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s-Sequence-Covering Mappings on Metric Spaces

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Abstract. In this paper, we introduce and study s-sequence-covering mappings and 1-s-sequence-covering mappings, obtain some characterizations of s-sequence-covering and compact images of metric spaces, and prove that every s-sequence-covering and compact mapping in first-countable spaces is a 1-s-sequence-covering mapping.

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1 Introduction

Statistical convergence as a generalization of the usual notion of convergence was introduced by H. Fast [1] and H. Steinhaus [2]. There is not doubt that the study of statistical convergence and its various generalizations has become an active research area [3–8]. The original notion of statistical convergence was introduced for the real space $\mathbb{R}$. Generally speaking, this notion was extended in two directions. One is to discuss statistical convergence in more general spaces, for example, locally convex spaces [9], Banach spaces with the weak topologies [6,10,11], and topological spaces [5,7,8]. The other is to consider generalized notions defined by various limit processes, for example, A-statistical convergence [12], lacunary statistical convergence [13], and λ-statistical convergence [14]. Perhaps, a most general notion of statistical convergence is ideal (or filter) convergence [15, 16]. On the other hand, to find the internal characterizations of certain images of metric spaces is one of the central questions in general topology. F. Siwiec [17] introduced the concept of sequence-covering mappings. Thereafter, the research in this area has been well developed [18–22].

As we know, sequence-covering mappings, 1-sequence-covering mappings and sequentially quotient mappings are one of the most important tools to study certain images

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of metric spaces [19]. Recently, V. Renukadevi and B. Prakash defined two new sequence-covering mappings about statistical convergence as follows: Let $f:X\to Y$ be a mapping. The mapping f is said to be a *statistically sequence-covering mapping* [23], if for a given sequence $\{y_n\}_{n\in\mathbb{N}}$ with $y_n\to y$ in Y, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ which statistically converges to a point $x\in f^{-1}(y)$ and each $x_n\in f^{-1}(y_n)$; the mapping f is said to be a *statistically sequentially quotient mapping* [24], if for a given sequence $y_n\to y$ in Y, there exists a sequence $x_k\to x\in f^{-1}(y)$ such that the sequence $\{f(x_k)\}_{k\in\mathbb{N}}$ is statistically dense in $\{y_n\}_{n\in\mathbb{N}}$. They discussed the relationship among sequence-covering mappings, statistically sequence-covering mappings and statistically sequentially quotient mappings, and studied their roles in the images of metric spaces.

Theorem 1.1 ([24]). *Let $f:X\to Y$ be a statistically sequentially quotient and boundary-compact map. If the space X is an open and compact-covering image of some metric space, then f is a 1-sequence-covering map.*

It is well known that we have the following result for the usual convergence.

Theorem 1.2 ([22]). *The following are equivalent for a topological space X:*
(1) *X is a sequence-covering and compact image of a metric space.*
(2) *X is a 1-sequence-covering and compact image of a metric space.*
(3) *X has a point-star network consisting of point-finite cs-covers.*
(4) *X has a point-star network consisting of point-finite sn-covers.*

We wonder if there are similar results for the case of statistical convergence? For this reason, this paper introduces and discusses s-sequence-covering mappings and 1-s-sequence-covering mappings. It is expected that s-sequence-covering mappings and 1-s-sequence-covering mappings shall also play an active role.

2 Preliminaries

In this paper, the set of all positive integers is denoted by $\mathbb{N}$, and the cardinality of the set B is denoted by $|B|$. The definition of statistical convergence of sequences is based on the notion of asymptotic density of a set $A\subset\mathbb{N}$.

Definition 2.1 ([25]). *Let $A\subset\mathbb{N}$ and $A(n)=\{k\in A: k\le n\}$ for each $n\in\mathbb{N}$. Then $\underline{\delta}(A)=\liminf_{n\to\infty}|A(n)|/n$ and $\overline{\delta}(A)=\limsup_{n\to\infty}|A(n)|/n$ are the lower and upper asymptotic density of the set A, respectively. If $\underline{\delta}(A)=\overline{\delta}(A)$, then $\delta(A)=\lim_{n\to\infty}|A(n)|/n$ is called the asymptotic density of A. A set $A\subset\mathbb{N}$ is said to be a statistically dense set if $\delta(A)=1$; a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of a sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be statistically dense in $\{x_n\}_{n\in\mathbb{N}}$ if the set $\{n_k:k\in\mathbb{N}\}$ is statistically dense in $\mathbb{N}$.*

Definition 2.2 ([5]). *Let X be a topological space.*

(1) *A sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* X *is said to converge statistically (or shortly, s-converge) to a point* $x\in X$, *if* $\delta(\{n\in\mathbb{N}: x_n\notin U\})=0$ *for each neighborhood* U *of* x *in* X, *which is denoted by* $s\text{-}\lim\limits_{n\to\infty} x_n = x$ *or* $x_n \xrightarrow{s} x$.

(2) *A sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* X *is said to* s^**-converge to a point* $x\in X$, *if there is* $A\subset\mathbb{N}$ *with* $\delta(A)=1$ *and* $\lim\limits_{A\ni n\to\infty} x_n = x$, *which is denoted by* $x_n \xrightarrow{s^*} x$.

Remark 2.1. (1) If $A\subset\mathbb{N}$ and $\delta(A)$ exists, then $\delta(\mathbb{N}\setminus A)=1-\delta(A)$ and $0\le\delta(A)\le 1$.

(2) The limit of a statistically convergent sequence is uniquely determined in Hausdorff spaces.

(3) If a sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x in the usual sense, then it statistically converges to x; but the converse is not true in general.

(4) A sequence $\{x_n\}_{n\in\mathbb{N}}$ is statistically convergent if and only if each statistically dense subsequences of its is statistically convergent.

Lemma 2.1 ([5]). *Let* X *be a first-countable space and a sequence* $\{x_n\}_{n\in\mathbb{N}}\subset X$. *Then* $x_n \xrightarrow{s} x$ *if and only if* $x_n \xrightarrow{s^*} x$.

Let X be a topological space, $P\subset X$ and $x\in X$. P is called a *sequential neighborhood* of x in X if whenever $\{x_n\}_{n\in\mathbb{N}}$ is a sequence converging to the point x, $\{x_n\}_{n\in\mathbb{N}}$ is eventually in P. A subset $F\subset X$ is called a *sequentially closed set* if F is closed with respect to the usual convergence of sequences in F, i.e., for each sequence $\{x_n\}_{n\in\mathbb{N}}\subset F$ with $x_n\to x\in X$, $x\in F$. X is called a *sequential space* [8, 26] if each sequentially closed subset of X is a closed set. A subset $U\subset X$ is called a *sequentially open set* if $X\setminus U$ is sequentially closed. Obviously, a subset $U\subset X$ is a sequentially open set if and only if for each sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to a point $x\in U$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is eventually in U; a space X is a sequential space if and only if each sequentially open subset of X is open. Every first-countable space is a sequential space [24].

Definition 2.3 ([8]). *Let* X *be a topological space.*

(1) *A subset* $F\subset X$ *is said to be an s-sequentially closed set if for each sequence* $\{x_n\}_{n\in\mathbb{N}}\subset F$ *with* $x_n\xrightarrow{s} x\in X$, $x\in F$.

(2) *A subset* $U\subset X$ *is said to be an s-sequentially open set if* $X\setminus U$ *is s-sequentially closed.*

(3) X *is called an s-sequential space if each s-sequentially closed subset of* X *is closed.*

Obviously, every sequential space is an s-sequential space.

Definition 2.4. *Let* X *be a topological space and* $P\subset X$. P *is called an s-sequential neighborhood of* x, *if for each sequence* $\{x_n\}_{n\in\mathbb{N}}$ *statistically converges to* $x\in P$, $\delta(\{n\in\mathbb{N}: x_n\notin P\})=0$.

Lemma 2.2. *Let* X *be a first-countable space and* $P\subset X$. *If* P *is a sequential neighborhood of* x *in* X, *then* P *is an s-sequential neighborhood of* x.

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X with $x_n \xrightarrow{s} x$. By Lemma 2.1, there exists a set $M = \{m_k : k\in\mathbb{N}\}\subset\mathbb{N}$ with $\delta(M)=1$ and $\lim_{k\to\infty} x_{m_k}=x$. Since P is a sequential neighborhood of x in X, there exists $k_0\in\mathbb{N}$ such that $\{x_{m_k}:k>k_0\}\cup\{x\}\subset P$, hence

$$\{n\in\mathbb{N}:x_n\notin P\}\subset(\mathbb{N}\setminus M)\cup\{m_1,m_2,\cdots,m_{k_0}\}.$$

Since $\delta(\mathbb{N}\setminus M)=1-\delta(M)=0$, it follows that $\delta((\mathbb{N}\setminus M)\cup\{m_1,m_2,\cdots,m_{k_0}\})=0$. Thus P is an s-sequential neighborhood of x. □

Definition 2.5. *Let X, Y be topological spaces and $f:X\to Y$ be a mapping.*

(1) *f is called a preserving s-convergence mapping provided for each sequence $\{x_n\}_{n\in\mathbb{N}}$ in X with $x_n\xrightarrow{s}x$, the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ s-converges to $f(x)$.*

(2) *f is called an s-continuous mapping provided U is an s-sequentially open set in Y, $f^{-1}(U)$ is an s-sequentially open set in X.*

Lemma 2.3 ([7]). *Every continuous mapping is a preserving s-convergence mapping. And every preserving s-convergence mapping is an s-continuous mapping.*

Definition 2.6. *Let X be a topological space and $\mathscr{P}$ be a cover of X.*

(1) *$\mathscr{P}$ is a cs-cover* [27] *of X if for any convergent sequence S in X, there exists $P\in\mathscr{P}$ such that S is eventually in P.*

(2) *$\mathscr{P}$ is an sn-cover* [22] *of X if each element of $\mathscr{P}$ is a sequential neighborhood of some point in X and for each $x\in X$, there exists $P\in\mathscr{P}$ such that P is a sequential neighborhood of x.*

(3) *$\mathscr{P}$ is an s-cs-cover of X if whenever $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X statistically converging to x, there exists $P\in\mathscr{P}$ such that $x\in P$ and $\delta(\{n\in\mathbb{N}:x_n\notin P\})=0$.*

(4) *$\mathscr{P}$ is an s-sn-cover of X if each element of $\mathscr{P}$ is an s-sequential neighborhood of some point of X and for each $x\in X$, there exists $P\in\mathscr{P}$ such that P is an s-sequential neighborhood of x.*

Lemma 2.4. *Let $n_0\in\mathbb{N}$ and $\{x_{m,n}\}_{n\in\mathbb{N}}$ be a sequence in X with $x_{m,n}\xrightarrow{s}x_0$ for each $m\in\{1,\cdots,n_0\}$. Put*

$$\{z_k\}_{k\in\mathbb{N}}=\{x_{1,1},\cdots,x_{n_0,1},x_{1,2},\cdots,x_{n_0,2},\cdots\},$$

where $k=(n-1)n_0+m, m<n_0, n\in\mathbb{N}$. Then the sequence $z_k\xrightarrow{s}x_0$.

Proof. For each neighborhood U of x_0 in X, it is not difficult to observe that

$$\begin{aligned}\{k\in\mathbb{N}:z_k\notin U\}&=\bigcup_{m=1}^{n_0}\{k=(n-1)n_0+m\in\mathbb{N}:z_k\notin U\}\\&=\bigcup_{m=1}^{n_0}\{(n-1)n_0+m\in\mathbb{N}:x_{m,n}\notin U\}.\end{aligned}$$

For each $m\in\{1,2,\cdots,n_0\}$, since each $x_{m,n}\xrightarrow{s}x_0$, it follows that $\delta(\{n\in\mathbb{N}:x_{m,n}\notin U\})=0$. Besides, for each $n,i\in\mathbb{N}$, it is easy to verify that $(n-1)n_0+m\le n_0 i\Leftrightarrow n\le i$. Hence, for each $m\in\{1,2,\cdots,n_0\}$

$$|\{n\in\mathbb{N}:x_{m,n}\notin U,n\le i\}|=|\{(n-1)n_0+m\in\mathbb{N}:x_{m,n}\notin U,(n-1)n_0+m\le n_0 i\}|.$$

Consequently,

$$\begin{aligned}
&\delta(\{k\in\mathbb{N}:z_k\notin U\})\\
\le&\sum_{m=1}^{n_0}\delta(\{k=(n-1)n_0+m\in\mathbb{N}:z_k\notin U\})\\
=&\sum_{m=1}^{n_0}\delta(\{(n-1)n_0+m\in\mathbb{N}:x_{m,n}\notin U\})\\
=&\sum_{m=1}^{n_0}\lim_{i\to\infty}\frac{|\{(n-1)n_0+m\in\mathbb{N}:x_{m,n}\notin U,(n-1)n_0+m\le n_0 i\}|}{n_0 i}\\
=&\sum_{m=1}^{n_0}\lim_{i\to\infty}\frac{|\{n\in\mathbb{N}:x_{m,n}\notin U,n\le i\}|}{n_0 i}=0.
\end{aligned}$$

Thus $z_k\xrightarrow{s}x_0$. □

Throughout this paper, all spaces are assumed to be Hausdorff, and all mappings are surjection and continuous. The readers may refer to [28,29] for notation and terminology not explicitly given here.

3 s-sequence-covering and compact images of metric spaces

In this section, we mainly discuss s-sequence-covering and compact images of metric spaces. Let X,Y be topological spaces and $f:X\to Y$ be a mapping. The mapping f is said to be a *sequence-covering mapping* if whenever $\{y_n\}_{n\in\mathbb{N}}$ is a convergent sequence in Y, there is a convergent sequence $\{x_n\}_{n\in\mathbb{N}}$ in X with each $x_n\in f^{-1}(y_n)$ [17]. The mapping f is *compact* if $f^{-1}(y)$ is a compact subset in X for each $y\in Y$.

Definition 3.1 ([7])**.** *Let X,Y be topological spaces and $f:X\to Y$ be a mapping. f is said to be an s-sequence-covering mapping if whenever $\{y_n\}_{n\in\mathbb{N}}$ is a statistically convergent sequence in Y, there is a statistically convergent sequence $\{x_n\}_{n\in\mathbb{N}}$ in X with each $x_n\in f^{-1}(y_n)$.*

Two examples below show that sequence-covering mappings and s-sequence-covering mappings are independent.

Example 3.1. There exists a sequence-covering mapping which is not an s-sequence-covering mapping.

Proof. Let $S=\{x_n:n\in\mathbb{N}\}$ be a countable set. Take $x\notin S$ and put $X=S\cup\{x\}$. The topology on X is defined as follows:

(1) each point x_n is isolated;

(2) each open neighborhood U of the point x is of the form $\{x\}\cup M$, where $M\subset S$ and $\delta(\{n\in\mathbb{N}:x_n\in M\})=1$.

It was obtained that the space X is a statistically sequential space but no sequence of S converges to the point x [8, Example 2.1] .

Now, let Z be the set X endowed with the discrete topology. Define a mapping $f: Z \to X$ to be the identity mapping. Obviously, f is a continuous mapping. Since there is no any non-trivial convergent sequence in X, f is a sequence-covering mapping. But f is not an s-sequence-covering mapping. In fact, the sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ s-converges to $x \in X$. But $\delta(\{n\in\mathbb{N}: x_n \neq x\}) = \delta(\mathbb{N}) = 1$. Consequently, $\{x_n\}_{n\in\mathbb{N}} \subset Z$ does not s-converge to $x\in Z$. $\square$

Example 3.2. There exists an s-sequence-covering mapping which is not a sequence-covering mapping.

Proof. Let $Y = \{0\}\cup\{1/n : n\in\mathbb{N}\}$ be a subspace of $\mathbb{R}$ with the usual topology. Denote

$$\{\overline{\{y_k : k\in\mathbb{N}\}} : \{y_k\}_{k\in\mathbb{N}} \subset Y \text{ is a convergent sequence}\} = \{Y_\alpha : \alpha\in\Lambda\}.$$

Obviously, $\{Y_\alpha : \alpha\in\Lambda\}$ is a cover of Y. For each $\alpha\in\Lambda$, the set Y_α is endowed with the following topology and denoted it by X_α: if Y_α is a finite set, then X_α is a discrete space; if Y_α is an infinite set, the topology on X_α is defined as Example 3.1 with $x_0 = 0$. Put the topological sum $X = \bigoplus_{\alpha\in\Lambda} X_\alpha \times \{\alpha\}$. Let $p: X \to Y$ be a natural mapping, that is, $p((y,\alpha)) = y$ for each $(y,\alpha)\in X_\alpha\times\{\alpha\}$ and $\alpha\in\Lambda$.

Assume that U is a neighborhood of 0 in Y. Then $Y\setminus U$ is a finite set, and further $(X_\alpha\times\{\alpha\})\cap p^{-1}(Y\setminus U)$ is a finite set for each $\alpha\in\Lambda$. Thus $p^{-1}(Y\setminus U)$ is closed in X, and hence $p^{-1}(U)$ is open in X. Therefore p is continuous.

In Example 3.1, it was mentioned that there is no any non-trivial convergent sequence in X_α for each $\alpha\in\Lambda$. Hence there is no any non-trivial convergent sequence in X. Consequently, p is not a sequence-covering mapping.

Let $\{y_k\}_{k\in\mathbb{N}} \subset Y$ be an s-convergent sequence. Without loss of generality, we can assume that $y_k \xrightarrow{s} 0$. Since Y is a first-countable space, by Lemma 2.1, there is $A\subset\mathbb{N}$ with $\delta(A) = 1$ such that $\lim\limits_{A\ni k\to\infty} y_k = 0$. Hence, there exists $\alpha\in\Lambda$ such that $\{y_k : k\in A\}\cup\{0\} = Y_\alpha$. Since the sequence $\{y_k\}_{k\in A}$ in X_α s-converges to 0, the sequence $\{(y_k,\alpha)\}_{k\in A}$ s-converges to $(0,\alpha)$. For each $k\in\mathbb{N}$, put $x_k\in p^{-1}(y_k)$ satisfying $x_k = (y_k,\alpha)\in Y_\alpha\times\{\alpha\}$ as $k\in A$. And because $\delta(A) = 1$, the sequence $\{x_k\}_{k\in\mathbb{N}}$ in X s-converges to $(0,\alpha)$. Thus p is an s-sequence-covering mapping. $\square$

Theorem 3.1. *Let $f: X\to Y$ be an s-sequence-covering and compact mapping. Then for each $y\in Y$, there exists $x\in f^{-1}(y)$ such that if U is an open neighborhood of x, then $f(U)$ is an s-sequential neighborhood of y.*

Proof. Suppose not, that is, there exists $y\in Y$ such that for each $x\in f^{-1}(y)$, there exists an open neighborhood U_x of x such that $f(U_x)$ is not an s-sequential neighborhood of y. Since $f^{-1}(y)\subset\bigcup_{x\in f^{-1}(y)} U_x$ and f is a compact mapping, there exists a finite subset $\{x_m : m\le n_0\}$ of $f^{-1}(y)$ such that $f^{-1}(y)\subset\bigcup_{m=1}^{n_0} U_{x_m}$. Since each $f(U_{x_m})$ is not an s-sequential neighborhood of y, choose a sequence $\{y_{m,n}\}_{n\in\mathbb{N}}$ in Y with $y_{m,n}\xrightarrow{s}y$ as $n\to\infty$, such that $\overline{\delta}(\{n\in\mathbb{N}:y_{m,n}\notin f(U_{x_m})\})\neq 0$ for each $m\in\{1,2,\cdots,n_0\}$ and $n\in\mathbb{N}$. Assume that

$$1\geq\overline{\delta}(\{n\in\mathbb{N}:y_{m,n}\notin f(U_{x_m})\})=\lambda_m>0$$

for each $m\in\{1,2,\cdots,n_0\}$. Now define a sequence

$$\{y_{1,1},y_{2,1}\cdots,y_{n_0,1},y_{1,2},\cdots,y_{n_0,2},\cdots\},$$

and denote it by $\{y_k\}_{k\in\mathbb{N}}$, where $k=(n-1)n_0+m,m\leq n_0$ and $n\in\mathbb{N}$. By Lemma 2.4, it follows that $y_k\xrightarrow{s}y$. Since f is an s-sequence-covering mapping, there exist $x\in f^{-1}(y)$ and $x_k\in f^{-1}(y_k)$ such that $x_k\xrightarrow{s}x$. Note that $x\in f^{-1}(y)\subset\bigcup_{m=1}^{n_0}U_{x_m}$, there exists $m_0\leq n_0$ such that $x\in U_{x_{m_0}}$. So that $\delta(\{k\in\mathbb{N}:x_k\notin U_{x_{m_0}}\})=0$, and hence

$$\delta(\{k\in\mathbb{N}:y_k\notin f(U_{x_{m_0}})\})=\delta(\{k\in\mathbb{N}:x_k\notin U_{x_{m_0}}\})=0.$$

But this contradicts to

$$\begin{aligned}
0<\frac{\lambda_{m_0}}{n_0}&=\frac{1}{n_0}\overline{\delta}(\{n\in\mathbb{N}:y_{m_0,n}\notin f(U_{x_{m_0}})\})\\
&=\limsup_{n\to\infty}\frac{|\{n\in\mathbb{N}:y_{m_0,n}\notin f(U_{x_{m_0}}),n\leq i\}|}{n_0 i}\\
&=\limsup_{n\to\infty}\frac{|\{(n-1)n_0+m_0\in\mathbb{N}:y_{m_0,n}\notin f(U_{x_{m_0}}),(n-1)n_0+m_0\leq n_0 i\}|}{n_0 i}\\
&=\overline{\delta}(\{k=(n-1)n_0+m_0\in\mathbb{N}:y_k\notin f(U_{x_{m_0}})\})\\
&\leq\overline{\delta}(\{k\in\mathbb{N}:y_k\notin f(U_{x_{m_0}})\})\\
&=\delta(\{k\in\mathbb{N}:y_k\notin f(U_{x_{m_0}})\})\\
&=0.
\end{aligned}$$

This completes the proof of the theorem. □

Lemma 3.1. *Let Γ be an index set and $\{x_{\gamma,n}\}_{n\in\mathbb{N}}$ be a sequence in X_γ for each $\gamma\in\Gamma$. Then the sequence $(x_{\gamma,n})_{\gamma\in\Gamma}\xrightarrow{s}(x_\gamma)_{\gamma\in\Gamma}\in\prod_{\gamma\in\Gamma}X_\gamma$ if and only if each $x_{\gamma,n}\xrightarrow{s}x_\gamma\in X_\gamma$ $(\gamma\in\Gamma)$.*

Proof. Sufficiency. For any neighborhood U of $(x_\gamma)_{\gamma\in\Gamma}$ in $\prod_{\gamma\in\Gamma}X_\gamma$, there exists a finite subset $\Gamma'\subset\Gamma$ and an open set U_γ in X_γ $(\gamma\in\Gamma')$ such that

$$(x_\gamma)_{\gamma\in\Gamma}\in\prod_{\gamma\in\Gamma'}U_\gamma\times\prod_{\gamma\in\Gamma\setminus\Gamma'}X_\gamma\subset U.$$

Since each $x_{\gamma,n}\xrightarrow{s}x_\gamma$, we have $\delta(\{n\in\mathbb{N}:x_{\gamma,n}\notin U_\gamma\})=0$ for each $\gamma\in\Gamma'$. Since

$$\{n\in\mathbb{N}:(x_{\gamma,n})_{\gamma\in\Gamma}\notin U\}\subset\bigcup_{\gamma\in\Gamma'}\{n\in\mathbb{N}:x_{\gamma,n}\notin U_\gamma\},$$

it follows that $\delta(\{n\in\mathbb{N}:(x_{\gamma,n})_{\gamma\in\Gamma}\notin U\})=0$. Thus $(x_{\gamma,n})_{\gamma\in\Gamma}\xrightarrow{s}(x_\gamma)_{\gamma\in\Gamma}$.

Necessity. Let $p_\gamma:\prod_{\gamma\in\Gamma}X_\gamma\to X_\gamma$ be the projection mapping. Since p_γ is continuous, by Lemma 2.3, it is a preserving s-convergence mapping. Hence, $x_{\gamma,n}\xrightarrow{s}x_\gamma\in X_\gamma$ for each $\gamma\in\Gamma$. $\square$

Let us recall the concept of point-star networks. Let $\mathscr{P}$ be a family of subsets of a set X and $x\in X$. Put $(\mathscr{P})_x=\{P\in\mathscr{P}:x\in P\}$ and denote $\mathrm{st}(x,\mathscr{P})=\bigcup(\mathscr{P})_x$. A family $\mathscr{P}$ of subsets of a space X is *point-finite* if $(\mathscr{P})_x$ is finite for each $x\in X$; $\mathscr{P}$ is called a *network* at a point $x\in X$ [30] if $x\in\bigcap\mathscr{P}$ and for each neighborhood U of x, there exists $P\in\mathscr{P}$ such that $P\subset U$. Let $\{\mathscr{P}_n\}$ be a sequence of covers of a space X. $\{\mathscr{P}_n\}$ is called a *point-star network* [22] of X if $\langle\mathrm{st}(x,\mathscr{P}_n)\rangle$ is a network at x in X for each $x\in X^\dagger$. Obviously, $\{\mathscr{P}_n\}$ is a point-star network of X if and only if for each $x\in X$ and for given $P_n\in(\mathscr{P}_n)_x$, $\langle P_n\rangle$ is a network at x in X [19].

Theorem 3.2. *The following are equivalent for a space X:*

(1) *X is an s-sequence-covering and compact image of a metric space.*
(2) *X has a point-star network consisting of point-finite s-sn-covers.*
(3) *X has a point-star network consisting of point-finite s-cs-covers.*

Proof. $(1)\Rightarrow(2)$. Suppose that $f:M\to X$ is an s-sequence-covering and compact mapping, where M is a metric space. Then there exists a sequence $\{\mathscr{B}_i\}_{i\in\mathbb{N}}$ of locally finite open covers of M such that for each compact subset K of M, $\langle\mathrm{st}(K,\mathscr{B}_i)\rangle$ is a neighborhood base of K in M. Put $\mathscr{P}_i=f(\mathscr{B}_i)$. As f being a compact mapping, $\mathscr{P}_i$ is a point-finite cover of X. For each $x\in X$, let V be an open neighborhood of x in X. Since $f^{-1}(x)$ is a compact subset in M and $f^{-1}(x)\subset f^{-1}(V)$, there exists $n\in\mathbb{N}$ such that $\mathrm{st}(f^{-1}(x),\mathscr{B}_n)\subset f^{-1}(V)$. Hence $\mathrm{st}(x,\mathscr{P}_n)\subset V$, thus $\langle\mathrm{st}(x,\mathscr{P}_i)\rangle$ is a network at x in X. This implies that $\{\mathscr{P}_n\}$ is a point-star network of X.

For each $x\in X$, there exists $b\in f^{-1}(x)$ satisfying the condition in Theorem 3.1. Since each $\mathscr{B}_i$ is an open cover of X, there exists $B\in\mathscr{B}_i$ such that $b\in B$. Put $P=f(B)$. By Theorem 3.1, P is an s-sequential neighborhood of x. Let

$$\mathscr{P}_i'=\{P\in\mathscr{P}_i:P\text{ is an }s\text{-sequential neighborhood of some point in }X\}.$$

Then $\mathscr{P}_i'$ is a point- finite cover of X and $\{\mathscr{P}_i'\}$ is a point-star network consisting of point-finite s-*sn*-covers of X.

$(2)\Rightarrow(3)$ is obvious by Definition 2.6.

$(3)\Rightarrow(1)$. Let $\{\mathscr{P}_i\}$ be a point-star network consisting of point-finite s-*cs*-covers of X. For each $i\in\mathbb{N}$, put $\mathscr{P}_i=\{\mathscr{P}_\alpha:\alpha\in\Lambda_i\}$ and each Λ_i is endowed with the discrete topology. Put

$$M=\{\alpha=(\alpha_i)\in\prod_{i\in\mathbb{N}}\Lambda_i:\langle P_{\alpha_i}\rangle\text{ forms a network at some point }x_\alpha\text{ in }X\},$$

†A set $\{x_n:n\in\mathbb{N}\}$ is simply expressed as $\langle x_n\rangle$ in this paper.

then M, which is a subspace of the product space $\prod_{i\in\mathbb{N}}\Lambda_i$, is a metrizable space and the point x_α is unique for each $\alpha\in M$. Define a function $f:M\to X$ by $f(\alpha)=x_\alpha$. Then f is a compact mapping [19, Lemma 3.3.2].

Next, we shall show that f is an s-sequence-covering mapping. Let $\{x_n\}_{n\in\mathbb{N}}\subset X$ be a sequence satisfying $x_n \xrightarrow{s} x_0\in X$. Since $\{\mathscr{P}_i\}$ is a point-finite s-cs-covers of X, we can choose $\alpha_j\in\Lambda_j$ such that $x_0\in P_{\alpha_j}$ and $\delta(\{n\in\mathbb{N}:x_n\notin P_{\alpha_j}\})=0$ for each $j\in\mathbb{N}$. Since $\{\mathscr{P}_i\}$ is a point-star network of X and $P_{\alpha_j}\in(\mathscr{P}_j)_{x_0}$, $\langle P_{\alpha_j}\rangle$ forms a network at x_0 in X. Let $\alpha=(\alpha_j)\in M$. Then $\alpha\in f^{-1}(x_0)$. Choose a sequence $\{(\alpha_{j,n})\}_{n\in\mathbb{N}}$ in M as follows: for each $j\in\mathbb{N}$,

$$\alpha_{j,n}=\begin{cases}\alpha_j, & x_n\in P_{\alpha_j},\\ \beta_j, & x_n\in P_{\beta_j}, \text{ for some } \beta_j\in\Lambda_j.\end{cases}$$

Then $\alpha_{j,n}\xrightarrow{s}\alpha_j$ for each $j\in\mathbb{N}$, because $\delta(\{n\in\mathbb{N}:\alpha_{j,n}\notin V_j\})\le\delta(\{n\in\mathbb{N}:\alpha_{j,n}\neq\alpha_j\})=\delta(\{n\in\mathbb{N}:x_n\notin P_{\alpha_j}\})=0$, if V_j is a neighborhood of α_j in Λ_j. It follows from Lemma 3.1 that $(\alpha_{j,n})\xrightarrow{s}(\alpha_j)$ in M. By the choice of $(\alpha_{j,n})$, it is easy to see that $P_{\alpha_{j,n}}\in(\mathscr{P}_j)_{x_n}$, hence $\langle P_{\alpha_{j,n}}\rangle$ forms a network at x_n in X, thus $(\alpha_{j,n})\in f^{-1}(x_n)$ for each $n\in\mathbb{N}$. Therefore, f is an s-sequence-covering mapping. □

Definition 3.2. *Let X,Y be topological spaces and $f:X\to Y$ be a mapping. f is said to be an s-quotient mapping provided $f^{-1}(U)$ is s-open in X, then U is s-open in Y.*

The following two theorems can be seen in [7].

Theorem 3.3. *Each s-sequence-covering mapping is an s-quotient mapping.*

Theorem 3.4. *Let X,Y be topological spaces and $f:X\to Y$ be a mapping.*

(1) *If X is an s-sequential space and f is a quotient mapping, then Y is an s-sequential space and f is an s-quotient mapping.*

(2) *If Y is an s-sequential space and f is an s-quotient mapping, then f is a quotient mapping.*

By Theorems 3.2, 3.3 and 3.4, we have the following corollary.

Corollary 3.1. *The following are equivalent for a topological space X:*

(1) *X is an s-sequence-covering, quotient and compact image of a metric space.*

(2) *X is a sequential space and has a point-star network consisting of point-finite s-sn-covers.*

(3) *X is a sequential space and has a point-star network consisting of point-finite s-cs-covers.*

4 1-s-sequence-covering mappings on first-countable spaces

The work of this section is a continuation of the previous section. In this section, we obtain that s-sequence-covering and compact mappings in first-countable spaces are 1-s-sequence-covering mappings. Recall the notion of 1-sequence-covering mappings in topological spaces. A mapping $f:X\to Y$ is a *1-sequence-covering mapping* if for each $y\in Y$, there is $x\in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to y in Y there is a sequence $\{x_n\}$ converging to x in X with each $x_n\in f^{-1}(y_n)$ [18].

Definition 4.1. *A mapping $f:X\to Y$ is called a 1-s-sequence-covering mapping if for each $y\in Y$, there is $x\in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence statistically converging to y in Y there is a sequence $\{x_n\}$ statistically converging to x in X with each $x_n\in f^{-1}(y_n)$.*

Obviously, if f is a 1-s-sequence-covering mapping, then f is an s-sequence-covering mapping. Two examples below show that 1-sequence-covering mappings and 1-s-sequence-covering mappings are independent.

Example 4.1. There exists a 1-sequence-covering mapping in a first-countable space which is not an s-sequence-covering mapping.

Proof. Let $f:Z\to X$ be the mapping in Example 3.2. Then Z is a first-countable space. Example 3.2 showed the mapping f is not an s-sequence-covering mapping. Since there is no any non-trivial convergent sequence in X, f is a 1-sequence-covering mapping. $\square$

Example 4.2. There exists a 1-s-sequence-covering mapping which is not a sequence-covering mapping.

Proof. Let $X=\{x\}\cup\{x_n:n\in\mathbb{N}\}$ be the topological space defined in Example 3.2. $Y=\{0\}\cup\{1/n:n\in\mathbb{N}\}$ be a subspace of $\mathbb{R}$ with the usual topology. Define a mapping $f:X\to Y$ by $f(x)=0$ and $f(x_n)=1/n$ for each $n\in\mathbb{N}$. Since there is no any non-trivial convergent sequence in X, f is not a sequence-covering mapping.

For each $y\in Y$, without loss of generality, we can assume that $y=0$. Take $x\in X$. If $\{y_n\}_{n\in\mathbb{N}}\subset Y$ is a sequence statistically converging to 0. Since Y is a first-countable space, by Lemma 2.1, there exists $A\subset\mathbb{N}$ with $\delta(A)=1$ and $\lim_{A\ni n\to\infty}y_n=0$. Assume that $\{y_n\}_{n\in\mathbb{N}}=\{1/n_i\}_{i\in\mathbb{N}}$. Then $f(x_{n_i})=1/n_i$ for each $i\in\mathbb{N}$. Since $\delta(A)=1$, $\{x_{n_i}\}_{i\in\mathbb{N}}$ contains a statistically dense subsequence. It follows from Remark 2.1(4) that $x_{n_i}\xrightarrow{s}x$. Thus f is a 1-s-sequence-covering mapping. $\square$

Lemma 4.1. *Let $f:X\to Y$ be a 1-s-sequence-covering mapping. Then for each $y\in Y$, there exists $x\in f^{-1}(y)$ such that whenever U is an open neighborhood of x in X, $f(U)$ is an s-sequential neighborhood of y in Y.*

Proof. Let $y\in Y$. Since f is a 1-s-sequence-covering, there is $x\in f^{-1}(y)$ satisfying the condition in Definition 4.1. Let U be an open neighborhood of x in X and $\{y_n\}$ be a sequence statistically converging to y in Y. There is a sequence $\{x_n\}$ statistically converging to x in X with each $x_n\in f^{-1}(y_n)$. Hence $\delta(\{n\in\mathbb{N}:x_n\notin U\})=0$, and further $\delta(\{n\in\mathbb{N}:y_n\notin f(U)\})\leq\delta(\{n\in\mathbb{N}:x_n\notin U\})=0$. Therefore, $f(U)$ is an s-sequential neighborhood of y in Y. $\square$

Lemma 4.2. *Let $f:X\to Y$ be a mapping and $\{B_n\}_{n\in\mathbb{N}}$ be a decreasing network at some point x in X. If $\{y_i\}_{i\in\mathbb{N}}$ is a sequence in Y statistically converging to $f(x)$ with each $\delta(\{i\in\mathbb{N}:y_i\notin f(B_n)\})=0$, then there is a sequence $\{x_i\}_{i\in\mathbb{N}}$ statistically converging to x in X with each $x_i\in f^{-1}(y_i)$.*

Proof. Let $\{y_i\}_{i\in\mathbb{N}}$ be a sequence in Y statistically converging to $f(x)$ and $\delta(\{i\in\mathbb{N}:y_i\notin f(B_n)\})=0$ for each $n\in\mathbb{N}$. Note that each $f(B_n)\supset f(B_{n+1})$. For each $i\in\mathbb{N}$, we can pick

$$x_i\in\begin{cases} f^{-1}(y_i), & y_i\notin f(B_1),\\ f^{-1}(y_i)\cap B_n, & y_i\in f(B_n)\setminus f(B_{n+1}),\ n\in\mathbb{N}.\end{cases}$$

For each $n\in\mathbb{N}$, if $x_i\notin B_n$, then $y_i\notin f(B_n)$. Otherwise, if $y_i\in f(B_n)$, then there exists $k\geq n$ such that $y_i\in f(B_k)\setminus f(B_{k+1})$, thus $x_i\in B_k\subset B_n$, a contradiction; if $y_i\notin f(B_n)$, then $f^{-1}(y_i)\cap B_n=\emptyset$, hence $x_i\notin B_n$. Hence, by the choosing of x_i, it follows that $x_i\notin B_n$ if and only if $y_i\notin f(B_n)$ for each $n\in\mathbb{N}$. Thus, $x_i\xrightarrow{s}x$. In fact, for each open neighborhood U of x, there exists $n_0\in\mathbb{N}$ such that $x\in B_{n_0}\subset U$. Therefore, $\{i\in\mathbb{N}:x_i\notin U\}\subset\{i\in\mathbb{N}:x_i\notin B_{n_0}\}=\{i\in\mathbb{N}:y_i\notin f(B_{n_0})\}$, hence $\delta(\{i\in\mathbb{N}:x_i\notin U\})=0$, and further $\{x_i\}_{i\in\mathbb{N}}$ statistically converges to x in X with each $x_i\in f^{-1}(y_i)$. □

Corollary 4.1. *Let $f:X\to Y$ be a mapping and $\{B_n\}_{n\in\mathbb{N}}$ be a decreasing network at some point x in X. If $\{y_i\}_{i\in\mathbb{N}}$ is a sequence in Y statistically converging to $f(x)$ and $f(B_n)$ is an s-sequential neighborhood of $f(x)$ in Y for each $n\in\mathbb{N}$. Then there is a sequence $\{x_i\}_{i\in\mathbb{N}}$ statistically converging to x in X with each $x_i\in f^{-1}(y_i)$.*

Theorem 4.1. *Let $f:X\to Y$ be an s-sequence-cover and compact mapping. If X is a first-countable space, then f is a 1-s-sequence-cover mapping.*

Proof. By Theorem 3.1, it follows that for each $y\in Y$, there exists $x\in f^{-1}(y)$ such that if U is an open neighborhood of x, $f(U)$ is an s-sequential neighborhood of y. Let $\langle B_n\rangle$ be a decreasing open neighborhood base at x in X. Then $f(B_n)$ is an s-sequential neighborhood of $f(x)$ in Y for each $n\in\mathbb{N}$. By Corollary 4.1, if $\{y_i\}_{i\in\mathbb{N}}$ is a sequence in Y statistically converging to y, there is a sequence $\{x_i\}_{i\in\mathbb{N}}$ statistically converging to x in X with each $x_i\in f^{-1}(y_i)$. Therefore, f is a 1-s-sequence-cover mapping. □

By Theorems 3.2 and 4.1, it is easy to obtain the following corollary.

Corollary 4.2. *The following are equivalent for a topological space X:*

(1) *X is a 1-s-sequence-covering and compact image of a metric space.*
(2) *X is an s-sequence-covering and compact image of a metric space.*
(3) *X has a point-star network consisting of point-finite s-sn-covers.*
(4) *X has a point-star network consisting of point-finite s-cs-covers.*

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