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Homotopy Perturbation Method for Time-Fractional Shock Wave Equation

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Abstract. A scheme is developed to study numerical solution of the time-fractional shock wave equation and wave equation under initial conditions by the homotopy perturbation method (HPM). The fractional derivatives are taken in the Caputo sense. The solutions are given in the form of series with easily computable terms. Numerical results are illustrated through the graph.

AMS subject classifications: 76L99, 76K99, 35L10

Key words: Partial differential equation, fractional derivative, shock wave equation, homotopy perturbation method.

1 Introduction

In recent years, considerable interest has been devoted to the study of the fractional calculus during the past decades and their numerous applications in the area of physics and engineering. Many important phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science, probability and statistics, electrochemistry of corrosion, chemical physics, and signal processing are well described by differential equations of fractional order [1–3]. The HPM is the new method for finding the approximate analytical solution of linear and nonlinear problems [4, 5] and successfully applied to solve nonlinear wave equation. The fractional diffusion equation with absorbent term and external force through HPM is analyzed in [6]. The proof of the existence of the attractor for the one-dimensional viscous Fornberg-Whitham equation is studied by [7]. The solution of shock wave equation is examined by ADM and HPM in [8, 9]. In 2010, Golbabai and Sayevand [10] applied the HPM to solve the

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multi-order time fractional differential equations and space-time fractional solidification in a finite slab solved by Singh et al. [11]. Recently, Gupta and Singh [12] used the HPM to solve the time-fractional Fornberg-Whitham equation. Recently, many new approaches for finding the exact solutions to nonlinear equations have been proposed, for example, Exp-function method [13], homotopy analysis method [14], and reduced differential transform method [15] and so on. All methods, mentioned above, have limitations in their applications.

In present article, we implement the Homotopy perturbation method for obtaining analytical and numerical solutions of the shock wave equation with time-fractional derivatives. This equation can be written in operator form as [16–18]

$$u_t^\alpha(x,t)=\Big(\frac{1}{c_0}-\frac{\gamma+1}{2}\frac{u}{c_0^2}\Big)u_x=0,\quad t>0,\ \ x\in R,\ \ 0<\alpha\le 1, \tag{1.1}$$

with initial condition

$$u_0(x,0)=\exp\Big(-\frac{x^2}{2}\Big), \tag{1.2}$$

where c_0 is constant and γ is specific heat.

In [8,9], it is shown that if $c_0\ge(\gamma+1)u/2$ then a series solution can be obtained and it is given by

$$u(x,t)=\sum_0^\infty\frac{(n+1)^{\frac{n}{2}}}{(n+1)!}H_n\big(\sqrt{n+1}\big)\exp\Big[-\frac{1}{2}\Big(x-\frac{t}{2}\Big)^2(n+1)\Big], \tag{1.3}$$

where $B=(\gamma+1)/2c_0^2$ and $H_n(\cdot)$ is the Hermit polynomial of order n.

2 Preliminaries and notations

In this section, we have given some definitions and properties of the fractional calculus [1] which are used further in this paper.

Definition 2.1. *A real function $f(t)$, $t>0$ is said to be in the space C_μ, $\mu\in\Re$, if there exists a real number $p>\mu$, such that $f(t)=t^pf_1(t)$, where $f_1(t)\in C(0,\infty)$, and it is said to be in the space C_μ^n if and only if $h^{(n)}\in C_\mu$, $n\in N$.*

Definition 2.2. *The Riemann-Liouville fractional integral operator (J_t^α) of order $\alpha\ge 0$, of a function $f\in C_\mu$, $\mu\ge -1$ is defined as* [2]

$$\begin{aligned}&J_t^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\xi)^{\alpha-1}f(\xi)d\xi,\quad \alpha>0,\ \ t>0,\\ &J_t^0f(t)=f(t),\end{aligned}$$

where $\Gamma(\alpha)$ is the well-known gamma function. Some of the properties of the operator J_t^α, which we will need here, are as follows: for $f\in C_\mu$, $\mu\ge-1$, $\alpha,\beta\ge 0$ and $\gamma\ge -1$,

(1) $J_t^\alpha J_t^\beta f(t) = J_t^{\alpha+\beta} f(t)$,

(2) $J_t^\alpha J_t^\beta f(t) = J_t^\beta J_t^\alpha f(t)$,

(3) $J_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

Definition 2.3. *The fractional derivative* (D_t^α) *of* $f(t)$*, in the Caputo sense is defined as*

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi,$$

for $n-1 < \alpha < n$*,* $n \in N$*,* $t > 0$*,* $f \in C_{-1}^n$*. The following are two basic properties of the Caputo fractional derivative* [1] *and* [3]

(1) *Let* $f \in C_{-1}^n$*,* $n \in N$*, then* $D_t^\alpha f$*,* $0 \le \alpha \le n$ *is well defined and* $D_t^\alpha f \in C_{-1}$.

(2) *Let* $n-1 \le \alpha \le n$*,* $n \in N$ *and* $f \in C_\mu^n$*,* $\mu \ge -1$*. Then*

$$(J_t^\alpha D_t^\alpha) f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}.$$

3 Solution of the first problem

We first consider the following time-fractional shock waves equation

$$D_t^\alpha u - \Big(\frac{1}{c_0} - \frac{\gamma+1}{2}\frac{u}{c_0^2}\Big) D_x u = 0, \tag{3.1}$$

with initial condition is

$$u_0(x,0) = \exp\Big(-\frac{x^2}{2}\Big). \tag{3.2}$$

According to the HPM [19–21] construct the following homotopy

$$D_t^\alpha u = p\Big(\frac{1}{c_0} - \frac{\gamma+1}{2}\frac{u}{c_0^2}\Big) D_x u, \tag{3.3}$$

where the homotopy parameter p is considered as a small parameter ($p \in [0,1]$). Now applying the classical perturbation technique, we can assume that the solution of Eq. (3.1) can be expressed as a power series in p as given below

$$u = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \cdots . \tag{3.4}$$

When $p \to 1$, Eq. (3.3) corresponding to Eqs. (3.1) and (3.4) becomes the approximate solution of (1.3), that is, of Eq. (1.1). Substituting Eq. (3.4) in Eq. (3.3) and comparing the like powers of p, we obtain the following set of linear differential equations

$$p^0: \; D_t^\alpha u_0 = 0, \tag{3.5a}$$

$$p^1: \; D_t^\alpha u_1 = \Big(\frac{1}{c_0} - \frac{\gamma+1}{2}\frac{u_0}{c_0^2}\Big) D_x u_0, \tag{3.5b}$$

$$p^2: \; D_t^\alpha u_2 = -\frac{\gamma+1}{2}\frac{u_1}{c_0^2} D_x u_0 + \Big(\frac{1}{c_{01}} - \frac{\gamma+1}{2c_0^2}\Big) D_x u_1, \tag{3.5c}$$

$$p^3: \; D_t^\alpha u_3 = \Big(\frac{1}{c_{01}} - \frac{\gamma+1}{2c_0^2}\Big) D_x u_2 - \frac{\gamma+1}{2}\frac{u_2}{c_0^2} D_x u_0 - \frac{\gamma+1}{2}\frac{u_1}{c_0^2} D_x u_1, \tag{3.5d}$$

and so on.

The method is based on applying the operator j_t^α (the inverse operator of Caputo derivative D_t^α) on both sides of Eqs. (3.5a)-(3.5d), then we get

$$u_0(x,t) = e^{-\frac{x^2}{2}}, \tag{3.6a}$$

$$u_1(x,t) = f_1(x)\frac{t^\alpha}{\Gamma(\alpha+1)}, \tag{3.6b}$$

$$u_2(x,t) = f_2(x) f_{1x}(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{3.6c}$$

$$u_3(x,t) = \Big(\frac{\gamma+1}{2c_0^2} x e^{-x^2} f_2(x) + \Big(\frac{1}{c_0} - \frac{\gamma+1}{2c_0^2}\Big) f_{2x} - \frac{\gamma+1}{2c_0^2} f_1(x) f_{1x}(x)\Big)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \tag{3.6d}$$

where

$$f_1(x) = -\Big(\frac{1}{c_0} - \frac{\gamma+1}{2c_0^2}\exp\Big(\frac{-x^2}{2}\Big)\Big) x \exp\Big(\frac{-x^2}{2}\Big),$$

$$f_2(x) = \frac{\gamma+1}{2c_0^2}\Big(\frac{1}{c_0} - \frac{\gamma+1}{2c_0^2}\Big)\Big(\frac{1}{c_0} - \frac{\gamma+1}{2c_0^2}\exp\Big(\frac{-x^2}{2}\Big)\Big) x \exp(-x^2).$$

Proceeding in this manner, the rest of the components u_n can be obtained and the series solutions are thus entirely determined.

Finally, we approximate the analytical solution $u(x,t)$ by the truncated series

$$u(x,t) = \lim_{N\to\infty} \Psi_N(x,t), \tag{3.7}$$

where

$$\Psi_N(x,t) = \sum_0^{N-1} u_n(x,t).$$

The above series solutions generally converge very rapidly. A classical approach of convergence of this type of series is already presented by Abbaoui and Cherruaul [22].

4 Second problem

We consider time-fractional wave equation in the following form [9]

$$u_t^\alpha + u u_x - u_{xxt} = 0, \tag{4.1}$$

with initial condition as

$$u(x,0) = 3\sec h^2\Big(\frac{x-15}{2}\Big). \tag{4.2}$$

The exact solution of Eq. (4.1) in the closed form is

$$u(x,t) = 3\sec h^2\Big(\frac{x-15-t}{2}\Big) \quad \text{at} \;\; \alpha = 1. \tag{4.3}$$

4.1 Solution of second problem

We second consider the following time-fractional shock wave equation

$$D_t^\alpha u + uD_x u - D_{xxt}u = 0, \tag{4.4}$$

with initial condition is

$$u(x,0) = 3\sec h^2\Big(\frac{x-15}{2}\Big). \tag{4.5}$$

We construct homotopy in the following equation

$$D_t^\alpha u = p\big[-uD_x u + D_{xxt}u\big], \tag{4.6}$$

where the homotopy parameter p is considered as a small parameter ($p \in [0,1]$). Applying the classical perturbation scheme, we can assume that the solution of Eq. (4.6) can be expressed as a power series in p as given below

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots, \tag{4.7a}$$
$$p^0: \; D_t^\alpha u_0 = 0, \tag{4.7b}$$
$$p^1: \; D_t^\alpha u_1 = -u_0 D_x u_0, \tag{4.7c}$$
$$p^2: \; D_t^\alpha u_2 = -u_1 D_x u_0 - u_0 D_x u_1 + D_{xxt}u_1, \tag{4.7d}$$
$$p^3: \; D_t^\alpha u_3 = -u_0 D_x u_2 - u_2 D_x u_0 - u_1 D_x u_1 + D_{xxt}u_2, \tag{4.7e}$$

and so on. The method is based on applying the operator j_t^α (the inverse operator of Caputo derivative D_t^α) on both sides of Eqs. (4.7b)-(4.7e), then we obtain

$$u(x,0) = 3\sec h^2\Big(\frac{x-15}{2}\Big), \tag{4.8a}$$
$$u_1(x,t) = -u_0 u_{0x}\frac{t^\alpha}{\Gamma(\alpha+1)}, \tag{4.8b}$$
$$u_2(x,t) = g_1(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + g_2(x)\frac{t^{2\alpha}}{\Gamma(\alpha+1)}, \tag{4.8c}$$
$$u_3(x,t) = g_3(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + (g_4(x) + 2\alpha g_{1xx}(x))\frac{t^{2\alpha}}{\Gamma(3\alpha+1)} + \alpha g_{2xx}(x)\frac{t^\alpha}{\Gamma(3\alpha+1)}, \tag{4.8d}$$

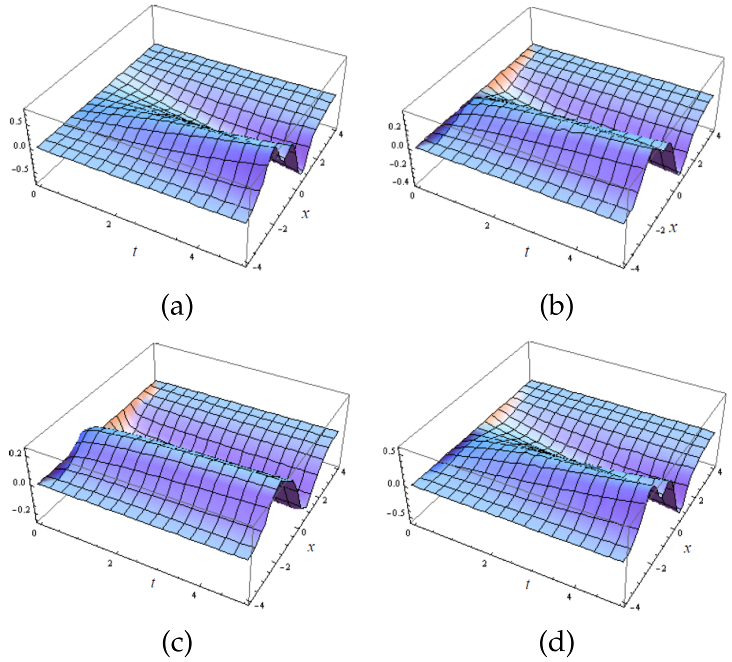

Figure 1: Explicit numerical solutions of Eq. (1.1) for $u(x,t)$ as in Eq. (3.7) at (a) $\alpha = 1$, $\gamma = 1.4$; (b) $\alpha = 1/2$, $\gamma = 1.4$; (c) $\alpha = 1/4$, $\gamma = 1.4$; (d) $\alpha = 4/5$, $\gamma = 1.4$ with $c_0 = 2$.

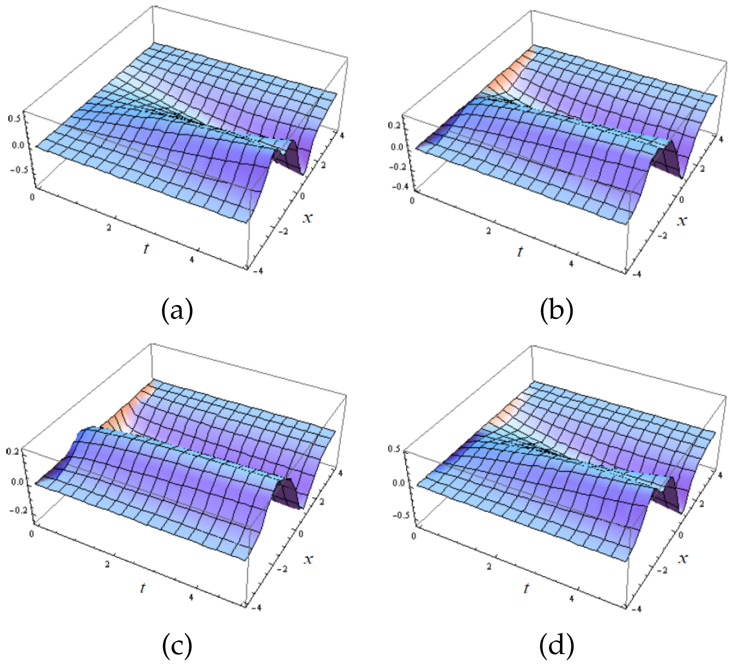

Figure 2: Explicit numerical solutions of Eq. (1.1) for $u(x,t)$ as in Eq. (3.7) at (a) $\alpha = 1$, $\gamma = 1.67$; (b) $\alpha = 1/2$, $\gamma = 1.67$; (c) $\alpha = 1/4$, $\gamma = 1.67$; (d) $\alpha = 4/5$, $\gamma = 1.67$ with $c_0 = 2$.

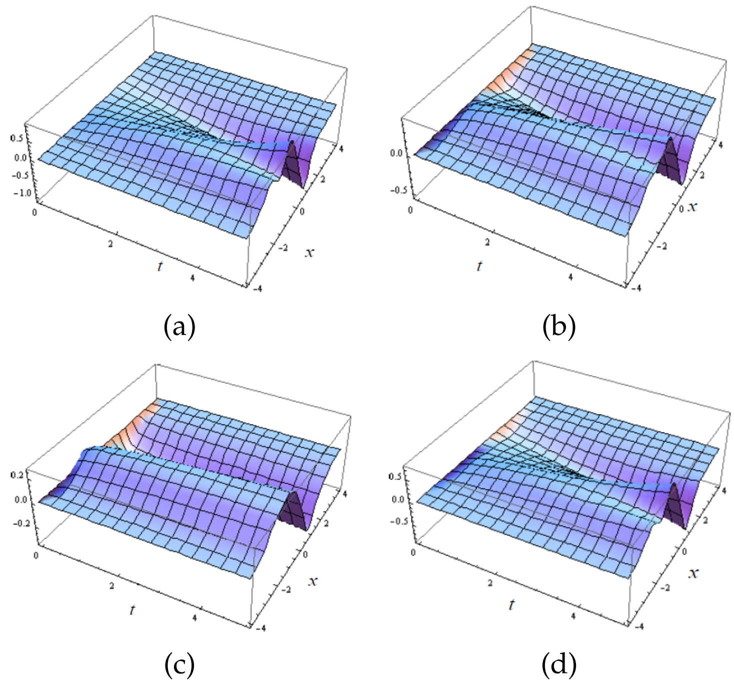

Figure 3: Explicit numerical solutions of Eq. (1.1) for $u(x,t)$ as in Eq. (3.7) at (a) $\alpha = 1$, $\gamma = 1.4$; (b) $\alpha = 1/2$, $\gamma = 1.4$; (c) $\alpha = 1/4$, $\gamma = 1.4$; (d) $\alpha = 4/5$, $\gamma = 1.4$ with $c_0 = 1.5$.

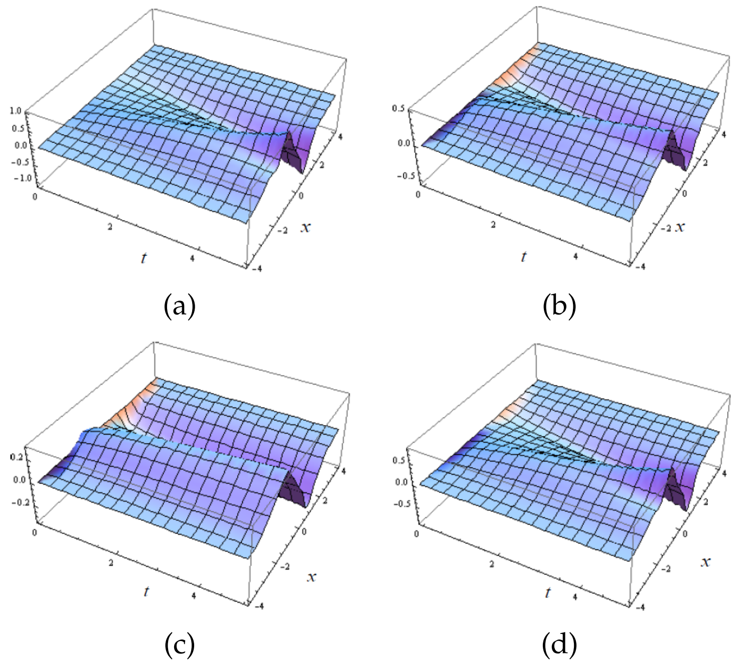

Figure 4: Explicit numerical solutions of Eq. (1.1) for $u(x,t)$ as in Eq. (3.7) at (a) $\alpha = 1$, $\gamma = 1.67$; (b) $\alpha = 1/2$, $\gamma = 1.67$; (c) $\alpha = 1/4$, $\gamma = 1.67$; (d) $\alpha = 4/5$, $\gamma = 1.67$ with $c_0 = 1.5$.

where

$$g_1(x) = u_0(2u_{0x}^2 + u_0), \qquad g_2(x) = (u_0 u_{0x})_{xx},$$
$$g_3(x) = -g_{1x}u_0 - u_{0x}g_1 - u_0 u_{0x}(u_0 u_{0x})_x, \qquad g_4(x) = -g_{2x}u_0 - u_{0x}g_2,$$

and so on. The approximate solution can be obtained by setting $p = 1$, in (4.7a) yields

$$u(x,t) = u_1 + u_2 + u_3 + \cdots . \tag{4.9}$$

This series has the closed form.

The evolution results for the exact solution

$$u(x,t) = 3\sec h^2 \frac{(x - 15 - t)}{2},$$

and the approximate solution (4.9), for the special case $\alpha = 1$. Then, we may conclude that we have achieved a good approximation with the exact solution of the equation by using the first few terms only of the linear equations derived above. It is evident that the overall errors can be made smaller by adding new terms of the decomposition series (4.9).

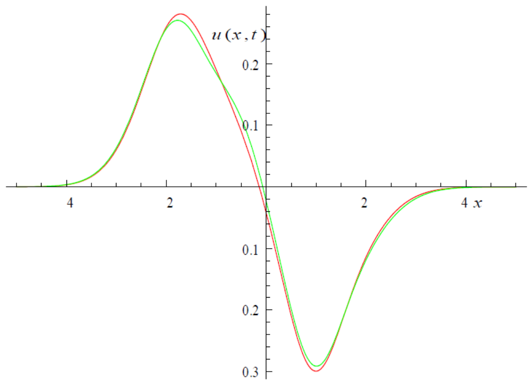

Figure 5: Plots of $u(x,t)$ vs. x at $t = 3$ the green line at $\alpha = 1/4$ and red line at $\alpha = 1/4$ with $\gamma = 1.4$, $c_0 = 2$.

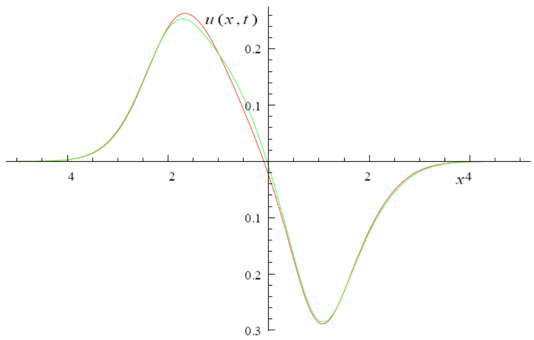

Figure 6: Plots of $u(x,t)$ vs. x at $t = 3$ the green line at $\alpha = 1/4$ and red line at $\alpha = 1/4$ with $\gamma = 1.67$, $c_0 = 2$.

5 Results and discussion

In this section, the numerical results of $u(x,t)$ for different time-fractional Brownian motions $\alpha = 1/2$, $1/4$, $4/5$ and standard $\alpha = 1$ with $\gamma = 1.4$, 1.67 and $c_0 = 1.5$, 2.0 are calculated for various values of x and t from Eqs. (3.7) and (4.9) by software MATHEMATICA 6.0. Here initial condition is taken as $u(x,0) = \exp(-x^2)/2$ for time-fractional shock wave equation and $u(x,0) = 3\sec h^2(x-15)/2$ for time-fractional wave equation. The numerical results of $u(x,t)$ for various values of x and t with $\gamma = 1.4$, 1.67 and $c_0 = 1.5$, 2.0 are illustrated through the Fig. 1(a)-Fig. 4(d) and those for different values of x, α, γ and c_0 are given in Figs. 5 and 6. It is seen from Figs. 1(a)-(d) the values of $u(x,t)$ decrease corresponding to α decreases with $\gamma = 1.4$ and $u(x,t)$ decrease corresponding to γ increases with $\alpha = 1/2, 1/4, 4/5$.

6 Conclusions

In this paper, the homotopy perturbation method is directly applied to derive approximate solutions of the fractional coupled nonlinear differential equations. We choose time-fractional shock equation and wave equation with initial conditions to illustrate our method. As results, we obtain the approximate solutions of fractional shock wave equation and also wave equation with high accuracy. The obtained results demonstrate the reliability of the algorithm and its wider applicability to nonlinear fractional differential equations. The HPM contains the homotopy parameter p, which provides us with a simple way to control the convergence region of solution series for large values of t. It is obvious to see that the HPM is a very powerful, easy and efficient technique for solving various kinds of nonlinear problems in science and engineering without many assumptions and restrictions.

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