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Derivative Estimates for the Solution of Hyperbolic Affine Sphere Equation

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Abstract: Considering the hyperbolic affine sphere equation in a smooth strictly convex bounded domain with zero boundary values, the sharp derivative estimates of any order for its convex solution are obtained.

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1 Introduction

In affine differential geometry, the classification of complete hyperbolic affine hyperspheres has attracted the attention of many geometers. By a Legendre transformation, the classification of Euclidean-complete hyperbolic hyperspheres is reduced to the study of the following boundary value problem

$$\begin{cases} \det\left(\dfrac{\partial^2 u(x)}{\partial x_i \partial x_j}\right) = (-u(x))^{-n-2} & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded convex domain. Calabi[1] conjectured that there is a unique convex solution to (1.1). Loewner and Nirenberg[2] solved (1.1) in the cases of domains in $\mathbf{R}^2$ with smooth boundary. Cheng and Yau[3] showed there always exists a convex solution $u \in C^\infty(\Omega) \cap C^0(\bar{\Omega})$, and the uniqueness follows from the maximum principal.

When $\Omega = B^n(1)$, the unit ball in $\mathbf{R}^n$, the convex solution of (1.1) is

$$u_0 = -\sqrt{1 - \sum_{1 \le k \le n} x_k^2}. \tag{1.2}$$

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When Ω is projectively homogeneous, Sasaki[4] found that the convex solution of (1.1) and the characteristic function χ of domain Ω have the following relation:

$$u = C_0\chi^{-\frac{1}{n+1}} \qquad \text{for a constant } C_0.$$

Also, Sasaki and Yagi[5] obtained an expansion of derivatives of the characteristic function χ along the boundary of the smooth convex bounded domain. Referring the Fefferman's expansion of the Bergman kernel on smooth strictly pseudoconvex domains (see [6]), Sasaki[7] obtained an asymptotic expansion form of χ with respect to the solution u:

$$\chi = C_0 u^{-(n+1)}\Big[1 + \frac{5}{24(n-1)}Fu^2 + \text{the higher orders of } u\Big], \tag{1.3}$$

where F is a smooth function on $\bar{\Omega}$.

In this paper, we confine ourselves to the case that Ω is a strictly convex bounded domain with smooth boundary. By the barrier functions on the balls, the convex solution of (1.1) has the bound:

$$\frac{1}{C}d(x)^{\frac{1}{2}} \le -u(x) \le Cd(x)^{\frac{1}{2}}, \tag{1.4}$$

where $d(x) =: \operatorname{dist}(x,\, \partial\Omega)$, and C is a positive constant depending on Ω and n.

By (1.4) and the convexity of u, the gradient estimate is given by:

$$\frac{1}{C}d(x)^{-\frac{1}{2}} \le |\operatorname{grad} u| \le Cd(x)^{-\frac{1}{2}}. \tag{1.5}$$

Loewner and Nirenberg[2] first obtained the sharp second order estimates in dimension two. Their methods and Pogorelov's calculations also gave bound for the higher dimensions (see [8]):

$$|u_{ij}| \le Cd(x)^{-\frac{3}{2}}, \qquad 1 \le i,\, j \le n. \tag{1.6}$$

Now we introduce the basic notations. For a multi-index $\alpha = (\alpha_1,\, \alpha_2,\, \cdots,\, \alpha_n)$, where α_i, $i = 1, 2, \cdots, n$, are non-negative integers with $|\alpha| = \sum\limits_{1\le i\le n} \alpha_i$, we define

$$D_i = \frac{\partial}{\partial x_i}, \quad D_i^{\alpha_i} = \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}, \qquad i = 1, 2, \cdots, n,$$

$$D^{\alpha} = D_1^{\alpha_1}\cdots D_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}.$$

In this paper, by the finite geometry of complete hyperbolic affine sphere as stated in Lemma 2.1, we obtain derivative estimates of any order:

Theorem 1.1 *For $n = 2$, the convex solution of* (1.1) *satisfies*

$$|D^{\boldsymbol{\alpha}}(u)| \le Cd(x)^{\frac{1}{2}-|\alpha|}, \qquad |\alpha| = 0, 1, 2, \cdots, \tag{1.7}$$

where C is a constant depending on Ω and $|\alpha|$.

Remark 1.1 For $|\alpha| = 3$, the estimate (1.7) holds for any dimension $n \ge 2$. The sharpness of exponent "$\frac{1}{2} - |\alpha|$" can be seen in the case that Ω is projectively homogeneous (see [5]).

Remark 1.2 As in [7], the function $v = -\dfrac{1}{2}u^2$ satisfies a real analogue of Fefferman equation

$$\begin{cases} \det \begin{pmatrix} v_{ij} & v_i \\ v_j & 2v \end{pmatrix} = -1 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where v_i, v_{ij} are the usual first and second derivatives. For the boundary behaviors of derivatives of the solution u, it is necessary to study the smoothness of v on the closure of Ω, and to derive a complete description of the boundary singularity.

2 Formulas for Hyperbolic Affine Hyperspheres

Let M be a locally strictly convex affine hypersurface in $\mathbf{R}^{n+1}$, given by a convex function f defined in a domain $D \subset \mathbf{R}^n$:

$$M = \{(y_1, \cdots, y_n, y_{n+1}) \mid y_{n+1} = f(y_1, \cdots, y_n),\ y = (y_1, \cdots, y_n) \in D\}.$$

The Blaschke metric is given by (see [9])

$$G = \sum_{1 \le i,j \le n} \rho \boldsymbol{f}_{ij} \mathrm{d}y_i \mathrm{d}y_j, \tag{2.1}$$

where $\boldsymbol{f}_{ij}$ $(1 \le i, j \le n)$ are the second derivatives of f with respect to y, $(\boldsymbol{f}^{ij})$ is the inverse of matrix $(\boldsymbol{f}_{ij})$, and

$$\rho = (\det(\boldsymbol{f}_{ij}))^{-\frac{1}{n+2}}.$$

The Fubini-Pick form is given by (see [10])

$$A_{ijk} = -\frac{1}{2}\Big(f_{kj}\frac{\partial\rho}{\partial y_i} + f_{ik}\frac{\partial\rho}{\partial y_j} + f_{ij}\frac{\partial\rho}{\partial y_k} + \rho\frac{\partial f_{ij}}{\partial y_k}\Big). \tag{2.2}$$

Consider the Legendre transformation relative to f

$$\begin{cases} x_i = \dfrac{\partial f}{\partial y_i}(y_1, \cdots, y_n), \\ u(x_1, x_2, \cdots, x_n) = \displaystyle\sum_{1 \le i \le n} y_i \frac{\partial f}{\partial y_i}(y_1, \cdots, y_n) - f(y_1, \cdots, y_n). \end{cases}$$

The Legendre transformation domain Ω of f is defined by

$$\Omega = \Big\{x = (x_1, x_2, \cdots, x_n) \mid x_i = \frac{\partial f}{\partial y_i},\ (y_1, y_2, \cdots, y_n) \in D\Big\}.$$

In the terms of coordinates $(x_1, x_2, \cdots, x_n)$, the Blaschke metric G is given by

$$G = \sum_{1 \le i,j \le n} \frac{1}{\tilde{u}} \boldsymbol{u}_{ij} \mathrm{d}x_i \mathrm{d}x_j.$$

Here and later we denote by u_i, u_{ij}, u_{ijk}, $\cdots$ the derivatives of u with respect to x, $(\boldsymbol{u}^{ij})$ the inverse of matrix $(\boldsymbol{u}_{ij})$, and

$$\tilde{u} = (\det(\boldsymbol{u}_{ij}))^{\frac{-1}{n+2}}, \qquad \tilde{u}_{ij} = \frac{\partial^2 \tilde{u}}{\partial x_i \partial x_j}.$$

By a direct calculation, the Fubini-Pick form can be represented in the following form:

$$\begin{aligned} A &= \sum_{1\le i,j,k\le n} A_{ijk}\mathrm{d}y_i\mathrm{d}y_j\mathrm{d}y_k \\ &= \sum_{1\le i,j,k\le n} \frac{1}{2\tilde{u}^2}\left(u_{ij}\frac{\partial \tilde{u}}{\partial x_k} + u_{ik}\frac{\partial \tilde{u}}{\partial x_j} + u_{jk}\frac{\partial \tilde{u}}{\partial x_i} + \tilde{u}u_{ijk}\right)\mathrm{d}x_i\mathrm{d}x_j\mathrm{d}x_k. \end{aligned} \tag{2.3}$$

Suppose that $M = \{(y,\, f(y))\}$ is a hyperbolic affine hypersphere with center at the origin. Then the Legendre function u of f satisfies (see [11])

$$\det(u_{ij}) = (-u)^{-n-2}. \tag{2.4}$$

It follows from (2.4) that the Blaschke metric and the Fubini-Pick form are given respectively:

$$G = \sum_{1\le i,j\le n} -\frac{1}{u}u_{ij}\mathrm{d}x_i\mathrm{d}x_j, \tag{2.5}$$

$$A = \sum_{1\le i,j,k\le n} -\frac{1}{2u^2}(u_{ij}u_k + u_{ik}u_j + u_{jk}u_i + uu_{ijk})\mathrm{d}x_i\mathrm{d}x_j\mathrm{d}x_k. \tag{2.6}$$

By using (2.4), the Laplacian with respect to the metric G is given by

$$\Delta = -u\sum_{1\le i,j\le n} u^{ij}\frac{\partial^2}{\partial x_i\partial x_j} - 2\sum_{1\le i,j\le n} u^{ij}u_i\frac{\partial}{\partial x_j}. \tag{2.7}$$

There exist two notions of completeness on affine hypersurfaces in $\mathbf{R}^{n+1}$: (1) Euclidean completeness, that is the completeness of the Riemannian metric induced from a Euclidean metric on $\mathbf{R}^{n+1}$; (2) Affine completeness, that is the completeness of the Blaschke metric G. But for hyperbolic affine hyperspheres, these two completeness are equivalent (see [11]). Now we state a corollary of Theorem 2 of [9].

Lemma 2.1[9] *Let M be a Euclidean-complete hyperbolic affine sphere in $\mathbf{R}^3$. Then M has finite geometry*:

$$\|A\|_G + \|\nabla A\|_G + \cdots + \|\nabla^k A\|_G \le C, \qquad k = 0,1,2,\cdots, \tag{2.8}$$

where C is a constant depending on k and ∇ is the covariant differentiation with respect to the Blaschke metric G.

We remark here every Euclidean-complete hyperbolic affine hypersphere in $\mathbf{R}^{n+1}$ has bounded Pick invariant $\|A\|_G$ (see [11]). Next, we give a Lemma due to Yau[12].

Lemma 2.2[12] *Let $(M,\, g)$ be a complete Riemannian manifold with Ricci curvature bounded from below. If a smooth positive function ϕ on M satisfies*

$$\Delta\phi = \lambda\phi, \tag{2.9}$$

where λ is a constant and Δ is the Laplacian with respect to g, then there exists a constant C such that

$$\frac{\|\nabla\phi\|_g}{\phi} \le C. \tag{2.10}$$

3 The Third Order Derivative Estimates

In this section, we give the third order derivative estimates for any dimension. Let u be the convex solution of boundary value problem (1.1) in a smooth strictly convex bounded domain Ω. Then the Blaschke metric

$$G=\sum_{1\le i,j\le n}G_{ij}\mathrm{d}x_i\mathrm{d}x_j=\sum_{1\le i,j\le n}-\frac{1}{u}u_{ij}\mathrm{d}x_i\mathrm{d}x_j \tag{3.1}$$

is a complete Riemannian metric, and the Pick invariant

$$\|A\|_G^2=\sum_{1\le i,j,k,r,s,t\le n}G^{ir}G^{js}G^{kt}A_{ijk}A_{rst}$$

is bounded. For any point $x\in\Omega$, we assume $u_{ij}(x)=\lambda_i\delta_j^i$. It follows from (2.6) that

$$\begin{aligned}\|A\|_G^2&=\sum_{1\le i,j,k\le n}\frac{-1}{4u\lambda_i\lambda_j\lambda_k}(3\lambda_i^2\delta_i^j u_k^2+u^2u_{ijk}^2+6\lambda_i^2\delta_j^i\delta_i^k u_k u_j+6uu_k\lambda_i\delta_i^j u_{ijk})\\&=\frac{-1}{4u}\left(\sum_{1\le i\le n}9\frac{u_i^2}{\lambda_i}+\sum_{1\le i,j,k\le n}u^2\frac{u_{ijk}^2}{\lambda_i\lambda_j\lambda_k}+\sum_{1\le i,k\le n}6u\frac{u_k u_{iik}}{\lambda_i\lambda_k}\right)\\&\le C.\end{aligned} \tag{3.2}$$

Here and later we use the same C for different constants. Differentiating equation (1.1) with respect to x_k, one has

$$\sum_{1\le i,j\le n}u^{ij}u_{ijk}=\sum_{1\le i\le n}\frac{u_{iik}}{\lambda_i}=-(n+2)\frac{u_k}{u}. \tag{3.3}$$

Inserting (3.3) into (3.2), we get

$$\sum_{1\le i,j,k\le n}\frac{u_{ijk}^2}{\lambda_i\lambda_j\lambda_k}\le -4C\frac{1}{u}+(6n+3)\frac{1}{u^2}\sum_{1\le k\le n}\frac{u_k^2}{\lambda_k}. \tag{3.4}$$

Combining (1.1) and (2.7), we have

$$\Delta(-u^{-1})=n(-u^{-1}). \tag{3.5}$$

Recall that the Ricci curvature of hyperbolic affine hypersphere is bounded from below (see [11]), by (3.5) and Lemma 2.2, we get

$$\frac{\|\nabla(-u^{-1})\|_G}{-u^{-1}}=\frac{\|\ \nabla u\ \|_G}{-u}\le C. \tag{3.6}$$

It follows that

$$\frac{\|\nabla u\|_G^2}{u^2}=-\frac{1}{u}\sum_{1\le i,j\le n}u^{ij}u_iu_j=-\frac{1}{u}\sum_{1\le k\le n}\frac{u_k^2}{\lambda_k}\le C. \tag{3.7}$$

By using (1.4), (3.4) and (3.7), we have

$$\frac{|u_{ijk}|}{\sqrt{\lambda_i\lambda_j\lambda_k}}\le Cd(x)^{-\frac{1}{4}}. \tag{3.8}$$

Applying (1.6), we have proved

$$|u_{ijk}|\le Cd(x)^{-\frac{5}{2}},\qquad 1\le i,j,k\le n. \tag{3.9}$$

Formula (3.7) gives a lower bound of the maximal eigenvalue of Hessian (u_{ij}). In fact,

$$\frac{|\mathrm{grad}\,u|^2}{-u}\frac{1}{\lambda_{\max}(u_{ij})}\le -\frac{1}{u}\sum_{1\le i,j\le n}u^{ij}u_iu_j\le C. \tag{3.10}$$

It follows from (1.4) and (1.5) that

$$\lambda_{\max}(u_{ij})\ge Cd(x)^{-\frac{3}{2}}. \tag{3.11}$$

Hence, by (1.6), we get

Corollary 3.1 *The convex solution of* (1.1) *satisfies*

$$\frac{1}{C}d(x)^{-\frac{3}{2}} \leq \lambda_{\max}(u_{ij})(x) \leq Cd(x)^{-\frac{3}{2}}, \tag{3.12}$$

where C is a constant depending on Ω and n.

4 The Higher Order Derivative Estimates

In this section, we show (1.7) under the condition (2.8). Hence our theorem follows from Lemma 2.1.

Let u be the convex solution of (1.1) in a smooth strictly convex bounded domain Ω, the Blaschke metric is given by (3.1). Then, by (2.6), the Christoffel symbols of G are given by

$$\begin{aligned}\Gamma_{ij}^t &= \frac{1}{2}\sum_{1\leq s\leq n} G^{ts}\left(\frac{\partial G_{sj}}{\partial x_i} + \frac{\partial G_{si}}{\partial x_j} - \frac{\partial G_{ij}}{\partial x_s}\right)\\ &= \frac{1}{2u}\sum_{1\leq s\leq n} u^{ts}(uu_{sij} + u_s u_{ij}) - \frac{u_i}{2u}\delta_j^t - \frac{u_j}{2u}\delta_i^t\\ &= -u\sum_{1\leq s\leq n} u^{ts}A_{sij} - \frac{u_i}{u}\delta_j^t - \frac{u_j}{u}\delta_i^t.\end{aligned} \tag{4.1}$$

We write

$$\nabla^k A = \sum_{1\leq i_1, i_2, \ldots, i_{k+3}\leq n} A_{i_1\cdots i_{k+3}} \mathrm{d}x_{i_1}\cdots \mathrm{d}x_{i_{k+3}}, \qquad k = 0, 1, 2, \cdots,$$

and assume that $u_{ij}(x) = \lambda_i \delta_j^i$.

To obtain (1.7), it suffices to prove the following estimates:

$$\frac{|D^\mu(u)|}{\sqrt{\lambda_1^{\mu_1}\cdots\lambda_n^{\mu_n}}} \leq Cd(x)^{\frac{2-|\mu|}{4}}, \qquad |\mu| = 3, 4, 5, \cdots, \tag{4.2}$$

where $\mu = (\mu_1, \cdots, \mu_n)$ and C is a constant depending on Ω, n and $|\mu|$.

We proceed by introduction on $|\mu|$. For $|\mu| = 3$, (4.2) is obtained in Section 3. Suppose that these estimates hold for $|\mu| \leq m-1$. To prove (4.2) for $|\mu| = m$, we first prove

Lemma 4.1 *For multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $1 \leq |\alpha| \leq m-3$,*

$$\frac{|D^\alpha(A_{i_1\cdots i_k})|}{\sqrt{\lambda_1^{\alpha_1}\cdots\lambda_n^{\alpha_n}\cdot\lambda_{i_1}\cdots\lambda_{i_k}}} \leq Cd(x)^{-\frac{k+|\alpha|}{4}}, \qquad k = 3, 4, \cdots, \tag{4.3}$$

where C is a constant depending on Ω, n, k and m.

Proof. We proceed by introduction on $|\alpha|$. For $\alpha = (1, 0, \cdots, 0)$, it is obvious that

$$D_1(A_{i_1\cdots i_k}) = A_{i_1\cdots i_k 1} + \sum_{1\leq p\leq n} A_{i_1\cdots i_s p i_{s+1}\cdots i_k}\cdot \Gamma_{i_s 1}^p. \tag{4.4}$$

Noting that for any integral $l > 0$, $\| \nabla^l A \|_G$ is bounded, we have

$$\|\nabla^l A\|_G^2 = (-u)^{l+3}\sum_{1\leq i_1, i_2, \ldots, i_{l+3}\leq n} \frac{(A_{i_1\cdots i_{l+3}})^2}{\lambda_{i_1}\cdots\lambda_{i_{l+3}}} \leq C. \tag{4.5}$$

Hence, by the estimate (1.4), we get
$$\frac{|A_{i_1\cdots i_k 1}|}{\sqrt{\lambda_1\lambda_{i_1}\cdots\lambda_{i_k}}}\le C\cdot d(x)^{-\frac{k+1}{4}}. \tag{4.6}$$
(4.1) gives
$$\begin{aligned}\sum_{1\le p\le n} A_{i_1\cdots i_s p i_{s+1}\cdots i_k}\mathit{\Gamma}^p_{i_s 1} = &-u\sum_{1\le p,t\le n} u^{pt}\cdot A_{ti_s 1}\cdot A_{i_1\cdots i_s p i_{s+1}\cdots i_k}\\ &-\frac{1}{u}\cdot u_{i_s}\cdot A_{i_1\cdots i_{s-1} 1 i_{s+1}\cdots i_k}\\ &-\frac{1}{u}\cdot u_1\cdot A_{i_1\cdots i_{s-1} i_s i_{s+1}\cdots i_k}.\end{aligned} \tag{4.7}$$
By (1.4) and (4.5) we have
$$\begin{aligned}&-u\frac{\left|\sum\limits_{1\le p,t\le n} u^{pt}A_{ti_s 1}A_{i_1\cdots i_s p i_{s+1}\cdots i_k}\right|}{\sqrt{\lambda_1\lambda_{i_1}\cdots\lambda_{i_k}}}\\ &\le -u\sum_{1\le t\le n}\frac{|A_{ti_s 1}|}{\sqrt{\lambda_t\lambda_{i_s}\lambda_1}}\cdot\frac{|A_{i_1\cdots i_{s-1} t i_{s+1}\cdots i_k}|}{\sqrt{\lambda_t\lambda_{i_1}\cdots\lambda_{i_{s-1}}\lambda_{i_{s+1}}\cdots\lambda_{i_k}}}\\ &\le C\cdot d(x)^{\frac{1}{2}}\cdot d(x)^{-\frac{3}{4}}\cdot d(x)^{-\frac{k}{4}}\\ &= C\cdot d(x)^{-\frac{k+1}{4}}.\end{aligned} \tag{4.8}$$
From (1.4), (3.7) and (4.5) we also have
$$-\frac{1}{u}\cdot\left|\frac{u_{i_s}}{\sqrt{\lambda_{i_s}}}\cdot\frac{A_{i_1\cdots i_{s-1}1 i_{s+1}\cdots i_k}}{\sqrt{\lambda_1\lambda_{i_1}\cdots\lambda_{i_{s-1}}\lambda_{i_{s+1}}\cdots\lambda_{i_k}}}\right|\le C\cdot d(x)^{-\frac{k+1}{4}}. \tag{4.9}$$
Combining the above estimates we have
$$\frac{|D_1(A_{i_1\cdots i_k})|}{\sqrt{\lambda_1\lambda_{i_1}\cdots\lambda_{i_k}}}\le C\cdot d(x)^{-\frac{k+1}{4}}. \tag{4.10}$$
This proves (4.3) for $|\alpha|=1$.

Now suppose that the estimate (4.3) holds for multi-index α with $|\alpha|\le t$. We need to prove that for $|\alpha|=t+1$ (4.3) holds. Without loss of generality, we assume that $D^\alpha=D^\beta D_1$, where $\beta=(\beta_1,\,\beta_2,\,\cdots,\,\beta_n)=(\alpha_1-1,\,\alpha_2,\,\cdots,\,\alpha_n)$. Then
$$D^\alpha A_{i_1\cdots i_k}=D^\beta D_1 A_{i_1\cdots i_k}=D^\beta\Big(A_{i_1\cdots i_k 1}+\sum_{1\le p\le n}A_{i_1\cdots i_{s-1}p i_{s+1}\cdots i_k}\mathit{\Gamma}^p_{i_s 1}\Big). \tag{4.11}$$
By using the Leibniz formula we have
$$D^\beta\Big(\sum_{1\le p\le n}A_{i_1\cdots i_{s-1}p i_{s+1}\cdots i_t}\mathit{\Gamma}^p_{i_s 1}\Big)=\sum_{1\le p\le n}\sum_{\gamma\le\beta}\binom{\beta}{\gamma}D^{\beta-\gamma}(A_{i_1\cdots i_{s-1}p i_{s+1}\cdots i_k})D^\gamma(\mathit{\Gamma}^p_{i_s 1}).$$
Noting the assumption that for $|\alpha|\le t$ the estimate (4.3) holds, we have
$$\frac{|D^{\beta-\gamma}(A_{i_1\cdots i_{s-1}p i_{s+1}\cdots i_k})|}{\sqrt{\lambda_1^{\beta_1-\gamma_1}\cdots\lambda_n^{\beta_n-\gamma_n}\lambda_{i_1}\cdots\lambda_{i_{s-1}}\lambda_p\lambda_{i_{s+1}}\cdots\lambda_{i_k}}}\le C\cdot d(x)^{-\frac{k+t-|\gamma|}{4}}. \tag{4.12}$$
Applying (4.1) we have
$$D^\gamma(\mathit{\Gamma}^p_{i_s 1})=D^\gamma\Big(-u\sum_{1\le t\le n}u^{pt}A_{ti_s 1}-\frac{u_{i_s}}{u}\delta^p_1-\frac{u_1}{u}\delta^p_{i_s}\Big). \tag{4.13}$$
Noting that $|\gamma|\le|\beta|=t\le m-3$ and the assumption for $|\mu|\le m-1$, we have
$$\frac{|D^\gamma(u_1\cdot u^{-1})|}{\sqrt{\lambda_1^{\gamma_1}\cdots\lambda_n^{\gamma_n}\cdot\lambda_1}}\le C\cdot d(x)^{-\frac{|\gamma|+1}{4}}. \tag{4.14}$$

For multi-index $\rho = (\rho_1, \cdots \rho_n)$ with $|\rho| \le t \le m-3$, the same reason gives

$$\frac{\sqrt{\lambda_i \lambda_1} \cdot |D^\rho(u^{i1})|}{\sqrt{\lambda_1^{\rho_1} \cdots \lambda_n^{\rho_n}}} \le C \cdot d(x)^{-\frac{|\rho|}{4}}. \tag{4.15}$$

It follows from (4.15) and the assumptions for $|\alpha| \le t$ and $|\mu| \le m-1$ that

$$\begin{aligned}
&\left|\frac{\sqrt{\lambda_p} \cdot D^\gamma(u \cdot u^{pt} \cdot A_{ti_s1})}{\sqrt{\lambda_1^{\gamma_1} \cdots \lambda_n^{\gamma_n} \cdot \lambda_1 \cdot \lambda_{i_s}}}\right| \\
&= \left| \sum_{\substack{\tau \le \gamma \\ \rho \le \tau}} \binom{\gamma}{\tau} \binom{\tau}{\rho} \cdot \frac{D^{\gamma-\tau}(u)}{\sqrt{\lambda_1^{\gamma_1-\tau_1} \cdots \lambda_n^{\gamma_n-\tau_n}}} \right. \\
&\quad \left. \cdot \frac{\sqrt{\lambda_p \cdot \lambda_t} \cdot D^\rho(u^{pt})}{\sqrt{\lambda_1^{\rho_1} \cdots \lambda_n^{\rho_n}}} \cdot \frac{D^{\tau-\rho}(A_{ti_s1})}{\sqrt{\lambda_1^{\tau_1-\rho_1} \cdots \lambda_n^{\tau_n-\rho_n} \cdot \lambda_{i_s} \cdot \lambda_1 \cdot \lambda_t}} \right| \\
&\le C \cdot d(x)^{\frac{2-|\gamma|+|\tau|}{4}} \cdot d(x)^{-\frac{|\rho|}{4}} \cdot d(x)^{-\frac{3+|\tau|-|\rho|}{4}} \\
&= C \cdot d(x)^{-\frac{|\gamma|+1}{4}}.
\end{aligned} \tag{4.16}$$

By the assumption for $|\alpha| \le t$ we have

$$\frac{|D^\beta(A_{i_1 \cdots i_k 1})|}{\sqrt{\lambda_1^{\beta_1} \cdots \lambda_n^{\beta_n} \cdot \lambda_{i_1} \cdots \lambda_{i_k} \cdot \lambda_1}} \le C \cdot d(x)^{-\frac{k+t+1}{4}}. \tag{4.17}$$

Combining (4.11)–(4.14) and (4.16)–(4.17) we have

$$\frac{|D^\alpha(A_{i_1 \cdots i_k})|}{\sqrt{\lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n} \cdot \lambda_{i_1} \cdots \lambda_{i_k}}} \le C \cdot d(x)^{-\frac{k+t+1}{4}}. \tag{4.18}$$

Now we prove that for $|\mu| = m$ (4.2) holds. By (4.3) we get

$$\frac{|D^\mu(A_{ijk})|}{\sqrt{\lambda_1^{\mu_1} \cdots \lambda_n^{\mu_n} \cdot \lambda_i \lambda_j \lambda_k}} \le C d(x)^{-\frac{3+|\mu|}{4}}, \qquad |\mu| \le m-3. \tag{4.19}$$

From (2.6), we get

$$D^\mu(A_{ijk}) = -\frac{1}{2} D^\mu\Big(\frac{u_{ijk}}{u}\Big) - \frac{1}{2} D^\mu\Big(\frac{u_{ij}u_k + u_{ik}u_j + u_{jk}u_i}{u^2}\Big). \tag{4.20}$$

By using (1.4), (3.7) and the assumption for $|\mu| \le m-1$, we have

$$\left|\frac{D^\mu(u_{ijk})}{u\sqrt{\lambda_1^{\mu_1} \cdots \lambda_n^{\mu_n} \cdot \lambda_i \lambda_j \lambda_k}}\right| \le \frac{2|D^\mu(A_{ijk})|}{\sqrt{\lambda_1^{\mu_1} \cdots \lambda_n^{\mu_n} \cdot \lambda_i \lambda_j \lambda_k}} + C d(x)^{-\frac{m}{4}}, \qquad |\mu| = m-3. \tag{4.21}$$

It follows from (1.4) and (4.19) that

$$\left|\frac{D^\mu(u_{ijk})}{\sqrt{\lambda_1^{\mu_1} \cdots \lambda_n^{\mu_n} \cdot \lambda_i \lambda_j \lambda_k}}\right| \le C d(x)^{\frac{2-m}{4}}, \qquad |\mu| = m-3. \tag{4.22}$$

This proves (4.2), furthermore, by using (1.6), we obtain (1.7).

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