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Eigenvalue Problem of Doubly Stochastic Hamiltonian Systems with Boundary Conditions*

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Abstract: In this paper, we investigate the eigenvalue problem of forward-backward doubly stochastic differential equations with boundary value conditions. We show that this problem can be represented as an eigenvalue problem of a bounded continuous compact operator. Hence using the famous Hilbert-Schmidt spectrum theory, we can characterize the eigenvalues exactly.

Key words: doubly stochastic Hamiltonian system, eigenvalue problem, spectrum theory

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1 Introduction

Stochastic Hamiltonian systems were introduced in the theory of stochastic optimal control as a necessary condition of an optimal control, known as the stochastic version of the maximum principle of Pontryagin's type (see [1]–[6]). In fact, those stochastic Hamiltonian systems with boundary conditions are forward-backward stochastic differential equations (FBSDE for short). These have been extensively investigated by Antonelli[7], Ma *et al.*[8], Hu and Peng[9], Peng and Wu[10], Yong[11]. Recently, combining the FBSDE and the backward doubly stochastic differential equations introduced by Pardoux and Peng[12], Peng and Shi[13] have investigated a type of time-symmetric FBSDE. They showed the uniqueness and existence of solutions for these equations under certain monotonicity conditions.

In this paper, we study a special type of time-symmetric FBSDE, namely doubly stochastic Hamiltonian systems (DSHS for short). We discuss the eigenvalue problem of this type

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of stochastic Hamiltonian system in a standard functional analysis way.

The rest of this paper is organized as follows. The next section begins with a general formulation of time-symmetric FBSDE, then a special case, DSHS with boundary conditions. In Section 3, we give the proof of the main results.

2 Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $T > 0$ be fixed throughout this paper. Let $\{W_t : 0 \le t \le T\}$ and $\{B_t : 0 \le t \le T\}$ be two mutually independent standard Brownian motions which are $\mathbf{R}^d$-valued processes defined on $(\Omega, \mathcal{F}, P)$. Without loss of generality, we assume that $d = 1$. Let $\mathcal{N}$ denote the class of P-null sets of $\mathcal{F}$. For each $t \in [0, T]$, we define

$$\mathcal{F}_t \triangleq \mathcal{F}_t^w \vee \mathcal{F}_{t,T}^B,$$

where

$$\mathcal{F}_t^w = \mathcal{N} \vee \sigma\{W_r - W_0 : 0 \le r \le t\},$$

$$\mathcal{F}_{t,T}^B = \mathcal{N} \vee \sigma\{B_r - B_t : t \le r \le T\}.$$

Note that the collection $\{\mathcal{F}_t : t \in [0, T]\}$ is neither increasing nor decreasing. Thus it does not constitute a filtration.

Let $M^2(0, T; \mathbf{R}^n)$ denote the set of all classes ($dt \times dP$ is equal a.e.) $\mathcal{F}_t$-measurable stochastic processes $\{\varphi_t : t \in [0, T]\}$ which satisfy

$$\mathrm{E} \int_0^T |\varphi_t|^2 \mathrm{d}t < +\infty.$$

For a given $\varphi_t, \psi_t \in M^2(0, T; \mathbf{R}^n)$, one can define the forward Itô integration $\int_0^{\cdot} \varphi_s \mathrm{d}W_s$ and the backward Itô integration $\int_{\cdot}^T \psi_s \mathrm{d}B_s$. They are both in $M^2(0, T; \mathbf{R}^n)$.

Let $H(y, Y, z, Z) : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ and $\Phi(y) : \mathbf{R}^n \to \mathbf{R}$ be C^1 functions. Find a triple

$$(y, Y, z, Z) \in M^2(0, T; \mathbf{R}^n)$$

such that a boundary problem for a doubly stochastic Hamiltonian system satisfies the following form

$$\begin{cases} dy_t = H_Y(t, y_t, Y_t, z_t, Z_t)dt + H_Z(t, y_t, Y_t, z_t, Z_t)dW_t - z_t dB_t, \\ y(0) = y_0, \\ -dY_t = H_y(t, y_t, Y_t, z_t, Z_t)dt + H_z(t, y_t, Y_t, z_t, Z_t)dB_t - Z_t dW_t, \\ Y_T = \Phi_y(y_T), \end{cases} \tag{2.1}$$

where H_y, H_Y, H_z, H_Z are gradients of the function H with respect to y, Y, z, Z respectively.

This is a sort of time-symmetric FBSDE introduced by Peng and Shi[13]. Let

$$\xi = (y, Y, z, Z)^\top,$$

$$\Lambda(t, \xi) = (-H_y, H_Y, -H_z, H_Z)^\top(t, \xi).$$

We assume the following:

(H1) For each $\xi \in \mathbf{R}^{4n}$, $\Lambda(\,\cdot\,,\xi)$ is an $\mathcal{F}_t$-measurable vector process defined on $[0,T]$ with $\Lambda(\,\cdot\,,0) \in M^2(0,T;\mathbf{R}^{4n})$, and for each $y \in \mathbf{R}^n$, $\Phi(y)$ is an $\mathcal{F}_T$-measurable random vector with $\Phi(0) \in L^2(\Omega,\mathcal{F}_T,P;\mathbf{R}^n)$.

We also assume that Λ and Φ satisfy Lipschitz condition respectively as follows:

(H2)
$$|\Lambda(t,\xi) - \Lambda(t,\xi')| \le c|\xi - \xi'|, \qquad \forall \xi,\xi' \in \mathbf{R}^{4n};$$
$$|\Phi(y) - \Phi(y')| \le c|y - y'|, \qquad \forall y,y' \in \mathbf{R}^n.$$

The main assumptions are the following monotonicity conditions

(H3)
$$\langle \Lambda(t,\xi) - \Lambda(t,\xi'),\ \xi - \xi' \rangle \le -\alpha|\xi - \xi'|^2,$$

where α is a constant and $\alpha > 0$, and

(H4)
$$\langle \Phi(y) - \Phi(y'),\ y - y' \rangle \ge 0, \qquad \forall y,y' \in \mathbf{R}^n.$$

The Theorem 2.2 in [13] is given as follows.

Proposition 2.1 *Under the assumptions* (H1)–(H4)*, there exists a unique solution*
$$(y,Y,z,Z)(\,\cdot\,) \in M^2(0,T;\mathbf{R}^{4n})$$
of equation (2.1)*.*

3 Eigenvalue Problem of Linear DSHS

We consider the FBDSHS as follows:
$$\begin{cases} dy_t = [H_Y(t,y_t,Y_t,z_t,Z_t) + \lambda h_2 h^\top(y_t,Y_t,z_t,Z_t)]dt \\ \qquad\quad +[H_Z(t,y_t,Y_t,z_t,Z_t) + \lambda h_4 h^\top(y_t,Y_t,z_t,Z_t)]dW_t - z_t dB_t, \\ -dY_t = [H_y(t,y_t,Y_t,z_t,Z_t) + \lambda h_1 h^\top(y_t,Y_t,z_t,Z_t)]dt \\ \qquad\quad +[H_z(t,y_t,Y_t,z_t,Z_t) + \lambda h_3 h^\top(y_t,Y_t,z_t,Z_t)]dB_t - Z_t dW_t, \\ y(0) = 0, \qquad Y_T = 0. \end{cases} \tag{3.1}$$

We assume that
$$H_\xi(\,\cdot\,,0) = 0, \quad h(\,\cdot\,,0) = 0, \qquad \text{for } \xi = (y_t,Y_t,z_t,Z_t)^\top.$$

Obviously, the system has an only trivial solution as $\lambda = 0$. The eigenvalue problem of DSHS is to find some $\lambda \ne 0$, such that this system has a nontrivial solution. The corresponding nontrivial solution is called eigenvalue function (the reader can see [14] for details of eigenvalue problem of stochastic differential equations).

Assume that

(H5) $h(\xi)$ is bounded and satisfies Lipschitz condition:
$$|h(\xi) - h(\xi')|^2 \le \mu|\xi - \xi'|^2, \qquad \forall \xi,\xi' \in R^{4n}.$$

We have the following main results.

Theorem 3.1 *Assume that* (H1)–(H5) *hold. Then the DSHS* (3.1) *has at most numerable eigenvalues. These eigenvalues are discrete, positive real numbers. Moreover,* $\dfrac{1}{\lambda} \ge 0$ *and has a limit 0.*

Let

$$\eta = (u, v, r, s) \in M^2(0, T; R^{4n}).$$

For the sake of proving Theorem 3.1, we investigate the forward backward doubly stochastic differential equations (FBDSDE for short) as follows:

$$\begin{cases} dy_t = [H_Y(t,\xi) + h_2(\eta)]dt + [H_Z(t,\xi) + h_4(\eta)]dW_t - z_t dB_t, \\ -dY_t = [H_y(t,\xi) + h_1(\eta)]dt + [H_z(t,\xi) + h_3(\eta)]dB_t - Z_t dW_t, \\ y(0) = 0, \qquad Y_T = 0. \end{cases} \tag{3.2}$$

We assume that (H1)–(H4) hold. By Proposition 2.1, for any $\eta \in M^2(0, T; R^{4n})$, we obtain that the FBDSDE (3.2) has a unique solution $\xi_\eta \in M^2(0, T; R^{4n})$. So we introduce the following map:

$$\begin{aligned} \mathcal{A}: \quad & \eta(\cdot) \in M^2(0, T; R^{4n}) \to \xi_\eta(\cdot) \in M^2(0, T; R^{4n}), \\ & \mathcal{A}(\eta(\cdot))(t) = h^\top(\eta)\xi_\eta(t). \end{aligned}$$

Firstly, for the map $\mathcal{A}$ we have as follows.

Lemma 3.1 *For any* $\eta, \eta' \in M^2(0, T; R^{4n})$,

$$\mathrm{E}\int_0^T \langle \xi_\eta - \xi_{\eta'},\ \Lambda(\xi_\eta) - \Lambda(\xi_{\eta'})\rangle \mathrm{d}t = -\mathrm{E}\int_0^T \langle \xi_\eta - \xi_{\eta'},\ h(\eta) - h(\eta')\rangle \mathrm{d}t, \tag{3.3}$$

where ξ_η, $\xi_{\eta'}$ *are the solutions of FBDSHS* (3.2) *with respect to* η, η' *respectively.*

Proof. Applying the generalized Itô formula (see the Lemma 1.3 of [12] for details) to $\langle y_\eta(t) - y_{\eta'}(t),\ Y_\eta(t) - Y_{\eta'}(t)\rangle$, we have

$$\begin{aligned} & d\langle y_\eta(t) - y_{\eta'}(t),\ Y_\eta(t) - Y_{\eta'}(t)\rangle \\ = {} & \langle y_\eta(t) - y_{\eta'}(t),\ d(Y_\eta(t) - Y_{\eta'}(t))\rangle + \langle d(y_\eta(t) - y_{\eta'}(t)),\ Y_\eta(t) - Y_{\eta'}(t)\rangle \\ & + \langle d(y_\eta(t) - y_{\eta'}(t)),\ d(Y_\eta(t) - Y_{\eta'}(t))\rangle \\ = {} & \left\langle \begin{pmatrix} y_\eta - y_{\eta'} \\ Y_\eta - Y_{\eta'} \\ z_\eta - z_{\eta'} \\ Z_\eta - Z_{\eta'} \end{pmatrix}, \begin{pmatrix} -[H_y(t,\xi_\eta) - H_y(t,\xi_{\eta'})] - [h_1(\eta) - h_1(\eta')] \\ [H_Y(t,\xi_\eta) - H_Y(t,\xi_{\eta'})] + [h_2(\eta) - h_2(\eta')] \\ -[H_z(t,\xi_\eta) - H_z(t,\xi_{\eta'})] - [h_3(\eta) - h_3(\eta')] \\ [H_Z(t,\xi_\eta) - H_Z(t,\xi_{\eta'})] + [h_4(\eta) - h_4(\eta')] \end{pmatrix} \right\rangle dt \\ & + \left\langle \begin{pmatrix} y_\eta - y_{\eta'} \\ Y_\eta - Y_{\eta'} \end{pmatrix}, \begin{pmatrix} -[H_z(t,\xi_\eta) - H_z(t,\xi_{\eta'})] - [h_3(\eta) - h_3(\eta')] \\ z_{\eta'}(t) - z_\eta(t) \end{pmatrix} \right\rangle dB_t \\ & + \left\langle \begin{pmatrix} y_\eta - y_{\eta'} \\ Y_\eta - Y_{\eta'} \end{pmatrix}, \begin{pmatrix} Z_{\eta'}(t) - Z_\eta(t) \\ [H_Z(t,\xi_\eta) - H_Z(t,\xi_{\eta'})] + [h_4(\eta) - h_4(\eta')] \end{pmatrix} \right\rangle dW_t. \end{aligned}$$

Noting that

$$y_\eta(0) = y_{\eta'}(0) = Y_\eta(T) = Y_{\eta'}(T) = 0,$$

we integrate it from 0 to T and take expectation on both sides. Then we have that

$$0 = \mathrm{E}\int_0^T \langle \xi_\eta - \xi_{\eta'},\ \Lambda(\xi_\eta) - \Lambda(\xi_{\eta'})\rangle \mathrm{d}t + \mathrm{E}\int_0^T \langle \xi_\eta - \xi_{\eta'},\ h(\eta) - h(\eta')\rangle \mathrm{d}t.$$

This completes the proof of Lemma 3.1.

Noting the assumption (H3) and (3.3), we have that

$$\begin{aligned} \mathrm{E}\int_0^T \langle \xi_\eta - \xi_{\eta'},\ h(\eta) - h(\eta')\rangle \mathrm{d}t &= -\mathrm{E}\int_0^T \langle \xi_\eta - \xi_{\eta'},\ \mathcal{A}(\xi_\eta) - \mathcal{A}(\xi_{\eta'})\rangle \mathrm{d}t \\ &\geq \alpha \mathrm{E}\int_0^T |\xi_\eta - \xi_{\eta'}|^2 \mathrm{d}t. \end{aligned} \tag{3.4}$$

Thus by assumption (H5) and Hölder inequality, we have

$$\begin{aligned} \mathrm{E}\int_0^T |\xi_\eta - \xi_{\eta'}|^2 \mathrm{d}t &\leq \frac{1}{\alpha}\mathrm{E}\int_0^T \langle \xi_\eta - \xi_{\eta'},\ h(\eta) - h(\eta')\rangle \mathrm{d}t \\ &\leq \frac{1}{\alpha}\Big(\mathrm{E}\int_0^T |\xi_\eta - \xi_{\eta'}|^2 \mathrm{d}t\Big)^{1/2} \cdot \Big(\mathrm{E}\int_0^T |h(\eta) - h(\eta')|^2 \mathrm{d}t\Big)^{1/2} \\ &\leq \frac{\mu}{\alpha}\Big(\mathrm{E}\int_0^T |\xi_\eta - \xi_{\eta'}|^2 \mathrm{d}t\Big)^{1/2} \cdot \Big(\mathrm{E}\int_0^T |\eta - \eta'|^2 \mathrm{d}t\Big)^{1/2}. \end{aligned}$$

Thus

$$\mathrm{E}\int_0^T |\xi_\eta - \xi_{\eta'}|^2 \mathrm{d}t \leq \frac{\mu^2}{\alpha^2}\mathrm{E}\int_0^T |\eta - \eta'|^2 \mathrm{d}t.$$

So

$$\begin{aligned} \|\mathcal{A}(\eta(\cdot)) - \mathcal{A}(\eta'(\cdot))\|^2 &= \mathrm{E}\int_0^T |h^\top(\eta)\xi_\eta - h^\top(\eta)\xi_{\eta'}|^2 \mathrm{d}t \\ &\leq \|h^\top(\eta)\|^2 \mathrm{E}\int_0^T |\xi_\eta - \xi_{\eta'}|^2 \mathrm{d}t \\ &\leq \frac{\mu^2\|h^*\|^2}{\alpha^2}\mathrm{E}\int_0^T |\eta - \eta'|^2 \mathrm{d}t. \end{aligned} \tag{3.5}$$

This shows that $\mathcal{A}(\eta(\cdot))$ is a bounded continuous map.

Now we assume that the original DSHS is linear, i.e.,

$$\begin{cases} dy_t = (H_{21}y_t + H_{22}Y_t + H_{23}z_t + H_{24}Z_t)dt \\ \qquad\quad +(H_{41}y_t + H_{42}Y_t + H_{43}z_t + H_{44}Z_t)dW_t - z_t dB_t, \\ -dY_t = (H_{11}y_t + H_{12}Y_t + H_{13}z_t + H_{14}Z_t)dt \\ \qquad\quad +(H_{31}y_t + H_{32}Y_t + H_{33}z_t + H_{34}Z_t)dB_t - Z_t dW_t, \\ y_0 = 0, \quad Y_T = 0. \end{cases} \tag{3.6}$$

The monotonicity condition (H3) is equivalent to which there exists $\beta > 0$ such that

$$\begin{bmatrix} -H_{11} & -H_{12} & -H_{13} & -H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ -H_{31} & -H_{32} & -H_{33} & -H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix} \leq -\mu I_{4n}. \tag{3.7}$$

Suppose (3.7) holds. Considering the preceding map $\mathcal{A}$, we obtain as follows.

Lemma 3.2 *The map $\mathcal{A}$ is a linear, bounded, self-adjoint, positive operator.*

Proof. It is easy to see that $\mathcal{A}$ is a linear operator. Noticing that $\mathcal{A}(0) = 0$ and (3.4), we have that

$$\begin{aligned} \mathrm{E}\int_0^T \langle \mathcal{A}(\eta(t)),\ \eta(t)\rangle \mathrm{d}t &= \mathrm{E}\int_0^T \langle \xi_\eta(t),\ h(\eta)\rangle \mathrm{d}t \\ &\geq \alpha \mathrm{E}\int_0^T |\xi_\eta|^2 \mathrm{d}t \\ &\geq 0. \end{aligned}$$

So $\mathcal{A}$ is positive.

We then prove $\mathcal{A}$ is self-adjoint. Applying the generalized Itô formula to $\langle y_\eta,\ Y_{\eta'}\rangle$, $\langle y_{\eta'},\ Y_\eta\rangle$, we have that

$$\begin{aligned} d\langle y_\eta,\ Y_{\eta'}\rangle = {}& \langle y_\eta,\ (-H_y\xi_{\eta'} - h_1\eta')dt - (H_z\xi_{\eta'} + h_3\eta')dB_t + Z_{\eta'}dW_t\rangle \\ &+ \langle Y_{\eta'},\ (H_Y\xi_\eta - h_2\eta)dt + (H_Z\xi_\eta + h_4\eta)dW_t - z_\eta dB_t\rangle \\ &+ \langle Z_{\eta'},\ H_Z\xi_\eta + h_4\eta\rangle dt - \langle z_\eta,\ H_z\xi_{\eta'} + h_3\eta'\rangle dt, \\ d\langle y_{\eta'},\ Y_\eta\rangle = {}& \langle y_{\eta'},\ (-H_y\xi_\eta - h_1\eta)dt - (H_z\xi_\eta + h_3\eta)dB_t + Z_\eta dW_t\rangle \\ &+ \langle Y_\eta,\ (H_Y\xi_{\eta'} - h_2\eta')dt + (H_Z\xi_{\eta'} + h_4\eta')dW_t - z_{\eta'}dB_t\rangle \\ &+ \langle Z_\eta,\ H_Z\xi_{\eta'} + h_4\eta'\rangle dt - \langle z_{\eta'},\ H_z\xi_\eta + h_3\eta\rangle dt. \end{aligned}$$

Noting that

$$y_\eta(0) = y_{\eta'}(0) = Y_\eta(T) = Y_{\eta'}(T) = 0,$$

we integrate it from 0 to T and take expectation on both sides. Then we have that

$$\begin{aligned} &\mathrm{E}\int_0^T \{\langle y_\eta,\ -H_y\xi_{\eta'} - h_1\eta'\rangle + \langle Y_{\eta'}, H_Y\xi_\eta - h_2\eta\rangle \\ &\quad + \langle z_\eta,\ H_z\xi_{\eta'} + h_3\eta'\rangle + \langle Z_{\eta'},\ H_Z\xi_\eta + h_4\eta\rangle\}\mathrm{d}t \\ &= \mathrm{E}\int_0^T \{\langle y_{\eta'},\ -H_y\xi_\eta - h_1\eta\rangle + \langle Y_\eta,\ H_Y\xi_{\eta'} - h_2\eta'\rangle \\ &\quad - \langle z_{\eta'},\ H_z\xi_\eta + h_3\eta\rangle + \langle Z_\eta,\ H_Z\xi_{\eta'} + h_4\eta'\rangle\}\mathrm{d}t. \end{aligned}$$

Noting that H is symmetric and the definition of $\mathcal{A}(\eta(\cdot))$, we have that

$$\mathrm{E}\int_0^T \langle \mathcal{A}(\eta(t)),\ \eta'(t)\rangle \mathrm{d}t = \mathrm{E}\int_0^T \langle \mathcal{A}(\eta'(t)),\ \eta(t)\rangle \mathrm{d}t.$$

This completes the proof of Lemma 3.2.

Now considering the eigenvalue problem of operator $\mathcal{A}$, we find some $\lambda \neq 0$ such that

$$\lambda\mathcal{A}(\eta) = \eta$$

has nontrivial solutions. By the definition of $\mathcal{A}$, we have that

$$\eta = \lambda h^\top \xi_\eta.$$

Substituting it into (3.2), we obtain (3.1). Hence the eigenvalue problem of DSHS (3.1) is equivalent to the eigenvalue problem of operator $\mathcal{A}$. By Lemmas 3.1, 3.2 and Hilbert-Schmidt spectrum theory, we get Theorem 3.1.

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