COMMUNICATIONS IN MATHEMATICAL RESEARCH
26(1)(2010), 85–96

A Sufficient Condition for the Genus of an Annulus Sum of Two 3-manifolds to Be Non-degenerate*

LI FENG-LING[1] AND LEI FENG-CHUN[2]

(*1. Department of Mathematics, Harbin Institute of Technology, Harbin, 150001*)

(*2. School of Mathematical Sciences, Dalian University of Technology, Dalian, 116024*)

Abstract: Let M_i be a compact orientable 3-manifold, and A_i a non-separating incompressible annulus on a component of ∂M_i, say F_i, $i = 1, 2$. Let $h : A_1 \to A_2$ be a homeomorphism, and $M = M_1 \cup_h M_2$, the annulus sum of M_1 and M_2 along A_1 and A_2. Suppose that M_i has a Heegaard splitting $V_i \cup_{S_i} W_i$ with distance $d(S_i) \geq 2g(M_i) + 2g(F_{3-i}) + 1$, $i = 1, 2$. Then $g(M) = g(M_1) + g(M_2)$, and the minimal Heegaard splitting of M is unique, which is the natural Heegaard splitting of M induced from $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$.

Key words: Heegaard genus, annulus sum, distance

2000 MR subject classification: 57M99

Document code: A

Article ID: 1674-5647(2010)01-0085-12

1 Introduction

Let M_i be a compact connected orientable bordered 3-manifold, and A_i an incompressible annulus on ∂M_i, $i = 1, 2$. Let $h : A_1 \to A_2$ be a homeomorphism. The manifold M obtained by gluing M_1 and M_2 along A_1 and A_2 via h is called an annulus sum of M_1 and M_2 along A_1 and A_2, and is denoted by $M_1 \cup_h M_2$ or $M_1 \cup_{A_1=A_2} M_2$.

Let $V_i \cup_{S_i} W_i$ be a Heegaard splitting of M_i for $i = 1, 2$, and

$$M = M_1 \cup_{A_1=A_2} M_2.$$

Then from Schultens[1], we know that M has a natural Heegaard splitting $V \cup_S W$ induced from $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ with genus

$$g(S) = g(S_1) + g(S_2).$$

So we always have

$$g(M) \leq g(M_1) + g(M_2).$$

*Received date: July 2, 2009.
Foundation item: The NSF (15071034) of China.

Let K_i be a knot in a closed 3-manifold N_i, $i = 1, 2$, and (N, K) the connected sum of pairs (N_1, K_1) and (N_2, K_2), i.e., $(N, K) = (N_1 \sharp N_2, K_1 \sharp K_2)$. Let $\eta(K)$ be an open regular neighborhood of K in N and the exterior $E(K) = N - \eta(K)$. Let A be the decomposing annulus in $E(K)$ which splits $E(K)$ into $E(K_1)$ and $E(K_2)$. Then

$$E(K) = E(K_1) \cup_{A_1 = A_2} E(K_2),$$

where A_1 is a copy of A in $E(K_1)$, and A_2 is a copy of A in $E(K_2)$. Thus

$$g(E(K)) \leq g(E(K_1)) + g(E(K_2)).$$

Note that

$$g(E(K)) = t(K) + 1,$$

where $t(K)$ is the tunnel number of K, so

$$t(K_1 \# K_2) \leq t(K_1) + t(K_2) + 1$$

always holds.

When $g(M) < g(M_1) + g(M_2)$, we say that the genus of the annulus sum is degenerate. Otherwise, it is non-degenerate. There exist examples which show that $g(M) < g(M_1) + g(M_2)$ could hold. For example, it has been shown in [2] and [3] that for any integer n, there exist infinitely many pairs of knots K_1, K_2 in S^3 such that

$$t(K_1 \# K_2) \leq t(K_1) + t(K_2) - n.$$

Note that for a knot K in S^3, $g(E(K)) = t(K) + 1$. So

$$g(E(K_1 \# K_2)) \leq g(E(K_1)) + g(E(K_2)) - n - 1.$$

In this paper, we give a sufficient condition for the genus of an annulus sum of two 3-manifolds to be non-degenerate in terms of distances of the factor Heegaard splittings.

The paper is organized as follows. In Section 2, we review some preliminaries which will be used later. The statement of the main result and its proof are included in Section 3. All 3-manifolds in this paper are assumed to be compact and orientable.

2 Preliminaries

In this section, we review some fundamental facts on surfaces in 3-manifolds. Definitions and terms which have not been defined are all standard; refer to, for examples, [4].

A Heegaard splitting of a 3-manifold M is a decomposition $M = V \cup_S W$ in which V and W are compression bodies such that

$$V \cap W = \partial_+ V = \partial_+ W = S$$

and

$$M = V \cup W.$$

S is called a Heegaard surface of M. The genus $g(S)$ of S is called the genus of the splitting $V \cup_S W$. We use $g(M)$ to denote the Heegaard genus of M, which is equal to the minimal genus of all Heegaard splittings of M. A Heegaard splitting $V \cup_S W$ for M is minimal if $g(S) = g(M)$. $V \cup_S W$ is said to be weakly reducible (see [5]) if there are essential disks $D_1 \subset V$ and $D_2 \subset W$ with $\partial D_1 \cap \partial D_2 = \emptyset$. Otherwise, $V \cup_S W$ is strongly irreducible.

Let $M = V \cup_S W$ be a Heegaard splitting, α and β be two essential simple closed curves in S. The distance $d(\alpha, \beta)$ of α and β is the smallest integer $n \geq 0$ such that there is a sequence of essential simple closed curves $\alpha = \alpha_0, \alpha_1, \cdots, \alpha_n = \beta$ in S with $\alpha_{i-1} \cap \alpha_i = \emptyset$, for $1 \leq i \leq n$. The distance of the Heegaard splitting $V \cup_S W$ is defined to be $d(S) = \min\{d(\alpha, \beta)\}$, where α bounds an essential disk in V and β bounds an essential disk in W. $d(S)$ was first defined by Hempel[6].

A properly embedded surface is essential if it is incompressible and not ∂-parallel.

Let P be a properly embedded separating surface in a 3-manifold M which cuts M into two 3-manifolds M_1 and M_2. Then P is bicompressible if P has compressing disks in both M_1 and M_2. P is strongly irreducible if it is bicompressible and each compressing disk in M_1 meets each compressing disk in M_2.

Now let P be a closed bicompressible surface in an irreducible 3-manifold M. Maximally compress P on both sides of P and remove the possible 2-sphere components, and denote the resulting surfaces by P_+ and P_-. Let H_1^P denote the closure of the region that lies between P and P_+ and similarly define H_2^P to denote the closure of the region that lies between P and P_-. Then H_1^P and H_2^P are compression bodies. If P is strongly irreducible in M, then the Heegaard splitting $H_1^P \cup_P H_2^P$ is strongly irreducible. Two strongly irreducible surfaces P and Q are said to be well-separated in M if $H_1^P \cup_P H_2^P$ is disjoint from $H_1^Q \cup_Q H_2^Q$ by isotopy.

Scharlemann and Thompson[7] showed that any irreducible and ∂-irreducible Heegaard splitting $M = V \cup_S W$ has an untelescoping

$$V \cup_S W = (V_1 \cup_{S_1} W_1) \cup_{F_1} (V_2 \cup_{S_2} W_2) \cup_{F_2} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m),$$

such that each $V_i \cup_{S_i} W_i$ is a strongly irreducible Heegaard splitting with

$$F_i = \partial_- W_i \cap \partial_- V_{i+1}, \qquad 1 \leq i \leq m-1,$$
$$\partial_- V_1 = \partial_- V, \qquad \partial_- W_m = \partial_- W,$$

and for each i, each component of F_i is a closed incompressible surface of positive genus, and only one component of $M_i = V_i \cup_{S_i} W_i$ is not a product. It is easy to see that

$$g(S) \geq g(S_i), g(F_i),$$

and when $m \geq 2$,

$$g(S) \geq g(S_i) + 1 \geq g(F_i) + 2$$

for each i. From $V_1 \cup_{S_1} W_1, \cdots, V_m \cup_{S_m} W_m$, we can get a Heegaard splitting of M by a process called amalgamation (see [8]).

The following are some basic facts and results on Heegaard splittings.

Lemma 2.1 [1] *Let F be an incompressible surface (not a 2-sphere, a 2-disk or a projective plane) properly embedded in $M = V \cup_S W$. If the Heegaard splitting $V \cup_S W$ is strongly irreducible, then F can be isotopic such that $S \cap F$ are essential loops in both F and S.*

Lemma 2.2 [1] *Let V be a compression body and F be an incompressible surface in V with $\partial F \subset \partial_+ V$. Then each component of $V \backslash F$ is a compression body.*

Lemma 2.3 [9] *Let V be a non-trivial compression body and $\mathcal{A}$ be a collection of essential annuli properly embedded in V. Then there is an essential disk D in V with $D \cap \mathcal{A} = \emptyset$.*

Lemma 2.4 [10, 11] *Let $V \cup_S W$ be a Heegaard splitting of M and F be an properly embedded incompressible surface (maybe not connected) in M. Then any component of F is parallel to ∂M or $d(S) \leq 2 - \chi(F)$.*

Lemma 2.5 [12] *Let $M = V \cup_S W$ be a Heegaard splitting such that $d(S) > 2g(M)$. Then $V \cup_S W$ is the unique minimal Heegaard splitting of M up to isotopy.*

Lemma 2.6 [13] *Let V be a non-trivial compression body and $\mathcal{A}$ be a collection of essential annuli properly embedded in V. If U is a component of $\overline{V - \mathcal{A}}$ with $U \cap \partial_- V \neq \emptyset$, then $\chi(U \cap \partial_- V) \geq \chi(U \cap \partial_+ V)$.*

Lemma 2.7 [14] *Let N be a compact orientable 3-manifold which is not a compression body, and F a component of ∂N. Suppose that Q is a properly embedded connected separating surface in N with $\partial Q \subset F$ and essential in F, and Q cuts N into two compression bodies N_1 and N_2 with $Q = \partial_+ N_1 \cap \partial_+ N_2$. If Q is compressible in both N_1 and N_2, and Q can be compressed to Q^* in some N_i such that any component of Q^* is parallel to a subsurface of ∂N, then N has a Heegaard splitting $V \cup_S W$ with $g(S) \leq g(F) - \dfrac{1}{2}\chi(Q)$ and $d(S) \leq 2$.*

Lemma 2.8 [15] *Let P and Q be bicompressible but strongly irreducible connected closed separating surfaces in a 3-manifold M. Then either*

(1) *P and Q are well-separated, or*

(2) *P and Q are isotopic, or*

(3) *$d(P) \leq 2g(Q)$.*

3 The Main Result and Its Proof

Let M_1 and M_2 be two 3-manifolds, and A_i be a non-separating incompressible annulus on a component of ∂M_i, say F_i for $i = 1, 2$. Let $M = M_1 \cup_{A_1 = A_2} M_2$. Let $F_i \times [0, 1]$ be a regular neighborhood of F_i in M_i with $F_i = F_i \times \{0\}$. We denote the surface $F_i \times \{1\}$ by F^i. Let

$$M^i = M_i - F_i \times [0, 1) \qquad \text{for } i = 1, 2,$$

and

$$M^0 = F_1 \times [0, 1] \cup_A F_2 \times [0, 1].$$

Then

$$M = M^1 \cup_{F^1} M^0 \cup_{F^2} M^2.$$

The following is the main result of the present paper:

Theorem 3.1 *Let M_i be a compact orientable 3-manifold, and A_i be a non-separating incompressible annulus on a component of ∂M_i, say F_i, $i = 1, 2$. If M_i has a Heegaard splitting $V_i \cup_{S_i} W_i$ with $d(S_i) \geq 2g(M_i) + 2g(F_{3-i}) + 1$ for $i = 1, 2$, then the minimal Heegaard splitting of M is the amalgamation of the minimal Heegaard splittings of M^1, M^0, and M^2 along F^1, F^2, and $g(M) = g(M_1) + g(M_2)$.*

Proof. Since
$$d(S_1) \geq 2g(M_1) + 2g(F_2) + 1,$$
M_1 is irreducible and not a compression body. By Lemma 2.5, $V_1 \cup_{S_1} W_1$ is the unique minimal Heegaard splitting of M_1. Similarly, M_2 is not a compression body and $V_2 \cup_{S_2} W_2$ is the unique minimal Heegaard splitting of M_2. Hence A, F^1 and F^2 are essential in M.

Now suppose that $V \cup_S W$ is a minimal Heegaard splitting of M. Then
$$g(S) \leq g(M_1) + g(M_2).$$

If $V \cup_S W$ is strongly irreducible. By Lemma 2.1, we may assume that $S \cap A$ is a collection of essential simple closed curves on both S and A. Furthermore, by the strong irreducibility of $V \cup_S W$ and Lemma 2.3, we may assume that $S \cap M_2$ is bicompressible while $S \cap M_1$ is incompressible. If each component of $S \cap M_1$ is ∂-parallel in M_1, $(S \cap M_1) \subset M^0$, then S can be isotoped to be disjoint from F^1, which means that a compression body contains a closed essential surface, a contradiction. Hence $S \cap M_1$ is essential in M_1, and by Lemma 2.4,
$$2 - \chi(S \cap M_1) \geq d(S_1) \geq 2g(M_1) + 2g(F_2) + 1.$$
Thus
$$\chi(S \cap M_1) \leq 1 - 2g(M_1) - 2g(F_2).$$

Now we denote the only bicompressible component of $S \cap M_2$ by P. If one of the incompressible component P' of $S \cap M_2$ is essential in M_2, then by Lemma 2.4, we have
$$\begin{aligned} 2 - \chi(P') &\geq d(S_2) \geq 2g(M_2) + 2g(F_1) + 1, \\ \chi(S) &= \chi(S \cap M_1) + \chi(S \cap M_2) \\ &\leq \chi(S \cap M_1) + \chi(P') + \chi(P) \\ &\leq -2g(M_1) - 2g(F_1) - 2g(M_2) - 2g(F_2), \\ g(S) &\geq g(M_1) + g(F_1) + g(M_2) + g(F_2) + 1, \end{aligned}$$
a contradiction. We may thus assume that any incompressible component of $S \cap M_2$ is ∂-parallel in M_2.

Let P^* be the surface obtained by maximally compressing P in W. Since any compressing disk of P is a compressing disk of S and S is strongly irreducible in M_2, P is strongly irreducible in M_2 and by [11], P^* is incompressible in M_2. By similar argument as above, we can show that each component of P^* is ∂-parallel in M_2.

Since A is an essential annulus in M and by Lemma 2.2, each component of $V \cap M_2$ and $W \cap M_2$ is a compression body. Let U_1 be the component of $V \cap M_2$ containing P and U_2 be the component of $W \cap M_2$ containing P. Since the incompressible components of $S \cap M_2$ are ∂-parallel in M_2, P separates M_2 into two compression bodies U_1 and U_2 with $\partial_+ U_1 \cap \partial_+ U_2 = P$. Since M_2 is not a compression body, by Lemma 2.7, there exists a Heegaard splitting $V^* \cup_{S^*} W^*$ for M_2 with $d(S^*) \leq 2$ and $g(S^*) \leq g(F_2) - \frac{1}{2}\chi(P)$. Since $d(S^*) \leq 2$, S^* is not isotopic to the unique minimal Heegaard surface S_2 of M_2, and we have that

$$g(S^*) \geq g(M_2) + 1.$$

Then

$$\chi(S \cap M_2) \leq \chi(P) \leq 2g(F_2) - 2g(S^*) \leq 2g(F_2) - 2g(M_2) - 2,$$

and

$$\chi(S) = \chi(S \cap M_1) + \chi(S \cap M_2) \leq -1 - 2g(M_1) - 2g(M_2),$$

i.e.,

$$g(S) \geq g(M_1) + g(M_2) + 2,$$

a contradiction.

Hence $V \cup_S W$ is weakly reducible, and $V \cup_S W$ has an untelescoping

$$V \cup_S W = (V_1' \cup_{S_1'} W_1') \cup_{H_1} (V_2' \cup_{S_2'} W_2') \cup_{H_2} \cdots \cup_{H_{n-1}} (V_n' \cup_{S_n'} W_n'),$$

where $n \geq 2$, each component of $\mathcal{F} = \{H_1, \cdots, H_{n-1}\}$ is a closed incompressible surface in M. First of all, we have

Claim 1 There are no two adjacent components H_i, H_{i+1} in $\mathcal{F}$ such that $H_i \cap M_1$ is essential in M_1 and $H_{i+1} \cap M_2$ is essential in M_2 whether with boundary or not.

Proof. Suppose that there exist two components of $\mathcal{F}$ such that $H_i \cap M_1$ is essential in M_1 and $H_{i+1} \cap M_2$ is essential in M_2. Then by Lemma 2.4, we have

$$2 - \chi(H_i \cap M_1) \geq d(S_1) \geq 2g(M_1) + 2g(F_2) + 1,$$
$$2 - \chi(H_{i+1} \cap M_2) \geq d(S_2) \geq 2g(M_2) + 2g(F_1) + 1.$$

Suppose that $V_i' \cup_{S_i'} W_i'$ is the Heegaard splitting in the untelescoping between them. Let

$$S_i^1 = S_i' \cap M_1, \qquad S_i^2 = S_i' \cap M_2.$$

If we denote the component of $V_i' \cap M_1$ or $W_i' \cap M_1$ which contains $H_i \cap M_1$ as part of boundary component by U_1, by Lemma 2.6, we have

$$\begin{aligned} \chi(S_i^1) &\leq \chi(U_1 \cap S_i^1) \\ &\leq \chi(U_1 \cap (H_i \cap M_1)) \\ &= \chi(H_i \cap M_1) \\ &\leq 1 - 2g(M_1) - 2g(F_2). \end{aligned}$$

If we denote the component of $V_i' \cap M_2$ or $W_i' \cap M_2$ which contains $H_{i+1} \cap M_2$ as part of boundary component by U_2, by Lemma 2.6,

$$\begin{aligned} \chi(S_i^2) &\leq \chi(U_2 \cap S_i^2) \\ &\leq \chi(U_2 \cap (H_{i+1} \cap M_2)) \\ &= \chi(H_{i+1} \cap M_2) \\ &\leq 1 - 2g(M_2) - 2g(F_1). \end{aligned}$$

Hence

$$\chi(S) \leq \chi(S_i') - 2 \leq -2g(M_1) - 2g(M_2) - 2g(F_1) - 2g(F_2),$$

and

$$g(S) \geq g(M_1) + g(M_2) + g(F_1) + g(F_2),$$

a contradiction.

This completes the proof of Claim 1.

We divide the proof of Theorem 3.1 into the following three cases to discuss.

Case 1. $A \cap \mathcal{F} \neq \emptyset$.

From now on, by Claim 1, we may assume that each component of $\mathcal{F} \cap M_2$ with boundary is essential in M_2 and each component of $\mathcal{F} \cap M_1$ with boundary is ∂-parallel in M_1. Among the surfaces of $\mathcal{F} \cap M_1$, let B be the innermost one, that is, B cuts M_1 into two pieces M_1' and M_1'', where $M_1' \cong M_1$ and $M_1'' \cong B \times I$, and the interior of M_1' contains no component of $\mathcal{F} \cap M_1$ with boundary. B lies in a component, say H_r, of $\mathcal{F}$. Hence $H_r \cap M_1$ is ∂-parallel in M_1 and $H_r \cap M_2$ is essential in M_2. Then

$$\chi(H_r \cap M_1) \leq \chi(F_1) = 2 - 2g(F_1),$$

and by Lemma 2.4,

$$2 - \chi(H_r \cap M_2) \geq d(S_2) \geq 2g(M_2) + 2g(F_1) + 1.$$

We have

$$g(H_r) \geq g(M_2) + 2g(F_1).$$

If there is another component F of $\mathcal{F}$ lying in M_1', then by Claim 1, it must be parallel to F^1 in M_1. By amalgamating the Heegaard splittings in the untelescoping along the surfaces in $\mathcal{F}$ besides F^1 and H_r, we get a generalized Heegaard splitting of M as follows:

$$M = (V_1^* \cup_{S_1^*} W_1^*) \cup_{F^1} (V_2^* \cup_{S_2^*} W_2^*) \cup_{H_r} (V_3^* \cup_{S_3^*} W_3^*),$$

where $V_1^* \cup_{S_1^*} W_1^*$ is a Heegaard splitting of M^1. Then we have

$$\begin{aligned} g(S) &= g(S_1^*) + g(S_2^*) + g(S_3^*) - g(F^1) - g(H_r) \\ &\geq g(S_1^*) + g(H_r) + 2 - g(F_1) \\ &\geq g(M_1) + g(M_2) + g(F_1) + 2, \end{aligned}$$

a contradiction. Hence there is no other component of $\mathcal{F}$ in M_1. We may assume that M_1' is contained in the submanifold $N' = V_r' \cup_{S_r'} W_r'$ of the untelescoping. Since B is innermost, N' is not a product.

$V_r' \cup_{S_r'} W_r'$ is a strongly irreducible Heegaard splitting of N'. By Lemma 2.1, we can isotope $A \cap N'$ and S_r' so that $(A \cap N') \cap S_r'$ is essential in both $A \cap N'$ and S_r', and $|(A \cap N') \cap S_r'|$ is minimal. Let

$$S_r^i = S_r' \cap M_i, \qquad i = 1, 2.$$

Since any component of $H_r \cap M_2$ is essential in M_2, if we denote the component of $V_r' \cap M_2$ or $W_r' \cap M_2$ which contains some component Q of $H_r \cap M_2$ as part of boundary by U', then by Lemma 2.4 and Lemma 2.6, we have

$$\begin{aligned}\chi(S_r^2) &\leq \chi(U' \cap S_r^2)\\ &\leq \chi(U' \cap H_r)\\ &\leq \chi(Q)\\ &\leq 1 - 2g(M_2) - 2g(F_1).\end{aligned}$$

By Lemma 2.3, there is only one component P of $S_r'\backslash A$ which is bicompressible in $N'\backslash A$, and all other components of $S_r'\backslash A$ are incompressible in $N'\backslash A$. In fact, P is strongly irreducible.

First assume $P \subset S_r^2$. Then S_r^1 is incompressible in M_1. If all components of S_r^1 are ∂-parallel in M_1, then F^1 is an essential closed surface in V_r' or W_r', a contradiction. Hence S_r^1 is essential in M_1. By Lemma 2.4 we have that

$$2 - \chi(S_r^1) \geq d(S_1) \geq 2g(M_1) + 2g(F_2) + 1,$$

and thus

$$\chi(S_r^1) \leq 1 - 2g(M_1) - 2g(F_2).$$

Then

$$\begin{aligned}\chi(S) &\leq \chi(S_r') - 2\\ &= \chi(S_r^1) + \chi(S_r^2) - 2\\ &\leq -2g(M_1) - 2g(M_2) - 2g(F_1) - 2g(F_2),\end{aligned}$$

i.e.,

$$g(S) \geq g(M_1) + g(M_2) + g(F_1) + g(F_2) + 1,$$

a contradiction.

Hence we have that $P \subset S_r^1$, and then any other component of S_r^1 is incompressible in M_1. Then by a similar argument as above and Lemma 2.7, we have

$$\chi(S_r^1) \leq 2g(F_1) - 2g(M_1) - 2,$$

$$\begin{aligned}\chi(S) &\leq \chi(S_r') - 2\\ &= \chi(S_r^1) + \chi(S_r^2) - 2\\ &\leq -2g(M_1) - 2g(M_2) - 3,\end{aligned}$$

$$g(S) \geq g(M_1) + g(M_2) + 3,$$

a contradiction.

Case 2. Any component of $\mathcal{F}$ is not ∂-parallel in M_1 or M_2, and $A \cap \mathcal{F} = \emptyset$.

In this case, by Claim 1 and the assumption, we may assume that any component of $\mathcal{F}$ is contained in M_1. Let H be an outermost component of $\mathcal{F}$ in M_1, H is essential in M_1. By Lemma 2.4, we have

$$2 - \chi(H) \geq d(S_1) \geq 2g(M_1) + 2g(F_2) + 1.$$

Suppose that

$$A \subset N = V_j' \cup_{S_j'} W_j'.$$

A is essential in M, so is in N. By Lemma 2.1, each component of $S'_j \cap A$ is essential in both S'_j and A, and we may assume that $|S'_j \cap A|$ is minimal. Set

$$S_j^1 = S'_j \cap M_1, \qquad S_j^2 = S'_j \cap M_2.$$

If we denote the component of $V'_j \cap M_1$ or $W'_j \cap M_1$ which contains H as a boundary component by U, then by Lemma 2.4 and Lemma 2.6, we have

$$\begin{aligned}\chi(S_j^1) &\le \chi(U \cap S_j^1)\\ &\le \chi(U \cap H)\\ &= \chi(H)\\ &\le 1 - 2g(M_1) - 2g(F_2).\end{aligned}$$

Since $V'_j \cup_{S'_j} W'_j$ is strongly irreducible, by Lemma 2.3, only one component, say P, of $S'_j \backslash A$ which is bicompressible in $N \backslash A$, and all other components of $S'_j \backslash A$ are incompressible in $N \backslash A$. In fact, P is strongly irreducible.

Suppose that $P \subset S_j^1$. Then S_j^2 is incompressible in M_2. If all components of S_j^2 are ∂-parallel in M_2, then F^2 is an essential closed surface in V'_j or W'_j, a contradiction. Hence S_j^2 is essential in M_2. By Lemma 2.4 we have that

$$2 - \chi(S_j^2) \ge d(S_2) \ge 2g(M_2) + 2g(F_1) + 1,$$

and thus

$$\chi(S_j^2) \le 1 - 2g(M_2) - 2g(F_1).$$

Then

$$\begin{aligned}\chi(S) &\le \chi(S'_j) - 2\\ &= \chi(S_j^1) + \chi(S_j^2) - 2\\ &\le -2g(M_1) - 2g(M_2) - 2g(F_1) - 2g(F_2),\end{aligned}$$

i.e.,

$$g(S) \ge g(M_1) + g(M_2) + g(F_1) + g(F_2) + 1,$$

a contradiction.

Hence $P \subset S_j^2$, and then any other component of S_j^2 is incompressible in M_2. By a similar argument as above and Lemma 2.7, $\chi(S_j^2) \le 2g(F_2) - 2g(M_2) - 2$, and we have

$$\begin{aligned}\chi(S) &\le \chi(S'_j) - 2\\ &= \chi(S_j^1) + \chi(S_j^2) - 2\\ &\le -2g(M_1) - 2g(M_2) - 3,\\ g(S) &\ge g(M_1) + g(M_2) + 3,\end{aligned}$$

a contradiction.

Case 3. There is one component of $\mathcal{F}$ which is ∂-parallel in M_1 or M_2, and $A \cap \mathcal{F} = \emptyset$.

In this case, we may assume that $F^1 \subset \mathcal{F}$. If there is another component H of $\mathcal{F}$ which is essential in M_1, since M^0 contains no essential closed surface, $H \subset \mathrm{int} M^1$. By Lemma 2.4, we have

$$g(H) \ge g(M_1) + g(F_2) + 1.$$

This gives a Heegaard splitting of M_1 with genus at least $g(M_1)+g(F_2)+2$, a contradiction to the minimality of $g(S_1)$.

Now we only need to consider the case that all components of $\mathcal{F}$ other than F^1 lie in M_2. If there is a component F of $\mathcal{F}$ which is essential in M_2, then by Lemma 2.4,

$$g(F) \geq g(M_2) + g(F_1) + 1.$$

By amalgamating the Heegaard splittings in the untelescoping along the surfaces in $\mathcal{F}$ besides F^1 and F, we get a generalized Heegaard splitting of M as follows:

$$M = (V_1^* \cup_{S_1^*} W_1^*) \cup_{F^1} (V_2^* \cup_{S_2^*} W_2^*) \cup_F (V_3^* \cup_{S_3^*} W_3^*),$$

where $V_1^* \cup_{S_1^*} W_1^*$ is a Heegaard splitting of M^1. Then we have

$$\begin{aligned} g(S) &= g(S_1^*) + g(S_2^*) + g(S_3^*) - g(F^1) - g(F) \\ &\geq g(S_1^*) + g(F) + 2 - g(F_1) \\ &\geq g(M_1) + g(M_2) + 2, \end{aligned}$$

a contradiction.

Hence each component of $\mathcal{F}$ can be isotoped to be parallel to F^1 or F^2, and the length n of the untelescoping is at most 3.

Now suppose that $n = 2$. Then

$$V \cup_S W = (V_1' \cup_{S_1'} W_1') \cup_{H_1} (V_2' \cup_{S_2'} W_2'),$$

and each of $V_1' \cup_{S_1'} W_1'$ and $V_2' \cup_{S_2'} W_2'$ is strongly irreducible. H_1 is isotopic to one of F^1 and F^2, and we may assume that H_1 is isotopic to F^2. We may further assume that $V_1' \cup_{S_1'} W_1'$ is a strongly irreducible Heegaard splitting of $M^1 \cup_{F^1} M^0$, and $V_2' \cup_{S_2'} W_2'$ is a Heegaard splitting of M^2. Since S' is a Heegaard surface of $M^1 \cup_{F^1} M^0 = M_1 \cup_A F_2 \times [0,1]$ and S_1 is a Heegaard surface of M_1, S' and S_1 are not well-separated. Furthermore, S' is not isotopic to S_1. By Lemma 2.8, we have

$$d(S_1) \leq 2g(S'),$$

and hence

$$g(S') \geq g(M_1) + g(F_2) + 1.$$

Then

$$\begin{aligned} g(S) &= g(S_1') + g(S_2') - g(H_1) \\ &\geq g(M_1) + g(M_2) + 1, \end{aligned}$$

a contradiction.

Hence $n = 3$, and now

$$V \cup_S W = (V_1' \cup_{S_1'} W_1') \cup_{H_1} (V_2' \cup_{S_2'} W_2') \cup_{H_2} (V_3 \cup_{S_3} W_3).$$

We may assume that H_1 is isotopic to F^1, and H_2 is isotopic to F^2. We may further assume that $V_1' \cup_{S_1'} W_1'$ is a Heegaard splitting of M^1, $V_2' \cup_{S_2'} W_2'$ is a Heegaard splitting of M^0, and $V_3' \cup_{S_3'} W_3'$ is a Heegaard splitting of M^2. Since A is non-separating on both F_1 and F_2, M^0 contains only three boundary components F^1, F^2 and $(F_1 \backslash A_1) \cup (F_2 \backslash A_2)$. We denote $(F_1 \backslash A_1) \cup (F_2 \backslash A_2)$ by F_3. Then

$$g(M^0) \geq \min\{g(F_1) + g(F_2),\ g(F_1) + g(F_3),\ g(F_2) + g(F_3)\}.$$

Note that

$$g(F_3) = g(F_1) + g(F_2) - 1, \qquad g(M^0) \geq g(F_1) + g(F_2).$$

Hence

$$g(S_2') \geq g(M^0) \geq g(F_1) + g(F_2).$$

Then we have that

$$\begin{aligned} g(S) &= g(S_1') + g(S_2') + g(S_3') - g(H_1) - g(H_2) \\ &\geq g(M_1) + g(M_2), \end{aligned}$$

which, combining with Schultens' results in [1], implies that $g(M) = g(M_1) + g(M_2)$, and the equality holds if and only if

$$\begin{aligned} g(S_1') &= g(M_1), \\ g(S_2') &= g(F_1) + g(F_2), \\ g(S_3') &= g(M_2). \end{aligned}$$

Hence the minimal Heegaard splitting of M is the amalgamation of the minimal Heegaard splittings of M^1, M^0 and M^2.

This completes the proof of Theorem 3.1.

As a direct consequence, we have

Corollary 3.1 *Let K_i be a knot in a closed 3-manifold N_i, $i = 1, 2$, and $(N, K) = (N_1 \sharp N_2, K_1 \sharp K_2)$. If $E(K_i)$ has a Heegaard splitting $V_i \cup_{S_i} W_i$ with $d(S_i) \geq 2t(K_i) + 5$ for $i = 1, 2$, then*

$$t(K) = t(K_1) + t(K_2) + 1$$

and the minimal Heegaard splitting of $E(K)$ is weakly reducible.

Remark 3.1 Schultens showed in [1] that for two small knots

$$K_1, K_2 \subset S^3, \qquad t(K_1 \sharp K_2) \geq t(K_1) + t(K_2);$$

Morimoto showed in [9] that for two m-small knots

$$K_1, K_2 \subset S^3, \qquad t(K_1 \sharp K_2) \geq t(K_1) + t(K_2).$$

References

[1] Schultens, J., Additivity of tunnel number for small knots, *Comment. Math. Helv.*, **75**(2000), 353–363.

[2] Kobayashi, T., A construction of arbitrarily high degeneration of tunnel numbers of knots under connected sum, *J. Knot Theory Ramifications*, **3**(1994), 179–186.

[3] Morimoto, K., There are knots whose tunnel numbers go down under connected sum, *Proc. Amer. Math. Soc.*, **123**(1995), 3527–3532.

[4] Jaco, W., Lectures on Three-Manifold Topology, CBMS Regional Conf. Ser. in Math. 43, Amer. Math. Soc., Providence, R.I., 1980.

[5] Casson, A. J. and Gordon, C. M., Reducing Heegaard splittings, *Topology Appl.*, **27**(1987), 275–283.

[6] Hempel, J., 3-manifolds as viewed from the curve complex, *Topology*, **40**(2001), 631–657.

[7] Scharlemann, M. and Thompson, A., Thin position for 3-manifolds, *AMS Contem. Math.*, **164**(1994), 231–238.

[8] Schultens, J., The classification of Heegaard splittings for (compact orientable surfaces)$\times S^1$, *Proc. London Math. Soc.*, **67**(1993), 425–448.

[9] Morimoto, K., Tunnel number, connected sum and meridional essential surface, *Topology*, **39**(2000), 469–485.

[10] Hartshorn, K., Heegaard splittings of Haken manifolds have bounded distance, *Pacific J. Math.*, **204**(2002), 61–75.

[11] Scharlemann, M., Proximity in the curve complex: boundary reduction and bicompressible surfaces, *Pacific J. Math.*, **228**(2006), 325–348.

[12] Kobayashi, T. and Qiu, R. F., The amalgamation of high distance Heegaard splittings is always efficient, *Math. Ann.*, **341**(2008), 707–715.

[13] Scharlemann, M. and Schultens, J., The tunnel number of the sum of n knots is at least n, *Topology*, **38**(1999), 265–270.

[14] Yang, G. Q. and Lei, F. C., Some sufficient conditions for tunnel numbers of connected sum of two knots not to go down, *preprint*.

[15] Scharlemann, M. and Tomova, M., Alternate Heegaard genus bounds distance, *Geom. Topology*, **10**(2006), 593–617.