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A PML Method for Electromagnetic Scattering from Two-dimensional Overfilled Cavities*

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Abstract: In this paper, we consider electromagnetic scattering problems for two-dimensional overfilled cavities. A half ringy absorbing perfectly matched layer (PML) is introduced to enclose the cavity, and the PML formulations for both TM and TE polarizations are presented. Existence, uniqueness and convergence of the PML solutions are considered. Numerical experiments demonstrate that the PML method is efficient and accurate for solving cavity scattering problems.

Key words: overfilled cavity, scattering, TM polarization, TE polarization, perfectly matched layer (PML), DtN operator

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1 Introduction

The problem of calculating electromagnetic scattering from open cavities, such as jet inlet and exit of airplane, is of high interest in certain military and defense applications, because open cavities usually cause strong radar echo and can be easily found. Researchers in the engineering community have raised many methods to solve scattering problems involving cavities filled with penetrable material, which include high and low frequency method (see [1]–[3]), the method of moment (see [4] and [5]) and Finite element-Boundary integral equation method (see [6]–[8]). Analysis of cavity scattering problems can also be found in mathematic area (see [9]–[12]). It is a common assumption that the cavity opening coincides with the aperture on an infinite ground plane, and hence simplifying the modelling of the exterior (to the cavity) domain. This limits the applications of these methods since many cavity openings are not planar. For determining the fields scattered by overfilled cavities, we have to find new ways.

In [13], a mathematical model characterizing the scattering by overfilled cavities was developed and proved to be well posed. For solving overfilled cavity scattering problems,

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[14]–[16] presented a method and [16] gave the corresponding mathematical analysis. Although this method reduces the infinite computational domain exactly to a finite one, the numerical implementation is difficult, because the boundary condition imposed on the artificial semicircle includes the DtN operator.

Zhang and Ma presented a PML method in [17] to solve cavity scattering problems which can also be used to solve overfilled cavity scattering problems. But they only consider the case of TM polarization. Basing on the works of Aihua Wood on overfilled cavity scattering (see [14]–[16]) and Chen *et al.*[18] on PML technique for obstacle scattering, by introducing symmetrical prolongation, we present a PML method to solve cavity scattering problems, for both TM and TE polarizations. Our method also reduces the infinite computational domain to a finite one that enclosed by a half ringy PML layer. The PML formulations for both TM polarization and TE polarization of cavity scattering are presented. We establish the existence and uniqueness of the PML solutions, and we also prove that the PML solutions converge exponentially to the exact solutions derived by the DtN method. Numerical computations demonstrate that our method is effective and its numerical computing is easy to be implemented on the PDE toolbox of MATLAB or other finite element softwares.

2 Problem Setting

Let $\varOmega \subset \mathbb{R}^2$ be the cross-section of a z-invariant trough in the infinite ground plane such that its fillings protrude above the ground plane. Denote S as the cavity wall, and $\varGamma$ the cavity aperture so that $\partial\varOmega = S \cup \varGamma$. The infinite ground plane excluding the cavity opening is denoted as $\varGamma_{\rm ext}$, the infinite homogenous region above the cavity as $\mathcal{U} = \mathbb{R}^2_+ \backslash \varOmega$.

Given the incident electromagnetic wave $(\mathbf{E}^{\rm i},\ \mathbf{H}^{\rm i})$, we wish to determine the resulting scattering field $(\mathbf{E}^{\rm s},\ \mathbf{H}^{\rm s})$.

Due to the uniformity in the z-axis, the scattering problem can be decomposed into two fundamental polarizations: transverse magnetic (TM) and transverse electric (TE). Its solution can be expressed as a linear combination of the solutions to TM and TE problems.

In the TM polarization, the magnetic field $\mathbf{H}$ is transverse to the z-axis so that $\mathbf{E}$ and $\mathbf{H}$ are of the form

$$\mathbf{E} = (0, 0, E_z), \qquad \mathbf{H} = (H_x, H_y, 0).$$

In this case, by setting $u = E_z$, we can determine $\mathbf{E}$ and $\mathbf{H}$ by u which satisfies the scattering problem with the following form:

$$(\text{TM}) \begin{cases} \Delta u + k^2 \varepsilon_r u = 0 & \text{in } \varOmega \cup \mathcal{U}, \\ u = 0 & \text{on } S \cup \varGamma_{\rm ext}. \end{cases}$$

u naturally satisfies the continuity conditions on $\varGamma$:

$$u|_{\varGamma+} = u|_{\varGamma-}, \qquad \frac{\partial u}{\partial n}\Big|_{\varGamma+} = \frac{\partial u}{\partial n}\Big|_{\varGamma-}.$$

$\varepsilon_r = \varepsilon/\varepsilon_0$ is the relative electric permittivity and k is the free space wave number. We assume $\mathrm{Re}\varepsilon_r \geq \alpha > 0$, $\mathrm{Im}\varepsilon_r \geq 0$, and $\varepsilon_r \in L^\infty(\varOmega)$. The homogeneous region $\mathcal{U}$ above the protruding cavity is assumed to be air and hence its permittivity is $\varepsilon_r = 1$. In $\mathcal{U}$, the total field can be decomposed as

$$u = u^i + u^r + u^s,$$

where u^i is the incident field, u^r the reflected field, and u^s the scattered field. The reflected field exists due to the presence of the infinite ground plane. For the TM polarization, the incident and reflected fields satisfy

$$u^i + u^r = 0 \qquad \text{on } \ \Gamma_{\text{ext}} \in \{(x, y) : y = 0\},$$

and the scattered field u^s of the scattering problem (TM) satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\Big(\frac{\partial u^s}{\partial r} - \mathrm{i}ku^s\Big) = 0, \qquad r = |x|.$$

Similarly, in the TE polarization, the electric field $\mathbf{E}$ is transverse to the z-axis so that $\mathbf{E}$ and $\mathbf{H}$ are of the form

$$\mathbf{E} = (E_x, E_y, 0), \qquad \mathbf{H} = (0, 0, H_z).$$

By setting $u = H_z$, we can determine $\mathbf{E}$ and $\mathbf{H}$ by u which satisfies the scattering problem with the following form:

$$(\text{TE}) \begin{cases} \nabla \cdot \Big(\dfrac{1}{\varepsilon_r}\nabla u\Big) + k^2 u = 0 & \text{in } \Omega \cup \mathcal{U}, \\ \partial u/\partial n = 0 & \text{on } S \cup \Gamma_{\text{ext}}. \end{cases}$$

u naturally satisfies the continuity conditions on Γ:

$$u|_{\Gamma+} = u|_{\Gamma-}, \qquad \frac{\partial u}{\partial n}\Big|_{\Gamma+} = \frac{1}{\varepsilon_r}\frac{\partial u}{\partial n}\Big|_{\Gamma-}.$$

In $\mathcal{U}$, the total field can be decomposed as

$$u = u^i + u^r + u^s,$$

where

$$\partial u^i/\partial n + \partial u^r/\partial n = 0 \qquad \text{on } \ \Gamma_{\text{ext}} \in \{(x, y) : y = 0\},$$

and the scattered field u^s of the scattering problem (TE) satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\Big(\frac{\partial u^s}{\partial r} - \mathrm{i}ku^s\Big) = 0, \qquad r = |x|.$$

3 The PML Method for TE Polarization

We introduce the PML layer

$$\Omega^+_{\text{PML}} = \{x \in \mathbb{R}^2_+;\ R < |x| < \rho\}.$$

Let

$$B_r = \{x; |x| \le r\}, \qquad \Gamma_r = \partial B_r,$$

$$\Omega_{\text{PML}} = \{x \in \mathbb{R}^2; R < |x| < \rho\}, \qquad \Gamma^+_r = \Gamma_r \cap \{(x, y); y \ge 0\},$$

$$B^+_r = B_r \cap \{(x, y);\ y \ge 0\}, \qquad \Omega_r = B^+_r \cup \bar{\Omega},$$

$$\Gamma_0 = \{(x, 0);\ R < |x| < \rho\}, \qquad S_1 = \Gamma_{\text{ext}} \backslash \{(x, 0); |x| > R\}.$$

See Fig. 3.1.

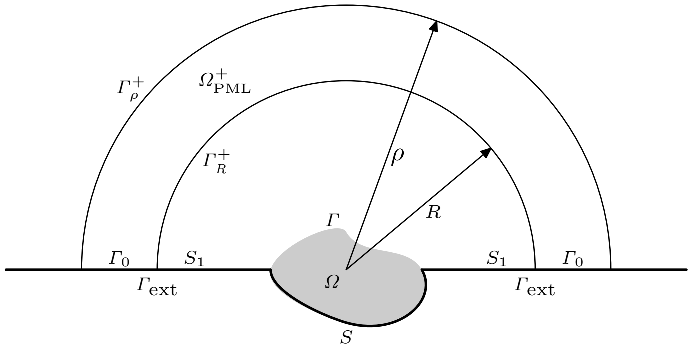

Fig. 3.1 Overfilled cavity with PML layer.

Throughout the paper we assume $\rho \le CR$ for some generic fixed constant $C > 0$.

Let $\alpha(r) = 1 + \mathrm{i}\sigma(r)$ be the model medium property of PML layer which satisfies

$$\sigma \in C(\mathbb{R}), \quad \sigma \ge 0 \quad \text{and} \quad \sigma = 0, \qquad r \le R.$$

Denote by $\widetilde{r}$ the complex radius defined by

$$\widetilde{r} = \widetilde{r}(r) = \begin{cases} r, & \text{if } r \le R; \\ \displaystyle\int_0^r \alpha(t)\mathrm{d}t = r\beta(r), & \text{if } r \ge R. \end{cases}$$

Following [19], we introduce the PML equation

$$\nabla \cdot (A\nabla w) + \alpha\beta k^2 w = 0 \qquad \text{in } \Omega_{\text{PML}}^+, \tag{3.1}$$

where $A = A(r, \theta)$ is a matrix as follows:

$$A(r,\theta) = \begin{pmatrix} \dfrac{\beta}{\alpha}\cos^2\theta + \dfrac{\alpha}{\beta}\sin^2\theta & \cos\theta\sin\theta\Big(\dfrac{\beta}{\alpha} - \dfrac{\alpha}{\beta}\Big) \\ \cos\theta\sin\theta\Big(\dfrac{\beta}{\alpha} - \dfrac{\alpha}{\beta}\Big) & \dfrac{\beta}{\alpha}\sin^2\theta + \dfrac{\alpha}{\beta}\cos^2\theta \end{pmatrix},$$

so in polar coordinates, A satisfies

$$\nabla \cdot (A\nabla) = \frac{1}{r}\frac{\partial}{\partial r}\Big(\frac{\beta r}{\alpha}\frac{\partial}{\partial r}\Big) + \frac{\alpha}{\beta r^2}\frac{\partial^2}{\partial\theta^2}.$$

The PML scattering solution $\hat{u}^s$ in Ω_ρ corresponding to the scattered field $\hat{u}$ of (TE) is defined as the solution of the system

$$\begin{cases} \nabla \cdot \Big(\dfrac{1}{\varepsilon_r} A\nabla \hat{u}^s\Big) + \alpha\beta k^2 \hat{u}^s = h(x) & \text{in } \Omega_\rho, \\ \partial\hat{u}^s/\partial n = p(x) \quad \text{on } S \cup S_1 \cup \Gamma_0, & \hat{u}^s = 0 \quad \text{on } \Gamma_\rho^+, \end{cases} \tag{3.2}$$

where

$$h(x) = (\varepsilon_r^{-1}k^2 - k^2)(u^i + u^s),$$

$$p(x) = -\partial(u^i + u^r)/\partial n \qquad \text{on } S$$

and

$$p(x) = 0 \qquad \text{on } \Gamma_0 \cup S_1.$$

The PML problem (3.2) in Ω_ρ can be reformulated in Ω_R by imposing the boundary condition

$$\frac{\partial \hat{u}^s}{\partial n}\Big|_{\Gamma_R^+} = \hat{T}_0 \hat{u}^s,$$

where the operator $\hat{T}_0 : H^{1/2}(\Gamma_R^+) \to H^{-1/2}(\Gamma_R^+)$ is defined as follows: given $f \in H^{1/2}(\Gamma_R^+)$,

$$\hat{T}_0 f = \frac{\partial \zeta}{\partial n}\Big|_{\Gamma_R^+},$$

where $\zeta \in H^1(\Omega_{\mathrm{PML}}^+)$ satisfies

$$(\mathrm{P}_0)\begin{cases}\nabla\cdot(A\nabla\zeta)+\alpha\beta k^2\zeta=0 & \text{in } \Omega_{\mathrm{PML}}^+,\\ \zeta=f \quad \text{on } \Gamma_R^+, \qquad \zeta=0 & \text{on } \Gamma_\rho^+,\\ \partial\zeta/\partial n=0 \qquad \text{on } \Gamma_0. \end{cases}$$

Denote by

$$\hat{u} := u^i + u^r + \hat{u}^s$$

the PML total solution. From the discussion above, the normal derivative of the total field can be decomposed into

$$\begin{aligned}\frac{\partial \hat{u}}{\partial r} &= \frac{\partial u^i}{\partial r}+\frac{\partial u^r}{\partial r}+\frac{\partial \hat{u}^s}{\partial r}\\ &= \frac{\partial u^i}{\partial n}+\frac{\partial u^r}{\partial n}+\hat{T}_0\hat{u}^s\\ &\equiv g(\theta)+\hat{T}_0\hat{u}-\hat{T}_0(u^i+u^r).\end{aligned}$$

We can now reduce the PML problem (3.2) in Ω_ρ to the following PML problem in Ω_R:

$$(\mathrm{TE}_{\mathrm{PML}})\begin{cases}\nabla\cdot\Big(\dfrac{1}{\varepsilon_r}\nabla\hat{u}^s\Big)+k^2\hat{u}=0 & \text{in } \Omega_R,\\ \dfrac{\partial\hat{u}}{\partial n}-\hat{T}_0\hat{u}=g(\theta)-\hat{T}_0(u^i+u^r) & \text{on } \Gamma_R^+,\\ \dfrac{\partial\hat{u}}{\partial n}=0 & \text{on } S_1\cup S.\end{cases}$$

As showed in [15], we can exactly reduce the problem (TE) defined in the infinite domain $\Omega\cup\mathcal{U}$ to the following problem in Ω_R:

$$(\mathrm{TE}_{\mathrm{EXA}})\begin{cases}\nabla\cdot\Big(\dfrac{1}{\varepsilon_r}\nabla\hat{u}^s\Big)+k^2u=0 & \text{in } \Omega_R,\\ \dfrac{\partial u}{\partial n}-T_0u=g(\theta)-T_0(u^i+u^r) & \text{on } \Gamma_R^+,\\ \dfrac{\partial u}{\partial n}=0 & \text{on } S_1\cup S,\end{cases}$$

where the DtN operator T_0 on Γ_R^+ is defined as follows: for any $u \in H^{1/2}(\Gamma_R^+)$

$$T_0u = k\frac{u_0}{2}\frac{H_0^{(1)'}(kR)}{H_0^{(1)}(kR)}+k\sum_{n=1}^{\infty}u_n\frac{H_n^{(1)'}(kR)}{H_n^{(1)}(kR)}\cos(n\theta),$$

where

$$u_n=\frac{2}{\pi}\int_0^{\pi}u\cos(n\theta')\mathrm{d}\theta', \qquad n=0,1,2,\cdots$$

To proceed, we introduce the following notations. For any function ξ defined on

$$\Gamma_a=\{x\in\mathbb{R}^2 : |x|=a\}$$

and ξ^+ defined on

$$\Gamma_a^+=\{x\in\mathbb{R}_+^2 : |x|=a\},$$

we define

$$\|\xi\|^2_{H^{\pm1/2}(\Gamma_a)}=2\pi\sum_{n\in\mathbb{Z}}(1+n^2)^{\pm1/2}|\hat{\xi}_n|^2,$$

$$\|\xi^+\|^2_{H^{\pm 1/2}(\Gamma_a^+)} = \pi|\hat{\xi}_0^+|^2 + \frac{\pi}{2}\sum_{n=1}^{\infty}(1+n^2)^{\pm 1/2}|\hat{\xi}_n^+|^2,$$

where

$$\hat{\xi}_n = 1/(2\pi)\int_0^{2\pi}\xi \mathrm{e}^{-\mathrm{i}n\theta}\mathrm{d}\theta$$

and

$$\hat{\xi}_0^+ = 1/\pi\int_0^{\pi}\xi^+\mathrm{d}\theta,$$

$$\hat{\xi}_n^+ = 2/\pi\int_0^{\pi}\xi^+\cos(n\theta)\mathrm{d}\theta.$$

The operator $\hat{T}: H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R)$ is defined as follows: Given $f \in H^{1/2}(\Gamma_R)$,

$$\hat{T}f = \frac{\partial\zeta}{\partial n}\Big|_{\Gamma_R},$$

where $\zeta \in H^1(\Omega_{\mathrm{PML}})$ satisfies

$$(\mathrm{P})\begin{cases}\nabla\cdot(A\nabla\zeta) + \alpha\beta k^2\zeta = 0 & \text{in } \Omega_{\mathrm{PML}},\\ \zeta = f \quad \text{on } \Gamma_R, \qquad \zeta = 0 & \text{on } \Gamma_\rho.\end{cases}$$

The Dirichlet-to-Neumann (DtN) operator T on Γ_R is defined as follows: for any $u \in H^{1/2}(\Gamma_R)$,

$$Tu = k\sum_{n\in\mathbb{Z}}\frac{H_n^{(1)'}(kR)}{H_n^{(1)}(kR)}u_n\mathrm{e}^{\mathrm{i}n\theta}, \qquad u_n = \frac{1}{2\pi}\int_0^{2\pi}u\mathrm{e}^{-\mathrm{i}n\theta}\mathrm{d}\theta.$$

We make the following assumptions:

(H1) $\sigma = \sigma_0\Big(\dfrac{r-R}{\rho-R}\Big)^m$ for some constant $\sigma_0 > 0$ and some integer $m \geq 1$;

(H2) There exists a unique weak solution to the Dirichlet PML problem (P).

For sufficiently large σ_0, (H2) can be proved to hold (see [18]).

Lemma 3.1 *Let* (H1) *and* (H2) *be satisfied. For $u \in H^{1/2}(\Gamma_R^+)$, we have*

$$\|T_0u - \hat{T}_0u\|_{H^{-1/2}(\Gamma_R^+)} \leq C\hat{C}^{-1}(1+kR)^2|\alpha_0|^2\mathrm{e}^{-k\mathrm{Im}(\tilde{\rho})\left(1-\frac{R^2}{|\tilde{\rho}|^2}\right)^{1/2}}\|u\|_{H^{1/2}(\Gamma_R^+)},$$

where

$$\alpha_0 = 1 + \mathrm{i}\sigma_0, \qquad \tilde{\rho} = \rho + \mathrm{i}\sigma_0(\rho-R)/(m+1),$$

$C > 0$ is a constant independent of k, R, ρ and σ_0, and $\hat{C} \leq 1$.

Proof. We denote by x_ρ the reflection of x about x-axis. We prolongate $u \in H^{1/2}(\Gamma_R^+)$ with

$$u(x) = u(x_\rho),$$

and the function after prolongation is denoted by $u_1 \in H^{1/2}(\Gamma_R)$. We conclude with simple calculation that

$$(T_0u)(x) = (Tu_1)(x), \quad x \in \Gamma_R^+ \ \text{ and } \ (Tu_1)(x) = (Tu_1)(x_\rho). \tag{3.3}$$

Let the boundary function

$$f(x) = u_1(x)$$

on Γ_R for the problem (P) which defines $\hat{T}$, and let

$$f(x) = u(x)$$

on Γ_R^+ for the problem (P_0) which defines $\hat{T}_0$. We can prove the equivalence of the two problems (P) and (P_0).

If a function $\zeta(x)$ solves (P), then by the assumption that the boundary function

$$f(x) = u_1(x)$$

is even with respect to y-axis and the symmetry of Ω_{PML}, clearly, the function $\zeta(x_\rho)$ also solves (P). Hence, from the assumption (H2), we know

$$\zeta(x) = \zeta(x_\rho)$$

which implies that

$$\partial\zeta(x)/\partial n = \partial\zeta(x_\rho)/\partial n \qquad \text{on } \Gamma_0.$$

But

$$\partial\zeta(x)/\partial n = -\partial\zeta(x_\rho)/\partial n \qquad \text{on } \Gamma_0.$$

Hence, we conclude that

$$\partial\zeta(x)/\partial n = 0 \qquad \text{on } \Gamma_0,$$

which implies that $\zeta(x)$ is the solution of (P_0). On the other hand, a solution $\zeta(x)$ for (P_0) can be defined to be a solution for (P) via

$$\zeta(x) = \zeta(x_\rho).$$

By the equivalence of (P) and (P_0) and the definitions of $\hat{T}$ and $\hat{T}_0$, we conclude that

$$(\hat{T}_0 u)(x) = (\hat{T} u_1)(x), \quad x \in \Gamma_R^+ \ \text{ and } \ (\hat{T} u_1)(x) = (\hat{T} u_1)(x_\rho). \tag{3.4}$$

Upon using Lemma 2.5 in [18] and (3.3) and (3.4), we have

$$\begin{aligned}
\|T_0 u - \hat{T}_0 u\|_{H^{-1/2}(\Gamma_R^+)} &= \|T u_1 - \hat{T} u_1\|_{H^{-1/2}(\Gamma_R^+)} \\
&= \frac{1}{2}\|T u_1 - \hat{T} u_1\|_{H^{-1/2}(\Gamma_R)} \\
&\le C\hat{C}^{-1}(1+kR)^2|\alpha_0|^2 \mathrm{e}^{-k\mathrm{Im}(\tilde{\rho})\left(1-\frac{R^2}{|\tilde{\rho}|^2}\right)^{1/2}} \|u_1\|_{H^{1/2}(\Gamma_R)} \\
&= 2C\hat{C}^{-1}(1+kR)^2|\alpha_0|^2 \mathrm{e}^{-k\mathrm{Im}(\tilde{\rho})\left(1-\frac{R^2}{|\tilde{\rho}|^2}\right)^{1/2}} \|u\|_{H^{1/2}(\Gamma_R^+)}.
\end{aligned}$$

We complete the proof by replacing $2C$ with C.

In the following part of this section, we will derive the variational formulation of the PML problem ($\mathrm{TE}_{\mathrm{PML}}$) and prove the existence and uniqueness of the PML solution $\hat{u}$. We will also consider the convergence of the PML solution $\hat{u}^s$.

Define the space $W := H^1(\Omega_R)$.

The variational formulation of ($\mathrm{TE}_{\mathrm{PML}}$) is as follows: find $\hat{u} \in W$ such that

$$\hat{b}_{\mathrm{TE}}(\hat{u}, w) = \hat{G}(w), \qquad \forall w \in W, \tag{3.5}$$

where

$$\hat{b}_{\mathrm{TE}}(\hat{u}, w) = \int_{\Omega_R} (\varepsilon_r^{-1}\nabla\hat{u}\cdot\nabla\bar{w} - k^2\hat{u}\bar{w})\mathrm{d}x\mathrm{d}y - \int_{\Gamma_R^+} \hat{T}_0(\hat{u})\bar{w}\mathrm{d}s,$$

$$\hat{G}(w) = \int_{\Gamma_R^+} [g(\theta) - \hat{T}_0(u^i + u^r)]\bar{w}\mathrm{d}s.$$

The variational formulation of ($\mathrm{TE}_{\mathrm{EXA}}$) is as follows: find $u \in W$ such that

$$b_{\mathrm{TE}}(u, w) = G(w), \qquad \forall w \in W, \tag{3.6}$$

where

$$b_{\mathrm{TE}}(u, w) = \int_{\Omega_R} (\varepsilon_r^{-1} \nabla u \cdot \nabla \bar{w} - k^2 u \bar{w}) \mathrm{d}x\mathrm{d}y - \int_{\Gamma_R^+} T_0(u) \bar{w} \mathrm{d}s,$$

$$G(w) = \int_{\Gamma_R^+} [g(\theta) - T_0(u^i + u^r)] \bar{w} \mathrm{d}s.$$

Lemma 3.2 *The variational problem* (3.6) *has a unique solution* $u \in W$.

Proof. We split $b_{\mathrm{TE}}(u, w)$ into $b_0(u, w) + b_1(u, w)$ with

$$b_0(u, w) = \int_{\Omega_R} (\varepsilon_r^{-1} \nabla u \cdot \nabla \bar{w} + u \bar{w}) \mathrm{d}x\mathrm{d}y - \int_{\Gamma_R^+} T_0'(u) \bar{w} \mathrm{d}s,$$

$$b_1(u, w) = -\int_{\Omega_R} (k^2 + 1) u \bar{w} \mathrm{d}x\mathrm{d}y + \int_{\Gamma_R^+} (T_0'(u) - T_0(u)) \bar{w} \mathrm{d}s,$$

where T_0' is defined as follows: for any $u \in H^{1/2}(\Gamma_R)$

$$T_0' u = -\frac{1}{\pi} \int_0^{\pi} u \mathrm{d}\theta - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{n \cos(n\theta)}{R} \int_0^{\pi} \cos(n\theta) u \mathrm{d}\theta.$$

We see directly from the orthogonality of $\cos(n\theta)$ that $-T_0'$ is strictly coercive, i.e.,

$$-\int_{\Gamma_R^+} T_0'(u) \bar{u} \mathrm{d}s \geq c \|u\|_{H^{1/2}(\Gamma_R^+)}.$$

So we can conclude directly that $b_0(u, w)$ is strictly coercive with respect to u.

Hankel functions $H_n^{(1)}(t)$ satisfy the recurrence relation

$$t H_n^{(1)\prime}(t) + n H_n^{(1)}(t) = t H_{n-1}^{(1)}(t)$$

(see [20]), i.e.,

$$\frac{t H_n^{(1)\prime}(t)}{H_n^{(1)}(t)} = -n + \frac{t H_{n-1}^{(1)}(t)}{H_n^{(1)}(t)}. \tag{3.7}$$

$H_n^{(1)}(t)$ also have the following asymptotic behavior (see [20]):

$$H_n^{(1)}(t) = \frac{2^n (n-1)!}{\pi i t^n} \Big(1 + O\Big(\frac{1}{n}\Big)\Big), \qquad n \to \infty. \tag{3.8}$$

By (3.7) and (3.8) we have

$$\frac{kR H_n^{(1)\prime}(kR)}{H_n^{(1)}(kR)} = -n + O\Big(\frac{1}{n}\Big), \qquad n \to \infty$$

which implies that the difference operator $T_0' - T_0$ is compact from $H^{1/2}(\Gamma_R^+)$ to $H^{-1/2}(\Gamma_R^+)$ since $T_0' - T_0$ is bounded from $H^{1/2}(\Gamma_R^+)$ to $H^{1/2}(\Gamma_R^+)$ and the imbedding from $H^{1/2}(\Gamma_R^+)$ into $H^{-1/2}(\Gamma_R^+)$ is compact. So from the compactness of $T_0' - T_0$ and the imbedding $W \in L^2(\Omega_R)$, we conclude that $b_1(u, w)$ is compact with respect to u.

By the theorem of Lax-Milgram and Fredholm alternative, we complete the proof.

The following theorem is the main result of this section.

Theorem 3.1 *Let* (H1) *and* (H2) *be satisfied. Then for sufficiently large* σ_0*, the variational formulation* (3.5) *of the problem* ($\mathrm{TE}_{\mathrm{PML}}$) *has a unique solution* $\hat{u} \in H^1(\Omega_R)$*. Moreover, we have the following estimate:*

$$\|u^s - \hat{u}^s\|_{H^1(\Omega_R)} \le C\hat{C}^{-1}(1+kR)^2|\alpha_0|^2 \mathrm{e}^{-k\mathrm{Im}(\tilde{\rho})\left(1-\frac{R^2}{|\tilde{\rho}|^2}\right)^{1/2}} \|\hat{u}^s\|_{H^{1/2}(\Gamma_R^+)}, \tag{3.9}$$

where

$$\alpha_0 = 1 + \mathrm{i}\sigma_0, \qquad \tilde{\rho} = \rho + \mathrm{i}\sigma_0(\rho - R)/(m+1),$$

$C > 0$ *is a constant independent of* k, ρ *and* σ_0, *and* $\hat{C} \le 1$.

Proof. For proving that the variational formulation (3.5) has a unique solution, we resort to the general existence and uniqueness result for sesquilinear forms in [21]. The key point is to show the inf-sup condition for the sesquilinear form $\hat{b}_{\mathrm{TE}} : W \times W \to \mathbb{C}$, i.e.,

$$\sup_{0\ne w\in W} \frac{|\hat{b}_{\mathrm{TE}}(\hat{u}, w)|}{\|w\|_{H^1}} \ge C\|\hat{u}\|_{H^1}, \qquad \forall u \in W. \tag{3.10}$$

The following inequality is obvious:

$$|\hat{b}_{\mathrm{TE}}(\hat{u}, w)| \ge |b_{\mathrm{TE}}(\hat{u}, w)| - \Big|\int_{\Gamma_R^+} (T_0 - \hat{T}_0)\hat{u}\bar{w}\mathrm{d}s\Big|. \tag{3.11}$$

We have showed in Lemma 3.2 that the variational formulation (3.6) of ($\mathrm{TE}_{\mathrm{EXA}}$) has a unique solution. The general theory in [21] implies that there exists a constant $C_0 > 0$ such that the following inf-sup condition holds:

$$\sup_{0\ne w\in W} \frac{|b_{\mathrm{TE}}(u, w)|}{\|w\|_{H^1}} \ge C_0\|u\|_{H^1}, \qquad \forall u \in W. \tag{3.12}$$

By Lemma 3.1, we have

$$\|(T_0 - \hat{T}_0)\hat{u}\|_{H^{-1/2}(\Gamma_R^+)} = \sup_{w\in H^{1/2}(\Gamma_R^+)} \frac{|\langle (T_0 - \hat{T}_0)\hat{u}, w\rangle_{\Gamma_R^+}|}{\|w\|_{H^{1/2}(\Gamma_R^+)}} \le C_{\sigma_0}\|u\|_{H^{1/2}(\Gamma_R^+)},$$

where C_{σ_0} is the coefficient in the inequality of Lemma 3.1. Although the coefficient depends not only on σ_0, it is sufficient small when σ_0 is sufficient large. We know from the equality above and the Trace theorem that there exists a constant $C' > 0$ such that

$$\begin{aligned} \Big|\int_{\Gamma_R^+} (T_0 - \hat{T}_0)\hat{u}w\mathrm{d}s\Big| &\le C_{\sigma_0}\|u\|_{H^{1/2}(\Gamma_R^+)}\|w\|_{H^{1/2}(\Gamma_R^+)} \\ &\le C_{\sigma_0}C'\|u\|_{H^1(\Omega_R)}\|w\|_{H^1(\Omega_R)}. \end{aligned} \tag{3.13}$$

Then by (3.11)–(3.13), we conclude that $\hat{b}_{\mathrm{TE}}$ satisfies the inf-sup condition (3.10).

We now prove the error estimate (3.9). By (3.5) and (3.6)

$$\begin{aligned} b_{\mathrm{TE}}(u - \hat{u}, w) &= \hat{b}_{\mathrm{TE}}(\hat{u}, w) - \int_{\Gamma_R^+} (T_0 - \hat{T}_0)(u^i + u^r)\bar{w}\mathrm{d}s - b_{\mathrm{TE}}(\hat{u}, w) \\ &= \int_{\Gamma_R^+} (T_0 - \hat{T}_0)\hat{u}\bar{w}\mathrm{d}s - \int_{\Gamma_R^+} (T_0 - \hat{T}_0)(u^i + u^r)\bar{w}\mathrm{d}s \\ &= \int_{\Gamma_R^+} (T_0 - \hat{T}_0)\hat{u}^s\bar{w}\mathrm{d}s. \end{aligned}$$

This implies (3.9) upon using Lemma 3.1 and (3.12). We thus complete the proof.

4 The PML Method for TM Polarization

We introduce the same PML layer Ω^+_{PML} in TM polarization as that in TE polarization. We also introduce the same model medium property α and PML equation (3.1).

The PML scattering solution $\hat{u}^s$ in Ω_ρ corresponding to the scattered field $u^s$of (TM) is defined as the solution of the following system:

$$\begin{cases} \nabla\cdot(A\nabla\hat{u}^s)+\alpha\beta k^2\varepsilon_r\hat{u}^s=h(x) \qquad \text{in } \Omega_\rho, \\ \hat{u}^s=q(x) \quad \text{on } S\cup S_1\cup\Gamma_0, \qquad \hat{u}^s=0 \quad \text{on } \Gamma^+_\rho, \end{cases} \tag{4.1}$$

where

$$h(x)=(k^2-k^2\varepsilon_r)(u^i+u^r),$$

$$q(x)=-(u^i+u^r) \qquad \text{on } S$$

and

$$q(x)=0 \qquad \text{on } \Gamma_0\cup S_1.$$

The PML problem (4.1) in Ω_ρ can be reformulated in Ω_R by imposing the boundary condition

$$\frac{\partial\hat{u}^s}{\partial n}\Big|_{\Gamma^+_R}=\hat{T}_1\hat{u}^s,$$

where the operator $\hat{T}_1: H^{1/2}(\Gamma^+_R)\to H^{-1/2}(\Gamma^+_R)$ is defined as follows: Given $f\in H^{1/2}(\Gamma^+_R)$,

$$\hat{T}_1 f=\frac{\partial\zeta}{\partial n}\Big|_{\Gamma^+_R},$$

where $\zeta\in H^1(\Omega^+_{\mathrm{PML}})$ satisfies

$$(\mathrm{P}_1)\begin{cases} \nabla\cdot(A\nabla\zeta)+\alpha\beta k^2\zeta=0 \qquad \text{in } \Omega^+_{\mathrm{PML}}, \\ \zeta=f \quad \text{on } \Gamma^+_R, \qquad \zeta=0 \quad \text{on } \Gamma^+_\rho\cup\Gamma_0. \end{cases}$$

Denote by $\hat{u}:=u^i+u^r+\hat{u}^s$ the PML total solution. Like the discussion in the last section, we can reduce the PML problem (4.1) in Ω_ρ to the following PML problem in Ω_R:

$$(\mathrm{TM}_{\mathrm{PML}})\begin{cases} \Delta\hat{u}+k^2\varepsilon_r\hat{u}=0 & \text{in } \Omega_R, \\ \dfrac{\partial\hat{u}}{\partial n}-\hat{T}_1\hat{u}=g(\theta)-\hat{T}_1(u^i+u^r) & \text{on } \Gamma^+_R, \\ \hat{u}=0 & \text{on } S_1\cup S. \end{cases}$$

As showed in [15], we can reformulate the problem (TM) defined in the infinite domain $\Omega\cup\mathcal{U}$ exactly to the following problem in Ω_R:

$$(\mathrm{TM}_{\mathrm{EXA}})\begin{cases} \Delta u+k^2\varepsilon_r u=0 & \text{in } \Omega_R, \\ \dfrac{\partial u}{\partial n}-T_1 u=g(\theta)-T_1(u^i+u^r) & \text{on } \Gamma^+_R, \\ u=0 & \text{on } S_1\cup S, \end{cases}$$

where the DtN operator T_1 on Γ^+_R is defined as follows: for any $u\in H^{1/2}(\Gamma^+_R)$

$$T_1 u=k\sum_{n=1}^{\infty}\frac{H_n^{(1)'}(kR)}{H_n^{(1)}(kR)}\sin(n\theta)u_n, \qquad u_n=\frac{2}{\pi}\int_0^\pi u\sin(n\theta')\mathrm{d}\theta'.$$

Here, we introduce another definition of $\|\cdot\|^2_{H^{\pm1/2}(\Gamma^+_a)}$. For any function ξ^+ defined on

$$\Gamma^+_a=\{x\in\mathbb{R}^2_+ : |x|=a\},$$

we can also define $\|\cdot\|_{H^{\pm 1/2}(\Gamma_a^+)}$ as follows:

$$\|\xi^+\|^2_{H^{1/2}(\Gamma_a^+)} = \frac{\pi}{2}\sum_{n=1}^{\infty}(1+n^2)^{1/2}|\hat{\xi}_n^+|^2,$$

$$\|\xi^+\|^2_{H^{-1/2}(\Gamma_a^+)} = \frac{\pi}{2}\sum_{n=1}^{\infty}(1+n^2)^{-1/2}|\hat{\xi}_n^+|^2,$$

where

$$\hat{\xi}_n^+ = 2/\pi\int_0^{\pi}\xi^+\sin(n\theta)\mathrm{d}\theta.$$

Lemma 4.1 *Let* (H1) *and* (H2) *be satisfied. For* $u \in H^{1/2}(\Gamma_R^+)$, *we have*

$$\|T_1u - \hat{T}_1u\|_{H^{-1/2}(\Gamma_R^+)} \le C\hat{C}^{-1}(1+kR)^2|\alpha_0|^2\mathrm{e}^{-k\mathrm{Im}(\tilde{\rho})\left(1-\frac{R^2}{|\tilde{\rho}|^2}\right)^{1/2}}\|u\|_{H^{1/2}(\Gamma_R^+)},$$

where

$$\alpha_0 = 1 + \mathrm{i}\sigma_0,$$

$$\tilde{\rho} = \rho + \mathrm{i}\sigma_0(\rho - R)/(m+1),$$

$C > 0$ *is a constant independent of* k, R, ρ *and* σ_0, *and* $\hat{C} \le 1$.

Proof. We also denote by x_ρ the reflection of x about x-axis. We prolongate $u \in H^{1/2}(\Gamma_R^+)$ with

$$u(x) = -u(x_\rho),$$

and the function after prolongation is denoted by $u_2 \in H^{1/2}(\Gamma_R)$. We conclude with simple calculation that

$$(T_1u)(x) = (Tu_2)(x), \quad x \in \Gamma_R^+ \ \text{ and } \ (Tu_2)(x) = -(Tu_2)(x_\rho). \tag{4.2}$$

Let the boundary function

$$f(x) = u_2(x)$$

on Γ_R for the problem (P) which defines $\hat{T}$, and let

$$f(x) = u(x)$$

on Γ_R^+ for the problem (P_1) which defines $\hat{T}_1$. Using the idea of [22], we can prove the equivalence of the two problems (P) and (P_1).

If a function $\zeta(x)$ solves (P), then by the assumption that the boundary function

$$f(x) = u_2(x)$$

is odd with respect to y-axis and the symmetry of Ω_{PML}, clearly, the function $-\zeta(x_\rho)$ also solves (P). Hence, from the assumption (H2), we conclude that

$$\zeta(x) = -\zeta(x_\rho),$$

that is,

$$\zeta(x) = 0 \qquad \text{on } \Gamma_0,$$

so $\zeta(x)$ is also the solution of (P_1). On the other hand, a solution $\zeta(x)$ for (P_1) can be defined to be a solution for (P) via

$$\zeta(x) = -\zeta(x_\rho).$$

By the equivalence of (P) and (P_1) and the definitions of $\hat{T}$ and $\hat{T}_1$, we conclude that

$$(\hat{T}_1 u)(x) = (\hat{T} u_2)(x), \quad x \in \Gamma_R^+ \ \text{ and } \ (\hat{T} u_2)(x) = -(\hat{T} u_2)(x_\rho). \tag{4.3}$$

Upon using Lemma 2.5 in [18] and (4.2) and (4.3), we have

$$\begin{aligned} \|T_1 u - \hat{T}_1 u\|_{H^{-1/2}(\Gamma_R^+)} &= \|T u_2 - \hat{T} u_2\|_{H^{-1/2}(\Gamma_R^+)} \\ &= \frac{1}{2}\|T u_2 - \hat{T} u_2\|_{H^{-1/2}(\Gamma_R)} \\ &\le C\hat{C}^{-1}(1+kR)^2|\alpha_0|^2 \mathrm{e}^{-k\mathrm{Im}(\tilde{\rho})\left(1-\frac{R^2}{|\tilde{\rho}|^2}\right)^{1/2}} \|u_2\|_{H^{1/2}(\Gamma_R)} \\ &= 2C\hat{C}^{-1}(1+kR)^2|\alpha_0|^2 \mathrm{e}^{-k\mathrm{Im}(\tilde{\rho})\left(1-\frac{R^2}{|\tilde{\rho}|^2}\right)^{1/2}} \|u\|_{H^{1/2}(\Gamma_R^+)}. \end{aligned}$$

We complete the proof by replacing $2C$ with C.

We also have the conclusion of Theorem 3.1 for the case of TM. We omit the deduction here, because it is almost the same as that in the TE polarization.

5 Numerical Experiment

In this section, we always take

$$k = 2\pi, \quad \rho = 6 \quad \text{and} \quad R = 2.$$

Let $m = 2$ in assumption (H1), so

$$\sigma = \sigma_0\left(\frac{r-R}{\rho-R}\right)^2.$$

According to the error estimates in Theorem 3.1 and Theorem 4.1, we choose σ_0 such that the exponentially decaying factor

$$\mathrm{e}^{-k\mathrm{Im}(\tilde{\rho})\left(1-\frac{R^2}{|\tilde{\rho}|^2}\right)^{1/2}} \approx 10^{-8}.$$

Example 5.1 This example will demonstrate the accuracy of the PML method for scattering problems in half-space. Considering the scattering problems for the perturbed half-plane showed in Fig. 5.1: find radiating solutions u_1 and u_2 satisfying

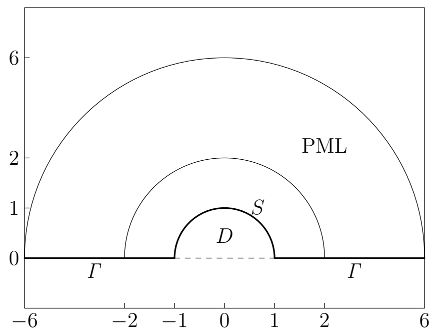

Fig. 5.1 Perturbed half plane

$$\begin{cases} \Delta u_1 + k^2 u_1 = 0, & \text{in } \mathbb{R}_+^2 \backslash \bar{D}, \\ u_1 = 0, & \text{on } \Gamma, \\ u_1 = 2H_1^{(1)}(k)\sin\theta, & \text{on } S \end{cases}$$

and

$$\begin{cases} \Delta u_2 + k^2 u_2 = 0, & \text{in } \mathbb{R}^2_+ \backslash \bar{D}, \\ \partial u_2/\partial n = 0, & \text{on } \mathit{\Gamma}, \\ u_2 = 2H_1^{(1)}(k)\cos\theta, & \text{on } S \end{cases}$$

respectively. The exact solutions of the two scattering problems are known as

$$u_1 = 2H_1^{(1)}(kr)\sin\theta$$

and

$$u_2 = 2H_1^{(1)}(kr)\cos\theta, \qquad r = |x|.$$

We take different values for σ_0 with 1, 2, 3. Fig. 5.2 shows that the PML solutions with different absorbing property σ_0 are close to the exact solutions u_1 and u_2 on $\mathit{\Gamma}_R^+$ for the two problems.

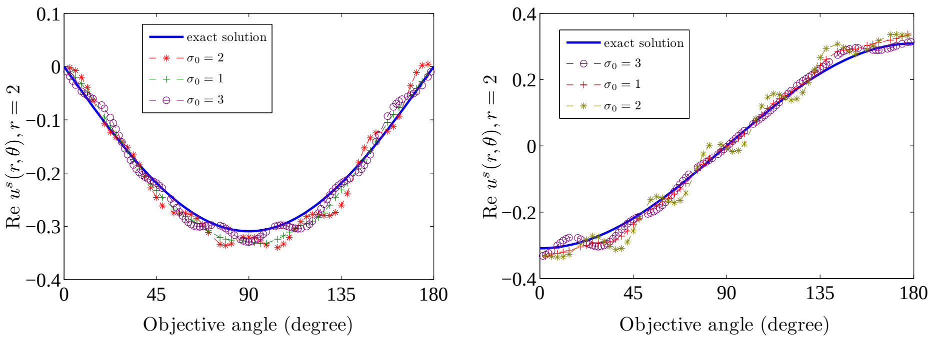

Fig. 5.2 The real part of the exact solutions and the PML solutions with different σ_0 on $\mathit{\Gamma}_R^+$ for the two problems in Example 5.1.

Example 5.2 In this example, we use the PML method to solve an overfilled cavity scattering problem. The cavity wall is a semicircle of radius 1 and the protruding portion is a segment of a circle centering at $(-1, 0)$ and of radius $\sqrt{2}$, as showed in Fig. 5.3. ε_r equals $4-\mathrm{i}$ for filled media in the cavity.

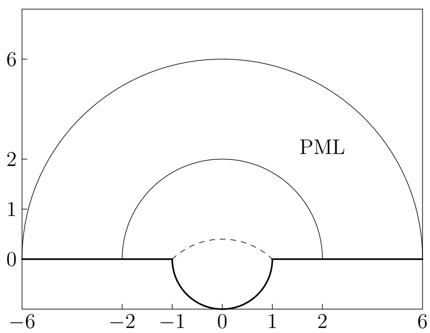

Fig. 5.3 Overfilled cavity

We assume

$$u^i = \mathrm{e}^{\alpha x + \beta y}.$$

Then

$$u^r = -\mathrm{e}^{\alpha x - \beta y}$$

for TM polarization and

$$u^r = \mathrm{e}^{\alpha x - \beta y}$$

for TE polarization, where $\alpha = \mathrm{i}k\cos(\theta^{\text{inc}})$, $\beta = \mathrm{i}k\sin(\theta^{\text{inc}})$, and θ^{inc} is the incident angle. As showed in Example 5.1, the PML solution with $\sigma_0 = 2$ is close to the exact solution u_1 and u_2, so we choose medium property $\sigma_0 = 2$ in this example too.

Fig. 5.4 shows that the real part of the scattered fields for TM and TE polarizations are both perfectly symmetric and both decay in the PML layer with $\theta^{\text{inc}} = \pi/2$. Fig. 5.5 shows the perfect linkage between the numerical solutions in the interior domain and the analytical solutions in the exterior. Fig. 5.6 shows the RCS for the TM polarization with $\theta^{\text{inc}} = \pi/2$. We compute the RCS for TM polarization by using the analytical formulation

$$\sigma(\theta) = \frac{16}{k\pi^2}\left|\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}n\pi/2}\sin(n\theta)}{H_n^{(1)}(kR)}\int_0^{\pi} u^s(R,\theta)\sin(n\theta)\mathrm{d}\theta\right|^2$$

and we compute the RCS for TE polarization by the analytical formulation

$$\sigma(\theta) = \frac{4}{k\pi^2}\left|\frac{1}{H_n^{(1)}(kR)}\int_0^{\pi} u^s(R,\theta)\mathrm{d}\theta + 2\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}n\pi/2}\cos(n\theta)}{H_n^{(1)}(kR)}\int_0^{\pi} u^s(R,\theta)\cos(n\theta)\mathrm{d}\theta\right|^2.$$

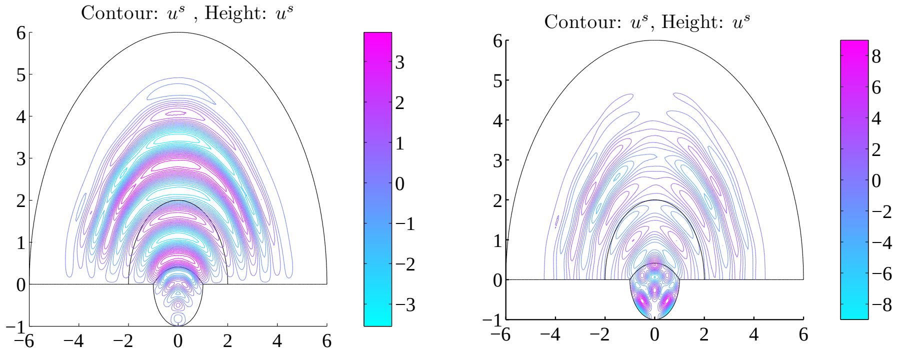

Fig. 5.4 Contours of the real part of the scattered field for TM polarization and for TE polarization in the computing domain, respectively. The incident angle $\theta^{\text{inc}} = \pi/2$.

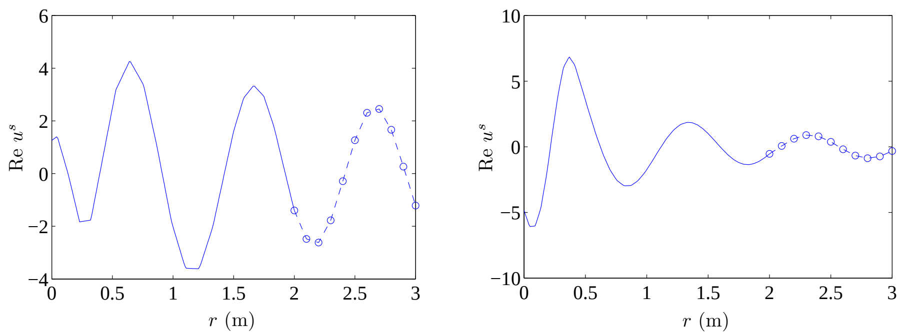

Fig. 5.5 Linkages of the real part of the scattered field on Γ_R^+ for TM polarization and for TE polarization, respectively. The observing angle $\theta = \theta^{\text{inc}} = \pi/2$.

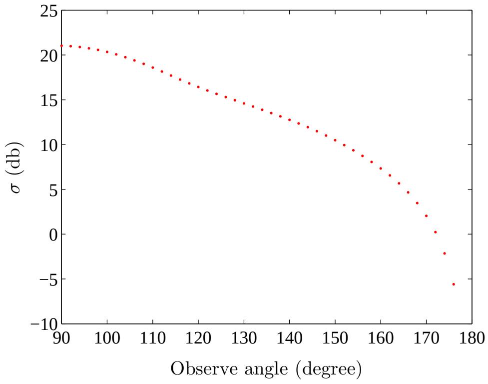

Fig. 5.6 RCS of the overfilled cavity for TM polarization.

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