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# Contact Finite Determinacy of Relative Map Germs*

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**Abstract:** The strong contact finite determinacy of relative map germs is studied by means of classical singularity theory. We first give the definition of a strong relative contact equivalence (or $\mathcal{K}_{S,T}$ equivalence) and then prove two theorems which can be used to distinguish the contact finite determinacy of relative map germs, that is, $f$ is finite determined relative to $\mathcal{K}_{S,T}$ if and only if there exists a positive integer $k$, such that $\mathcal{M}^k(n)\mathcal{E}(S;n)^p \subset T\mathcal{K}_{S,T}(f)$.
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## 1 Introduction

A basic idea in the classical singularity theory is that local topological properties of a generic differentiable mapping are determined by finite terms of its Taylor series, i.e., finite determinacy of map germs. It relates to the most important local characteristics of the singularity theory. Therefore, finite determinacy is always an active research subject in the singularity theory. When we treat various spaces of differentiable mappings with several constraints, depending on situations, we need to study the corresponding finite determinacy, whose validity depends on given mapping spaces.

In the present paper we treat the space of differentiable mappings between manifolds with the constraint that a fixed submanifold is mapped into another fixed submanifold. Then naturally we need to study the "relative finite determinacy".

There are several works studying relative finite determinacy of function germs, for instance [1]–[7]. However, the study of relative finite determinacy of map germs such as [8]

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is very few. This paper is a sequel to [8]. The purpose of the present paper is to give an algebraic criterion for relative contact finite determinacy of relative smooth map germs. It is a generalization of the algebraic criteria for $\mathcal{K}$ finite determinacy of smooth map germs originated in [9].

This paper is organized as follows: In Section 2 we give some notations, definitions and other related knowledge. Section 3 is the main part of this paper. We prove the main results in a similar way to [9].

# 2 Preliminaries

Let $S$, $T$ be submanifolds without boundary of $\mathbf{R}^n$ and $\mathbf{R}^p$ respectively, both contaning the origin. Denote by $M(R^n, S; R^p, T)$ the set of relative smooth mappings $f : (\mathbf{R}^n, S) \to (\mathbf{R}^p, T)$ with $f(S) \subset T$ and $f(0) = 0$. For $f, g \in M(\mathbf{R}^n, S; \mathbf{R}^p, T)$, we call $f$, $g$ equivalent near the origin if $f = g$ on some neighborhood of the origin in $R^n$. This equivalence class of $f$ is called a relative map germ and denoted by $[f]$. In this paper we also denote it by $f$ for convenience. We only consider the local case, so we may assume that

$$S = \{0\} \times \mathbf{R}^{n-s}, \qquad T = \{0\} \times \mathbf{R}^{p-t}.$$

Denote by $M^* = M(n, s; p, t)$ the set of relative map germs and $E_f{}^* = E(f, n, s; p, t)$ the set of relative map germs $g$ which satisfies $g(S) = f(S)$ for a given $f \in M^*$. Let

$$C_S(\mathbf{R}^n) = \{h : (\mathbf{R}^n, S) \to R \mid h|_S = \text{Constant}\}$$

be a local ring and

$$\mathcal{E}(S; n) = \{h \in C_S(\mathbf{R}^n) \mid h|_S = 0\}$$

be the maximal ideal of $C_S(\mathbf{R}^n)$. Similarly, we can define the set $C_T(\mathbf{R}^p)$ and $\mathcal{E}(T; p)$. For $f \in M^*$, it induces a homomorphism $f^* : C_T(\mathbf{R}^p) \to C_S(\mathbf{R}^n)$ defined by $f^*(h) = h \circ f$ with $f^*\mathcal{E}(T; p) \subseteq \mathcal{E}(S; n)$. Every $C_S(\mathbf{R}^n)$-module can be viewed as a $C_T(\mathbf{R}^p)$-module through $f^*$. We denote

$$L_n = \{h : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0) \mid h \text{ is a diffeomorphism germ at the origin}\}$$

and

$$\mathcal{K} = \{H \mid H \in L_{n+p}\}.$$

We can also denote by

$$\mathcal{K}_{S,T} = \{H \in \mathcal{K} \mid H(x, y) = (x, y),\ \forall x \in S,\ y \in T\}$$

the strong relative contact equivalent group, where $\mathcal{K}_{S,T}$ acts on $M^*$ in a natural way: If $f \in M^*$, $H \in \mathcal{K}_{S,T}$, then $H \cdot f$ is defined by

$$(\mathbf{1}, H \cdot f) \circ h = H \circ (\mathbf{1}, f),$$

where $h : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$, $h(x) = x$, for any $x \in S$ is a diffeomorphism of $\mathbf{R}^n$ decided only by $H$ and $\mathbf{1}$: $(\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ is the identity map germ. We also observe that $\mathcal{K}_{S,T}$ is a subgroup of the general contact equivalent group $\mathcal{K}$ in [9]. More details can be found in [8] and [9].

**Definition 2.1** *We say that two relative map germs $f, g \in M^*$ are strong relative contact equivalent if there exists $H \in \mathcal{K}_{S,T}$ such that $g = H \cdot f$. We also call that $f$ and $g$ are $\mathcal{K}_{S,T}$ equivalent.*

**Definition 2.2** *Let $f \in M^*$. The tangent space of $\mathcal{K}_{S,T}$ orbit of $f$ is the vector space*
$$T\mathcal{K}_{S,T}(f) := \mathcal{E}(S;n)\langle df\rangle + f^*\mathcal{E}(T;p)\mathcal{E}_{n,p}.$$

**Definition 2.3** *Let $f \in M^*$. We call the map germ $F : (\mathbf{R} \times \mathbf{R}^n, \mathbf{R} \times S) \to (\mathbf{R}^p, T)$, $(t, x) \mapsto F(t, x)$ a 1-parameter relative deformation of $f$ if $F$ satisfies*
$$F(0, x) = f(x)$$
*and*
$$F(t, x) = f(x), \qquad \forall x \in S, t \in \mathbf{R}.$$

**Definition 2.4** *We call $F : (\mathbf{R}^n \times \mathbf{R}, S \times \mathbf{R}) \to (\mathbf{R}^p \times \mathbf{R}, T \times \mathbf{R})$ a level preserving map germ if $F(x, t) = (F_a(x, t), t)$, where $F_a : (\mathbf{R}^n \times \mathbf{R}, S \times \mathbf{R}) \to (\mathbf{R}^p, T)$ satisfies*
$$F_a(x, t) = F_a(x, 0), \qquad \forall x \in S.$$

**Definition 2.5** *Suppose $k$ is a nonnegative integer. Then $f \in M^*$ is $k$ determined relative to $\mathcal{K}_{S,T}$ (abbreviated as k.det.rel.$\mathcal{K}_{S,T}$) if for any $g \in E_f{}^*$ with $j^k g(0) = j^k f(0)$, the $\mathcal{K}_{S,T}$ orbit of $f$ contains $g$, i.e., $f$ and $g$ are $\mathcal{K}_{S,T}$ equivalent.*

# 3 Finite Determinacy Relative to $\mathcal{K}_{S,T}$

In this section, we give two algebraic criterions for judging whether or not a relative map germ is finite determined relative to $\mathcal{K}_{S,T}$.

**Lemma 3.1**[8] *Let $f, g \in M^*$ with $j^k f(0) = j^k g(0)$. Then*
$$T\mathcal{K}_{S,T}(f) + \mathcal{M}^k(n)\mathcal{E}(S;n)^p = T\mathcal{K}_{S,T}(g) + \mathcal{M}^k(n)\mathcal{E}(S;n)^p.$$

**Theorem 3.1** *Let $f \in M^*$, $S = \{0\} \times \mathbf{R}^{n-s}$, and $T = \{0\} \times \mathbf{R}^{p-t}$. If $f$ is $k$ determined relative to $\mathcal{K}_{S,T}$, then there exists a positive integer $k$ such that*
$$\mathcal{M}^{k+1}(n)\mathcal{E}(S;n)^p \subset T\mathcal{K}_{S,T}(f).$$

*Proof.* Let
$$J_0{}^l(f, S, T; n, p) = \overline{J_0{}^l}$$
denote the set of 1-jets at 0 of elements in $E_{f^*}$. We denote by
$$\overline{\pi^l} : \mathcal{M}(n)\mathcal{E}(S;n)^p \to T_{j^l f}\overline{J_0{}^l}$$
the canonical projection.

Since $f$ is $k$ determined relative to $\mathcal{K}_{S,T}$, for all $g \in E_{f^*}$, with $j^k g(0) = j^k f(0)$, the $\mathcal{K}_{S,T}$ orbit of $f$ contains $g$. We have
$$\{g \in E_{f^*} |\ j^k g(0) = j^k f(0)\} \subset \{g \in E_{f^*} |\ g \text{ and } f \text{ are } \mathcal{K}_{S,T} \text{ equivalent}\};$$
for any $l > k$, taking $l$-jets on both sides, we have
$$\{j^l g \in \overline{J_0{}^l} |\ j^k g(0) = j^k f(0)\} \subset \{j^l g \in \overline{J_0{}^l} |\ g \text{ and } f \text{ are } \mathcal{K}_{S,T} \text{ equivalent}\};$$

and then taking tangent spaces at $j^l f$ on both sides, we have

$$T_{j^l f}\{j^l g \in \overline{J_0{}^l} |\ j^k g(0) = j^k f(0)\} \subset T_{j^l f}\{j^l g \in \overline{J_0{}^l} |\ g \text{ and } f \text{ are } \mathcal{K}_{S,T} \text{ equivalent}\}.$$

From

$$T_{j^l f}\{j^l g \in \overline{J_0{}^l} | j^k g(0) = j^k f(0)\} = \overline{\pi^l}(\mathcal{M}^{k+1}(n)\mathcal{E}(S;n)^p),$$

$$T_{j^l f}\{j^l g \in \overline{J_0{}^l} |\ g \text{ and } f \text{ are } \mathcal{K}_{S,T} \text{ equivalent}\} = T_{j^l f}(\mathcal{K}^l_{S,T} \cdot j^l f) = \overline{\pi^l}(T\mathcal{K}_{S,T}(f))$$

and

$$\ker \overline{\pi^l} = \mathcal{M}^{l+1}(n)\mathcal{E}(S;n)^p$$

it follows that

$$\mathcal{M}^{k+1}(n)\mathcal{E}(S;n)^p \subset T\mathcal{K}_{S,T}(f) + \mathcal{M}^{l+1}(n)\mathcal{E}(S;n)^p.$$

Obviously, the right hand side of the last formula is a $C_S(\mathbf{R}^n)$-module. Let $l = k+1$. By the Nakayama Lemma, we have

$$\mathcal{M}^{k+1}(n)\mathcal{E}(S;n)^p \subset T\mathcal{K}_{S,T}(f).$$

**Theorem 3.2** *If there exists a positive integer $k$ such that $\mathcal{M}^k(n)\mathcal{E}(S;n)^p \subset T\mathcal{K}_{S,T}(f)$, then $f$ is $(k+1)$ determined relative to $\mathcal{K}_{S,T}$.*

*Proof.* Let $g \in M^*$ satisfy $j^l g(0) = j^l f(0)$. For any $t_0 \in R$, let

$$F = (I_{1+n}, \widetilde{F}) : (\mathbf{R} \times \mathbf{R}^n, t_0 \times 0) \to (\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^p, t_0 \times 0 \times 0), \qquad (t,x) \mapsto (t, x, \widetilde{F}(t,x)),$$

where

$$I_{1+n} : (\mathbf{R} \times \mathbf{R}^n, t_0 \times 0) \to (\mathbf{R} \times \mathbf{R}^n, t_0 \times 0), \qquad (t,x) \mapsto (t,x)$$

is the identity map germ, and

$$\widetilde{F} : (\mathbf{R} \times \mathbf{R}^n, \mathbf{R} \times S) \to (\mathbf{R}^p, T), \qquad (t,x) \mapsto (1-t)f(x) + tg(x)$$

is the 1-parameter relative deformation of $f$.

Let

$$F_t = \widetilde{F}_t, \quad \frac{\partial F}{\partial t} = \frac{\partial \widetilde{F}}{\partial t}, \quad \frac{\partial F}{\partial x} = \frac{\partial \widetilde{F}}{\partial x}.$$

Obviously, for any $x \in \mathbf{R}^n$, we have

$$F_0 = \widetilde{F}_0 = f, \qquad F_1 = \widetilde{F}_1 = g.$$

In the following, we prove that $F_t$ and $F_{t_0}$ are $\mathcal{K}_{S,T}$ equivalent for any $t$ in a neighborhood $J$ of $t_0$. To see this, we only need to prove that there exists two level preserving map germs $H$ and $H'$ which satisfy the following conditions:

(1) $H'_t(x,y) = (x,y)$ for any $x \in S$, $y \in T$, $t \in J$, $H_t(x) = x$ for any $x \in S$, $t \in J$;

(2) $H'_{t_0}(x,y) = (x,y)$, $H_{t_0}(x) = x$ for any $x \in \mathbf{R}^n$, $y \in \mathbf{R}^p$;

(3) ${H'_t}^{-1} \circ (I_n, F_t) \circ H_t = (I_n, F_{t_0})$, i.e., ${H'}^{-1}(F(H(t,x))) = F(t_0, x)$,

where

$$H' : (\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^p, t_0 \times 0 \times 0) \to (\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^p, t_0 \times 0 \times 0),$$

$$H : (\mathbf{R} \times \mathbf{R}^n, t_0 \times 0) \to (\mathbf{R} \times \mathbf{R}^n, t_0 \times 0).$$

We observe that $H_t$ is determined only by $H'_t$, so our problem turns to construct a smooth map germ $H'_t$ which satisfies conditions (1)–(3). We claim that it suffice to construct a smooth vector field germ $\Theta$ at the origin of $\mathbf{R}^n \times \mathbf{R}^p$ for which

(A) $\xi_i(x) = 0$, for any $x \in S$, $\eta_j(x,y) = 0$, for any $x \in \mathbf{R}^n$, $y \in T$;

(B) $\dfrac{\partial F}{\partial t} = \sum\limits_{i=1}^{n} \xi_i(x) \dfrac{\partial F}{\partial x_i} + \eta(x, F_t(x))$,

where

$$\Theta : (\mathbf{R}^n \times \mathbf{R}^p, 0) \to (\mathbf{R}^n \times \mathbf{R}^p, 0),$$

$$(x,y) \mapsto \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i} + \sum_{j=1}^{p} \eta_j(x,y) \frac{\partial}{\partial y_j}.$$

Since if such a vector field germ $\Theta$ exists, by integrating this vector field, it has a flow, i.e., a map germ $H'_t$ of the required type, which can be supposed to satisfy the "initial condition" (1). A minor computation shows that the conditions (2), (3) are also satisfied.

It remains to prove the existence of a vector field germ $\Theta$ which having the properties (A), (B). Since $\mathcal{M}^k(n)\mathcal{E}(S;n)^p \subset T\mathcal{K}_{S,T}(f)$, $g \in {E_f}^*$ and $j^l g(0) = j^l f(0)$, by Lemma 1.1 we have

$$T\mathcal{K}_{S,T}(g) + \mathcal{M}^l(n)\mathcal{E}(S;n)^p = T\mathcal{K}_{S,T}(f) + \mathcal{M}^l(n)\mathcal{E}(S;n)^p.$$

And then

$$\mathcal{M}^k(n)\mathcal{E}(S;n)^p \subset T\mathcal{K}_{S,T}(g) + \mathcal{M}^l(n)\mathcal{E}(S;n)^p.$$

Let $l = k+1$. By the Nakayama Lemma, we have

$$\mathcal{M}^k(n)\mathcal{E}(S;n)^p \subset T\mathcal{K}_{S,T}(g). \qquad (*)$$

Next we have

$$\widetilde{F}(t,x) = (1-t)f(x) + tg(x).$$

Thus $\widetilde{F}$ and $f$ have the same $l$-jet, for any $t \in J$. By $(*)$, we have

$$\mathcal{M}^k(n)\mathcal{E}(S;n)^p \subset T\mathcal{K}_{S,T}(\widetilde{F}_t),$$

and then we have

$$\frac{\partial F}{\partial t} = \frac{\partial \widetilde{F}}{\partial t} = \frac{\partial \widetilde{F}_t}{\partial t} = g(x) - f(x),$$

$$g(x) - f(x) \in \mathcal{M}^k(n)\mathcal{E}(S;n)^p.$$

Thus

$$\frac{\partial F}{\partial t} = \frac{\partial \widetilde{F}}{\partial t} = \frac{\partial \widetilde{F}_t}{\partial t} = g(x) - f(x) \in T\mathcal{K}_{S,T}(\widetilde{F}_t).$$

According to the definition of $T\mathcal{K}_{S,T}(\widetilde{F}_t)$, the result is proved.

**Corollary 3.1** *Let $f \in M^*$, $S = \{0\} \times \mathbf{R}^{n-s}$ and $T = \{0\} \times \mathbf{R}^{p-t}$. $f$ is finite determined relative to $\mathcal{K}_{S,T}$ if and only if there exists a positive integer $k$ such that*

$$\mathcal{M}^{k+1}(n)\mathcal{E}(S;n)^p \subset T\mathcal{K}_{S,T}(f).$$

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