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A Riesz Product Type Measure on the Cantor Group*

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Abstract: A Riesz type product as
$$P_n = \prod_{j=1}^{n}(1 + a\omega_j + b\omega_{j+1})$$
is studied, where a, b are two real numbers with $|a| + |b| < 1$, and $\{\omega_j\}$ are independent random variables taking values in $\{-1, 1\}$ with equal probability. Let $\mathrm{d}\omega$ be the normalized Haar measure on the Cantor group $\Omega = \{-1, 1\}^N$. The sequence of probability measures $\left\{\dfrac{P_n\mathrm{d}\omega}{E(P_n)}\right\}$ is showed to converge weakly to a unique continuous measure on Ω, and the obtained measure is singular with respect to $\mathrm{d}\omega$.

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1 Introduction

The Riesz product is a kind of lacunary series of trigonometric. It is an important topic in the field of harmonic analysis. The classical Riesz product measure is first introduced on the circle group $T = R/Z$ by Riesz, and later generalized by Zygmund[1] as the weak limit of finite Riesz products
$$\prod_{1}^{N}(1 + a_n \cos(2\pi\lambda_n t))$$
as N tends to infinity, where a_n's are bounded by 1 and the integers λ_n's are lacunary in the sense $\lambda_{n+1}/\lambda_n \geq 3$. In other words, there is a Radon measure μ such that
$$\lim_{N\to\infty}\int_T f(t)\prod_{1}^{N}(1 + a_n\cos(2\pi\lambda_n t))\mathrm{d}t = \int_T f(t)\mathrm{d}\mu(t), \qquad \forall f \in C(T).$$
Moreover, this measure is continuous, that is,
$$\mu(\{t\}) = 0, \qquad \forall t \in T.$$

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Later, Hewitt and Zuckerman[2] defined Riesz products on a general non-discrete compact abelian group. A short description of their approach is as follows.

Let G be a nondiscrete compact abelian group with discrete dual group Γ, Λ be a subset of Γ, and $W(\Lambda)$ be the set of all elements $\gamma \in \Gamma$ in the form of

$$\gamma = \lambda_1^{\epsilon_1} \lambda_2^{\epsilon_2} \cdots \lambda_n^{\epsilon_n}, \tag{1.1}$$

where $\epsilon_k \in \{-1, 1\}$ and λ_k are distinct elements of Λ. Suppose that Λ satisfies the requirement that each element of $W(\Lambda)$ has a unique representation of the form (1.1) up to the order of the factors, and let α be any complex function on Λ bounded by 1. For any finite set $\Phi \subset \Lambda$, define a Riesz product on G as follows:

$$P(\Phi, \alpha) = \prod \{1 + \mathrm{Re}[\alpha(\lambda)\lambda] : \lambda \in \Phi\}.$$

Hewitt and Zuckerman[2] showed that there exists a unique continuous probability measure $\mu_{\alpha,\lambda}$ on G which is the weak limit of $P(\Phi, \alpha)\mathrm{d}m$ in the topology of $M(G)$, where $M(G)$ is the convolution algebra of all Radon measures on G and m is the normalized Haar measure on G. A famous theorem of Kakutani[3] says that $\mu_{\alpha,\lambda}$ is either absolutely continuous or singular with respect to the Lebesgue-Haar measure on G, according to whether $\alpha \in l^2(\Lambda)$ or not.

The Riesz product is proved to be a source of powerful idea that can be used to produce concrete examples of measures with desired properties, such as singularity and multifractal structure. For the latter topic, refer to Peyriere[4] and Fan[5].

In this paper, we study a Riesz product type measure on the Cantor group. Throughout this paper, let

$$\Omega = \prod_1^\infty \Omega_j = \{-1, 1\}^N$$

be the cartesian product with all factors equal to

$$\Omega_j = \{-1, 1\}, \qquad \forall j \geq 1,$$

and write its elements

$$\varepsilon = (\varepsilon_n)_{n \in \mathbf{N}}$$

or

$$\varepsilon = \varepsilon_1 \varepsilon_2 \cdots$$

Ω is well known as an abelian group under the operation of pointwise product. With the discrete topology on each factor, the product topology on Ω makes it a compact abelian group, the so-called Cantor group. This topology can also be induced by a metric that the distance between two elements $\varepsilon = (\varepsilon_n)_{n \in \mathbf{N}}$, $\delta = (\delta_n)_{n \in \mathbf{N}}$ in Ω equals to

$$2^{-\inf\{n : \varepsilon_i = \delta_i, 0 \leq i \leq n, \varepsilon_{n+1} \neq \delta_{n+1}\}}.$$

Denote the projection $\omega_n : \Omega \to \{-1, 1\}$ by

$$\omega_n(\varepsilon) = \varepsilon_n.$$

Elements in the dual group Γ of Ω, which are continuous group homomorphisms from Ω into the multiplicative group of complex numbers of modulus 1, are provided by the projection functions. Precisely, let

$$\mathcal{R} = \{\omega_n : n \in \mathbf{N}\} \subset \Gamma.$$

Then each nontrivial element of Γ can be uniquely written as

$$\omega_{j_1}\omega_{j_2}\cdots\omega_{j_k}, \qquad 1 \le j_1 < j_2 < \cdots < j_k < \infty.$$

Note that for the normalized Haar measure m on Ω, $\{\omega_j\}$ may be viewed as independent random variables taking values in $\{-1, 1\}$ with equal probability. We write dm as dω, and the Haar measure on $\Omega_j = \{-1, 1\}$ by dω_j in the sequel.

Let $M(\Omega)$ be the convolution algebra of all Radon measure on Ω. As usual, we define the Fourier transform of $\mu \in M(\Omega)$ by

$$\hat{\mu}(\gamma) = \int_\Omega \gamma \mathrm{d}\mu, \qquad \gamma \in \Gamma.$$

The following result due to Lévy is needed in the next section.

Theorem 1.1 *Let G be a nondiscrete metrizable compact abelian group with discrete dual group Γ and let $\{\mu_n\}$ be a sequence of probability measures on G. If $\hat{\mu}_n$ converges everywhere in Γ and defines a limit function f, then μ_n converges weakly to a probability measure μ on G, and $f = \hat{\mu}$.*

The classical Riesz product measure on Ω is of the form

$$\prod_{j\ge1}(1 + a_j\omega_j)\mathrm{d}\omega, \qquad a_j \in \mathbf{R},\ |a_j| < 1. \tag{1.2}$$

As we have known, it is a continuous probability measure, and is either absolutely continuous or singular with respect to the normalized Haar measure m on Ω according to whether $\{a_j\}$ is square summable or not. Moreover, if a_j are all constants, the dimension and multifractal structure of μ are completely known (see [6]). Now it is natural to consider the following products:

$$P_n = \prod_{j=1}^{n}(1 + a_j\omega_j + b_j\omega_{j+1}), \qquad n \ge 1, \tag{1.3}$$

where a_j, b_j are real numbers and

$$|a_j| + |b_j| < 1, \qquad \forall j \ge 1.$$

They are generalization of classical Riesz product measure on Ω and give birth to essentially different properties compared with the classical ones.

In the present we consider the case that a_j, b_j are constants. The article is arranged as follows: in Section 2, we show that $\left\{\dfrac{P_n \mathrm{d}\omega}{E(P_n)}\right\}$ converges to a certain probability measure in the weak topology of $M(\Omega)$, and that the measure is continuous. Singularity with respect to the normalized Haar measure on Ω is studied in Section 3. A brief discussion is given in Section 4.

2 A Measure

Consider the finite products on Ω

$$P_n = \prod_{j=1}^{n}(1 + a\omega_j + b\omega_{j+1}), \qquad n \ge 1, \tag{2.1}$$

where a, b are two real numbers with

$$|a| + |b| < 1.$$

Denote

$$p_n = E(P_n) = \int_\Omega P_n \mathrm{d}\omega.$$

We wish to prove that the sequence of probability measures $\left\{\mu_n = \dfrac{P_n}{p_n}\mathrm{d}\omega\right\}$ converges to a measure in the weak topology of $M(\Omega)$.

For $1 \le k \le n$, let

$$P_{k,n} = \prod_{j=k}^{n}(1 + a\omega_j + b\omega_{j+1}).$$

Then

$$\int_{\Omega_k} \cdots \int_{\Omega_1} P_n \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_k = (u_k + v_k\omega_{k+1})P_{k+1,n}, \qquad 1 \le k \le n-1, \tag{2.2}$$

where u_k, v_k are real numbers independent of ω, and satisfy the following relations:

$$u_{k+1} = u_k + av_k, \qquad v_{k+1} = bu_k, \qquad u_1 = 1, \qquad v_1 = b. \tag{2.3}$$

To see this, first we have

$$\int_{\Omega_1} P_n \mathrm{d}\omega_1 = \int_{\Omega_1} (1 + a\omega_1 + b\omega_2)P_{2,n}\mathrm{d}\omega_1 = (1 + b\omega_2)P_{2,n}.$$

If (2.2) holds for $k \le n-2$, then

$$\begin{aligned}
&\int_{\Omega_{k+1}} \cdots \int_{\Omega_1} P_n \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_{k+1} \\
&= \int_{\Omega_{k+1}} (u_k + v_k\omega_{k+1})(1 + a\omega_{k+1} + b\omega_{k+2})P_{k+2,n}\mathrm{d}\omega_{k+1} \\
&= [(u_k + av_k) + bu_k\omega_{k+2}]P_{k+2,n},
\end{aligned}$$

by induction we have (2.2).

(2.3) is equivalent to

$$u_{k+1} = u_k + abu_{k-1}, \qquad u_1 = 1, \qquad u_2 = 1 + ab, \tag{2.4}$$

and thus

$$u_k = \lambda^k \cdot \frac{1 - t^{k+1}}{1 - t}, \tag{2.5}$$

where

$$\lambda = \frac{1}{2}(1 + \sqrt{1 + 4ab}), \qquad \lambda' = \frac{1}{2}(1 - \sqrt{1 + 4ab}), \qquad t = \frac{\lambda'}{\lambda}. \tag{2.6}$$

Furthermore,

$$p_k = u_k = \lambda^k \cdot \frac{1 - t^{k+1}}{1 - t}, \tag{2.7}$$

which follows from that

$$\begin{aligned}
p_k &= \int_{\Omega} P_k \mathrm{d}\omega \\
&= \int_{\Omega_{k+1}} \cdots \int_{\Omega_1} P_k \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_{k+1} \\
&= \int_{\Omega_{k+1}} \int_{\Omega_k} (u_{k-1} + v_{k-1}\omega_k) P_{k,k} \mathrm{d}\omega_k \mathrm{d}\omega_{k+1} \\
&= \int_{\Omega_{k+1}} \int_{\Omega_k} (u_{k-1} + v_{k-1}\omega_k)(1 + a\omega_k + b\omega_{k+1}) \mathrm{d}\omega_k \mathrm{d}\omega_{k+1} \\
&= u_{k-1} + a v_{k-1} \\
&= u_k.
\end{aligned}$$

For convenience, we denote

$$p_0 = 1, \qquad p_{-1} = 0$$

which can be seen from the formula (2.7), though they are not defined in the beginning of this section.

Lemma 2.1 *For $1 \leq k \leq n$, we have*

(i) $E(P_{k,n}) = p_{n-k+1}$;

(ii) $E(\omega_{k+1} P_k) = b p_{k-1}$;

(iii) $E(\omega_k P_{k,n}) = a p_{n-k}$;

(iv) $E(\omega_k \omega_{n+1} P_n) = b E(\omega_k P_{n-1})$;

(v) $E(\omega_k P_n) = \dfrac{\lambda^{n-1}}{(1-t)^2}[a + b - bt^{k-1} - at^k - at^{n-k+1} - bt^{n-k+2} + (a+b)t^{n+1}]$.

Proof. (i)–(iv) These four formulas are easy to be established.

(v) For convenience, we denote

$$P_{n+1,n} = 1.$$

By (i)–(iii) we have

$$\begin{aligned}
E(\omega_k P_n) &= E(\omega_k P_{k-1}(1 + a\omega_k + b\omega_{k+1}) P_{k+1,n}) \\
&= E(\omega_k P_{k-1}) E(P_{k+1,n}) + a E(P_{k-1}) E(P_{k+1,n}) + b E(\omega_k P_{k-1}) E(\omega_{k+1} P_{k+1,n}) \\
&= b p_{k-2} p_{n-k} + a p_{k-1} p_{n-k} + a b^2 p_{k-2} p_{n-k-1}.
\end{aligned}$$

Substituting (2.7) into the right hand side of the above equation and using the equation

$$\lambda^2 = \lambda + ab,$$

we have the desired result.

By this lemma, we have

Proposition 2.1 $\left\{\mu_n = \dfrac{P_n}{p_n} \mathrm{d}\omega\right\}$ *converges to a probability measure μ in the weak topology of $M(\Omega)$.*

Proof. We prove that $\hat{\mu}_n$ converges everywhere in Γ and then apply Lévy theorem. Noticing that

$$|t| < 1, \qquad \frac{p_{n-k}}{p_n} \to \frac{1}{\lambda^k}$$

and

$$\frac{1}{p_n} E(\omega_k P_{k,n}) \to \frac{a}{\lambda^k} \qquad (\text{as } n \to \infty),$$

we have

$$\hat{\mu}_n(\omega_1) = \int_\Omega \omega_1 \mathrm{d}\mu_n = \frac{1}{p_n} \int_\Omega \omega_1 P_{1,n} \mathrm{d}\omega \to \frac{a}{\lambda} \qquad (\text{as } n \to \infty).$$

For $j \geq 2$,

$$\begin{aligned}
\hat{\mu}_n(\omega_j) &= \int_\Omega \omega_j \mathrm{d}\mu_n \\
&= \frac{1}{p_n} \int_\Omega \omega_j P_n \mathrm{d}\omega \\
&= \frac{1}{p_n} \int_\Omega P_{j-1} \omega_j (1 + a\omega_j + b\omega_{j+1}) P_{j+1,n} \mathrm{d}\omega \\
&= \frac{1}{p_n} \int_\Omega P_{j-1} (a + \omega_j + b\omega_j \omega_{j+1}) P_{j+1,n} \mathrm{d}\omega \\
&= \frac{1}{p_n} [a p_{j-1} p_{n-j} + E(P_{j-1}\omega_j) p_{n-j} + bE(P_{j-1}\omega_j) E(\omega_{j+1} P_{j+1,n})] \\
&\to \frac{1}{\lambda(1-t)} (a + b - bt^{j-1} - at^j) \quad (\text{as } n \to \infty).
\end{aligned}$$

For $1 \leq i < j$,

$$\begin{aligned}
\hat{\mu}_n(\omega_i \omega_j) &= \frac{1}{p_n} \int_\Omega \omega_i \omega_j P_n \mathrm{d}\omega \\
&= \frac{1}{p_n} \int_\Omega \omega_i \omega_j P_{j-1} (1 + a\omega_j + b\omega_{j+1}) P_{j+1,n} \mathrm{d}\omega \\
&= \frac{1}{p_n} \{p_{n-j} E(\omega_i \omega_j P_{j-1}) + a p_{n-j} E(\omega_i P_{j-1}) + bE(\omega_i \omega_j P_{j-1}) E(\omega_{j+1} P_{j+1,n})\} \\
&\to \frac{1}{\lambda^2 (1-t)^2} \{(a+b)^2 - (ab + b^2) t^{i-1} - (a^2 + ab) t^i - abt^{j-i-1} \\
&\quad - (a^2 + b^2) t^{j-i} - abt^{j-i+1} + (ab + b^2) t^{j-1} + (a^2 + ab) t^j\} \quad (\text{as } n \to \infty).
\end{aligned}$$

And similarly, for $1 \leq j_1 < j_2 < \cdots < j_k$,

$$\begin{aligned}
&\hat{\mu}_n(\omega_{j_1} \omega_{j_2} \cdots \omega_{j_k}) \\
&= \frac{1}{p_n} \int_\Omega \omega_{j_1} \omega_{j_2} \cdots \omega_{j_k} P_n \mathrm{d}\omega \\
&= \frac{1}{p_n} \int_\Omega P_{j_k - 1} \omega_{j_1} \omega_{j_2} \cdots \omega_{j_k} (1 + a\omega_{j_k} + b\omega_{j_k+1}) P_{j_k+1,n} \mathrm{d}\omega \\
&= \frac{1}{p_n} [p_{n-j_k} (E(\omega_{j_1} \cdots \omega_{j_k} P_{j_k - 1}) + aE(\omega_{j_1} \cdots \omega_{j_{k-1}} P_{j_k - 1})) \\
&\quad + bE(\omega_{j_1} \cdots \omega_{j_k} P_{j_k - 1}) E(\omega_{j_k+1} P_{j_k+1,n})] \\
&\to \frac{1}{\lambda^{j_k}} \left[E(\omega_{j_1} \cdots \omega_{j_k} P_{j_k - 1}) + aE(\omega_{j_1} \cdots \omega_{j_{k-1}} P_{j_k - 1}) + \frac{ab}{\lambda} E(\omega_{j_1} \cdots \omega_{j_k} P_{j_k - 1}) \right]
\end{aligned}$$

$$= \frac{1}{\lambda^{j_k}}[\lambda E(\omega_{j_1}\cdots\omega_{j_k}P_{j_k-1}) + aE(\omega_{j_1}\cdots\omega_{j_{k-1}}P_{j_k-1})] \quad (\text{as } n\to\infty).$$

Thus, we have proved that $\hat{\mu}_n$ converges everywhere and defines a limit function f in Γ. By Theorem 1.1,

$$f = \hat{\mu}$$

for some probability measure $\mu \in M(\Omega)$, and

$$\mu_n \xrightarrow{w} \mu.$$

Moreover, we have the following result.

Proposition 2.2 *μ is a continuous measure.*

Proof. Notice that μ is continuous if and only if

$$\mu(\{\varepsilon\}) = 0, \qquad \forall\varepsilon\in\Omega,$$

if and only if

$$\mu(\Omega_{\varepsilon|_n}) \to 0 \qquad \text{as } n\to\infty,\ \forall\varepsilon\in\Omega,$$

where

$$\varepsilon|_n = \varepsilon_1\varepsilon_2\cdots\varepsilon_n$$

for any $\varepsilon = \varepsilon_1\varepsilon_2\cdots\in\Omega$, and

$$\Omega|_{u_1u_2\cdots u_n} = \{\varepsilon = \varepsilon_1\varepsilon_2\cdots\in\Omega : \varepsilon_1 = u_1, \varepsilon_2 = u_2, \cdots, \varepsilon_n = u_n\}.$$

Since $\Omega_{\varepsilon|_n}$ is open and closed in Ω, we have

$$\begin{aligned}
\mu(\Omega_{\varepsilon|_n}) &= \lim_{m\to\infty}\mu_m(\Omega_{\varepsilon|_n})\\
&= \lim_{m\to\infty}\int_{\Omega_{\varepsilon|_n}}\frac{P_m}{p_m}\mathrm{d}\omega\\
&= \lim_{m\to\infty}\int_{\Omega_{\varepsilon|_n}}\frac{P_{n-1}P_{n,m}}{p_m}\mathrm{d}\omega\\
&= \lim_{m\to\infty}\frac{1}{p_m}\prod_{j=1}^{n-1}(1+a\varepsilon_j+b\varepsilon_{j+1})\int_{\Omega_{\varepsilon|_n}}(1+a\omega_n+b\omega_{n+1})P_{n+1,m}\mathrm{d}\omega\\
&= \frac{1}{2^n}\prod_{j=1}^{n-1}(1+a\varepsilon_j+b\varepsilon_{j+1})\lim_{m\to\infty}\frac{1}{p_m}\left[(1+a\varepsilon_n)p_{m-n}+bE(\omega_{n+1}P_{n+1,m})\right]\\
&= \frac{1}{(2\lambda)^n}(\lambda+a\varepsilon_n)\prod_{j=1}^{n-1}(1+a\varepsilon_j+b\varepsilon_{j+1}).
\end{aligned}$$

We only consider the case of $ab\neq 0$ because, in the case a or b is zero the corresponding measure is continuous, which has been discussed in Section 1.

If a, b have the same signs. Since

$$|a|+|b|<1<\lambda,$$

the right hand side in (2.8) tends to 0 as $n\to\infty$.

If a, b have the different signs, without loss of generality, we assume that

$$|a| \geq |b|.$$

Then

$$\max\left\{\prod_{j=1}^{n-1}(1+a\varepsilon_j+b\varepsilon_{j+1}) : \varepsilon \in \Omega\right\} = (1+|a|-|b|)^{n-2}(1+|a|+|b|).$$

But

$$1+|a|-|b| < 1+\sqrt{1-4|ab|} = 2\lambda,$$

and thus the right hand side in (2.8) also tends to 0 as $n \to \infty$. This completes the proof.

3 Singularity

In this section, we show that μ defined in the previous section is singular with respect to the normalized Haar measure m on Ω in case $a+b \neq 0$.

Set

$$g_0 = 1, \qquad g_1 = \omega_1 - \frac{a}{\lambda}, \qquad g_j = \omega_j - t\omega_{j-1} - \frac{a+b}{\lambda} \quad (j = 2, 3, \cdots). \tag{3.1}$$

Lemma 3.1 $\{g_j\}_{j\geq 0}$ *forms an orthogonal system in* $L^2(\mu)$*, and*

$$\int_\Omega |g_j|^2 \mathrm{d}\mu \leq M, \qquad j = 0, 1, 2, \cdots \tag{3.2}$$

for some constant $M > 0$.

Proof. By the calculation in Proposition 2.1, we have

$$\begin{aligned}
\hat{\mu}(\omega_1) &= \frac{a}{\lambda},\\
\hat{\mu}(\omega_j) &= \frac{1}{\lambda(1-t)}\{a+b-bt^{j-1}-at^j\} \quad (j \geq 2),\\
\hat{\mu}(\omega_i\omega_j) &= \frac{1}{\lambda^2(1-t)^2}\{(a+b)^2-(ab+b^2)t^{i-1}-(a^2+ab)t^i-abt^{j-i-1}\\
&\quad -(a^2+b^2)t^{j-i}-abt^{j-i+1}+(ab+b^2)t^{j-1}+(a^2+ab)t^j\}
\end{aligned}$$

for $1 \leq i \leq j$.

By the above formulas, a straightforward calculation gives

$$\int_\Omega g_i g_j \mathrm{d}\mu = 0, \qquad i, j \geq 0,\ i \neq j,$$

which indicates that $\{g_j\}_{j\geq 0}$ forms an orthogonal system in $L^2(\mu)$.

Now we prove (3.2). Since

$$\int_\Omega |g_0|^2 \mathrm{d}\mu = 1,$$

$$\int_\Omega |g_1|^2 \mathrm{d}\mu = 1 - \frac{a^2}{\lambda^2} \leq 1,$$

$$\begin{aligned}\int_{\Omega}|g_j|^2\mathrm{d}\mu &= \int_{\Omega}(\omega_j - t\omega_{j-1})^2\mathrm{d}\mu - \left(\frac{a+b}{\lambda}\right)^2\\ &= 1+t^2 - \left(\frac{a+b}{\lambda}\right)^2 - 2t\hat{\mu}(\omega_{j-1}\omega_j)\\ &\to 1+t^2 - \left(\frac{a+b}{\lambda}\right)^2 - \frac{2t}{\lambda^2(1-t)}[a^2+b^2+ab(1+t)] \quad (\text{as } j\to\infty)\\ &= \frac{1+t}{\lambda^2(1-t)}(\sqrt{1+4ab}+a+b)(\sqrt{1+4ab}-a-b)\\ &> 0, \qquad j=2,3,\cdots,\end{aligned}$$

we have

$$\int_{\Omega}|g_j|^2\mathrm{d}\mu \le M, \qquad j=0,1,2,\cdots$$

for some $M>0$.

Proposition 3.1 *If $a+b\neq 0$, the measures μ and the normalized Haar measure m on Ω are mutually singular.*

Proof. Let $\{c_j\}_{j\ge 1}$ be a sequence in l^2, but not in l^1; for example, take $c_j = \frac{1}{j}$. Then the series $\sum\limits_{j\ge 1} c_j g_j$ converges in $L^2(\mu)$. We also have $\left\{d_j = c_j - \frac{\lambda'}{\lambda}c_{j+1}\right\}_{j\ge 1} \in l^2$ since l^2 is a vector space, whence the series $\sum\limits_{j\ge 1} d_j\omega_j$ converges in $L^2(m)$. So we can choose a subsequence of positive integers $\{N_k\}$ such that $\sum\limits_{1\le j\le N_k} c_j g_j$ converges for μ-a.e. $\varepsilon\in\Omega$ and $\sum\limits_{1\le j\le N_k} d_j\omega_j$ converges for m-a.e. $\varepsilon\in\Omega$, respectively, as k tends to infinity. If μ and m are not mutually singular, then there exists $\varepsilon\in\Omega$ such that these two series converge at the same time. But

$$\sum_{1\le j\le N_k} c_j g_j - \sum_{1\le j\le N_k} d_j\omega_j = \frac{bc_1}{\lambda} + tc_{N_k+1}\omega_{N_k} - \frac{a+b}{\lambda}\sum_{1\le j\le N_k} c_j,$$

which implies that the limit $\lim\limits_{k\to\infty}\sum\limits_{1\le j\le N_k} c_j$ exists and is finite, a contradiction.

4 Discussions

(i) By formula (2.8) we know that μ is a quasi-Bernoulli measure on Ω. The fractal analysis and the validity of multifractal formalism of such a measure were studied extensively by Brown *et al.*[7].

(ii) Our approach may be applied to the products

$$P_n = \prod_{j=1}^{n}(1+a\omega_j + b\omega_{j+1} + c\omega_{j+2}),$$

which can have even more items in the bracket, where a, b, c are real numbers with

$$|a|+|b|+|c|<1.$$

(iii) For the general case of

$$P_n = \prod_{j=1}^{n}(1+a_j\omega_j + b_j\omega_{j+1}),$$

where $\{a_j\}$, $\{b_j\}$ are two sequences of real numbers with $|a_j| + |b_j| < 1$ and additional conditions such as periodicity, uniformly distribution, etc., does $\left\{\mu_n = \dfrac{P_n}{p_n}\mathrm{d}\omega\right\}$ also converge to certain measures in the weak topology of $M(\Omega)$? If these measures exist, what properties do they possess? These are left to be discussed in future publications.

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