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Empirical Bayes Test for the Parameter of Rayleigh Distribution with Error of Measurement*

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Abstract: For the data with error of measurement in historical samples, the empirical Bayes test rule for the parameter of Rayleigh distribution is constructed, and the asymptotically optimal property is obtained. It is shown that the convergence rate of the proposed EB test rule can be arbitrarily close to $O(n^{-\frac{1}{2}})$ under suitable conditions.

Key words: error of measurement, empirical Bayes, asymptotic optimality, convergence rate

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1 Introduction

Empirical Bayes (EB) approach has been studied extensively by the researchers, and the readers are referred to literature [1]–[8].

Data with error of measurement take place in many fields, including biology, ecology, geology and medicine (see [9]–[10]). Up to now, empirical Bayes test problem for the parameter of distribution with error of measurement has not been studied by any researcher. Rayleigh distribution plays an important role in reliability analysis. In this paper, we discuss the empirical Bayes test for the parameter of Rayleigh distribution with error data of measurement.

Let X have a conditional density function

$$f(x \mid \theta) = \frac{x}{\theta^2} \mathrm{e}^{-\frac{x^2}{2\theta^2}}, \tag{1.1}$$

where θ is an unknown parameter. Denote the sample space by $x \in \Omega = \{x \mid x > 0\}$ and

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parameter space by $\Theta = \{\theta \mid \theta > 0\}$. In this paper, we discuss the one-sided test problem

$$\mathrm{H}_0: \ \theta \le \theta_0 \Longleftrightarrow \mathrm{H}_1: \ \theta > \theta_0, \tag{1.2}$$

where θ_0 is a given positive constant.

To construct EB test function, we have firstly loss functions

$$L_0(\theta, d_0) = \begin{cases} 0, & \theta \le \theta_0; \\ a\Big[1 - \Big(\dfrac{\theta_0}{\theta}\Big)^2\Big], & \theta > \theta_0, \end{cases} \qquad L_1(\theta, d_1) = \begin{cases} a\Big[\Big(\dfrac{\theta_0}{\theta}\Big)^2 - 1\Big], & \theta \le \theta_0; \\ 0, & \theta > \theta_0, \end{cases}$$

where $a > 0$, $d = \{d_0, \ d_1\}$ is action space, d_0 and d_1 imply acceptance and rejection of H_0.

Assume that the prior distribution $G(\theta)$ of θ is unknown. Then we have the randomized decision function

$$\delta(x) = P(\text{accept } \mathrm{H}_0 \mid X = x). \tag{1.3}$$

And the risk function of $\delta(x)$ is shown by

$$\begin{aligned} R(\delta(x), G(\theta)) &= \int_\Theta \int_\Omega [L_0(\theta, d_0) f(x \mid \theta)\delta(x) + L_1(\theta, d_1) f(x \mid \theta)(1 - \delta(x))]\mathrm{d}x\mathrm{d}G(\theta) \\ &= a \int_\Omega \beta(x)\delta(x)\mathrm{d}x + C_G, \end{aligned} \tag{1.4}$$

where

$$C_G = \int_\Theta L_1(\theta, d_1)\mathrm{d}G(\theta), \qquad \beta(x) = \int_\Theta \Big[1 - \Big(\frac{\theta_0}{\theta}\Big)^2\Big] f(x \mid \theta)\mathrm{d}G(\theta). \tag{1.5}$$

The marginal density function of X is given by

$$f_G(x) = \int_\Theta f(x \mid \theta)\mathrm{d}G(\theta) = \int_\Theta \frac{x}{\theta^2}\mathrm{e}^{-\frac{x^2}{2\theta^2}}\mathrm{d}G(\theta).$$

By (1.5) and simple calculations, we have

$$\beta(x) = u(x) f_G(x) + v(x) f_G^{(1)}(x), \tag{1.6}$$

where $f_G^{(1)}(x)$ is the first order derivative of $f_G(x)$, and

$$u(x) = 1 - \frac{1}{4}\theta_0 x^{-2}, \qquad v(x) = \frac{1}{2}\theta_0 x^{-1}.$$

Using (1.4), we obtain the Bayes test function as follows:

$$\delta_G(x) = \begin{cases} 1, & \beta(x) \le 0; \\ 0, & \beta(x) > 0. \end{cases} \tag{1.7}$$

Further, we can get the minimum Bayes risk

$$R(G) = \inf_\delta R(\delta, G) = R(\delta_G, G) = a \int_\Omega \beta(x)\delta_G(x)\mathrm{d}x + C_G. \tag{1.8}$$

When the prior distribution of $G(\theta)$ is known and $\delta(x) = \delta_G(x)$, $R(G)$ can be obtained. However, when $G(\theta)$ is unknown, so that $\delta_G(x)$ cannot be made use of, we need to introduce EB method.

2 Construction of EB Test Function

Under the following assumptions, we are to construct the EB test function. Let (X_1, θ_1), (X_2, θ_2), $\cdots$, (X_n, θ_n) and $(X_{n+1}, \theta_{n+1}) \widehat{=} (X, \theta)$ be independent random vectors, where θ_i $(i = 1, \cdots, n)$ and θ are indepently identically distributed (i.i.d.) and have common prior distribution $G(\theta)$. Let X_1, X_2, $\cdots$, X_n, X be sequence of mutually independent random variables, where X_1, X_2, $\cdots$, X_n are historical samples and X is the present sample. Due to some factors, historical samples X_1, X_2, $\cdots$, X_n cannot be observed, which are sufferred from interruption of random error variables ε_1, ε_2, $\cdots$, ε_n. Hence, we can only observe the data with error of measurement X_1', X_2', $\cdots$, X_n', and X_i' is determined by

$$X_i = X_i' + \varepsilon_i, \qquad 1 \le i \le n,$$

where ε_i $(i = 1, 2, \cdots, n)$ are mutually independent random variables having the identical normal distribution $N(0,\ \sigma^2)$, and the variance σ^2 is known.

Assume that X_1', X_2', $\cdots$, X_n' and ε_1, ε_2, $\cdots$, ε_n are mutually independent. Let X_1, X_2, $\cdots$, X_n have the common marginal density function $f_G(x)$, and X_1', X_2', $\cdots$, X_n' have the common marginal density function $f_{G'}(x)$. Assume $f_G(x) \in C_{s,\alpha}$, $x \in R^1$, where

$$C_{s,\alpha} = \{g(x) \mid g(x) \text{ is the probability density function and has continuous } s\text{-th order derivative } g^{(s)}(x) \text{ with } |g^{(s)}(x)| \le \alpha,\ s \ge 2,\ \alpha > 0\}.$$

First we construct the estimator of $\beta(x)$.

Let $K_r(x)$ be a Borel measurable bounded function vanishing off $(0, 1)$ and such that

(A1) $$\frac{1}{t!}\int_0^1 y^t K_r(y)\mathrm{d}y = \begin{cases} 1, & t = r; \\ 0, & t \ne r,\ t = 0, 1, 2, \cdots, s-1. \end{cases}$$

By the convolution formula, we get

$$f_G(x) = \int_{-\infty}^{+\infty} f_{G'}(x-y)\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{y^2}{2\sigma^2}}\mathrm{d}y. \tag{2.1}$$

The kernel estimation of $f_G^{(r)}(x)$ is defined by

$$f_G^{(r)}(x) = \int_{-\infty}^{+\infty} f_{G'}^{(r)}(x-y)\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{y^2}{2\sigma^2}}\mathrm{d}y. \tag{2.2}$$

Denote

$$f_G^{(0)}(x) = f_G(x).$$

Since $f_{G'}^{(r)}(x)$ is unknown, the kernel-type density estimation (see [5]) of $f_{G'}^{(r)}(x)$ is defined by

$$\widehat{f}_{G'}^{(r)}(x) = \frac{1}{nh_n^{(1+r)}}\sum_{i=1}^n K_r\Big(\frac{X_i' - x}{h_n}\Big). \tag{2.3}$$

Substituting (2.3) into (2.2), we get

$$\widehat{f}_G^{(r)}(x) = \frac{1}{nh_n^{(1+r)}}\sum_{i=1}^n \int_{-\infty}^{+\infty} K_r\Big(\frac{X_i' - x + y}{h_n}\Big)\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{y^2}{2\sigma^2}}\mathrm{d}y, \tag{2.4}$$

where $r = 0, 1$, h_n is the smoothing bandwidth, $h_n > 0$, and $\lim\limits_{n\to\infty} h_n = 0$.

Write

$$f_{G'}^{(0)}(x) = f_{G'}(x).$$

Then, the estimator of $\beta(x)$ is obtained from

$$\beta_n(x) = u(x)\widehat{f}_G(x) + v(x)\widehat{f}_G^{(1)}(x). \tag{2.5}$$

Hence, the EB test function is defined by

$$\delta_n(x) = \begin{cases} 1, & \beta_n(x) \le 0; \\ 0, & \beta_n(x) > 0. \end{cases} \tag{2.6}$$

Let E_n denote the mathematical expectation with respect to the joint distribution of $X_1', X_2', \cdots, X_n'$. We get the overall Bayes risk of $\delta_n(x)$ as

$$R(\delta_n, G) = a \int_\Omega \beta(x) E_n[\delta_n(x)]\mathrm{d}x + C_G. \tag{2.7}$$

If

$$\lim_{n\to\infty} R(\delta_n, G) = R(\delta_G, G),$$

then $\{\delta_n(x)\}$ is the asymptotic optimality of EB test function; if

$$R(\delta_n, G) - R(\delta_G, G) = O(n^{-q}), \qquad q > 0,$$

then $O(n^{-q})$ is the asymptotic optimality convergence rate of EB test function of $\{\delta_n(x)\}$.

We give two lemmas in the following.

Let c, c_1, c_2, c_3, c_4 be constants which can be different in different cases even in the same expression.

Lemma 2.1 *Let* $\widehat{f}_G^{(r)}(x)$ *be defined by* (2.4). *Assume that* (A1) *holds, and* $x \in \Omega$.

(I) *If* $f_G^{(r)}(x)$ *is a continuous function,*

$$\lim_{n\to\infty} h_n = 0, \qquad \lim_{n\to\infty} n h_n^{2r+2} = \infty,$$

then

$$\lim_{n\to\infty} E|\widehat{f}_G^{(r)}(x) - f_G^{(r)}(x)|^2 = 0;$$

(II) *If* $f_G(x) \in C_{s,a}$, *taking* $h_n = n^{-\frac{1}{2+2s}}$, *then, for* $0 < \lambda \le 1$, *we have*

$$E|\widehat{f}_G^{(r)}(x) - f_G^{(r)}(x)|^{2\lambda} \le c \cdot n^{-\frac{\lambda(s-r)}{1+s}}.$$

Proof. (I) By C_r inequality, we get

$$E|\widehat{f}_G^{(r)}(x) - f_G^{(r)}(x)|^2 \le 2|E\widehat{f}_G^{(r)}(x) - f_G^{(r)}(x)|^2 + 2\mathrm{Var}(\widehat{f}_G^{(r)}(x)) := 2(I_1^2 + I_2). \tag{2.8}$$

Then

$$\begin{aligned} E\widehat{f}_G^{(r)}(x) &= \frac{1}{h_n^{(1+r)}} \int_{-\infty}^{+\infty} \Big\{ \Big[\int_{-\infty}^{+\infty} K_r\Big(\frac{s-x+y}{h_n}\Big) f_{G'}(s)\mathrm{d}s \Big] \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{y^2}{2\sigma^2}} \Big\} \mathrm{d}y \\ &= \frac{1}{h_n^r} \Big\{ \int_{-\infty}^{+\infty} \Big[\int_{-\infty}^{+\infty} K_r(u) f_{G'}(x-y+h_n u)\mathrm{d}u \Big] \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{y^2}{2\sigma^2}} \Big\} \mathrm{d}y \\ &= \frac{1}{h_n^r} \int_{-\infty}^{+\infty} I_1(x,y) \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{y^2}{2\sigma^2}} \mathrm{d}y, \end{aligned}$$

where

$$I_1(x,y) = \int_{-\infty}^{+\infty} K_r(u) f_{G'}(x-y+h_n u)\mathrm{d}u.$$

Since $f_G(x) \in C_{s,\alpha}$, it is easy to see that $f_{G'}(x) \in C_{s,\alpha}$. We obtain the following Taylor expansion of $f_{G'}(x-y+h_n u)$ in $x-y$:

$$\begin{aligned} &f_{G'}(x-y+h_n u) - f_{G'}(x-y) \\ =\ &\frac{f'_{G'}(x-y)}{1!} h_n u + \frac{f''_{G'}(x-y)}{2!}(h_n u)^2 + \cdots + \frac{f_{G'}^{(s)}(\xi^*)}{s!}(h_n u)^s, \end{aligned}$$

where $\xi^* \in (x-y, x+y+h_n u)$.

Due to (A1) and $f_G(x) \in C_{s,\alpha}$, it is easy to see that

$$I_1(x,y) = f_{G'}^{(r)}(x-y) + o(h_n^{s-r}).$$

Hence

$$\begin{aligned} E\widehat{f}_G^{(r)}(x) &= \int_{-\infty}^{+\infty} f_{G'}^{(r)}(x-y)\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{y^2}{2\sigma^2}}\mathrm{d}y + o(h_n^{s-r}) \\ &= f_G^{(r)}(x) + o(h_n^{s-r}). \end{aligned}$$

Furthermore, we have

$$I_1 = |E\widehat{f}_G^{(r)}(x) - f_G^{(r)}(x)|^2 \le ch_n^{2(s-r)}. \tag{2.9}$$

When $h_n \to 0$, we get

$$\lim_{n\to\infty} I_1^2 = \lim_{n\to\infty} |E\widehat{f}_G^{(r)}(x) - f_G^{(r)}(x)|^2 = 0. \tag{2.10}$$

It follows that

$$\begin{aligned} I_2 &= n^{-1}h_n^{-2(r+2)}\mathrm{Var}\Big[\int_{-\infty}^{+\infty} K_r\Big(\frac{X_i'-x+y}{h_n}\Big)\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{y^2}{2\sigma^2}}\mathrm{d}y\Big] \\ &\le n^{-1}h_n^{-2(r+2)}E\Big\{\int_{-\infty}^{+\infty}\Big[K_r\Big(\frac{X_1'-x+y}{h_n}\Big)\Big]\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{y^2}{2\sigma^2}}\mathrm{d}y\Big\}^2. \end{aligned}$$

Since $K_r(x)$ is a Borel measurable bounded function, we get

$$\int_{-\infty}^{+\infty}\Big|K_r\Big(\frac{X_1'-x+y}{h_n}\Big)\Big|\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{y^2}{2\sigma^2}}\mathrm{d}y < c.$$

Hence

$$I_2 \le cn^{-1}h_n^{-(2r+2)}. \tag{2.11}$$

When $h_n \to 0$, $nh_n^{2r+2} \to \infty$, we get

$$\lim_{n\to\infty} I_2 = \lim_{n\to\infty} \mathrm{Var}(\widehat{f}_G^{(r)}(x)) = 0. \tag{2.12}$$

By substituting (2.10), (2.12) into (2.8), the proof of (I) is completed.

(II) Similar to (2.8), we can show that

$$\begin{aligned} E|\widehat{f}_G^{(r)}(x) - f_G^{(r)}(x)|^{2\lambda} &\le 2[E\widehat{f}_G^{(r)}(x) - f_G^{(r)}(x)]^{2\lambda} + 2[\mathrm{Var}(\widehat{f}_G^{(r)}(x))]^{\lambda} \\ &:= 2(J_1^{2\lambda} + J_2^{\lambda}). \end{aligned} \tag{2.13}$$

By (2.9), when $h_n = n^{-\frac{1}{2+2s}}$, we get

$$J_1^{2\lambda} = |E\widehat{f}_G^{(r)}(x) - f_G^{(r)}(x)|^{2\lambda} \le c\cdot n^{-\frac{\lambda(s-r)}{s+1}}. \tag{2.14}$$

By (2.11), when $h_n = n^{-\frac{1}{2+2s}}$, we have

$$J_2^{\lambda} \le [c_1(nh_n^{2r+2})^{-1}]^{\lambda} \le c\cdot n^{-\frac{\lambda(s-r)}{1+s}}. \tag{2.15}$$

By substituting (2.14), (2.15) into (2.13), the proof of (II) is completed.

Lemma 2.2[2] *Let $R(\delta_G, G)$ and $R(\delta_n, G)$ be defined by* (1.8) *and* (2.7). *Then*

$$0 \le R(\delta_n, G) - R(\delta_G, G) \le a\int_{\Omega} |\beta(x)|P(|\beta_n(x) - \beta(x)| \ge |\beta(x)|)\mathrm{d}x.$$

3 Asymptotic Optimality and Convergence Rates

Theorem 3.1 *Let $\widehat{f}_G^{(r)}(x)$ be defined by* (2.4). *Assume that* (A1) *and the following regularity conditions hold*:

(i) $h_n > 0$, $\lim_{n\to\infty} h_n = 0$, $\lim_{n\to\infty} nh_n^4 = \infty$;

(ii) $\int_{\Theta} \theta^{-2}\mathrm{d}G(\theta) < +\infty$;

(iii) $f_G^{(1)}(x)$ *is a continuous function.*

Then we have

$$\lim_{n\to\infty} R(\delta_n, G) = R(\delta_G, G).$$

Proof. By Lemma 2.2 we have

$$0 \le R(\delta_n, G) - R(\delta_G, G) \le a \int_\Omega |\beta(x)| P(|\beta_n(x) - \beta(x)| \ge |\beta(x)|)\mathrm{d}x.$$

Write

$$Q_n(x) = |\beta(x)| P(|\beta_n(x) - \beta(x)| \ge |\beta(x)|).$$

Then

$$Q_n(x) \le |\beta(x)|.$$

Again by (1.6) and the Fubini theorem, we can get

$$\begin{aligned}\int_\Omega |\beta(x)|\mathrm{d}x = &\int_\Omega\int_\Theta \Big[1 - \Big(\frac{\theta_0}{\theta}\Big)^2\Big] f(x \mid \theta)\mathrm{d}G(\theta)\mathrm{d}x \\ &= 1 + \theta_0^2 \int_\Omega\int_\Theta \theta^{-2} f(x \mid \theta)\mathrm{d}G(\theta)\mathrm{d}x \\ &= 1 + \theta_0^2 \int_\Theta \theta^{-2}\mathrm{d}G(\theta) \\ &< +\infty.\end{aligned}$$

Applying the dominant convergence theorem, we have

$$0 \le \lim_{n\to\infty} R(\delta_n, G) - R(\delta_G, G) \le \int_\Omega [\lim_{n\to\infty} Q_n(x)]\mathrm{d}x. \tag{3.1}$$

To prove that Theorem 3.1 holds, we only need to prove

$$\lim_{n\to\infty} Q_n(x) = 0 \qquad a.s.x.$$

By Markov's and Jensen's inequalities, one has

$$\begin{aligned}Q_n(x)) &\le E|\beta_n(x) - \beta(x)| \\ &\le |u(x)|E|\widehat{f}_G(x) - f_G(x)| + |v(x)|E|\widehat{f}_G^{(1)}(x) - f_G^{(1)}(x)| \\ &\le |u(x)|[E|\widehat{f}_G(x) - f_G(x)|^2]^{1/2} + |v(x)|[E|\widehat{f}_G^{(1)}(x) - f_G^{(1)}(x)|^2]^{1/2}.\end{aligned}$$

Again by Lemma 2.1(I), for fixed $x \in \Omega$, when $r = 0, 1$, we have

$$\begin{aligned}0 &\le \lim_{n\to\infty} Q_n(x) \\ &\le |u(x)|[\lim_{n\to\infty} E|\widehat{f}_G(x) - f_G(x)|^2]^{1/2} + |v(x)|[\lim_{n\to\infty} E|\widehat{f}_G^{(1)}(x) - f_G^{(1)}(x)|^2]^{1/2} \\ &= 0.\end{aligned} \tag{3.2}$$

By substituting (3.2) into (3.1), the proof of Theorem 3.1 is completed.

Theorem 3.2 *Let* $\widehat{f}_G^{(r)}(x)$ *be defined by* (2.4). *Assume* (A1) *and the following regularity conditions hold*:

(B1) $f_G(x) \in C_{s,\alpha}$;

(B2) $\int_\Omega |x|^{-m\lambda}|\beta(x)|^{1-\lambda}\mathrm{d}x < +\infty$, *where* $0 < \lambda \le 1$, $m = 0, 1, 2$.

Then, if $h_n = n^{-\frac{1}{2s+2}}$, *we have*

$$R(\delta_n, G) - R(\delta_G, G) = O\left(n^{-\frac{\lambda(s-1)}{2(s+1)}}\right),$$

where $s \ge 2$.

Proof. By Lemma 2.2 and Markov's inequality, we have

$$\begin{aligned}
0 &\le R(\delta_n, G) - R(\delta_G, G) \\
&\le \int_{\Omega} |\beta(x)|^{1-\lambda} E|\beta_n(x) - \beta(x)|^{\lambda} \mathrm{d}x \\
&\le c_1 \int_{\Omega} |\beta(x)|^{1-\lambda} |u(x)| E|\widehat{f}_G(x) - f_G(x)| \mathrm{d}x \\
&\quad + c_2 \int_{\Omega} |\beta(x)|^{1-\lambda} |v(x)| E|\widehat{f}_G^{(1)}(x) - f_G^{(1)}(x)| \mathrm{d}x \\
&= A_n + B_n. \qquad (3.3)
\end{aligned}$$

By Lemma 2.1(II) and condition (B2), we get

$$A_n \le c_1 n^{-\frac{\lambda s}{2s+2}} \int_{\Omega} |\beta(x)|^{1-\lambda} |u(x)|^{\lambda} \mathrm{d}x \le c_3 n^{-\frac{\lambda s}{2s+2}}, \qquad (3.4)$$

$$B_n \le c_2 n^{-\frac{\lambda(s-1)}{2s+2}} \int_{\Omega} |\beta(x)|^{1-\lambda} |v(x)|^{\lambda}(x) \mathrm{d}x \le c_4 n^{-\frac{\lambda(s-1)}{2s+2}}. \qquad (3.5)$$

Substituting (3.4) and (3.5) into (3.3), we get

$$R(\delta_n, G) - R(\delta_G, G) = O\left(n^{-\frac{\lambda(s-1)}{2(s+1)}}\right).$$

The proof of Theorem 3.2 is completed.

Remark 3.1 When $\lambda \to 1$ and $s \to \infty$, $O\left(n^{-\frac{\lambda(s-1)}{2(s+1)}}\right)$ is arbitrarily close to $O(n^{-\frac{1}{2}})$.

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