Communications in Mathematical Research
25(1)(2009), 1–8

The Lie Algebras in which Every Subspace Is Its Subalgebra*

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Communicated by Liu Jian-ya

Abstract: In this paper, we study the Lie algebras in which every subspace is its subalgebra (denoted by HB Lie algebras). We get that a nonabelian Lie algebra is an HB Lie algebra if and only if it is isomorphic to $g\dot{+}\mathbb{C}id_g$, where g is an abelian Lie algebra. Moreover we show that the derivation algebra and the holomorph of a nonabelian HB Lie algebra are complete.

Key words: HB Lie algebra, complete Lie algebra, holomorph
2000 MR subject classification: 17B05, 17B40, 17B55
Document code: A
Article ID: 1674-5647(2009)01-0001-08

1 Introduction

The classification of Lie algebras is the most important work in Lie theory. There are two ways to get the classification of Lie algebras: by dimension, or by structure. The dimension approach has got a lot of useful results and some interesting applications in general relativity. However, it seems to be neither feasible, nor fruitful to proceed by dimension in the classification of Lie algebras when its dimension is beyond 6. We then turn to the structure approach. In this paper we study a special class of Lie algebras.

A subspace η of a Lie algebra is its subalgebra with $[\eta,\ \eta] \subset \eta$. The algebras in which every subalgebra is its ideal have been studied in [1], and the algebras in which every subspace is a subalgebra have been studied in [2]. In this paper, we study the Lie algebras in which every subspace is its subalgebra. We also study the derivation algebra and the holomorph of an HB Lie algebra.

Complete Lie algebras (i.e., centerless with only inner derivations: $H^0(g,\ g) = H^1(g,\ g) = 0$) first appeared in 1951, in the context of Schenkman's theory of subinvariant Lie algebras (see [3]). In recent years, different authors have concentrated on classifications and structural properties of complete Lie algebras (see [4]–[9]). We prove that the holomorph of an HB Lie algebra is complete.

*Received date: April 27, 2007.

In this paper, all Lie algebras discussed are finite dimensional complex Lie algebras.

2 The Structure of an HB Lie Algebra

Lemma 2.1 *If H is a Lie algebra, then the following assertions are equivalent:*

(1) *H is an HB Lie algebra;*

(2) *For any basis $\{x_1, x_2, \cdots, x_n\}$ of H, $[x_i,\ x_j] \in \mathbb{C}x_i + \mathbb{C}x_j$, $1 \le i, j \le n$.*

Proof. (1)⇒(2). By the definition of an HB Lie algebra, it is obvious.

(2)⇒(1). For any basis $\{x_1, x_2, \cdots, x_k\}$ of a subspace H_1, let $\{x_1, x_2, \cdots, x_k, x_{k+1}, \cdots, x_n\}$ be a basis of H. By (2), $[x_i, x_j] \in \mathbb{C}x_i + \mathbb{C}x_j$, $1 \le i, j \le k$, and we may assume $[x_i, x_j] = a_{ij}x_i + b_{ij}x_j$. For any $x = \sum\limits_{i=1}^{k} a_i x_i$, $y = \sum\limits_{j=1}^{k} b_j x_j$ in this subspace, we have

$$[x, y] = \Big[\sum_{i=1}^{k} a_i x_i,\ \sum_{j=1}^{k} b_j x_j\Big] = \sum_{i=1}^{k}\sum_{j=1}^{k} a_i b_j [x_i, x_j] = \sum_{i=1}^{k}\sum_{j=1}^{k} (a_i b_j a_{ij} x_i + a_i b_j b_{ij} x_j).$$

Hence this subspace is a subalgebra. By the definition of an HB Lie algebra, H is an HB Lie algebra.

Let L be a 2-dimensional Lie algebra. For any basis $\{x,\ y'\}$ of L, there exists another basis $\{x,\ y\}$ of L, such that $[x,\ y] \in \mathbb{C}y$ or $[x,\ y] \in \mathbb{C}x$. In fact, if L is abelian, then $0 = [x,\ y'] = 0y$. If L is nonabelian, as $[x,\ y'] \in \mathbb{C}x + \mathbb{C}y'$, we may assume $[x,\ y'] = ax + by'$. If $b \ne 0$, let $y = ax + by'$, and then

$$[x, y] = [x,\ ax + by'] = b(ax + by') = by;$$

if $b = 0$, let $y = y'$, and then

$$[x, y] = ax.$$

Lemma 2.2 *Let H be an HB Lie algebra. Then H has a decomposition:*

$$H = H_1 \dot{+} H_2 \dot{+} \cdots \dot{+} H_s,$$

where H_i is a subspace of H which has a basis $\{x_{i1},\ x_{i2},\ \cdots,\ x_{in_i}\}$, such that

$$\begin{aligned} &[x_{ip},\ x_{jq}] = \lambda(ip,\ jq)x_{ip}, && i < j,\ 1 \le p \le n_i,\ 1 \le q \le n_j; \\ &[x_{ip_1},\ x_{ip_2}] = 0, && 1 \le p_1, p_2 \le n_i. \end{aligned}$$

Proof. When $\dim H = 2$, the lemma holds.

In fact, there exists a basis $\{x_1,\ x_2\}$ of 2-dimensional Lie algebra such that

$$[x_1,\ x_2] = \lambda x_2$$

or

$$[x_1,\ x_2] = \lambda x_1.$$

We assume that the lemma holds for $\dim H < n$ to prove the lemma holds for $\dim H = n$. For any basis $\{x_1,\ x_2,\ \cdots,\ x_n\}$ of H, by the definition of an HB Lie algebra, we obtain that $\mathbb{C}x_1 + \mathbb{C}x_i$, $2 \le i \le n$, is a 2-dimensional Lie algebra. We choose a basis $\{x_1,\ y_i\}$ of this Lie algebra such that

$$[x_1,\ y_i] = \lambda_i y_i$$

or

$$[x_1,\ y_i] = \lambda_i x_1.$$

Obviously, $\{x_1,\ y_2,\ \cdots,\ y_n\}$ is also a basis of H. Denoting

$$A = \{y_i \mid 0 \neq [x_1, y_i] \in \mathbb{C}x_1]\},$$
$$B = \{y_i \mid [x_1,\ y_i] = 0\},$$
$$C = \{y_i \mid 0 \neq [x_1, y_i] \in \mathbb{C}y_i]\},$$

and assuming

$$A = \{y_2,\ y_3,\ \cdots,\ y_k\},$$
$$B = \{x_1,\ y_{k+1},\ y_{k+2},\ \cdots,\ y_m\},$$
$$C = \{y_{m+1},\ y_{m+2},\ \cdots,\ y_n\},$$

we have

$$[x_1,\ y_i] = \lambda_i x_1 \neq 0, \qquad 2 \leq i \leq k;$$
$$[x_1,\ y_i] = \lambda_i y_i = 0, \qquad k+1 \leq i \leq m;$$
$$[x_1,\ y_i] = \lambda_i y_i \neq 0, \qquad m+1 \leq i \leq n.$$

As $\mathbb{C}y_i + \mathbb{C}y_j$ is a Lie algebra, we may assume

$$[y_i,\ y_j] = a_{ij}y_i + b_{ij}y_j.$$

For any $k+1 \leq i, j \leq n$,

$$[x_1,\ [y_i,\ y_j]] = [[x_1,\ y_i],\ y_j] + [y_i,\ [x_1,\ y_j]] = (\lambda_i + \lambda_j)(a_{ij}y_i + b_{ij}y_j);$$
$$[x_1,\ [y_i,\ y_j]] = [x_1,\ a_{ij}y_i + b_{ij}y_j] = \lambda_i a_{ij}y_i + \lambda_j b_{ij}y_j.$$

By comparing the coefficients, we have

$$\lambda_i b_{ij} = \lambda_j a_{ij} = 0.$$

Hence

$$[y_i,\ y_j] = 0, \qquad m+1 \leq i, j \leq n;$$
$$[y_i,\ y_j] = b_{ij}y_j, \qquad k+1 \leq i \leq m,\ m+1 \leq j \leq n.$$

For any $2 \leq i \leq k$, $m+1 \leq j \leq n$,

$$\begin{aligned}[x_1,\ [y_i,\ y_j]] &= [[x_1,\ y_i],\ y_j] + [y_i,\ [x_1,\ y_j]] \\ &= \lambda_i[x_1,\ y_j] + \lambda_j[y_i,\ y_j] \\ &= \lambda_i\lambda_j y_j + \lambda_j[y_i, y_j],\end{aligned}$$

and by $[x_1,\ [y_i,\ y_j]] \in \mathbb{C}x_1 + \mathbb{C}[y_i, y_j]$, we have

$$\lambda_i\lambda_j y_j + \lambda_j(a_{ij}y_i + b_{ij}y_j) \in \mathbb{C}x_1 + \mathbb{C}(a_{ij}y_i + b_{ij}y_j).$$

If $a_{ij} \neq 0$, then

$$\lambda_i\lambda_j = 0,$$

a contradiction. So

$$a_{ij} = 0, \quad [y_i,\ y_j] = b_{ij}y_j, \qquad 2 \leq i \leq k,\ m+1 \leq j \leq n.$$

We denote

$$H_1 = \mathrm{span}(y_{m+1},\ y_{m+2},\ \cdots,\ y_n), \qquad H_2 = \mathrm{span}(x_1,\ y_2,\ \cdots,\ y_m).$$

If $H_1 \neq \mathbf{0}$, then $\dim H_2 < n$, and H_2 is an HB Lie algebra, too. By induction hypothesis, H_2 has a decomposition:

$$H_2 = H'_1 \dot{+} H'_2 \dot{+} \cdots \dot{+} H'_t,$$

where H'_i is a subspace of H_2 which has a basis $\{x_{i1},\ x_{i2},\ \cdots,\ x_{in_i}\}$ such that

$$[x_{ip},\ x_{jq}] = \lambda(ip,\ jq)x_{ip}, \qquad i < j,\ 1 \leq p \leq n_i,\ 1 \leq q \leq n_j,$$

$$[x_{ip_1},\ x_{ip_2}] = 0, \qquad 1 \leq p_1, p_2 \leq n_i.$$

Obviously, $H_1 \dot{+} H'_1 \dot{+} H'_2 \dot{+} \cdots \dot{+} H'_t$ is the decomposition required in the lemma.

If $H_1 = \mathbf{0}$, it means $[x_1,\ y_i] = 0$ or $[x_1,\ y_i] \in \mathbb{C}x_1$, $2 \leq i \leq n$. Denoting

$$H_1 = \mathrm{span}(x_1), \qquad H_2 = \mathrm{span}(y_2,\ y_3,\ \cdots,\ y_n),$$

we also have that $H_1 \dot{+} H'_1 \dot{+} H'_2 \dot{+} \cdots \dot{+} H'_t$ is the decomposition required in the lemma.

Theorem 2.1 *H is a nonabelian HB Lie algebra if and only if $H \cong g \dot{+} \mathbb{C}id_g$, where g is an abelian Lie algebra.*

Proof. $\Rightarrow$. Let the basis of H in Lemma 2.2 be

$$\Phi = \{x_{11},\ x_{12},\ \cdots,\ x_{1n_1},\ x_{21},\ x_{22},\ \cdots,\ x_{2n_2},\ \cdots,\ x_{s1},\ x_{s2},\ \cdots,\ x_{sn_s}\}.$$

For any $1 \leq i < j < s$, $1 \leq p \leq n_i$, $1 \leq q \leq n_j$, $1 \leq r \leq n_s$,

$$[x_{jq},\ x_{sr} + x_{ip}] = \lambda(jq,\ sr)x_{jq} - \lambda(ip,\ jq)x_{ip} \in \mathrm{span}(x_{jq},\ x_{sr} + x_{ip}),$$

so we obtain

$$\lambda(ip,\ jq) = 0.$$

Hence

$$[H_i,\ H_j] = 0, \qquad 1 \leq i, j \leq s - 1.$$

For any $1 \leq j \leq s - 1$, $1 \leq t_1, t_2 \leq n_s$, $1 \leq q \leq n_j$,

$$[x_{st_1}, x_{st_2} + x_{jq}] = -\lambda(jq, st_1)x_{jq} \in \mathrm{span}(x_{st_1},\ x_{st_2} + x_{jq}).$$

If x_{st_1}, x_{st_2} is linearly independent, then

$$\lambda(jq,\ st_1) = 0.$$

Similarly,

$$\lambda(jq,\ st_2) = 0.$$

So H is abelian, a contradiction.

$$\dim H_s = 1.$$

By $1 \leq i < j < s$, $1 \leq p_1, p_2 \leq n_i$, $1 \leq q \leq n_j$,

$$[x_s,\ x_{ip_1} + x_{ip_2}] = -\lambda(ip_1,\ s)x_{ip_1} - \lambda(ip_2,\ s)x_{ip_2} \in \mathrm{span}(x_s,\ x_{ip_1} + x_{ip_2}),$$

$$[x_s,\ x_{ip_1} + x_{jq}] = -\lambda(ip_1,\ s)x_{ip_1} - \lambda(jq,\ s)x_{jq} \in \mathrm{span}(x_s,\ x_{ip_1} + x_{jq}),$$

so we have

$$\lambda(ip_1,\ s) = \lambda(ip_2,\ s), \qquad \lambda(ip_1,\ s) = \lambda(jq,\ s),$$

and then

$$[x_s,\ x_{it}] = \lambda x_{it}, \qquad \forall x_{it} \in \Phi \backslash \{x_s\}.$$

Hence

$$H \cong g \dot{+} \mathbb{C}id_g,$$

where g is an abelian Lie algebra.

$\Leftarrow$. If $H \cong g \dot{+} \mathbb{C}id_g$, where g is an abelian Lie algebra, then H has a basis $\{Id_g,\ x_2,\ x_3,\ \cdots,\ x_n\}$. For any basis $\{y_1,\ y_2,\ \cdots,\ y_n\}$ of H, where

$$y_i = a_{1i}Id_g + \sum_{k=2}^{n} a_{ki}x_k,$$

we have

$$\begin{aligned}[y_i,\ y_j] &= \Big[a_{1i}Id_g + \sum_{k=2}^{n} a_{ki}x_k,\ a_{1j}Id_g + \sum_{k=2}^{n} a_{kj}x_k\Big] \\ &= a_{1i}\sum_{k=2}^{n} a_{kj}x_k - a_{1j}\sum_{k=2}^{n} a_{ki}x_k \\ &= a_{1i}y_j - a_{1j}y_i \in \mathbb{C}y_i + \mathbb{C}y_j.\end{aligned}$$

From Lemma 2.1 we know that H is an HB Lie algebra.

Corollary 2.1 *If H_1 and H_2 are HB Lie algebras, then $H_1 \cong H_2$ if and only if* $\mathrm{dim}H_1 = \mathrm{dim}H_2$.

As all HB Lie algebras are isomorphic to $g \dot{+} \mathbb{C}id_g$, where g is an abelian Lie algebra with dimension $\mathrm{dim}H - 1$. So all n-dimensional HB Lie algebras are isomorphic to each other.

Corollary 2.2 *Let L be a Lie algebra. If* $\mathrm{dim}[L,\ L] = \mathrm{dim}L - 1$, $[[L,\ L],\ [L,\ L]] = 0$, *and for any $x \in L\backslash[L,\ L]$, $y \in [L,\ L]$, $[x,\ y] = \lambda y$, then L is an HB Lie algebra.*

3 The Derivation Algebra and Holomorph of HB Lie Algebra

Lemma 3.1[10] *Let L be a centerless Lie algebra, $L^{\omega} := \bigcap_{i=1}^{\infty} L^i$ and $\eta(L^{\omega})$ be the holomorph (i.e., a Lie algebra with its derivation algebra) of L^{ω}. If there exists an ideal ζ of $\eta(L^{\omega})$ such that $\zeta \cong L$ and $L^{\omega} \subseteq \zeta$, then*

$$\mathrm{dimDer}L \le \mathrm{dimDer}L^{\omega} + \mathrm{dim}C(L^{\omega}),$$

and $\mathrm{Der}L$ *is a complete Lie algebra.*

Theorem 3.1 *The derivation algebra of a nonabelian HB Lie algebra is complete.*

Proof. By Theorem 2.1, we obtain $H \cong g \dot{+} \mathbb{C}id_g$, where g is an abelian Lie algebra. Obviously,

$$H^{\omega} = \bigcap_{i=1}^{\infty} H^i = \bigcap_{i=1}^{\infty} (g \dot{+} \mathbb{C}id_g)^i = g,$$

$$\eta(H^\omega) = \eta(g) = g \dot{+} \mathrm{Der} g = g \dot{+} gl(g),$$

and then

$$[H,\ \eta(H^\omega)] = [g \dot{+} \mathbb{C} id_g,\ g \dot{+} gl(g)] = [g,\ gl(g)] + [\mathbb{C} id_g,\ g] \subseteq g \subseteq H.$$

Hence

$$H \lhd \eta(H^\omega).$$

And obviously, $C(H) = 0$, H is isomorphic to H, $H^\omega = g \subseteq H$. We have that H is the ideal in Lemma 3.1, so $\mathrm{Der}H$ is a complete Lie algebra.

Lemma 3.2 *Let H be a nonabelian HB Lie algebra, and A be the matrix of linear transformation of φ with respect to the basis*

$$\phi = \{x_1,\ x_2,\ \cdots,\ x_n\},$$

where

$$[x_1,\ x_i] = x_i, \quad [x_i,\ x_j] = 0, \qquad 2 \le i, j \le n.$$

Then $\varphi \in \mathrm{Der}H$ if and only if A has the following form:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Proof. $\Rightarrow$. Since $\varphi \in \mathrm{Der}H$,

$$\varphi[x_1,\ x_j] = [\varphi x_1,\ x_j] + [x_1, \varphi x_j],$$

we obtain

$$\sum_{k=1}^{n} a_{kj} x_k = a_{11} x_j + \sum_{k=2}^{n} a_{kj} x_k.$$

Hence

$$a_{11} = a_{1j} = 0.$$

$\Leftarrow$. For any $x = \sum\limits_{i=1}^{n} a_i x_i$, $y = \sum\limits_{j=1}^{n} b_j x_j \in H$, we have

$$\begin{aligned} \varphi[x,\ y] &= \varphi\Big[\sum_{i=1}^{n} a_i x_i,\ \sum_{j=1}^{n} b_j x_j\Big] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j \varphi[x_i,\ x_j] \\ &= \sum_{j=2}^{n} a_1 b_j \varphi x_j - \sum_{i=2}^{n} a_i b_1 \varphi x_i, \\ [\varphi x, y] + [x, \varphi y] &= \Big[\sum_{i=1}^{n} a_i \varphi x_i,\ \sum_{j=1}^{n} b_j x_j\Big] + \Big[\sum_{i=1}^{n} a_i x_i,\ \sum_{j=1}^{n} b_j \varphi x_j\Big] \\ &= \sum_{j=2}^{n} a_1 b_j \varphi x_j - \sum_{i=2}^{n} a_i b_1 \varphi x_i. \end{aligned}$$

So

$$\varphi[x,\ y] = [\varphi x,\ y] + [x,\ \varphi y], \qquad \varphi \in \mathrm{Der}H.$$

Lemma 3.3[10] *Let g be a Lie algebra, and η be a Cartan subalgebra of g. If g and η satisfy*:

(1) *η is abelian*;

(2) *The decomposition of g with respect to η is*

$$g = \eta \dot{+} \sum_{\alpha\in\Delta} g_\alpha,$$

where $\Delta \subset \eta^*\backslash(0)$,

$$g_\alpha = \{x \in g \mid [h,x] = \alpha(h)x,\ h \in \eta\};$$

(3) *There exists a basis $\{\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_l \mid \alpha_i \in \Delta\}$ of η^*, such that*

$$\mathrm{dim} g_{\pm\alpha_j} \leq 1, \qquad [g_{\alpha_j}, g_{-\alpha_j}] \neq 0, \quad -\alpha_j \in \Delta;$$

(4) *η and $\{g_{\pm\alpha_j}, 1 \leq j \leq l\}$ generate g,*

then g is a complete Lie algebra.

Theorem 3.2 *The holomorph of a nonabelian HB Lie algebra is a complete Lie algebra.*

Proof. By Theorem 1.1, H has a basis $\{x_1,\ x_2,\ \cdots,\ x_n\}$ such that

$$[x_1,\ x_i] = x_i, \quad [x_i,\ x_j] = 0, \qquad 2 \leq i, j \leq n.$$

We view a transformation of H as its matrix with respect to $\{x_1,\ x_2,\ \cdots,\ x_n\}$. Let

$$\eta = \mathbb{C}x_1 \dot{+} \mathbb{C}E_{22} \dot{+} \mathbb{C}E_{33} \dot{+} \cdots \dot{+} \mathbb{C}E_{nn},$$

where E_{ij} is the elementary matrix in which the element in the j-th column and i-th row is 1 and the others are 0. Denoting by e_i the linear function which extracts the i-th entry of a diagonal matrix; by α the linear functional

$$\alpha(h) = \alpha\Big(a_1x_1 \dot{+} \sum_{i=2}^{n} a_iE_{ii}\Big) = a_1, \qquad \forall h \in \eta,$$

we have

$$[\eta,\ \eta] = 0;$$

$$[h,\ E_{i1}] = e_i(h)E_{i1};$$

$$[h,\ x_i] = (e_i + \alpha)(h)x_i, \qquad i \geq 2;$$

$$[h,\ E_{ij}] = (e_i - e_j)(h)E_{ij}, \qquad i, j \geq 2.$$

Now we let

$$\alpha_1 = e_2 + \alpha, \quad \alpha_2 = e_2, \quad \alpha_3 = e_3 - e_2, \quad \cdots, \quad \alpha_n = e_n - e_{n-1}.$$

Obviously,

$$\eta(H)_{\alpha_1} = \mathbb{C}x_2, \quad \eta(H)_{\alpha_2} = \mathbb{C}E_{21}\eta(H)_{\alpha_3} = \mathbb{C}E_{32}, \quad \cdots, \quad \eta(H)_{\alpha_n} = \mathbb{C}E_{n,n-1}.$$

If

$$\beta = \sum_{i=1}^{n} a_i\alpha_i = 0,$$

by

$$\beta(x_1) = a_1, \quad \beta(E_{22}) = a_1 + a_2 - a_3, \quad \beta(E_{33}) = a_3 - a_4,$$

$$\beta(E_{44}) = a_4 - a_5, \quad \cdots, \quad \beta(E_{n-1,n-1}) = a_{n-1} - a_n, \quad \beta(E_{nn}) = a_n,$$

we have

$$a_1 = a_2 = \cdots = a_n = 0.$$

Hence $\{\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n\}$ is linearly independent. By

$$\dim\eta^* = \dim\eta = n,$$

we obtain that $\{\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n\}$ is a basis of η^*. As

$$\eta(H)_{-\alpha_1} = 0, \qquad \eta(H)_{-\alpha_2} = 0,$$

and

$$\eta(H)_{-\alpha_i} = \mathbb{C}E_{i-1,i}, \qquad i \geq 3,$$

$$[E_{i,i-1},\ E_{i-1,i}] = E_{ii} - E_{i-1,\ i-1} \neq 0,$$

we have

$$[\eta(H)_{\alpha_j},\ \eta(H)_{-\alpha_j}] \neq 0, \qquad -\alpha_j \in \Delta.$$

By

$$E_{it} = [\cdots[[E_{i,i-1},\ E_{i-1,i-2}],\ E_{i-2,i-3}],\ \cdots,\ E_{t+1,t}], \qquad t < i,\ 3 \leq i;$$

$$E_{it} = [\cdots[[E_{i,i+1},\ E_{i+1,i+2}],\ E_{i+2,i+3}],\ \cdots,\ E_{t-1,t}], \qquad i < t,\ 2 \leq i,$$

we know that $\{E_{ij} \mid i, j \geq 2\}$ can be generated by $\{\eta(H)_{\pm\alpha_i} \mid i \geq 3\}$. By

$$[E_{i2},\ E_{21}] = E_{i1}, \quad [E_{i2},\ x_2] = x_i, \qquad i \geq 2,$$

$\eta(H)$ can be generated by η and $\{\eta(H)_{\pm\alpha_j},\ 1 \leq j \leq l\}$.

Hence by Lemma 3.3, $\eta(H)$ is a complete Lie algebra.

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