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Dynamics of a Diffusive SIR Epidemic Model with Time Delay*

Bounsanong Sounvoravong[1] and Shangjiang Guo[2,†]

Abstract This paper is devoted to a reaction-diffusion system for a SIR epidemic model with time delay and incidence rate. Firstly, the nonnegativity and boundedness of solutions determined by nonnegative initial values are obtained. Secondly, the existence and local stability of the disease-free equilibrium as well as the endemic equilibrium are investigated by analyzing the characteristic equations. Finally, the global asymptotical stability are obtained via Lyapunov functionals.

Keywords Diffusion, SIR epidemic model, time delay, basic reproduction number, stability, Lyapunov functional.

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1. Introduction

In this paper, we consider the following SIR epidemic model:

$$\begin{cases} S_t(x,t) - d_S\Delta S(x,t) = am - aS(x,t) - S(x,t)f(I(x,t-\tau)), & x \in \Omega, \\ I_t(x,t) - d_I\Delta I(x,t) = S(x,t)f(I(x,t-\tau)) - (a+c)I, & x \in \Omega, \\ R_t(x,t) - d_R\Delta R(x,t) = cI(x,t) - aR(x,t), & x \in \Omega, \\ \dfrac{\partial S}{\partial \mathbf{n}} = \dfrac{\partial I}{\partial \mathbf{n}} = \dfrac{\partial R}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

for $t \geq 0$, where d_S, d_I, d_R, a, c, m and τ are positive constants, the density functions $S(x,t)$, $I(x,t)$ and $R(x,t)$ represent the numbers of susceptible, infective and recovered individuals at position x and time t, respectively, and the parameters d_S, d_I, and d_R are their diffusion coefficients. The constant am is the recruitment rate of the susceptible population, a is a natural death rate for all the susceptible, infective and recovered population, c is the recovery rate of the infective individuals, and τ is the latent period of the disease. The constant m can be interpreted as a

†the corresponding author.
Email address: tear1284@hotmail.com(B. Sounvoravong), shangjguo@hnu.edu.cn(S. Guo)
[1]College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, China (Permanent address: Department of Mathematics, Faculty of Science, National University of Laos, Vientiane Capital, DongDok Campus, P.O. Box 7322, Laos)
[2]College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, China
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carrying capacity, or maximum possible population size. Δ denotes the Laplacian operator on $\mathbb{R}^N$, $\mathbf{n}$ is the outward unit normal vector on $\partial\Omega$. The homogeneous Neumann boundary condition means that the two species have zero flux across the boundary $\partial\Omega$. In practical use, there are various types of the incidence term $Sf(I)$. The common types include bilinear incidence (or mass action incidence) bSI (see, for example, [4,6,7,16,18,22,27]), standard incidence bSI/m (see, for example, [11]), and saturated incidence $bSI/(1+\alpha I)$ (see, for example, [5,11,12,17,19,20,24,26,28]), where b and α are positive constants. Throughout this paper, we always assume that the function $f(\cdot)$ is strictly monotone increasing, positive, and continuously differentiable on $[0,\infty)$ and satisfies the conditions

$$f(0)=0, \quad f(x)\le f'(0)x$$

for all $x>0$, and

$$\left[\frac{f(x)}{x}-\frac{f(y)}{y}\right][f(x)-f(y)]\le 0 \tag{1.2}$$

for all x, $y>0$. The initial conditions of system (1.1) are given as

$$\begin{cases} S(x,0)=S_0(x), \quad R(x,0)=R_0(x), \\ I(x,\theta)=I_0(x,\theta) \quad \text{for all} \quad \theta\in[-\tau,0]. \end{cases} \tag{1.3}$$

For a SIR epidemic model without diffusion (i.e., $d_S=d_I=d_R=0$), Wang [13] studied the existence, uniqueness and some estimates of a global solution, and also investigated the long time behavior of solutions to an initial-boundary value problem in a half space. Similarly, in this paper we can define a number R_0 (so-called the basic reproduction number) such that the disease-free equilibrium is stable when $R_0<1$. But for the case $R_0>1$ the endemic equilibrium is asymptotically stable. Kumar, Narayan and Reddy [14] studied the local asymptotical stability of the disease-free equilibrium and endemic equilibrium, and obtained the existence of the Hopf bifurcation at the positive equilibrium, Greenhalgh [8] studied the some SEIBS epidemiological models with vaccination and temporary immunity are considered. First of all, previously published work is reviewed. A general model with a constant contact rate and a density dependent death rate is examined. The model is reformulated in terms of the proportions of susceptible, incubating, infectious, and immune individuals. The equilibrium and stability properties of this model are examined, assuming that the average duration of immunity exceeds the infectious period. There is a threshold parameter R_0, and the disease can persist if and only if R_0, exceeds one. The disease-free equilibrium always exists and is locally stable if $R_0<1$ and unstable if $R_0>1$. Conditions are derived for the global stability of the disease-free equilibrium. For $R_0>1$, the endemic equilibrium is unique and locally asymptotically stable.

This paper is organized as follows. In section 2 we consider the nonnegativity and boundedness and show that all solutions of system (1.1) is nonnegative and bounded for all $t\ge 0$; Sections 3 is devoted to the local stability of equilibria of model (1.1). The global stability of the endemic equilibrium when $R_0<1$ and $R_0>1$ is proved in section 4, Numerical simulations are provided in section 5. In the paper, we denote by $\mathbb{N}$ (respectively, $\mathbb{R}_+$) the set of all the positive integers (respectively, nonnegative real numbers), and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. Denote by $H^k(\Omega,\mathbb{R}_+)$ ($k\ge 0$) the Sobolev space of the nonnegative L^2-functions $f(x)$ defined on Ω whose derivatives

$\frac{\mathrm{d}^n}{\mathrm{d}x^n} f$ $(n = 1, \ldots, k)$ also belong to $L^2(\Omega)$. Denote the spaces $\mathbb{X} = \{\psi \in H^2(\Omega, \mathbb{R}_+) : \partial\psi/\partial\mathbf{n} = 0 \text{ on } \partial\Omega\}$ and $\mathbb{Y} = L^2(\Omega, \mathbb{R}_+)$. Denote by $C^k_\tau = C^k([-\tau, 0], \mathbb{X})$ the Banach space of k-times continuously differentiable mappings from $[-\tau, 0]$ into $\mathbb{X}$ equipped with the supremum norm $\|\phi\| = \sup\{\|\phi^{(j)}(\theta)\|_{\mathbb{X}} : \theta \in [-\tau, 0],\ j = 0, 1, ..., k\}$ for $\phi \in C^k_\tau$.

2. Nonnegativity and boundedness

From biological meaning, it is necessary to show that all solutions of system (1.1) is nonnegative and bounded for all $t \geq 0$. Generally speaking, the local existence of solution of (1.1) and (1.3) is guaranteed, but the global existence of solution for (1.1) and (1.3) depends on the fact that the solution does not become infinite in a finite time. Since the growth functions are sufficiently smooth, the standard parabolic equation theory (see Ladyzenskaja, Solonnikov and Ural¡ceva [15]) implies that the solution of (1.1) and (1.3) is unique and continuous for all $t \geq 0$ in $\overline{\Omega}$. Furthermore, we have the following result.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$ with $1 \leq N \leq 3$ be a bounded domain with smooth boundary $\partial\Omega$. For each initial value $(S_0, I_0, R_0) \in \mathbb{X} \times C^0_\tau \times \mathbb{X}$ satisfying S_0, I_0, $R_0 \geq 0$ (but not identically equal to 0), system (1.1) has a unique global classical solution $(S(x,t), I(x,t), R(x,t))$ satisfying that $S(x,t) \geq 0$, $I(x,t) \geq 0$, and $R(x,t) \geq 0$ for $t \geq 0$ and $x \in \overline{\Omega}$, and that $\limsup_{t\to+\infty} S(x,t) \leq m$ for all $x \in \Omega$.*

Proof. We first give the existence of local solutions of (1.1), which can be readily proved by the Amann¡s theorem [1,2]. Namely, there exists $T_{\max} > 0$ such that the problem (1.1) has a unique classical solution $(S, I, R) \in C(\overline{\Omega} \times [0, T_{\max}), \mathbb{R}^3_+)$ satisfying $S(x,t) \geq 0$, $I(x,t) \geq 0$, and $R(x,t) \geq 0$ for all $(x,t) \in \overline{\Omega} \times [0, T_{\max})$. Moreover, if $T_{\max} < \infty$ then $\|S(\cdot,t)\|_{L^\infty} + \|I(\cdot,t)\|_{L^\infty} + \|R(\cdot,t)\|_{L^\infty} \to \infty$. Using the facts that S, I, R, and $f(\cdot)$ are non-negative, then we have

$$S_t(x,t) - d_S\Delta S(x,t) = am - aS(x,t) - S(x,t)f(I(x,t-\tau)) \leq am - aS(x,t)$$

for all $(x,t) \in \Omega \times (0, T_{\max})$, and $\frac{\partial S}{\partial \mathbf{n}} = 0$ for all $x \in \partial\Omega$, and $S(x,0) = S_0(x)$ for all $x \in \Omega$. Let $S^*(t)$ be the solution of the following ordinary differential equation problem

$$\begin{cases} \dfrac{\mathrm{d}S^*(t)}{\mathrm{d}t} = am - aS^*, \quad t > 0, \\ S^*(0) = \|S_0\|_{L^\infty}. \end{cases} \tag{2.1}$$

It is easy to see that $S^*(t) \leq S_{\max} \triangleq \max\{m, \|S_0\|_{L^\infty}\}$ and that $S^*(t)$ is a super-solution of the following partial differential equation problem

$$\begin{cases} \widetilde{S}_t(x,t) - d_S\Delta\widetilde{S}(x,t) = am - a\widetilde{S}(x,t), \quad x \in \Omega,\ t > 0, \\ \dfrac{\partial\widetilde{S}}{\partial\mathbf{n}} = 0, \quad x \in \partial\Omega,\ t > 0, \\ \widetilde{S}(x,0) = S_0(x), \quad x \in \Omega, \end{cases}$$

and hence it holds that

$$0 < \widetilde{S}(x,t) \leq S^*(t) \quad \text{for all} \quad (x,t) \in \overline{\Omega} \times (0, \infty),$$

where $\widetilde{S} > 0$ results from the strong maximum principle. Therefore, using the comparison principle, one has

$$0 < S(x,t) \le \widetilde{S}(x,t) \le S^*(t) \le S_{\max} \tag{2.2}$$

for all $(x,t) \in \overline{\Omega} \times (0, T_{\max})$. Let $M(x,t) = S(x,t) + I(x,t) + R(x,t)$, then we have

$$M_t(x,t) = \Delta[d_S S(x,t) + d_I I(x,t) + d_R R(x,t)] - aN(x,t) + am.$$

Integrating the above equation over Ω, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega M(x,t)\mathrm{d}x + a\int_\Omega M(x,t)\mathrm{d}x = am|\Omega|.$$

and hence

$$\int_\Omega M(x,t)\mathrm{d}x \le \max\left\{m|\Omega|, \int_\Omega M(x,0)\mathrm{d}x\right\} \tag{2.3}$$

for all $t \in (0, T_{\max})$. Using the variation-of-constants formula, we get

$$I(x,t) \le e^{(d_I\Delta - a - c)t} I(\cdot, 0) + S_{\max}\int_0^t e^{(d_I\Delta - a - c)(t-s)} f(I(\cdot, s-\tau))ds,$$

which implies $\|I(\cdot,t)\|_{L^\infty} \le I_1(t) + I_2(t)$ with

$$\begin{aligned} I_1(t) =& \|e^{(d_I\Delta - a - c)t} I(\cdot, 0)\|_{L^\infty}, \\ I_2(t) =& S_{\max} f'(0)\int_0^t \|e^{(d_I\Delta - a - c)(t-s)} I(\cdot, s-\tau)\|_{L^\infty} ds. \end{aligned}$$

Using the same argument in [3, Lemma 3.2] has shown that there is a constant $c_1 > 0$ such that $I_1(t) \le c_1$ for all $t \in (0, T_{\max})$. Letting $\widetilde{I}(t) = \frac{1}{|\Omega|}\int_\Omega I(x,t)\mathrm{d}x$, then it follows from (2.3) that there is a constant $c_2 > 0$ such that $\widetilde{I}(t) \le c_2$ for all t. Multiplying by $2I(x,t)$ the second equation of (1.1) and then integrating the resulted equation over Ω and using the fact that $0 < S(x,t) \le S_{\max}$ and $f(x) \le f'(0)x$, we have

$$\begin{aligned} &\frac{d}{dt}\|I(\cdot,t)\|_{L^2}^2 + 2d_I\|\nabla I(\cdot,t)\|_{L^2}^2 \\ \le& 2S_{\max} f'(0)\int_\Omega I(x,t)I(x,t-\tau)\mathrm{d}x - 2(a+c)\|I(\cdot,t)\|_{L^2}^2. \end{aligned} \tag{2.4}$$

It follows from the Young's inequality that

$$\int_\Omega I(x,t)I(x,t-\tau)\mathrm{d}x \le \frac{S_{\max} f'(0)}{2(a+c)}\|I(\cdot,t)\|_{L^2}^2 + \frac{a+c}{2S_{\max} f'(0)}\|I(\cdot,t-\tau)\|_{L^2}^2. \tag{2.5}$$

The Gagliardo-Nirenberg inequality and Young's inequality and (2.3) can give us that and

$$\frac{S_{\max}^2 f'^2(0)}{a+c}\|I(\cdot,t)\|_{L^2}^2 \le c_1\left[\|\nabla I(\cdot,t)\|_{L^2}\|I(\cdot,t)\|_{L^1} + \|I(\cdot,t)\|_{L^1}^2\right] \le d_I\|\nabla I(\cdot,t)\|_{L^2}^2 + c_2,$$

which, together with (2.4) and (2.5), implies that

$$\frac{d}{dt}\|I(\cdot,t)\|_{L^2}^2 + 2(a+c)\|I(\cdot,t)\|_{L^2}^2 \le c_2 + (a+c)\|I(\cdot,t-\tau)\|_{L^2}^2.$$

This together with the Gronwall's inequality implies that $\|I(\cdot,t)\|_{L^2}< c_3$ for all t. Using the smoothing properties of $\{e^{t\Delta} : t \geq 0\}$ ([23, Lemma 1.3]) again and noting $t - t_0 \leq 1$, we obtain

$$\begin{aligned}
I_2(t) \leq & S_{\max} f'(0) \int_0^t e^{-(a+c)(t-s)} \left\| e^{d_I(t-s)\Delta} \left[I(\cdot, s-\tau) - \widetilde{I}(s-\tau) \right] \right\|_{L^\infty} ds \\
& + S_{\max} f'(0) \int_0^t e^{-(a+c)(t-s)} \left\| e^{d_I(t-s)\Delta} \widetilde{I}(s-\tau) \right\|_{L^\infty} ds \\
\leq & c_4 \int_0^t e^{-(a+c)(t-s)} \left[1 + (t-s)^{-N/4} \right] \left\| I(\cdot, s-\tau) - \widetilde{I}(s-\tau) \right\|_{L^2} ds \\
& + c_2 S_{\max} f'(0) \int_0^t e^{-(a+c)(t-s)} ds \\
\leq & c_5 \int_0^t e^{-(a+c)(t-s)} (t-s)^{-N/4} ds + c_5 \int_0^t e^{-(a+c)(t-s)} ds \\
= & c_5 \int_0^t e^{-(a+c)s} s^{-N/2} \mathrm{d}s + c_5 \int_0^t e^{-(a+c)s} \mathrm{d}s,
\end{aligned}$$

and hence

$$\begin{aligned}
\|I(\cdot,t)\|_{L^\infty} \leq & I_1(t) + I_2(t) \\
\leq & c_1 + c_5 \int_0^t e^{-(a+c)s} s^{-N/2} \mathrm{d}s + c_5 \int_0^t e^{-(a+c)s} \mathrm{d}s \leq c_6
\end{aligned} \tag{2.6}$$

for all $t \in (0, T_{\max})$, where we have used (2.3) and the fact that $\int_0^t e^{-(a+c)s} \mathrm{d}s \leq \frac{1}{a+c}$ and

$$\begin{aligned}
\int_0^t e^{-(a+c)s} s^{-N/4} \mathrm{d}s = & \int_1^t e^{-(a+c)s} s^{-N/4} \mathrm{d}s + \int_0^1 e^{-(a+c)s} s^{-N/2} \mathrm{d}s \\
\leq & \int_1^t e^{-(a+c)s} \mathrm{d}s + \int_0^1 s^{-N/4} \mathrm{d}s \\
\leq & \frac{1}{a+c} + \frac{4}{4-N}.
\end{aligned}$$

Using a similar argument as above, we can show that there is a positive constant c_7 independently of t such that $\|R(\cdot,t)\|_{L^\infty} \leq c_7$ for all $t \in (0, T_{\max})$. This, together with (2.2) and (2.6) and the local existence of the solution $(S(x,t), I(x,t), R(x,t))$ of (1.1) with initial value $(S_0, I_0, R_0) \in \mathbb{X} \times C_\tau^0 \times \mathbb{X}$, implies that system (1.1) has a unique global classical solution $(S, I, R) \in C(\overline{\Omega} \times [0, \infty), \mathbb{R}^3)$. We further have from (2.1) that $\limsup_{t\to\infty} S^*(t) = m$, which along with the above inequalities gives

$$\limsup_{t\to\infty} S(x,t) \leq m \quad \text{for all} \quad x \in \overline{\Omega}.$$

The proof is completed. □

In what follows, we analyze the existence and stability of the disease-free equilibrium and endemic equilibria of model (1.1). Note that the first two equations of model (1.1) about (S, I) don't contain R, and the third equation is a linear equation of R. Therefore, the dynamical behaviors of model (1.1) is equivalent to those of the following model

$$\begin{cases} S_t(x,t) - d_S\Delta S(x,t) = am - aS(x,t) - S(x,t)f(I(x,t-\tau)), & x \in \Omega, \\ I_t(x,t) - d_I\Delta I(x,t) = S(x,t)f(I(x,t-\tau)) - (a+c)I(x,t), & x \in \Omega, \\ \dfrac{\partial S}{\partial \mathbf{n}} = \dfrac{\partial I}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, \end{cases} \tag{2.7}$$

which only contains S and I. Obviously, solutions of system (2.7) with positive initial values are positive and bounded, and system (2.7) has an equilibrium at $(m,0)$ in the S-axis for all permissible parameters. What we are interested in is the equilibria in the interior of the first quadrant, so we need to seek conditions ensuring such equilibria exist. In fact, if (S,I) is an interior equilibrium of system (2.7), then the positive numbers S and I satisfy

$$S = \frac{am - (a+c)I}{a}, \quad a(a+c) = F(I), \tag{2.8}$$

where the continuous function $F(\cdot)$ is given by

$$F(x) = \frac{[am - (a+c)x]f(x)}{x}.$$

It follows from (1.2) that $F(x)$ is monotone decreasing on $(0,\infty)$ and so $F(x) \leq amf'(0)$. Hence, the second equation of (2.8) is solvable for $x > 0$ when $R_0 \triangleq \frac{mf'(0)}{a+c} > 1$. Then the following results are obtained immediately.

Lemma 2.2. **(i)** *If the basic reproductive rate $R_0 < 1$, then model (2.7) has exactly one disease-free equilibrium $E_0(m,0)$;*

(ii) *If the basic reproductive rate $R_0 > 1$, then model (2.7) has two equilibria: a disease-free equilibrium $E_0(m,0)$ and an endemic equilibrium $E^*(S^*,I^*)$.*

3. Local stability

In this section, we discuss the local stability of the positive equilibrium of the following model:

$$\begin{cases} S_t - d_S S_{xx} = am - aS - Sf(I(x,t-\tau)), & x \in (0,l\pi), \ t > 0, \\ I_t - d_I I_{xx} = Sf(I(x,t-\tau)) - (a+c)I, & x \in (0,l\pi), \ t > 0, \\ S_x = I_x = 0, & x = 0, l\pi, \ t \geq 0. \end{cases} \tag{3.1}$$

We first consider the local stability of the constant steady-state of (3.1). Notice that $\mu = \sigma_n \triangleq (n/l)^2$, $n \in \mathbb{N}_0$ are the eigenvalues of the linear eigenvalue problem $U'' + \mu U = 0$ subject to the homogeneous Neumann boundary condition $U'(0) = U'(l\pi) = 0$. Let φ_n be the eigenfunction associated with the eigenvalue σ_n, then we have

$$\varphi_0(x) = \frac{1}{\sqrt{l\pi}}, \quad \varphi_n(x) = \sqrt{\frac{2}{l\pi}} \cos\frac{nx}{l}$$

for $n \in \mathbb{N}$. We have the following results on the local stability of steady-state solutions of (3.1)

Theorem 3.1. **(i)** *The steady-state solution* $(m,0)$ *is locally asymptotically stable if* $R_0 < 1$, *and is unstable if* $R_0 > 1$;

(ii) *If* $R_0 > 1$ *then the steady state solution* (S^*, I^*) *of (3.1) is locally asymptotically stable for all* $\tau \geq 0$.

Proof. The linearization of (3.1) at the steady-state solution $E = (S, I)$ takes the form

$$\begin{cases} u_t - d_S u_{xx} = -[a + f(I)]u - Sf'(I)\upsilon(x, t-\tau), \\ \upsilon_t - d_I \upsilon_{xx} = Sf'(I)\upsilon(x, t-\tau) + f(I)u - (a+c)\upsilon. \end{cases} \tag{3.2}$$

If system (3.2) has a solution of the form $u(x,t) = P(x)e^{\lambda t}$, then we have

$$\begin{bmatrix} \lambda - \frac{\partial^2}{\partial x^2} + a + f(I) & Sf'(I)e^{-\lambda\tau} \\ -f(I) & \lambda - \frac{\partial^2}{\partial x^2} + (a+c) - Sf'(I)e^{-\lambda\tau} \end{bmatrix} P = 0. \tag{3.3}$$

Let $P = \sum_{n=0}^{\infty}(c_n, d_n)^T \varphi_n$, then we have

$$\sum_{n=0}^{\infty} \begin{bmatrix} \lambda - \sigma_n + a + f(I) & Sf'(I)e^{-\lambda\tau} \\ -f(I) & \lambda - \sigma_n + (a+c) - Sf'(I)e^{-\lambda\tau} \end{bmatrix} \begin{bmatrix} c_n \\ d_n \end{bmatrix} = 0,$$

and hence that $(c_n, d_n) \neq 0$ for some $n \in \mathbb{N} \cup \{0\}$ if and only if

$$\mathcal{P}_n^{S,I}(\lambda) \triangleq \det \begin{bmatrix} \lambda + \sigma_n + a + f(I) & Sf'(I)e^{-\lambda\tau} \\ -f(I) & \lambda + \sigma_n + (a+c) - Sf'(I)e^{-\lambda\tau} \end{bmatrix} = 0. \tag{3.4}$$

Thus, the steady-state solution $E = (S, I)$ is locally asymptotically stable if all zeros of $\mathcal{P}_n^{S,I}(\cdot)$ have negative real parts for all $n \in \mathbb{N}_0$ and is unstable if there exists some $n \in \mathbb{N}_0$ such that $\mathcal{P}_n^{S,I}(\cdot)$ has at least one zero with positive real parts (see also [9, 10, 21]).

We first consider the boundary steady-state solutions $E_0(m, 0)$. Notice that

$$\mathcal{P}_n^{m,0}(\lambda) = (\lambda + \sigma_n + a)[\lambda + \sigma_n + (a+c) - mf'(0)e^{-\lambda\tau}],$$

one of whose zeros is $\lambda = -(a + \sigma_n) < 0$ for $n \in \mathbb{N}_0$, the others satisfy

$$\lambda + a + c + \sigma_n = mf'(0)e^{-\lambda\tau}. \tag{3.5}$$

Substituting $\lambda = \mathrm{i}\omega$ with $\omega > 0$ into (3.5) yields

$$a + c + \sigma_n = mf'(0)\cos\tau\omega, \quad -\omega = mf'(0)\sin\tau\omega,$$

from which it follows that $\omega^2 = b^2m^2 - (a + c + \sigma_n)^2$. If $R_0 < 1$ then $\omega^2 = [mf'(0)]^2 - (a + c + \sigma_n)^2 < 0$ and hence (3.5) has no purely imaginary solutions. This implies that all the solutions to (3.5) have negative real parts for all $\tau \geq 0$ and $n \in \mathbb{N} \cup \{0\}$ and hence that the disease-free equilibrium E_0 of model (3.1) is locally asymptotically stable for all $\tau \geq 0$.

If $R_0 > 1$ then $\omega = \omega_n \triangleq \sqrt{[mf'(0)]^2 - (a + c + \sigma_n)^2}$ for some $n \in \mathbb{N}_0$ (for example, $n = 0$) and we can obtain the following sequence of critical values of τ:

$$\tau_{k,n} = \frac{2k\pi - \arcsin \frac{\omega_n}{mf'(0)}}{\omega_n}, \quad k \in \mathbb{N}.$$

By mean of the implicit function theorem, for each $k \in \mathbb{N}$ there exists $\delta > 0$ and a smooth mapping λ: $(\tau_{k,n} - \delta, \tau_{k,n} + \delta) \to \mathbb{C}$ such that $\lambda(\tau_{k,n}) = \mathrm{i}\omega_n$ and that for each $\tau \in (\tau_{k,n} - \delta, \tau_{k,n} + \delta)$, $\lambda = \lambda(\tau)$ is a solution to (3.5). Differentiating (3.5) with respect to τ yields

$$\lambda'(\tau) + mf'(0)e^{-\lambda(\tau)\tau}\left[\tau\lambda'(\tau) + \lambda(\tau)\right] = 0,$$

and hence

$$\lambda'(\tau) = -\frac{mf'(0)\lambda(\tau)e^{-\lambda(\tau)\tau}}{1 + \tau mf'(0)e^{-\lambda(\tau)\tau}} = -\frac{\lambda(\tau)\left[\lambda(\tau) + a + c + \sigma_n\right]}{1 + \tau[\lambda + a + c + \sigma_n]}.$$

In particular, we have

$$\begin{aligned}\lambda'(\tau_{k,n}) =& -\frac{\mathrm{i}\omega_n\left(\mathrm{i}\omega_n + a + c + \sigma_n\right)}{1 + \tau_{k,n}(\mathrm{i}\omega_n + a + c + \sigma_n)}\\ =& -\frac{\mathrm{i}\omega_n\left[\mathrm{i}\omega_n + a + c + \sigma_n\right]\left[1 + \tau_{k,n}(-\mathrm{i}\omega_n + a + c + \sigma_n)\right]}{\left|1 + \tau_{k,n}(\mathrm{i}\omega_n + a + c + \sigma_n)\right|^2},\end{aligned}$$

and hence

$$\begin{aligned}\mathrm{sgnRe}\lambda'(\tau_{k,n}) =& -\mathrm{sgnRe}\left\{\mathrm{i}\left[\mathrm{i}\omega_n + a + c + \sigma_n\right]\left[1 + \tau_{k,n}(-\mathrm{i}\omega_n + a + c + \sigma_n)\right]\right\}\\ =& -\mathrm{sgnRe}\left\{\mathrm{i}\left[\mathrm{i}\omega_n + a + c + \sigma_n + \tau_{k,n}\left|\mathrm{i}\omega_n + a + c + \sigma_n\right|^2\right]\right\}\\ =&\omega_n > 0.\end{aligned}$$

This implies that as τ increases and passes through each critical value $\tau_{k,n}$, two more zero-points of $\mathcal{P}_n^{m,0}(\cdot)$ vary from a pair of conjugate complex numbers with negative real parts to a pair of conjugate purely imaginary numbers and then to a pair of conjugate complex numbers with positive real parts. This, together with the fact that $\mathcal{P}_n^{m,0}(\cdot)$ has exactly one positive real zero-point $mf'(0) - a - c$, implies that $\mathcal{P}_n^{m,0}(\cdot)$ has at least one zero-point with a positive real part for all $\tau \geq 0$, and hence that the disease-free equilibrium E_0 of model (3.1) is locally asymptotically stable for all $\tau \geq 0$.

Now, we consider the stability of steady-state solution $E^*(S^*, I^*)$. In this case, we have

$$\mathcal{P}_n^{S^*,I^*}(\lambda) = \lambda^2 + B_n\lambda + C_n - S^*f'(I^*)(\lambda + E_n)e^{-\lambda\tau}. \tag{3.6}$$

with

$$\begin{aligned}B_n =&2\sigma_n + 2a + c + f(I^*),\\ C_n =&[\sigma_n + a + f(I^*)](\sigma_n + a + c),\\ E_n =&\sigma_n + a.\end{aligned}$$

We need to seek the necessary and sufficient condition ensuring that every zero of $\mathcal{P}_n^{S^*,I^*}(\cdot)$ has negative real parts. We start with the case where $\tau = 0$. In this case, we have

$$\begin{aligned}\mathcal{P}_n^{S^*,I^*}(\lambda) =&\lambda^2 + [2\sigma_n + 2a + c + f(I^*) - S^*f'(I^*)]\lambda\\ &+ (\sigma_n + a + f(I^*))(\sigma_n + a + c) - (\sigma_n + a)S^*f'(I^*).\end{aligned} \tag{3.7}$$

It is easy to see that all zeros of $\mathcal{P}_n^{S^*,I^*}(\lambda)$ given in (3.7) have negative real parts for all $n \in \mathbb{N}_0$, which implies that $E^*(S^*, I^*)$ is locally asymptotically stable when $R_0 > 1$ and $\tau = 0$.

In what follows, we consider the case where $\tau > 0$. Assume that $\mathcal{P}_n^{S^*,I^*}(\cdot)$ has a zero $\mathrm{i}\omega$, then we have

$$-\omega^2 + B_n \mathrm{i}\omega + C_n - S^* f'(I^*)(\mathrm{i}\omega + E_n)(\cos\omega\tau - \mathrm{i}\sin\omega\tau) = 0. \tag{3.8}$$

Separating the real and imaginary parts, we get

$$\begin{aligned} C_n - \omega^2 =& S^* f'(I^*)\left(\omega\sin\omega\tau + E_n\cos\omega\tau\right), \\ B_n\omega =& S^* f'(I^*)\left(-E_n\sin\omega\tau + \omega\cos\omega\tau\right), \end{aligned} \tag{3.9}$$

and hence

$$\omega^4 + [B_n^2 - 2C_n - (S^* f'(I^*))^2]\omega^2 + C_n^2 - (S^* f'(I^*))^2 E_n^2 = 0. \tag{3.10}$$

Note that

$$a + c = \frac{S^* f(I^*)}{I^*} \geq S^* f'(I^*),$$

then we have

$$\begin{aligned} B_n^2 - 2C_n - (S^* f'(I^*))^2 =& [\sigma_n + a + f(I^*)]^2 + (\sigma_n + a + c)^2 - (S^* f'(I^*))^2 \\ >& (a+c)^2 - m^2 f'^2(I^*) > 0 \end{aligned}$$

and

$$\begin{aligned} C_n - S^* f'(I^*) E_n =& [\sigma_n + a + f(I^*)](\sigma_n + a + c) - (\sigma_n + a) S^* f'(I^*) \\ \geq& (\sigma_n + a)(\sigma_n + a + c) - (\sigma_n + a) S^* f'(I^*) \\ =& (\sigma_n + a)[\sigma_n + a + c - S^* f'(I^*] > 0. \end{aligned}$$

This implies that there is no positive solutions to (3.10). Note that (S^*, I^*) is stable when $\tau = 0$, then it remains stable for all $\tau \geq 0$. Thus, the proof is completed. $\square$

4. Global asymptotical stability

In the previous section, we see that the disease-free equilibrium $E_0(m, 0)$ (respectively, the endemic equilibrium $E^*(S^*, I^*)$) of model (3.1) is locally asymptotically stable for all $\tau \geq 0$ when $R_0 < 1$ (respectively, $R_0 > 1$). In this section, we shall study the global asymptotical stability. For this purpose, we introduce the following two functions:

$$F(u) = \frac{f(I^* u)}{f(I^*)}, \quad g(u) = u - 1 - \ln u$$

for $u > 0$. It is easy to see that function $g(u)$ is strictly decreasing on $(0, 1)$, is strictly increasing on $(1, +\infty)$, and has a global minimum 0 at $u = 1$.

Lemma 4.1 ([25]). *$g(F(u)) \leq g(u)$ for $\omega > 0$.*

Theorem 4.1. *The disease-free equilibrium $E_0(m, 0)$ is globally asymptotically stable for all $\tau \geq 0$ when $R_0 < 1$.*

Proof. Since E_0 is locally asymptotically stable for all $\tau \geq 0$ when $R_0 < 1$, then in order to show it's global asymptotical stability, we just need to consider it's global attractivity. Define Lyapunov functional as follows,

$$V(S(x,t), I(x,t)) = \int_\Omega \left[mg\left(\frac{S(x,t)}{m}\right) + I(x,t) + m\int_{t-\tau}^{t} f(I(x,s))\mathrm{d}s \right] \mathrm{d}x.$$

Obviously, $V(m,0)=0$, and V is positive definite with respect to $(S,I)\in\mathbb{R}_+^2$ and has the property $V(S,I)\to+\infty$ as $\|(S,I)\|\to+\infty$. The derivative of V along the solutions of system (3.1) is

$$\begin{aligned}
&V_t(S(x,t),I(x,t))\\
=&\int_\Omega\left(1-\frac{m}{S(x,t)}\right)(am-aS(x,t)-S(x,t)f(I(x,t-\tau)))\,\mathrm{d}x\\
&+\int_\Omega[S(x,t)f(I(x,t-\tau))-(a+c)I(x,t)]\,\mathrm{d}x\\
&+m\int_\Omega f(I(x,t))\mathrm{d}x-m\int_\Omega f(I(x,t-\tau))\mathrm{d}x\\
&+\int_\Omega\left(1-\frac{m}{S(x,t)}\right)d_S\Delta S(x,t)\mathrm{d}x+\int_\Omega d_I\Delta I(x,t)\mathrm{d}x\\
=&-\int_\Omega\left(1-\frac{m}{S(x,t)}\right)^2aS(x,t)\mathrm{d}x-\int_\Omega S(x,t)f(I(x,t-\tau))\mathrm{d}x\\
&+\int_\Omega mf(I(x,t-\tau))\mathrm{d}x+\int_\Omega S(x,t)f(I(x,t-\tau))\mathrm{d}x\\
&-\int_\Omega(a+c)I(x,t)\mathrm{d}x+\int_\Omega mf(I(x,t))\mathrm{d}x-\int_\Omega mf(I(x,t-\tau))\mathrm{d}x\\
&-\int_\Omega\frac{m}{S^2(x,t)}d_S|\nabla S(x,t)|^2\mathrm{d}x\\
\le&\int_\Omega mf(I(x,t))\mathrm{d}x-\int_\Omega(a+c)I(x,t)\mathrm{d}x\\
\le&\int_\Omega[mf'(0)-(a+c)]\,I(x,t)\mathrm{d}x\\
=&\frac{1}{a+c}\int_\Omega(R_0-1)\,I(x,t)\mathrm{d}x.
\end{aligned}$$

This implies that $V_t(S(x,t),I(x,t))\le 0$ along an orbit $(S(x,t),I(x,t))$ of system (3.1) with any non-negative initial value when $R_0<1$. This, together with Theorem 4.1, implies that E_0 is globally asymptotically stable if $R_0<1$.

Theorem 4.2. *The endemic equilibrium $E^*(S^*,I^*)$ of model (3.1) is globally asymptotically stable for all $\tau\ge 0$ when $R_0>1$.*

Proof. In this case we consider the global attractivity of E^*. Define Lyapunov functional as follows,

$$\begin{aligned}
V(S(x,t),I(x,t))=\int_\Omega\Bigg[&S^*g\left(\frac{S(x,t)}{S^*}\right)+I^*g\left(\frac{I(x,t)}{I^*}\right)\\
&+S^*f(I^*)\int_{t-\tau}^t g\left(\frac{I(x,s)}{I^*}\right)\mathrm{d}s\Bigg]\,\mathrm{d}x,
\end{aligned}$$

where function $g(\cdot)$ is defined as that in the previous discussion. Obviously, $V(S^*,I^*)=0$, and V is positive definite with respect to $(S,I)\in\mathbb{R}_+^2$ and has the property $V(S,I)\to+\infty$ as $\|(S,I)\|\to+\infty$. The derivative of V along the solutions of system (3.1) is

$$V_t(S(x,t),I(x,t))$$

$$
\begin{aligned}
&= \int_\Omega \left[\left(1 - \frac{S^*}{S(x,t)}\right)(am - aS(x,t) - S(x,t)f(I(x,t-\tau)))\right] \mathrm{d}x \\
&\quad + \int_\Omega \left[\left(1 - \frac{I^*}{I(x,t)}\right)(S(x,t)f(I(x,t-\tau)) - (a+c)I(x,t))\right] \mathrm{d}x \\
&\quad + S^* f(I^*) \int_\Omega \left[g\left(\frac{I(x,t)}{I^*}\right) - g\left(\frac{I(x,t-\tau)}{I^*}\right)\right] \mathrm{d}x \\
&\quad + \int_\Omega \left(1 - \frac{S^*}{S}\right) d_S \Delta S(x,t)\mathrm{d}x + \int_\Omega \left(1 - \frac{I^*}{I}\right) d_I \Delta I(x,t) \mathrm{d}x \\
&= \int_\Omega \left[\left(1 - \frac{S^*}{S(x,t)}\right)(aS^* + S^* f(I^*) - aS(x,t) - S(x,t)f(I(x,t-\tau)))\right] \mathrm{d}x \\
&\quad + \int_\Omega \left[\left(1 - \frac{I^*}{I(x,t)}\right)\left(S(x,t)f(I(x,t-\tau)) - \frac{S^* f(I^*)}{I^*} I(x,t)\right)\right] \mathrm{d}x \\
&\quad + S^* f(I^*) \int_\Omega \left[g\left(\frac{I(x,t)}{I^*}\right) - g\left(\frac{I(x,t-\tau)}{I^*}\right)\right] \mathrm{d}x \\
&\quad - \int_\Omega \frac{S^*}{S^2} d_S |\nabla S(x,t)|^2 \mathrm{d}x - \int_\Omega \frac{I^*}{I^2} d_I |\nabla I(x,t)|^2 \mathrm{d}x \\
&\le \int_\Omega \left(1 - \frac{S^*}{S(x,t)}\right)(aS^* - aS(x,t))\, \mathrm{d}x \\
&\quad + \int_\Omega \left(1 - \frac{S^*}{S(x,t)}\right)(S^* f(I^*) - S(x,t)f(I(x,t-\tau)))\, \mathrm{d}x \\
&\quad + S^* f(I^*) \int_\Omega \left(1 - \frac{I^*}{I(x,t)}\right)\left(\frac{S(x,t)f(I(x,t-\tau))}{S^* f(I^*)} - \frac{I(x,t)}{I^*}\right) \mathrm{d}x \\
&\quad + S^* f(I^*) \int_\Omega \left[g\left(\frac{I(x,t)}{I^*}\right) - g\left(\frac{I(x,t-\tau)}{I^*}\right)\right] \mathrm{d}x \\
&\le - a \int_\Omega \left(1 - \frac{S^*}{S(x,t)}\right)^2 S(x,t) \mathrm{d}x \\
&\quad + S^* f(I^*) \int_\Omega \left[1 - \frac{S(x,t)f(I(x,t-\tau))}{S^* f(I^*)} - \frac{S^*}{S(x,t)} + \frac{f(I(x,t-\tau))}{f(I^*)}\right] \mathrm{d}x \\
&\quad + S^* f(I^*) \int_\Omega \left[\frac{S(x,t)f(I(x,t-\tau))}{S^* f(I^*)} - \frac{I(x,t)}{I^*} - \frac{S(x,t)I^* f(I(x,t-\tau))}{S^* f(I^*) I(x,t)} + 1\right] \mathrm{d}x \\
&\quad + S^* f(I^*) \int_\Omega \left[\frac{I(x,t)}{I^*} - \frac{I(x,t-\tau)}{I^*} - \ln \frac{I(x,t)}{I^*} + \ln \frac{I(x,t-\tau)}{I^*}\right] \mathrm{d}x \\
&\le - a \int_\Omega \left(1 - \frac{S^*}{S(x,t)}\right)^2 S(x,t) \mathrm{d}x \\
&\quad + S^* f(I^*) \int_\Omega \left[1 - \left(\frac{S^*}{S(x,t)}\right) + \left(\frac{f(I(x,t-\tau))}{f(I^*)}\right) + 1 - \left(\frac{S(x,t)I^* f(I(x,t-\tau))}{S^* I(x,t) f(I^*)}\right)\right. \\
&\quad \left. + 1 - \left(\frac{I(x,t-\tau)}{I^*}\right) - \ln \frac{I(x,t)}{I^*} + \ln \frac{I(x,t-\tau)}{I^*}\right] \mathrm{d}x \\
&= - a \int_\Omega \left(1 - \frac{S^*}{S(x,t)}\right)^2 S(x,t) \mathrm{d}x \\
&\quad + S^* f(I^*) \int_\Omega \left[1 - g\left(\frac{S^*}{S(x,t)}\right) - 1 - \ln \frac{S^*}{S(x,t)} + g\left(\frac{f(I(x,t-\tau))}{f(I^*)}\right) + 1 \right.
\end{aligned}
$$

$$
\begin{aligned}
&+\ln\frac{f(I(x,t-\tau))}{f(I^*)}-g\left(\frac{S(x,t)I^*f(I(x,t-\tau))}{S^*I(x,t)f(I^*)}\right)+1-\ln\left(\frac{S(x,t)I^*f(I(x,t-\tau))}{S^*I(x,t)f(I^*)}\right)\\
&+1-g\left(\frac{I(x,t-\tau)}{I^*}\right)-1-\ln\frac{I(x,t-\tau)}{I^*}-\ln\frac{I(x,t)}{I^*}+\ln\frac{I(x,t-\tau)}{I^*}\Bigg]\mathrm{d}x\\
=&-a\int_\Omega\left(1-\frac{S^*}{S(x,t)}\right)^2S(x,t)\mathrm{d}x\\
&+S^*f(I^*)\int_\Omega\left[-g\left(\frac{S^*}{S(x,t)}\right)+g\left(\frac{f(I(x,t-\tau))}{f(I^*)}\right)-g\left(\frac{S(x,t)I^*f(I(x,t-\tau))}{S^*I(x,t)f(I^*)}\right)\right.\\
&\left.-g\left(\frac{I(x,t-\tau)}{I^*}\right)\right]\mathrm{d}x\\
&\le S^*f(I^*)\int_\Omega\left[g\left(\frac{f(I(x,t-\tau))}{f(I^*)}\right)-g\left(\frac{I(x,t-\tau)}{I^*}\right)\right]\mathrm{d}x\\
=&S^*f(I^*)\int_\Omega\left[g(f(\omega(x,t)))-g(\omega(x,t))\right]\mathrm{d}x
\end{aligned}
$$

where $\omega(x,t)=I(x,t-\tau)/I^*>0$. Using the fact in Lemma 4.1 , that

$$V_t(S(x,t),I(x,t))\le 0$$

along an orbit $(S(x,t),I(x,t))$ of system (3.1) with any non-negative initial value when $R_0>1$. This, together with Theorem 4.2, implies that E^* is globally asymptotically stable if $R_0>1$.

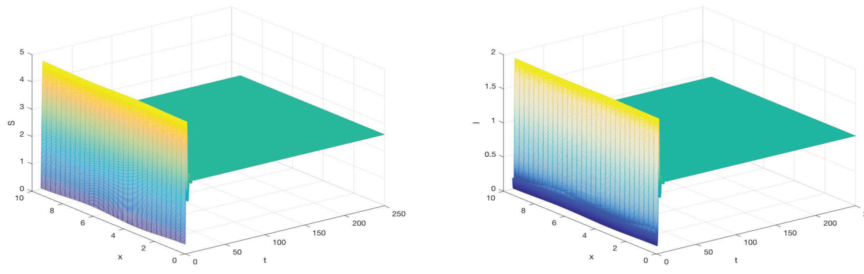
5. Conclusion and numerical simulations

In this paper, we have investigated the dynamic behavior of a delayed SIR epidemic model with diffusion. The global dynamical behaviour of the model is studied and the threshold value R_0 of the system is defined to determine the behaviours of the system. More precisely, the disease free equilibrium $(m,0)$ is asymptotically stable if $R_0<1$ and unstable if $R_0>1$. But, the endemic equilibrium (S^*,I^*) is asymptotically stable if $R_0>1$ for all $\tau\ge 0$. In what follows, we present some numerical simulations to support and supplement the our analytic results.

We first consider system (2.7) the case where $f(I)=I$, $a=0.01$, $c=2.6$, $\Omega=(0,3\pi)$, and initial values $S(x,0)=0.3+0.05\cos(x)$, $I(x,0)=0.3+0.05\cos(x)$. Namely, we consider the following system

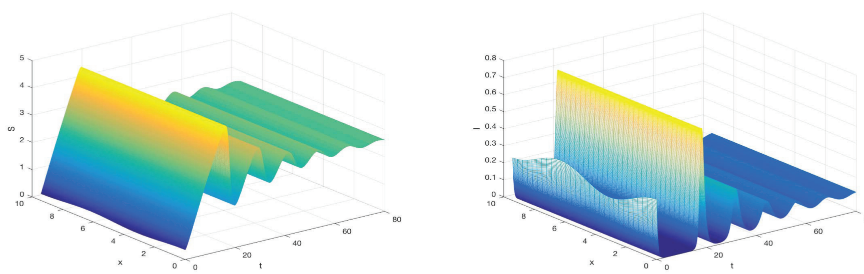
$$
\begin{cases}
S_t(x,t)-d_S\Delta S(x,t)=0.01m-0.01S(x,t)-S(x,t)I(x,t-\tau), & x\in(0,3\pi),\\
I_t(x,t)-d_I\Delta I(x,t)=S(x,t)I(x,t-\tau)-2.61I(x,t), & x\in(0,3\pi),\\
S_x(0,t)=S_x(3\pi,t)=I_x(0,t)=I_x(3\pi,t)=0.
\end{cases}
\tag{5.1}
$$

Choose $m=2.7$, then we have $R_0=1.03>1$ and hence it follows from Theorem 4.2 that the endemic equilibrium $E^*(S^*,I^*)$ of model (5.1) is asymptotically stable (see Figure 1). Choose $m=0.3$, then we have $R_0=0.11<1$ and hence the disease-free equilibrium $E_0(m,0)$ of model (5.1) is stable (see Figure 2)



(a) (b)

Figure 1. Numerical simulations of system (5.1) with $m = 2.7$ showing that the endemic equilibrium is asymptotically stable.



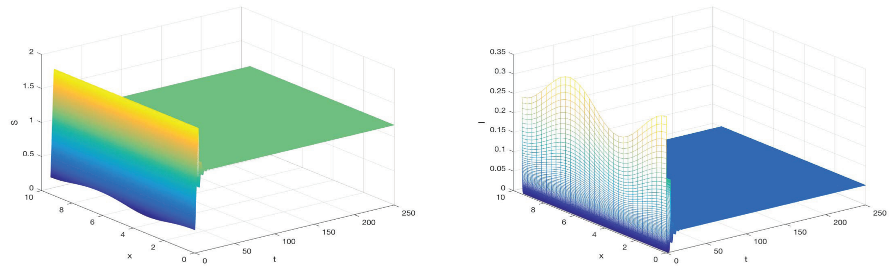
(a) (b)

Figure 2. Numerical simulations of system (5.1) with $m = 0.3$ showing that the disease-free equilibrium is asymptotically stable.

And then we consider system (2.7) the case where $f(I) = \frac{6.8I}{1+I}$, $a = 0.01$, $c = 7.6$, $\Omega = (0, 3\pi)$, and initial values $S(x,t) = 0.3 + 0.05\cos(x)$, $I(x,t) = 0.3 + 0.05\cos(x)$. Namely, consider the following system

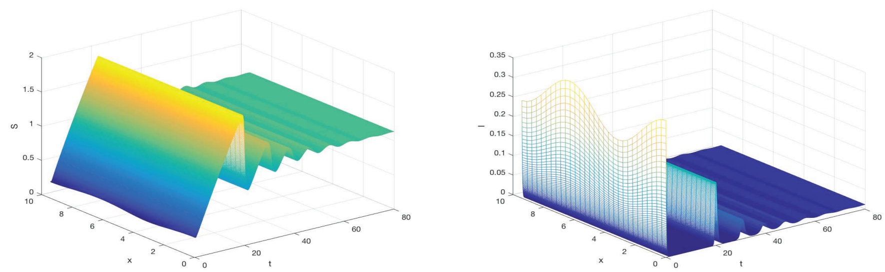
$$\begin{cases} S_t(x,t) - d_S\Delta S(x,t) = 0.01m - 0.01S(x,t) - \dfrac{6.8S(x,t)I(x,t-\tau)}{1+I(x,t-\tau)}, \quad x \in (0,3\pi), \\ I_t(x,t) - d_I\Delta I(x,t) = \dfrac{6.8S(x,t)I(x,t-\tau)}{1+I(x,t-\tau)} - 7.61I(x,t), \quad x \in (0,3\pi), \\ S_x(0,t) = S_x(3\pi,t) = I_x(0,t) = I_x(3\pi,t) = 0. \end{cases} \tag{5.2}$$

Choose $m = 1.13$, then we have $R_0 = 1.009 > 1$ and hence it follows from Theorem 4.2 that the endemic equilibrium $E^*(S^*, I^*)$ of model (5.2) is asymptotically stable (see Figure 3). Choose $m = 0.1$, then we have $R_0 = 0.08 < 1$ and hence the disease-free equilibrium $E_0(m, 0)$ of model (5.2) is stable (see Figure 4)



(a) (b)

Figure 3. Numerical simulations of system (5.2) with $m = 1.13$ showing that the endemic equilibrium is asymptotically stable..



(a) (b)

Figure 4. Numerical simulations of system (5.2) with $m = 0.1$ showing that the disease-free equilibrium is asymptotically stable.

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