}^{t} \dot{x}^T(s)Q_2\dot{x}(s)\mathrm{d}s. \tag{3.7}$$

It follows from Lemma 2.1 that

$$\begin{aligned} & -\int_{t-\delta}^{t} e^{s-t}x^T(s)\hat{D}^T P_2\hat{D}x(s)\mathrm{d}s \\ & \le -\frac{e^{-\delta}}{\delta}\int_{t-\delta}^{t} x^T(s)\mathrm{d}s\hat{D}^T P_2\hat{D}\int_{t-\delta}^{t} x(s)\mathrm{d}s. \end{aligned} \tag{3.8}$$

According to (2.1), we have

$$\begin{aligned} 0 =& \left(2\dot{x}^T(t)M_1 + 2x^T(t-\delta)M_2\right)[-\dot{x}(t) - \hat{D}x(t-\delta) + \hat{A}f(x(t)) \\ & + \hat{B}f(x(t-\tau(t))) - \Delta Dx(t-\delta) + \Delta Af(x(t)) \\ & + \Delta Bf(x(t-\tau(t))) + U(t)]. \end{aligned}$$

By Lemma 2.2, we can get that

$$\begin{aligned} & -\left(2\dot{x}^T(t)M_1 + 2x^T(t-\delta)M_2\right)\big(-\Delta Dx(t-\delta) + \Delta Af(x(t)) \\ & + \Delta Bf(x(t-\tau(t)))\big) \\ \le & x(t-\delta)\left[\varepsilon_2^{-1}M_2H_3H_3^TM_2^T + (\varepsilon_1+\varepsilon_2)E_3E_3^T\right]x(t-\delta) \\ & + \varepsilon_1^{-1}\dot{x}^T(t)M_1H_3H_3^TM_1^T\dot{x}(t) \\ & + \varepsilon_3^{-1}\dot{x}^T(t)M_1H_1H_1^TM_1^T\dot{x}(t) + \varepsilon_4^{-1}x^T(t-\delta)M_2H_1H_1^TM_2^Tx(t-\delta) \\ & + (\varepsilon_3+\varepsilon_4)f^T(x(t))E_1E_1^Tf(x(t)) \\ & + \varepsilon_5^{-1}\dot{x}^T(t)M_1H_2H_2^TM_1^T\dot{x}(t) + \varepsilon_6^{-1}x^T(t-\delta)M_2H_2H_2^TM_2^Tx(t-\delta) \\ & + (\varepsilon_5+\varepsilon_6)f^T(x(t-\tau(t)))E_2E_2^Tf(x(t-\tau(t))). \end{aligned} \tag{3.9}$$

By Lemma 2.3, we obtain

$$\begin{aligned}&\left(2\dot{x}^T(t)M_1+2x^T(t-\delta)M_2\right)U(t)\\ \le&\dot{x}^T(t)M_1Q_3^{-1}M_1^T\dot{x}(t)+x^T(t-\delta)M_2Q_4^{-1}M_2^Tx(t-\delta)\\ &+U^T(t)\left(Q_3+Q_4\right)U(t),\end{aligned}\tag{3.10}$$

By Newton-Leibniz formulation, we have

$$\begin{aligned}0=&-2x^T(t)M_3(x(t)-x(t-\delta)-\int_{t-\delta}^{t}\dot{x}(s)\mathrm{d}s),\\ 0=&2x^T(t-\tau(t))M_4(x(t)-x(t-\tau(t))-\int_{t-\tau(t)}^{t}\dot{x}(s)\mathrm{d}s).\end{aligned}\tag{3.11}$$

By Lemmas 2.1 and 2.3 and noting that $0\le\tau_i(t)\le\tau, i=1,2,\cdots,n$, it can be derived from (3.11) that

$$\begin{aligned}0\le&x^T(t)\left(-2M_3+\delta M_3Q_1^{-1}M_3^T\right)x(t)+2x^T(t)M_3x(t-\delta)\\ &+\int_{t-\delta}^{t}\dot{x}^T(s)Q_1\dot{x}(s)\mathrm{d}s,\\ 0\le&x^T(t-\tau(t))(-2M_4+\tau M_4Q_2^{-1}M_4^T)x(t-\tau(t))+\int_{t-\tau}^{t}\dot{x}^T(s)Q_2\dot{x}(s)\mathrm{d}s\\ &+2x^T(t-\tau(t))M_4x(t).\end{aligned}$$

For positive diagonal matrices K_1, K_2, it follows from assumption **(A1)** and the proof of Theorem 3.1 in [31] that

$$\begin{aligned}0\le&\begin{bmatrix}x(t)\\ f(x(t))\end{bmatrix}^T\begin{bmatrix}-F_1K_1 & F_2K_1\\ F_2K_1 & -K_1\end{bmatrix}\begin{bmatrix}x(t)\\ f(x(t))\end{bmatrix},\\ 0\le&\begin{bmatrix}x(t-\tau(t))\\ f(x(t-\tau(t)))\end{bmatrix}^T\begin{bmatrix}-F_1K_2 & F_2K_2\\ F_2K_2 & -K_2\end{bmatrix}\begin{bmatrix}x(t-\tau(t))\\ f(x(t-\tau(t)))\end{bmatrix}.\end{aligned}\tag{3.12}$$

Then it follows from (3.3) to (3.12) that

$$\begin{aligned}D^+V(t)\le&-V(t)+\eta^T(t)\Xi\eta(t)-\eta^T(t)\Gamma_1\Theta_1^{-1}\Gamma_1^T\eta(t)\\ &+U^T(t)\left(Q_3+Q_4\right)U(t),\end{aligned}\tag{3.13}$$

where

$$\eta(t)=\left[x^T(t),x^T(t-\tau(t)),x^T(t-\delta),\int_{t-\delta}^{t}x^T(s)ds\hat{D}^T,f^T(x(t)),f^T(x(t-\tau(t))),\dot{x}^T(t)\right]^T,$$

$$\Xi = \begin{bmatrix} \hat{\Sigma}_{11} & M_4 & \hat{\Sigma}_{13} & \hat{\Sigma}_{14} & F_2K_1 & 0 & \hat{\Sigma}_{17} \\ * & \hat{\Sigma}_{22} & 0 & 0 & 0 & F_2K_2 & 0 \\ * & * & \hat{\Sigma}_{33} & -P_1\hat{D} & M_2\hat{A} & M_2\hat{B} & \hat{\Sigma}_{37} \\ * & * & * & \hat{\Sigma}_{44} & 0 & 0 & -P_1 \\ * & * & * & * & \hat{\Sigma}_{55} & 0 & M_1\hat{A} \\ * & * & * & * & * & \hat{\Sigma}_{66} & M_1\hat{B} \\ * & * & * & * & * & * & \hat{\Sigma}_{77} \end{bmatrix},$$

in which

$$\hat{\Sigma}_{11} = P_1 + P_4 - M_3 - M_3^T - F_1K_1 + (1+\delta e^{\delta})P_3 + \delta \hat{D}^T P_2 \hat{D} - P_1\hat{D} - \hat{D}^T P_1^T,$$
$$\hat{\Sigma}_{22} = -F_1K_2 - M_4 - M_4^T, \hat{\Sigma}_{13} = P_1\hat{D} + M_3,$$
$$\hat{\Sigma}_{33} = -P_3 - M_2\hat{D} - \hat{D}^T M_2^T + (\varepsilon_1 + \varepsilon_2) E_3E_3^T,$$
$$\hat{\Sigma}_{14} = P_1\hat{D} - P_1, \hat{\Sigma}_{44} = P_1 - \tfrac{P_2}{\delta e^{\delta}}, \hat{\Sigma}_{55} = -K_1 + (\varepsilon_3 + \varepsilon_4) E_1E_1^T,$$
$$\hat{\Sigma}_{66} = -K_2 + (\varepsilon_5 + \varepsilon_6) E_2E_2^T, \hat{\Sigma}_{17} = P_1 + P_4, \hat{\Sigma}_{37} = -M_1\hat{D} - M_2,$$
$$\hat{\Sigma}_{77} = e^{\delta}\delta Q_1 + e^{\tau}\tau Q_2 - M_1 - M_1^T.$$

We can derive from (2.5) and (3.1) that

$$\sum_{i=1}^{2^n} \Psi_i(t)\Lambda_i < 0.$$

By Lemma 2.4, we have

$$\Xi - \Gamma_1\Theta_1^{-1}\Gamma_1^T < 0. \tag{3.14}$$

Substituting (3.14) into (3.13) yields

$$D^+V(t) \le -V(t) + U^T(Q_3 + Q_4)U.$$

By Lemma 2.5, we can get when $t \ge 0, V(x(t)) \ge U^T(Q_3 + Q_4)U$,

$$V(x(t)) - U^T(Q_3 + Q_4)U \le \left(V(x(0)) - U^T(Q_3 + Q_4)U\right)e^{-t},$$

so that network (2.1) is globally exponentially attractive. From the following inequality

$$\lambda_{\min}(P_4)\|x(t)\|^2 \le V(x(t)) \le V(x(0)) + U^T(Q_3 + Q_4)U,$$

we have

$$\|x(t)\|^2 \le \frac{V(x(0)) + U^T(Q_3 + Q_4)U}{\lambda_{\min}(P_4)},$$

network (2.1) is uniformly bounded. Then (2.1) is globally exponentially stable in Lagrange sense. By Definition 2.2 and solving the following inequalities

$$\lambda_{\min}(P_1)\left\|x(t) - D\int_{t-\delta}^{t} x(s)ds\right\|^2 \le V_1(x(t),t) \le U^T(Q_3 + Q_4)U,$$
$$\lambda_{\min}(P_4)\|x(t)\|^2 \le V_4(x(t),t) \le U^T(Q_3 + Q_4)U,$$

we obtain that

$$\Omega_1 = \left\{ x(t) \in \mathbb{R}^n, \|x(t)\| \le \sqrt{\frac{U^T\left(Q_3 + Q_4\right)U}{\lambda_{\min}(P_1)}e^{\delta d_{\max}}} \right\} \cap \left\{ x(t) \in \mathbb{R}^n, \|x(t)\| \le \sqrt{\frac{U^T\left(Q_3 + Q_4\right)U}{\lambda_{\min}(P_4)}} \right\}.$$

This completes the proof. □

Remark 3.1. In Theorem 3.1, when there are no external inputs, the original point is the globally exponentially attractive set, the neural network is globally exponentially stable in Lyapunov sense. Meanwhile, the criterion is dependent on both leakage delay and the upper bound of transmission delays, which implicates that the information on the sizes of delays is sufficiently utilized.

Theorem 3.2. *Under assumption (A2), if there exist positive definite matrices $P_1, P_2, P_3, P_4, Q_1, Q_2, Q_3$, matrices M_1, M_2 and positive scalars $\varepsilon_1, \varepsilon_2, \varepsilon_3$ such that the following LMIs hold:*

$$\Pi_i = \begin{bmatrix} \mathrm{Z}_i & \Gamma_2 \\ \Gamma_2^T & \Theta_2 \end{bmatrix} < 0, i = 1, 2, \cdots, 2^n, \tag{3.15}$$

where

$$\Gamma_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ M_2H_3 & M_2H_1 & M_2H_2 & M_2A_i & M_2B_i & M_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ M_1H_3 & M_1H_1 & M_1H_2 & M_1A_i & M_1B_i & M_1 \end{bmatrix},$$

$$\Theta_2 = diag(-\varepsilon_1 I, -\varepsilon_2 I, -\varepsilon_3 I, -Q_1, -Q_2, -Q_3),$$

$$\mathrm{Z}_i = \begin{bmatrix} \Sigma_1 & P_1D_i & P_1D_i - P & P_1 + P_4 \\ * & \Sigma_2 & -P_1D_i & -M_1D_i - M_2 \\ * & * & \Sigma_3 & -P_1 \\ * & * & * & -M_1 - M_1^T \end{bmatrix},$$

in which

$$\Sigma_1 = P_1 + P_4 + (1 + \delta e^{\delta})P_3 + \delta D_i^T P_2 D_i - P_1 D_i - D_i^T P_1^T,$$
$$\Sigma_2 = -P_3 - M_2 D_i - D_i^T M_2^T + \varepsilon_1 E_3 E_3^T,$$
$$\Sigma_3 = P_1 - \frac{P_2}{\delta e^{\delta}},$$

then (2.1) *is globally exponentially stable in Lagrange sense, and*

$$\Omega_2 = \left\{ x(t) \in \mathbb{R}^n, \|x(t)\| \le \sqrt{\frac{W}{\lambda_{\min}(P_1)}e^{\delta d_{\max}}} \right\} \cap \left\{ x(t) \in \mathbb{R}^n, \|x(t)\| \le \sqrt{\frac{W}{\lambda_{\min}(P_4)}} \right\}$$

is a globally exponentially attractive set of network (2.1)*, in which* $W = \varepsilon_2 H^T E_1^T E_1 H + \varepsilon_3 H^T E_2^T E_2 H + H^T Q_1 H + H^T Q_2 H + U^T Q_3 U$.

Proof. Employ the following Lyapunov-Krasovskii functional:

$$V(t) = V_1(t) + V_2(t) + V_3(t), \tag{3.16}$$

where

$$\begin{aligned} V_1(t) =& \left(x(t) - \hat{D}\int_{t-\delta}^{t} x(s)ds\right)^T P_1 \left(x(t) - \hat{D}\int_{t-\delta}^{t} x(s)ds\right), \\ V_2(t) =& \int_{-\delta}^{0}\int_{t+\theta}^{t} e^{s-t}\big(x(s)^{\mathrm{T}}\hat{D}^{\mathrm{T}}P_2\hat{D}x(s) + e^{\delta}x(s)^{\mathrm{T}}P_3x(s)\big)\mathrm{d}s\mathrm{d}\theta, \\ V_3(t) =& x^T(t)P_4x(t) + \int_{t-\delta}^{t} x^T(s)P_3x(s)\mathrm{d}s. \end{aligned} \tag{3.17}$$

The upper right Dini derivative of $V(t)$ along the solutions of network (2.1) can be computed as the sum of $D^+V_1(t), D^+V_2(t)$ and $D^+V_4(t)$ in the proof of Theorem 1. We also bring in free-weighting matrices M_1 and M_2 as we did in the proof of Theorem 3.1,

$$\begin{aligned} 0 =& \left(2\dot{x}^T(t)M_1 + 2x^T(t-\delta)M_2\right)[-\dot{x}(t) - \hat{D}x(t-\delta) + \hat{A}f(x(t)) \\ &+ \hat{B}f(x(t-\tau(t))) - \Delta Dx(t-\delta) + \Delta Af(x(t)) + \Delta Bf(x(t-\tau(t))) + U(t)]. \end{aligned}$$

Denote $\xi(t) = \left[x^T(t), x^T(t-\delta), \int_{t-\delta}^{t} x^T(s)ds\hat{D}^T, \dot{x}^T(t)\right]^T$. By Lemma 2.2, we can get that

$$\begin{aligned} &-\left(2\dot{x}^T(t)M_1 + 2x^T(t-\delta)M_2\right)\left(-\Delta Dx(t-\delta) + \Delta Af(x(t)) + \Delta Bf(x(t-\tau(t)))\right) \\ \leq& \varepsilon_1^{-1}\xi^T(t)\begin{bmatrix} 0 \\ M_2 \\ 0 \\ M_1 \end{bmatrix} H_3H_3^T \begin{bmatrix} 0 \\ M_2 \\ 0 \\ M_1 \end{bmatrix}^T \xi(t) + \varepsilon_2^{-1}\xi^T(t)\begin{bmatrix} 0 \\ M_2 \\ 0 \\ M_1 \end{bmatrix} H_1H_1^T \begin{bmatrix} 0 \\ M_2 \\ 0 \\ M_1 \end{bmatrix}^T \xi(t) \\ &+ \varepsilon_1 x(t-\delta)E_3E_3^Tx(t-\delta) + \varepsilon_2 f^T(x(t))E_1E_1^Tf(x(t)) \end{aligned} \tag{3.18}$$

$$+ \varepsilon_3^{-1}\xi^T(t)\begin{bmatrix} 0 \\ M_2 \\ 0 \\ M_1 \end{bmatrix} H_2H_2^T \begin{bmatrix} 0 \\ M_2 \\ 0 \\ M_1 \end{bmatrix}^T \xi(t) + \varepsilon_3 f^T(x(t-\tau(t)))E_2E_2^Tf(x(t-\tau(t))).$$

By Lemma 2.3, we can derive that

$$\begin{aligned}
&\left(2\dot{x}^T(t)M_1 + 2x^T(t-\delta)M_2\right)\left[\hat{A}f(x(t)) + \hat{B}f(x(t-\tau(t))) + U(t)\right] \\
\leq&\xi^T(t)\begin{bmatrix} 0 \\ M_2 \\ 0 \\ M_1 \end{bmatrix}\hat{A}Q_1^{-1}\hat{A}^T\begin{bmatrix} 0 \\ M_2 \\ 0 \\ M_1 \end{bmatrix}^T\xi(t) + \xi^T(t)\begin{bmatrix} 0 \\ M_2 \\ 0 \\ M_1 \end{bmatrix}\hat{B}Q_2^{-1}\hat{B}^T\begin{bmatrix} 0 \\ M_2 \\ 0 \\ M_1 \end{bmatrix}^T\xi(t) \\
&+ f^T(x(t))Q_1f(x(t)) + f^T(x(t-\tau(t)))Q_2f(x(t-\tau(t))) \\
&+ \xi^T(t)\begin{bmatrix} 0 \\ M_2 \\ 0 \\ M_1 \end{bmatrix}Q_3^{-1}\begin{bmatrix} 0 \\ M_2 \\ 0 \\ M_1 \end{bmatrix}^T\xi(t) + U^T(t)Q_3U(t),
\end{aligned} \tag{3.19}$$

Substituting (3.17)-(3.19) into $D^+V(t)$ yields

$$\begin{aligned}
D^+V(t) \leq& -V(t) + \xi^T(t)\begin{bmatrix} \hat{\Sigma}_1 & P_1\hat{D} & P_1\hat{D} - P_1 & P_1 + P_4 \\ * & \hat{\Sigma}_2 & P_1\hat{D} & -M_1\hat{D} - M_2 \\ * & * & \Sigma_3 & -P_1 \\ * & * & * & -M_1 - M_1^T \end{bmatrix}\xi(t) \\
&- \xi^T(t)\hat{\Gamma}_2\Theta_2{}^{-1}\hat{\Gamma}_2^T\xi(t) + U^T(t)Q_3U(t) \\
&+ \varepsilon_3 f^T(x(t-\tau(t)))E_2E_2^Tf(x(t-\tau(t))) \\
&+ \varepsilon_2 f^T(x(t))E_1E_1^Tf(x(t)) + f^T(x(t))Q_1f(x(t)) \\
&+ f^T(x(t-\tau(t)))Q_2f(x(t-\tau(t))),
\end{aligned} \tag{3.20}$$

where

$$\hat{\Gamma}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ M_2H_3 & M_2H_1 & M_2H_2 & M_2\hat{A} & M_2\hat{B} & M_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ M_1H_3 & M_1H_1 & M_1H_2 & M_1\hat{A} & M_1\hat{B} & M_1 \end{bmatrix},$$

$$\begin{aligned}
\hat{\Sigma}_1 &= P_1 + P_4 + (1 + \delta e^{\delta})P_3 + \delta\hat{D}^TP_2\hat{D} - P_1\hat{D} - \hat{D}^TP_1^T, \\
\hat{\Sigma}_2 &= -P_3 - M_2\hat{D} - \hat{D}^TM_2^T + \varepsilon_1E_3E_3^T.
\end{aligned}$$

We can derive from (2.5) and (3.15) that

$$\sum_{i=1}^{2^n}\Psi_i(t)\Pi_i < 0. \tag{3.21}$$

According to Lemma 2.4, (3.21) is equivalent to

$$\begin{bmatrix} \hat{\Sigma}_1 & P_1\hat{D} & P_1\hat{D}-P_1 & -P_1+P_4 \\ * & \hat{\Sigma}_2 & -P_1\hat{D} & -M_1\hat{D}-M_2 \\ * & * & \hat{\Sigma}_3 & 0 \\ * & * & * & -M_1-M_1^T \end{bmatrix} - \hat{\Gamma}_2{\Theta_2}^{-1}\hat{\Gamma}_2^T < 0. \tag{3.22}$$

It follows from (3.20),(3.22) and assumption (A2) that

$$\begin{aligned} D^+V(t) \leq & -V(t) + \varepsilon_2 H^T E_1^T E_1 H + H^T Q_1 H \\ & + \varepsilon_3 H^T E_2^T E_2 H + H^T Q_2 H + U^T Q_3 U. \end{aligned} \tag{3.23}$$

By Lemma 2.5, we can get when $t \geq 0, V(x(t)) \geq W$,

$$V(x(t)) - W \leq (V(x(0)) - W)\, e^{-t}. \tag{3.24}$$

(3.24) shows that network (2.1) is globally exponentially attractive.

Finally, by a similar discussion as the proof of Theorem 3.1, we can get that network (2.1) is globally exponentially stable in Lagrange sense. Moreover,

$$\Omega_2 = \left\{ x(t) \in \mathbb{R}^n, \|x(t)\| \leq \sqrt{\frac{W}{\lambda_{\min}(P_1)}} e^{\delta d_{\max}} \right\} \cap \left\{ x(t) \in \mathbb{R}^n, \|x(t)\| \leq \sqrt{\frac{W}{\lambda_{\min}(P_4)}} \right\}$$

is a globally exponentially attractive set of network (2.1). This completes the proof. □

Remark 3.2. Obviously, Theorems 2.1 and 2.2 also work for traditional neural networks. Moreover, our results generalized some work of stability for neural networks without parameter uncertainties. The model in this paper is more general than the models in [6, 8, 9], so our results can extend these work.

Remark 3.3. The criteria in Theorems 2.1 and 2.2 are given in terms of LMI. They can be easily checked by Matlab LMI Toolbox. Actually, in this paper the criteria essentially ensure the Lagrange stability of each subsystem (neural network with a possible group of parameters D_i, A_i and $B_i, i = 1, 2, \cdots, 2^n$). For each LMI, if there exist feasible solutions, then the corresponding subsystem is globally exponentially stable in Lagrange sense. Meanwhile, we can get a globally exponentially attractive set for the subsystem. If all of the LMIs can be satisfied by different groups of feasible solutions, the memristive neural network is also globally exponentially stable in Lagrange sense. And its globally exponentially attractive set is the union of the globally exponentially attractive sets for the subsystems. In this way, the conditions can be relaxed.

4. Numerical simulations

In this section, two numerical examples are provided to demonstrate the effectiveness of the theorems.

We consider the following two-dimensional memristive neural networks,

$$\begin{aligned}\dot{x}(t) = &-(\hat{D}+\Delta D)x(t-\delta)+(\hat{A}+\Delta A)f(x(t))+(\hat{B}+\Delta B)f(x(t-\tau(t)))\\ &+U(t),\end{aligned} \tag{4.1}$$

where $\hat{D} = \operatorname{diag}\left(\hat{d}_1(x_1(t)), \hat{d}_2(x_2(t))\right), \hat{A} = (\hat{a}_{ij}(x_i(t)))_{2\times 2}, B = \left(\hat{b}_{ij}(x_i(t))\right)_{2\times 2}$.

Example 4.1. Set

$$\begin{aligned}
&\hat{d}_1(x_1(t)) = \begin{cases} 1.8,\ |x_1(t)| \le 1, \\ 1.6, |x_1(t)| > 1, \end{cases} \hat{d}_2(x_2(t)) = \begin{cases} 1.8, |x_2(t)| \le 1, \\ 2, |x_2(t)| > 1, \end{cases}\\
&\hat{a}_{11}(x_1(t)) = \begin{cases} 0.3, |x_1(t)| \le 1, \\ 0.5, |x_1(t)| > 1, \end{cases} \hat{a}_{12}(x_1(t)) = \begin{cases} 0.3, |x_1(t)| \le 1, \\ -1.2, |x_1(t)| > 1, \end{cases}\\
&\hat{a}_{21}(x_1(t)) = \begin{cases} -0.2, |x_2(t)| \le 1, \\ -0.3, |x_2(t)| > 1, \end{cases} \hat{a}_{22}(x_2(t)) = \begin{cases} 0.1, |x_2(t)| \le 1, \\ -0.1, |x_2(t)| > 1, \end{cases}\\
&\hat{b}_{11}(x_1(t)) = \begin{cases} 0.6, |x_1(t)| \le 1, \\ 0.3, |x_1(t)| > 1, \end{cases} \hat{b}_{12}(x_1(t)) = \begin{cases} -1,\ |x_1(t)| \le 1, \\ -0.8, |x_1(t)| > 1, \end{cases}\\
&\hat{b}_{21}(x_2(t)) = \begin{cases} -0.3, |x_2(t)| \le 1, \\ -0.1, |x_2(t)| > 1, \end{cases} \hat{b}_{22}(x_2(t)) = \begin{cases} 0.3, |x_2(t)| \le 1, \\ 0.2, |x_2(t)| > 1, \end{cases}
\end{aligned}$$

$\delta = 0.05, \tau_1(t) = 0.2+0.1\sin(t), \tau_2(t) = 0.2+0.1\cos(t)$. The parameter uncertainties are selected as

$$\begin{aligned}
&H1 = H2 = H3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, E_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}, E_2 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix},\\
&E_3 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}, T_1(t) = \begin{bmatrix} -\tanh\,(t) & 0 \\ 0 & -\tanh\,(t) \end{bmatrix}, T_2(t) = \begin{bmatrix} \cos(10t) & 0 \\ 0 & \cos(10t) \end{bmatrix},\\
&T_3(t) = \begin{bmatrix} \tanh\,(t) & 0 \\ 0 & \tanh\,(t) \end{bmatrix}.
\end{aligned}$$

In this example we choose the extern inputs as $U(t) = (0.2\cos(t), 0.5\sin(t))^T$, and the activation functions are $f_1(x) = f_2(x) = \tanh\,(x) - x$. Then we solve the LMIs in Theorem 3.1. The following feasible solutions can be obtained:

$\varepsilon_1 = 5.1768, \varepsilon_2 = 5.0711, \varepsilon_3 = 11.1292, \varepsilon_4 = 11.1275, \varepsilon_5 = 7.4632, \varepsilon_6 = 7.4606,$

$$P_1 = \begin{bmatrix} 0.1025 & -0.0528 \\ -0.0528 & 0.4342 \end{bmatrix}, P_2 = \begin{bmatrix} 0.9991 & -0.0585 \\ -0.0585 & 1.5315 \end{bmatrix}, P_3 = \begin{bmatrix} 12.8185 & 1.2202 \\ 1.2202 & 33.9103 \end{bmatrix},$$

$$P_4 = \begin{bmatrix} 5.1890 & 0.7918 \\ 0.7918 & 16.0388 \end{bmatrix}, Q_1 = \begin{bmatrix} 6.6130 & 1.5160 \\ 1.5160 & 12.2393 \end{bmatrix}, Q_2 = \begin{bmatrix} 1.1963 & 0.4767 \\ 0.4767 & 2.8155 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 45.1196 & 4.7090 \\ 4.7090 & 77.5548 \end{bmatrix}, Q_4 = \begin{bmatrix} 49.8750 & 9.8697 \\ 9.8697 & 64.3301 \end{bmatrix}, K_1 = \begin{bmatrix} 5.5458 & 0 \\ 0 & 11.9692 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 5.5458 & 0 \\ 0 & 11.9692 \end{bmatrix}, M_1 = \begin{bmatrix} 1.5604 & 0.5779 \\ 0.5779 & 5.8099 \end{bmatrix}, M_2 = \begin{bmatrix} 3.5388 & 1.0666 \\ 1.0666 & 9.5785 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 17.4469 & 2.2483 \\ 2.2483 & 50.5540 \end{bmatrix}, M_4 = \begin{bmatrix} 2.0798 & -0.0198 \\ -0.0198 & 4.7619 \end{bmatrix}.$$

Therefore, network (4.1) is globally exponentially stable in Lagrange sense according to Theorem 3.1. Moreover, $\Omega_1 = \left\{x \in \mathbb{R}^2 \,|\|x\| < 2.8672\right\}$ is a globally exponentially attractive set. By choosing 20 random initial values $(x_1(t) = r_1, x_2(t) = r_2, t \in [-\rho, 0], \rho = \max\{\delta, \tau\}$, r_1, r_2 are random constants), the state trajectories of network (4.1) are shown in Figure 1.

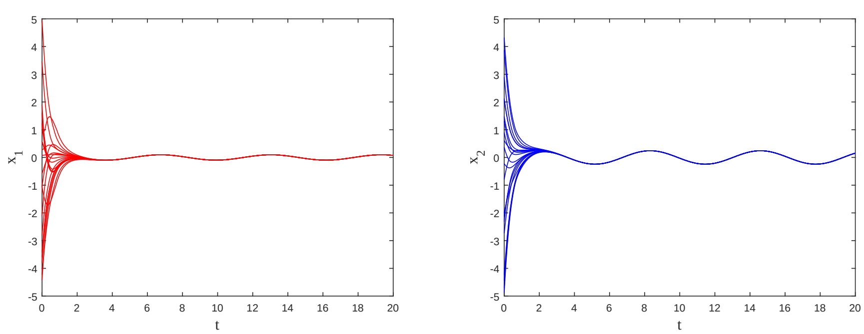

Figure 1. The state trajectories of network (4.1) in Example 4.1

To show that the leakage delay and parameter uncertainties can cause the neural networks instability and poor performance, we change the value of δ and E_1 in Example 4.1 and simulate the changed neural networks as comparisons.

Example 4.2. For neural network (4.1), select $\delta = 0.7$ and other parameters the same as they were in Example 4.1. Choosing 20 random initial values, the state trajectories of network (4.1) are shown in Figure 2.

Example 4.3. For neural network (4.1), select $E_1 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and other parameters

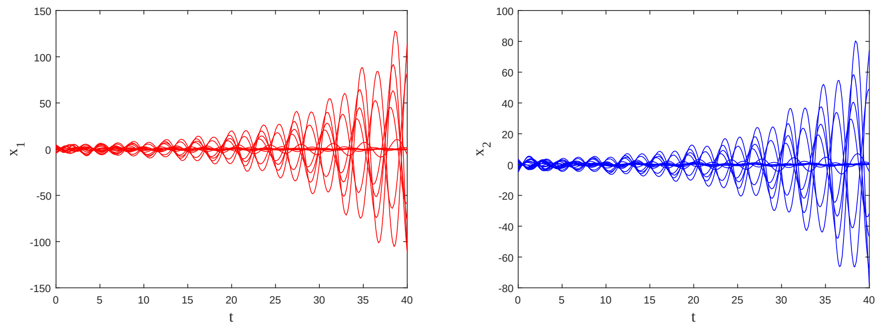

Figure 2. The state trajectories of network (4.1) in Example 4.2

the same as they were in Example 4.1. Choosing 20 random initial values, the state trajectories of network (4.1) are shown in Figure 3.

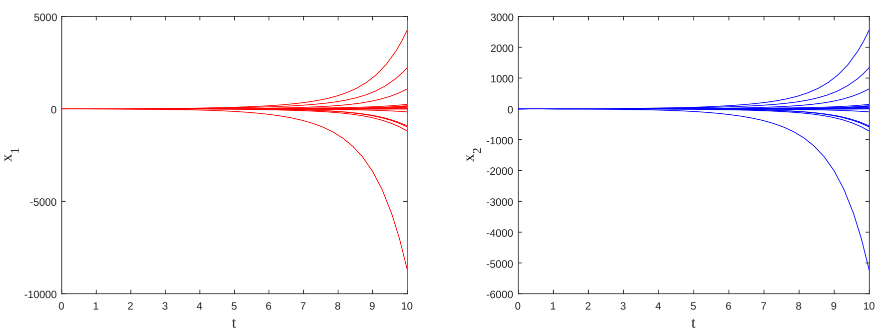

Figure 3. The state trajectories of network (4.1) in Example 4.3

Example 4.4. For neural network (4.1) we set $\delta = 0.1, \tau_1(t) = 1.5, \tau_2(t) = 1.3$. The extern inputs $U(t)$ are chosen as $(0,0)^T$. In this example we select a bounded nonmonotonic piecewise function as the activation function. The function takes the following form

$$f(t) = \begin{cases} 1 & t > 2, \\ -t+1 & 0.5 < t \leq 2, \\ t & 0 < t \leq 0.5, \\ t+1 & -0.5 < t \leq 0, \\ 1 & t \leq -0.5. \end{cases}$$

Other parameters are the same they were in Example 4.1. Solving the LMIs in the criterion of Theorem 3.2, we can get the following feasible solutions:

$$\varepsilon_1 = 3.4885, \varepsilon_2 = 3.4885, \varepsilon_3 = 3.4885,$$

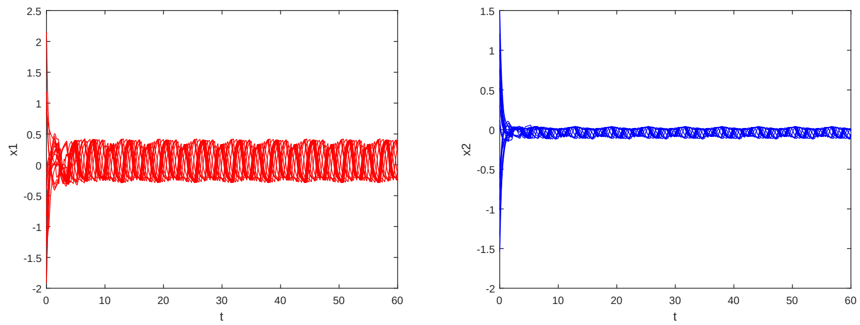

Figure 4. The state trajectories of network (4.1) in Example 4.4

$$P_1 = \begin{bmatrix} 1.1582 & 0.0599 \\ 0.0599 & 1.9003 \end{bmatrix}, P_2 = \begin{bmatrix} 0.7264 & 0.0109 \\ 0.0109 & 0.8853 \end{bmatrix}, P_3 = \begin{bmatrix} 0.6039 & 0.0643 \\ 0.0643 & 1.7358 \end{bmatrix},$$
$$P_4 = \begin{bmatrix} 0.1411 & 0.0144 \\ 0.0144 & 0.3232 \end{bmatrix}, Q_1 = \begin{bmatrix} 3.9186 & -0.6847 \\ -0.6847 & 5.2634 \end{bmatrix}, Q_2 = \begin{bmatrix} 3.6138 & -0.3716 \\ -0.3716 & 4.4454 \end{bmatrix},$$
$$Q_3 = \begin{bmatrix} 4.8484 & 0.0037 \\ 0.0037 & 4.7143 \end{bmatrix}, M_1 = \begin{bmatrix} 1.1326 & 0.0469 \\ 0.0469 & 1.3545 \end{bmatrix}, M_2 = \begin{bmatrix} 1.3115 & 0.0317 \\ 0.0317 & 1.5515 \end{bmatrix},$$

By Theorem 3.2, network (4.1) is globally exponentially stable in Lagrange sense, and $\Omega_2 = \left\{x \in \mathbb{R}^2 \,|\|x\| < 4.5789\right\}$ is a globally exponentially attractive set. Choosing 20 random initial values, the state trajectories of network (4.1) are shown in Figure 4.

The simulation results of Example 4.1 illustrated that Theorem 3.1 is feasible. As comparisons, the results of Examples 4.2 and 4.3 showed the necessity of taking leakage delay and parameter uncertainties into consideration in modeling. The simulation results of Example 4.4 demonstrated that Theorem 3.2 works. The trajectories showed that neural network (4.1) in Example 4.4 is not stable in Lyapunov sense. However, by Theorem 3.2, we know the network is globally exponentially stable in Lagrange sense, which means that no positive invariant sets outside the globally attractive set.

5. Summary

In this paper, a class of memristive neural networks with leakage delay and time-varying transmission delays as well as parameter uncertainties was investigated. We made qualitative analysis of a relevant differential inclusion under the framework of Filippov's solution to solve the discontinuity caused by using memristor in the neural networks. Then, by constructing suitable Lyapunov-Krasovskii functionals and using the free-weighting matrix method, sufficient conditions in terms of LMIs were given to ascertain the neural network with Lipschitz continuous activation functions and bounded activation functions to be stable in Lagrange sense. Finally, we made four numerical simulations, the simulations' results showed the necessity of taking leakage delay and parameter uncertainties into consideration in modeling and illustrated that our theoretical results are effective.

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