Communications in Mathematical Research
26(1)(2010), 76–84

The Sufficient and Necessary Condition of Lagrange Stability of Quasi-periodic Pendulum Type Equations*

Cong Fu-zhong[1,2], Liang Xin[1] and Han Yue-cai[1]
(*1. Fundamental Department, Aviation University of Air Force, Changchun, 130022*)
(*2. Institute of Mathematics, Jilin University, Changchun, 130012*)

Communicated by Li Yong

Abstract: The quasi-periodic pendulum type equations are considered. A sufficient and necessary condition of Lagrange stability for this kind of equations is obtained. The result obtained answers a problem proposed by Moser under the quasi-periodic case.

Key words: Lagrange stability, pendulum type equation, KAM theorem
2000 MR subject classification: 37J40
Document code: A
Article ID: 1674-5647(2010)01-0076-09

1 Introduction

The Lagrange stability of pendulum type equations is an important topic, which is proposed by Moser[1]. Moser[2], Levi[3] and You[4] investigated such topic for the periodic situation, respectively. In particular, You obtained a sufficient and necessary condition for Lagrange stability of the equation (1.1) in [4].

Recently, Bibikov[5] developed a KAM theorem for nearly integrable Hamiltonian systems with one degree of freedom under the quasi-periodic perturbation. In fact, his KAM theorem is of parameter type. Using this theorem he discussed the stability of equilibrium of a class of the second order nonlinear differential equations.

In this note we study quasi-periodic pendulum type equations. Under the standard Diophantine condition of frequency ω, a sufficient and necessary condition of Lagrange stability for quasi-periodic pendulum type equations is obtained. This answers Moser's problem under the quasi-periodic case.

*Received date: April 21, 2009.
Foundation item: Partially supported by the NSF (10871203, 10601019) of China and the NCET (07-0386) of China.

We consider a nonlinear pendulum type equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+p(t,x)=0, \tag{1.1}$$

where

$$p(t,x+1)=p(t,x),$$

and $p(t,x)$ is a quasi-periodic function in t with basic frequencies $\omega=(\omega_1,\cdots,\omega_n)$, that is,

$$p(t,x)=f(\omega t,x) \tag{1.2}$$

for some function $f(\theta,x)$ defined on $T^n\times T^1$. Here $T^n=R^n/Z^n$ is an n-dimensional torus.

Assume that $f(\theta,x)$ is a real analytic function on $T^n\times T$ and the frequency ω satisfies Diophantine condition as follows:

$$|\langle k,\omega\rangle|\geq\gamma|k|^{-(n+1)},\qquad 0\neq k\in Z^n \tag{1.3}$$

for a given $\gamma>0$, where $\langle\,\cdot\,,\cdot\,\rangle$ denotes the usual inner product.

We are in a position to state the main result of this paper.

Theorem 1.1 *Assume that* (1.3) *holds. Then system* (1.1) *is Lagrange stable if and only if*

$$\int_{T^n\times T^1}f(\theta,x)\mathrm{d}\theta\mathrm{d}x=0. \tag{1.4}$$

Moreover, if (1.3) *and* (1.4) *hold, equation* (1.1) *possesses infinitely many quasi-periodic solutions with* $n+1$ *basic frequencies* (*including* $\omega_1,\cdots,\omega_n$).

- Diophantine condition (1.3) can be replaced by a general form
$$|\langle k,\omega\rangle|\geq\gamma|k|^{-\tau_*},\qquad 0\neq k\in Z^n \tag{1.5}$$
with some constant $\tau_*>n$. Here we assume (1.3) for the convenience of the proof of Theorem 1.1.

- Huang[6] considered a class of almost periodic pendulum-type equations. He proved the existence of unbounded solutions of the equations. Summing up the works developed by Mose[2], Levi[3], You[4] and Huang[6], respectively, and Theorem 1.1, we can obtain a satisfactory answer to Moser's problem.

- Recently, Lin and Wang[7] have concerned with a dual quasi-periodic system as follows:
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\frac{\partial g}{\partial x}(t,x)=0, \tag{1.6}$$
where $g(t,x)$ is quasi-periodic in t and x with frequencies $\Omega^1=(\omega_1,\cdots,\omega_n)$ and $\Omega^2=(\omega_{n+1},\cdots,\omega_{n+m})$, respectively. Under the assumptions
$$\begin{aligned}(\Omega^1,\Omega^2)\in O_\gamma=\big\{(\Omega^1,\Omega^2)\in R^{n+m}:|\langle k,\Omega^1\rangle+\langle l,\Omega^2\rangle|\geq\gamma(|k|+|l|)^{-\tau_*},\\ \forall\ 0\neq(k,l)\in Z^{n+m},\ \tau_*>n+m\big\}\end{aligned}$$
and
$$\forall\ j\in N,\ \exists\ A(j)\geq j,\qquad \text{s.t. }(\Omega^1,A(j)\Omega^2)\in O_\gamma,$$
they proved that all the solutions of (1.6) are bounded (see [7]). It is easy to find that as $m=1$, their modified Diophantine condition is stronger than (1.5); in addition, the result of [7] is a sufficient condition to ensure Lagrange stability. This differs from Theorem 1.1.

- In [8] and [9], the authors developed a quasi-periodic monotonic twist theorem. For the sake of simplicity, we should apply Bibikov's lemma to prove our theorem, but apply the monotonic twist theorem.

2 A KAM Theorem under Quasi-periodic Perturbations

In this section we give a KAM theorem with a quasi-periodic perturbation by using Bibikov's lemma (see [5]).

Let us consider a Hamiltonian system with Hamiltonian

$$H(x,y,\omega t)=\frac{1}{2}y^2+P(x,y,\omega t), \tag{2.1}$$

where $P(x,y,\theta)$ is a function defined on $T^1\times R^1\times T^n$, and ω satisfies (1.3). Assume that P is real analytic, that is, there is $\delta>0$ such that P is analytic on $(T^1\times R^1\times T^n)+\delta$. Here $D+\delta$ is a complex neighborhood of D in C^r for any given subset D in R^r and fixed $\delta>0$. Let

$$A_\gamma=\big\{\tau\in R^1:|\langle k,\omega\rangle+l\tau|>\gamma(|k|+|l|)^{-(n+1)},\,0\neq(k,l)\in Z^n\times Z\big\}.$$

Theorem 2.1 *There exists $\varepsilon_0>0$ depending only on γ, δ and n such that, for any interval (a,b), if $|P|\le\varepsilon_0$ on $(T^1\times(a,b)\times T^n)+\delta$, the following conclusions hold*:

1) $\mathrm{meas}(A_\gamma\bigcap(a,b))\to b-a$, *as* $\gamma\to 0^+$;

2) *for every $\tau\in A_\gamma\bigcap(a,b)$, system* (2.1) *possesses an invariant torus $I_{(\tau,\omega)}$, which is full of quasi-periodic motions with frequency (τ,ω). Moreover, this torus is a drift of $T^1\times\{y=\tau\}\times T^n$ under some nearly identical transformation of coordinates.*

Consider an auxiliary Hamiltonian of the form

$$H(\varphi,r,\omega t,\tau)=\tau r+P(\varphi,r,\omega t,\tau) \tag{2.2}$$

with a parameter τ. Assume that $P(\varphi,r,\theta,\tau)$ is real analytic on

$$D_0=\big\{(\varphi,r,\theta,\tau):|\mathrm{Im}(\varphi,\theta)|<\iota_0,\ \mathrm{Re}(\varphi,\theta)\in T\times T^n,\ |r|<\delta_0,\tau\in A_\gamma+\frac{1}{2}\gamma\delta_0\big\}.$$

In order to prove Theorem 2.1 we need the following Bibikov's lemma.

Bibikov's Lemma[5] 1) *There exits $\epsilon_0>0$ depending only on δ_0, ι_0 and n such that if $|P|<\epsilon_0$ on D_0, then there exist a function $\tau_0:A_\gamma\to R$ and a nearly identical transformation of coordinates*

$$\varphi=\psi+u(\psi,\omega t,\alpha),\qquad r=\rho+v(\psi,\rho,\omega t,\alpha),\ \alpha\in A_\gamma,$$

which reduces Hamiltonian system

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t}=\tau+\frac{\partial P}{\partial r}(\varphi,r,\omega t,\tau),\qquad \frac{\mathrm{d}r}{\mathrm{d}t}=-\frac{\partial P}{\partial\varphi}(\varphi,r,\omega t,\tau)$$

to the following form:

$$\frac{\mathrm{d}\psi}{\mathrm{d}t}=\alpha+\varPsi(\psi,\rho,\omega t,\alpha),\qquad \frac{\mathrm{d}\rho}{\mathrm{d}t}=B(\psi,\rho,\omega t,\alpha)$$

with $\tau=\tau_0(\alpha)$ to satisfy

$$B(\psi,0,\omega t,\alpha)=\frac{\partial B}{\partial\rho}(\psi,0,\omega t,\alpha)=\varPsi(\psi,0,\omega t,\alpha)=0.$$

2) $\mathrm{meas}(R^1\backslash A_\gamma)\to 0$, *as* $\gamma\to 0^+$.

Proof of Theorem 2.1 Let $\varepsilon = \sup\limits_{(x,y,\theta)\in(T^1\times(a,b)\times T^n)+\delta} |P(x,y,\theta)|$. Taking a parameter $\tau \in (a,b)$, we introduce a transformation

$$x = \widetilde{x}, \qquad y = \tau + \sqrt{\varepsilon}\widetilde{y} \tag{2.3}$$

and construct a new Hamiltonian

$$\begin{aligned}\widetilde{H}(\widetilde{x},\widetilde{y},\omega t,\tau) &= \frac{1}{\sqrt{\varepsilon}}\left(H(x,y,\omega t) - \frac{1}{2}\tau^2\right)\\ &= \tau\widetilde{y} + \sqrt{\varepsilon}\left(\frac{1}{2}\widetilde{y}^2 + \frac{1}{\varepsilon}P(\widetilde{x},\tau+\sqrt{\varepsilon}\widetilde{y},\omega t)\right)\\ &= \tau\widetilde{y} + \sqrt{\varepsilon}\widetilde{P}(\widetilde{x},\widetilde{y},\omega t,\tau),\end{aligned} \tag{2.4}$$

where

$$\widetilde{P}(\widetilde{x},\widetilde{y},\omega t,\tau) = \frac{1}{2}\widetilde{y}^2 + O(1).$$

By applying Bibikov's lemma to (2.4) on $\left(\left(T^1 \times \left(-\frac{\delta}{2},\frac{\delta}{2}\right)\times T^n\right) + \frac{\delta}{2}\right) \times \left((A_\gamma \bigcap (a,b)) + \frac{\gamma\delta}{4}\right)$, we can prove Theorem 2.1.

3 Some Lemmas

This section is devoted to established some lemmas which will be used in the proof of Theorem 1.1.

Write

$$h(\theta) = -\int_{T^1} f(\theta,x)\mathrm{d}x, \qquad G(\theta,x) = \int_0^x f(\theta,s)\mathrm{d}s + h(\theta)x. \tag{3.1}$$

Then

$$G(\theta+e_i,x) = G(\theta,x) = G(\theta,x+1), \qquad e_i = (\delta_1^i,\delta_2^i,\cdots,\delta_n^i), \quad i=1,2,\cdots,n.$$

Here $\delta_i^i = 1$ and $\delta_j^i = 0$ as $i \neq j$. By using these notations, (1.1) can be rewritten as the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{\partial G}{\partial x}(\omega t,x) = h(\omega t). \tag{3.2}$$

Equation (3.2) is equivalent to the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y + \int_0^t h(\omega s)\mathrm{d}s, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{\partial G}{\partial x}(\omega t,x), \tag{3.3}$$

which is a Hamiltonian system with Hamiltonian

$$\widetilde{H}(x,y,t) = \frac{1}{2}y^2 + y\int_0^t h(\omega s)\mathrm{d}s + G(\omega t,x). \tag{3.4}$$

Because of the real analyticity of $f(\theta,x)$ there is a positive constant δ such that $h(\theta)$ and $G(\theta,x)$ are analytic on $T^n+\delta$ and on $(T^n\times T^1)+\delta$, respectively. It is clear that there is $M_0 > 0$ satisfying

$$\max\left\{|h(\theta)|,|G(\theta,x)|,\left|\frac{\partial G}{\partial\theta}(\theta,x)\right|\right\} < M_0, \qquad \forall(\theta,x)\in(T^n\times T^1)+\delta. \tag{3.5}$$

Lemma 3.1 *Assume* (1.3) *and* (1.4). *Then, Hamiltonian* (3.4) *is the following*

$$\widetilde{H}(x,y,t) = H(x,y,\omega t) = \frac{1}{2}y^2 + y\widetilde{h}(\omega t) + G(\omega t, x), \tag{3.6}$$

and for all $(\theta, x) \in (T^n \times T^1) + \dfrac{\delta}{2}$,

$$\max\{|\widetilde{h}(\theta)|, |G(\theta,x)|\} < M \tag{3.7}$$

with some positive constant M. *Here*

$$\widetilde{h}(\omega t) = \int_0^t h(\omega t)\mathrm{d}t.$$

Proof. Let the Fourier's expansion of h be

$$h(\theta) = \sum_{k\in Z^n} h_k \mathrm{e}^{2\pi\sqrt{-1}\langle k,\theta\rangle}. \tag{3.8}$$

By (3.1) and (1.4),

$$h_0 = \int_{T^n} h(\theta)\mathrm{d}\theta = 0.$$

Hence,

$$\int_0^t h(\omega t)\mathrm{d}t = \sum_{0\neq k\in Z^n} \frac{h_k}{2\pi\sqrt{-1}\langle k,\omega\rangle}\left(\mathrm{e}^{2\pi\sqrt{-1}\langle k,\omega t\rangle} - 1\right) :\overset{\theta=\omega t}{=} \widetilde{h}(\theta). \tag{3.9}$$

From (3.9), (1.3), (3.5) and Cauchy's formula, on $T^n + \dfrac{\delta}{2}$,

$$\begin{aligned} |\widetilde{h}(\theta)| &\le \sum_{0\neq k\in Z^n} \frac{|h_k|}{|\langle k,\omega\rangle|}\left|\mathrm{e}^{2\pi\sqrt{-1}\langle k,\omega t\rangle} - 1\right| \\ &\le \frac{2M_0}{\gamma}\sum_{j=1}^{\infty}\frac{2^n j^{2n}}{\mathrm{e}^{\frac{\pi\delta j}{2}}} \\ &\le \frac{2^{n+1}}{\gamma}\left(\frac{4n+4}{\delta \mathrm{e}\pi}\right)^{2n+2}\sum_{j=1}^{\infty}\frac{1}{j^2}M_0. \end{aligned} \tag{3.10}$$

Here the inequality

$$j^{2n+2}\mathrm{e}^{\dfrac{-\pi\delta j}{2}} \le \left(\frac{4n+4}{\delta\mathrm{e}\pi}\right)^{2n+2}$$

which is obtained by finding the maximum of the function $l(x) = x^{2n+2}\mathrm{e}^{\frac{-\pi\delta x}{2}}$ $(0 < x < \infty)$, is used.

From (3.10) and (3.5), it follows that, on $(T^n \times T^1) + \dfrac{\delta}{2}$, (3.7) holds. Thus, we end the proof of Lemma 3.1.

Let

$$\sigma_1 = 2\sqrt{2M+1}, \qquad \sigma_2 = \frac{\sigma_1 + \sqrt{\sigma_1^2 - 8M}}{2}.$$

Lemma 3.2 *There exists a canonical coordinate transformation* Φ *depending periodically on parameter* $\theta \in T^n$ *of the form*

$$\Phi : x = X + u(X,Y,\theta), \qquad y = Y - [\tilde{h}]_\theta + v(X,Y,\theta)$$

such that if $y \notin (-\sigma_1, \sigma_1)$, *one has*

$$|y - Y + [\tilde{h}]_\theta| < \frac{2M}{|Y - [\tilde{h}]_\theta|}, \tag{3.11}$$

where

$$[\widetilde{h}]_\theta = \int_{T^n} \widetilde{h}(\theta)\mathrm{d}\theta.$$

Moreover, (3.6) *is changed to*

$$H_{++}(X, Y, \theta) = \frac{1}{2}Y^2 + P(X, Y, \theta) \tag{3.12}$$

with the estimate

$$|P| < \frac{c}{|Y - [\tilde{h}]_\theta|}$$

on $(T^1 \times (R^1 \backslash (-\sigma_2 + [\tilde{h}]_\theta, \sigma_2 + [\tilde{h}]_\theta)) \times T^n) + \frac{\delta}{3}$. *Here* c *is a positive constant depending only on* δ, M, M_0 *and* ω.

Proof. First we construct a symplectic coordinate transformation Φ_1 by a generating function $xY_* + S_1(x, Y_*, \theta)$, that is,

$$\Phi_1 : X_* = x + \frac{\partial S_1}{\partial Y_*}, \qquad y = Y_* + \frac{\partial S_1}{\partial x}.$$

Then the new Hamiltonian is

$$\begin{aligned} H_+ &= H \circ \Phi_1 + \left\langle \frac{\partial S_1}{\partial \theta}, w \right\rangle \\ &= \frac{1}{2}\left(Y_* + \frac{\partial S_1}{\partial x}\right)^2 + \left(Y_* + \frac{\partial S_1}{\partial x}\right)\tilde{h} + G + \left\langle \frac{\partial S_1}{\partial \theta}, w \right\rangle \\ &= \frac{1}{2}Y_*^2 + Y_*\tilde{h} + \frac{\partial S_1}{\partial x}\tilde{h} + \frac{1}{2}\left(\frac{\partial S_1}{\partial x}\right)^2 + Y_*\frac{\partial S_1}{\partial x} + G + \left\langle \frac{\partial S_1}{\partial \theta}, w \right\rangle. \end{aligned}$$

Let

$$Y_*\frac{\partial S_1}{\partial x} + G(\theta, x) = [G]_x(\theta), \tag{3.13}$$

where

$$[G]_x(\theta) = \int_{T^1} G(\theta, x)\mathrm{d}x.$$

This leads to

$$S_1(x, Y_*, \theta) = -\frac{1}{Y_*}\int_0^x (G(\theta, s) - [G]_x(\theta))\mathrm{d}s. \tag{3.14}$$

By (3.14), (3.5) and the definition of Φ_1,

$$|y| \le |Y_*| + \frac{2M}{|Y_*|}.$$

Hence, as $|y| > \sigma_1$, we have

$$|Y_*| > \sigma_2. \tag{3.15}$$

According to

$$\int_0^{x+1} (G(\theta, s) - [G]_x(\theta))\mathrm{d}s = \int_0^x (G(\theta, s) - [G]_x(\theta))\mathrm{d}s \tag{3.16}$$

and (3.15), we assert that S_1 is defined on $(T^1\times(R^1\backslash(-\sigma_2,\sigma_2))\times T^n)+\dfrac{\delta}{2}$ for a small positive number δ. Denote

$$\widetilde{P}(X_*,Y_*,\theta)=\frac{\partial S_1}{\partial x}\widetilde{h}(\theta)+\frac{1}{2}\left(\frac{\partial S_1}{\partial x}\right)^2+\left\langle\frac{\partial S_1}{\partial\theta},\omega\right\rangle,\tag{3.17}$$

which and (3.13) imply that

$$H_+=\frac{1}{2}Y_*^2+Y_*\widetilde{h}(\theta)+\widetilde{P}(X_*,Y_*,\theta),\tag{3.18}$$

where we ignore $[G]_x$ in Hamiltonian because that $[G]_x$ is independent of X_* and Y_*. By (3.5), (3.7), (3.14) and (3.17), we have

$$|y-Y_*|\le\frac{2M}{|Y_*|},\qquad|\widetilde{P}|\le\frac{c}{|Y_*|},\tag{3.19}$$

where c is a positive constant depending only on δ, M, M_0 and ω.

Now introduce the second transformation $\Phi_2:(X,Y,\theta)\to(X_*,Y_*,\theta)$ by a generating function $X_*(Y-[\tilde{h}]_\theta)+(Y-[\tilde{h}]_\theta)S_2(\theta)$. This shows that Φ_2 satisfies the following formula:

$$\Phi_2:X=X_*+S_2(\theta),\qquad Y_*=Y-[\tilde{h}]_\theta.\tag{3.20}$$

Inserting (3.20) into (3.18) we reduce H_+ into

$$\begin{aligned}H_{++}&=H_+\circ\Phi_2+\left(Y-[\tilde{h}]_\theta\right)\left\langle\frac{\partial S_2}{\partial\theta},w\right\rangle\\&=\frac{1}{2}\left(Y-[\tilde{h}]_\theta\right)^2+\tilde{P}(X-S_2(\theta),Y-[\tilde{h}]_\theta,\theta)\\&\quad+\left(Y-[\tilde{h}]_\theta\right)\left(\tilde{h}(\theta)+\left\langle\frac{\partial S_2}{\partial\theta},w\right\rangle\right).\end{aligned}$$

Denote $\Phi=\Phi_1\circ\Phi_2$. Let

$$\widetilde{h}(\theta)+\left\langle\frac{\partial S_2}{\partial\theta},\omega\right\rangle=[\widetilde{h}]_\theta.\tag{3.21}$$

Write $\widetilde{h}-[\widetilde{h}]_\theta$ and S_2 in the Fourier series form

$$\begin{aligned}\widetilde{h}(\theta)-[\widetilde{h}]_\theta&=\sum_{0\neq k\in Z^n}\widetilde{h}_k\mathrm{e}^{2\pi\sqrt{-1}\langle k,\theta\rangle},\\S_2(\theta)&=\sum_{0\neq k\in Z^n}S_{2k}\mathrm{e}^{2\pi\sqrt{-1}\langle k,\theta\rangle}.\end{aligned}$$

By comparing the coefficients in the Fourier expansions of $\widetilde{h}-[\widetilde{h}]_\theta$ and S_2, we derive that (3.21) has a unique real analytic solution

$$S_2(\theta)=-\sum_{0\neq k\in Z^n}\frac{\widetilde{h}_k}{2\pi\langle k,\omega\rangle}\mathrm{e}^{2\pi\sqrt{-1}\langle k,\theta\rangle}$$

with

$$S_2(0)=[S_2]_\theta=0.$$

Similar to proving (3.10), we have

$$\max\left\{|S_2|,\left|\frac{\partial S_2}{\partial\theta}\right|\right\}<c_1\tag{3.22}$$

on $T^{n+1}+\dfrac{\delta}{3}$ for some positive constant c_1. Put

$$P(X,Y,\theta)=\widetilde{P}(X-S_2(\theta),Y-[\widetilde{h}]_\theta,\theta).$$

From (3.22), (3.20) and (3.19), we get the conclusion of the lemma.

4 Proof of Theorem 1.1

We first prove the sufficiency. For any $x_0 \in T^1$ and $y_0 \in R^1$, let $X_0 \in T^1$ and $Y_0 \in R^1$ be the corresponding coordinates under change Φ in Lemma 3.2. We denote by $(X(t, X_0, Y_0)$, $Y(t, X_0, Y_0))$ a solution of (3.12) with $X(0) = X_0, Y(0) = Y_0$. Consider Hamiltonian (3.6) on $(T^1 \times (Y_0 - 1, Y_0 + 1) \times T^n) + \dfrac{\delta}{3}$. Assume that $|Y_0|$ is large enough so that $(Y_0 - 1, Y_0 + 1) \subset R^1 \backslash (-\sigma_2 + [\widetilde{h}]_\theta, \sigma_2 + [\widetilde{h}]_\theta)$ and

$$|P| < \varepsilon_0. \tag{4.1}$$

By Lemma 3.2 and Theorem 2.1, there are Y_1, $Y_2 \in (Y_0 - 1, Y_0 + 1)$ with $Y_1 < Y_0 < Y_2$ such that two invariant tori $I_{(Y_1,\omega)}$ and $I_{(Y_2,\omega)}$ confine the solution $(X(t, X_0, Y_0), Y(t, X_0, Y_0))$ in the domain enclosed by them (in fact, in the coordinates (X, Y, t), the invariant torus is a drift of the elliptic cylinder with t-axis). Thus, for all time t,

$$M_1 < |Y(t, X_0, Y_1)| \le |Y(t, X_0, Y_0)| \le |Y(t, X_0, Y_2)| < M_2 \tag{4.2}$$

for some positive constants M_1 and M_2. By (3.11), we have

$$|y(t, x_0, y_0)| < M_3 \tag{4.3}$$

with some positive constant $M_3 > 0$. According to (3.3), (4.2) and (3.10),

$$|x'(t, x_0, y_0)| < |y(t, x_0, y_0)| + \left| \int_0^t h(\omega t) \mathrm{d}t \right| < M_4$$

for some positive constant M_4 depending on Y_0, which implies that (1.4) is a sufficient condition for the Lagrange stability of (1.1).

If Y_0 cannot ensure (4.1) to hold, we choose Y_+ such that $|Y_+| > |Y_0|$ and (4.1) holds on $(T^1 \times (Y_+ - 1, Y_+ + 1) \times T^n) + \dfrac{\delta}{3}$. A discussion similar to the above shows that $Y(t, X_0, Y_+)$ is bounded. Hence, $Y(t, X_0, Y_0)$ is also bounded from the uniqueness of solutions. This also proves the sufficient part of the theorem.

Now return to prove the necessary. Assume that (1.1) is Lagrange stable and

$$\int_{T^{n+1}} f(\theta, x) \mathrm{d}\theta \mathrm{d}x = h_0 \ne 0.$$

Without loss of generality, let $h_0 > 0$. By (3.9) and (3.10), we have

$$\left| \int_0^t (h(\omega t) - h_0) \mathrm{d}t \right| \le M_5 \tag{4.4}$$

for some positive constant M_5.

Note that (3.2) is also equivalent to another system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{\partial G}{\partial x}(\omega t, x) + h(\omega t). \tag{4.5}$$

Similar to [4], we can find a symplectic transformation, depending periodically on $\theta (= \omega t)$, of the form

$$x = u + U(u, v, \theta), \qquad y = v + V(u, v, \theta)$$

with $U = O(v^{-2})$, $V = O(v^{-1})$, which reduces (4.5) into the system

$$\frac{\mathrm{d}u}{\mathrm{d}t} = v + q_1(u, v, \omega t), \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = h(\omega t) + q_2(u, v, \omega t)$$

with $q_1 = O(v^{-2})$, $q_2 = O(v^{-1})$. Here v is sufficiently large.

Choose a sufficiently large v_* such that $|q_2| \le \frac{1}{2}h_0$ and $|V| \le h_0$ when $v \ge v_*$. Hence, as $v(0) \ge v_* + h_0 + M_5$, we have

$$\begin{aligned} v(t) &= v(0) + \int_0^t h(\omega s)\mathrm{d}s + \int_0^t q_2(\omega s, u, v)\mathrm{d}s \\ &\ge v(0) + h_0 t + \int_0^t (h(\omega s) - h_0)\mathrm{d}s - \frac{1}{2}h_0 t \\ &\ge v_* + h_0 + \frac{1}{2}h_0 t. \end{aligned}$$

Hence,

$$y(t) \ge v_* + h_0 + \frac{1}{2}h_0 t - |V(\omega t, u, v)| \ge v_* + \frac{1}{2}h_0 t, \tag{4.6}$$

which implies that $y(t) \to +\infty$ when $t \to +\infty$. This leads to a contradiction. The necessary is proved.

The proof of Theorem 1.1 is completed.

References

[1] Moser, J., Stable and Random Motions in Dynamical Systems, Ann. Math. Stud., Princeton Tokyo, 1973.

[2] Moser, J., A stability theorem for minimal foliations on a torus, *Ergodic Theory Dynam. Systems*, **8**(1988), 251–281.

[3] Levi, M., KAM theory for particles in periodic potentials, *Ergodic Theory Dynam. Systems*, **10**(1990), 777–785.

[4] You, J. G., Invariant tori and Lagrange stability of pendulum-type equations, *J. Differential Equations*, **85**(1990), 54–65.

[5] Bibikov, Yu. N., On stability of zero solution of essential nonlinear Hamiltonian and reversible systems with one degree of freedom, *Differentsial'nye Uravneniya*, **38**(2002), 579–584.

[6] Huang, H., Unbounded solutions of almost periodically forced pendulum-type equations, *Acta Math. Sinica*, **17**(2001), 391–396.

[7] Lin, S. S. and Wang, Y. Q., Lagrangian stability of a nonlinear quasi-periodic system, *J. Math. Anal. Appl.*, **293**(2004), 258–268.

[8] Levi, M. and Zehnder, E., Boundedness of solutions for quasiperiodic potentials, *SIAM J. Math. Anal.*, **26**(1995), 1233–1256.

[9] Zharnitsky, V., Invariant curve theorem for quasiperiodic twist mappings and stability of motion in the Fermi-Ulam problem, *Nonlinearity*, **13**(2000), 1123–1136.