

## GLOBAL DYNAMICS OF A PREDATOR-PREY MODEL WITH PREY REFUGE AND DISEASE\*<sup>†</sup>

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### Abstract

In this paper, we study a predator-prey model with prey refuge and disease. We study the local asymptotic stability of the equilibriums of the system. Further, we show that the equilibria are globally asymptotically stable if the equilibria are locally asymptotically stable. Some examples are presented to verify our main results. Finally, we give a brief discussion.

**Keywords** predator-prey model; prey refuge; disease; stability

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## 1 Introduction

Predator-prey model is one of the basic models between different species in nature. These models have been studied extensively and many excellent results have been obtained (see [1, 2]). On the other hand, the effect of disease in ecological system is an important topic from mathematical as well as ecological point of view. After the work of Kermack-McKendrick [3] on SIRS (susceptible-infected-removed-susceptible) systems, many authors have investigated the dynamical behavior of epidemiological models. Chattopadhyay and Arino [4] proposed a predator-prey epidemiological model with disease spreading in prey. They assumed that the sound prey population grows according to a logistic law involving the whole prey population, and discussed the positivity, uniqueness, boundedness of the solutions and the existence of supercritical Hopf bifurcation. Haque et al [5] investigated a Lotka-Volterra type predator-prey model with a transmissible disease in the predator species. They assumed that the sound and infected predators can hunt the prey and studied the stability of system.

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Sun and Yuan [6] proposed the following predator-prey model with disease in the predator

$$\begin{aligned}x' &= x(a - bx) - cxS, \\S' &= exS - d_1S - \beta SI, \\I' &= \beta SI - d_2I,\end{aligned}\tag{1.1}$$

where  $x(t)$ ,  $S(t)$  and  $I(t)$  represent the densities of the prey, susceptible (sound) predator and the infected predator population at time  $t$ , respectively. They assumed that there is a spread of disease in predator and only the susceptible predators have ability to capture prey. They investigated the boundedness of solution and global asymptotical stability of the equilibriums.

On the other hand, prey species makes use of refuges to decrease predation risk, here refuge means a places or situations where predation risk is somehow reduced. Ma et al [7] studied the following predator-prey model with prey refuges and a class of functional responses

$$\begin{aligned}X' &= rx\left(1 - \frac{x}{K}\right) - p\varphi(X - X_R)Y, \\Y' &= (q\varphi(X - X_R) - d)Y,\end{aligned}\tag{1.2}$$

where the term  $\varphi(X)$  represents the functional response of the predator population. They obtained the local asymptotical stability of equilibrium point and showed that the refuges used by prey can increase the equilibrium density of prey population but decrease that of predator. Ma et al. [8] further studied the influence of prey refuge and density dependent of predator species on the traditional Lotka-Volterra model. Huang, Chen and Li [9] studied the influence of prey refuge on a predator-prey model with Holling type III response function. In [10], a global analysis of a Holling type II predator-prey model with a constant prey refuge was presented. Ma et al [11] and Chen, Chen and Wang [12] studied a Lotka-Volterra predator-prey model incorporating a prey refuge and predator mutual interference. For more details in this direction, please see [13, 14].

However, there are still seldom scholars investigating the predator-prey model with prey refuge and disease in predator. More precisely, we study the global stability of the following model

$$\begin{aligned}x' &= rx\left(1 - \frac{x}{K}\right) - c(1 - m)xS, \\S' &= e(1 - m)xS - d_1S - \beta SI, \\I' &= \beta SI - d_2I,\end{aligned}\tag{1.3}$$

where  $x(t)$ ,  $S(t)$  and  $I(t)$  represent the densities of the prey, susceptible (sound) predator and the infected predator population at time  $t$ , respectively with initial

conditions  $x(0) > 0$ ,  $S(0) > 0$ ,  $I(0) > 0$ . All the parameters are positive constants and  $0 < m < 1$ . The disease incidence follows the simple law of mass action incidence  $\beta S(t)I(t)$  with  $\beta$  being the transmission coefficient;  $d_1 \leq d_2$ ;  $mK$  is the capacity of a refuge at time  $t$ .

The organization of this paper is as follows. In Section 2, we study the stability of the equilibriums of system (1.3). In Section 3, numerical simulation is presented to illustrate the feasibility of our main results. In the last section, we give a brief discussion.

## 2 Main Results

In this section, we investigate the local and global stability of system (1.3).

Similar to the proof of [6], we have the following lemma.

**Lemma 2.1** *All solutions of system (1.3) with a positive initial data will remain positive and uniformly bounded.*

We can easily calculate that system (1.3) always has a trivial equilibrium  $E_0(0,0,0)$  and a predator-extinction equilibrium  $E_1(K, 0, 0)$ . If

$$1 - m > \frac{d_1}{Ke} \quad (2.1)$$

holds, then system (1.3) has a disease-free equilibrium  $E_2(x_1, S_1, 0)$ , where  $x_1 = \frac{d_1}{e(1-m)}$ ,  $S_1 = \frac{r}{c(1-m)^2}(1 - m - \frac{d_1}{Ke})$ .

Further, for the unique endemic-coexistence equilibrium  $E^*(x^*, S^*, I^*)$  of system (1.3),  $x^*$ ,  $S^*$  and  $I^*$  satisfy the following equations

$$\begin{aligned} r\left(1 - \frac{x^*}{K}\right) - c(1-m)S^* &= 0, \\ e(1-m)x^* - d_1 - \beta I^* &= 0, \\ \beta S^* - d_2 &= 0. \end{aligned} \quad (2.2)$$

From the above equation, if

$$\begin{aligned} 1 - m < \min\left\{\frac{r\beta}{cd_2}, 1\right\}, \\ cd_2(1-m)^2 - r\beta(1-m) + \frac{d_1r\beta}{eK} < 0 \end{aligned} \quad (2.3)$$

hold, then  $x^*$ ,  $S^*$  and  $I^*$  are positive and satisfy

$$\begin{aligned} x^* &= \frac{Kcd_2}{r\beta}\left(\frac{r\beta}{cd_2} - (1-m)\right), \quad S^* = \frac{d_2}{\beta}, \\ I^* &= \frac{eK}{r\beta^2}\left(-cd_2(1-m)^2 + r\beta(1-m) - \frac{d_1r\beta}{eK}\right). \end{aligned}$$

Let  $\tilde{E}(\tilde{x}, \tilde{S}, \tilde{I})$  be an arbitrary equilibrium of system (1.3). Then the Jacobian matrix about  $\tilde{E}$  is given by

$$\begin{pmatrix} r - \frac{2r\tilde{x}}{K} - c(1-m)\tilde{S} & -c(1-m)\tilde{x} & 0 \\ e(1-m)\tilde{S} & e(1-m)\tilde{x} - d_1 - \beta\tilde{I} & -\beta\tilde{S} \\ 0 & \beta\tilde{I} & \beta\tilde{S} - d_2 \end{pmatrix}. \quad (2.4)$$

It is easy to prove that the equilibrium  $E_0(0, 0, 0)$  is always unstable.

For the predator-extinction equilibrium  $E_1(K, 0, 0)$ , the Jacobian matrix is given by

$$\begin{pmatrix} -r & 0 & 0 \\ 0 & e(1-m)K - d_1 & 0 \\ 0 & 0 & -d_2 \end{pmatrix}.$$

Hence, if  $\frac{d_1}{eK} \geq 1$ , then  $E_1(K, 0, 0)$  is locally asymptotically stable for any  $0 < m < 1$ . If  $\frac{d_1}{eK} < 1$ , then  $E_1(K, 0, 0)$  is locally asymptotically stable if and only if  $1 - m < \frac{d_1}{eK}$ .

For the disease-free equilibrium  $E_2(x_1, S_1, 0)$ , the Jacobian matrix is given by

$$\begin{pmatrix} -\frac{r}{K}x_1 & -c(1-m)x_1 & 0 \\ e(1-m)S_1 & 0 & -\beta S_1 \\ 0 & 0 & \beta S_1 - d_2 \end{pmatrix}.$$

Then the characteristic equation is

$$(\lambda - \beta S_1 + d_2) \left( \lambda^2 + \frac{r}{K}x_1\lambda + ce(1-m)^2x_1S_1 \right) = 0.$$

By calculation,  $\beta S_1 - d_2 < 0$  is equivalent to  $cd_2(1-m)^2 - r\beta(1-m) + \frac{d_1r\beta}{eK} > 0$ . Hence, the disease-free equilibrium  $E_2(x_1, S_1, 0)$  is locally asymptotically stable if and only if  $cd_2(1-m)^2 - r\beta(1-m) + \frac{d_1r\beta}{eK} > 0$ .

Again, for the unique endemic-coexistence equilibrium  $E^*(x^*, S^*, I^*)$ , the Jacobian matrix is given by

$$\begin{pmatrix} -\frac{r}{K}x^* & -c(1-m)x^* & 0 \\ e(1-m)S^* & 0 & -\beta S^* \\ 0 & \beta I^* & 0 \end{pmatrix}.$$

Then the characteristic equation is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

where  $a_1 = \frac{r}{K}x^* > 0$ ,  $a_2 = S^*(\beta^2 I^* + ec(1-m)^2 X^* S^*) > 0$ ,  $a_3 = \frac{r}{K}\beta^2 x^* S^* I^* > 0$ . Noting that  $a_1 a_2 - a_3 = \frac{rec}{K}(1-m)^2 (X^*)^2 S^* > 0$ , by Routh-Hurwitz criterion, the unique endemic-coexistence equilibrium  $E^*(x^*, S^*, I^*)$  is always locally asymptotically stable if and only if  $E^*$  exists.

Then we have the following result.

**Lemma 2.2**

- (i) *The trivial equilibrium  $E_0(0, 0, 0)$  is unstable.*
- (ii) *If  $\frac{d_1}{eK} \geq 1$ , then the predator-extinction equilibrium  $E_1(K, 0, 0)$  is always locally asymptotically stable, but the disease-free equilibrium  $E_2(x_1, S_1, 0)$  and the unique endemic-coexistence equilibrium  $E^*(x^*, S^*, I^*)$  do not exist.*
- (iii) *If  $\frac{d_1}{eK} < 1$ , then the predator-extinction equilibrium  $E_1(K, 0, 0)$  is locally asymptotically stable if  $0 < 1 - m < \frac{d_1}{eK}$ . The disease-free equilibrium  $E_2(x_1, S_1, 0)$  is locally asymptotically stable if  $\frac{d_1}{eK} < 1 - m < 1$  and  $cd_2(1 - m)^2 - r\beta(1 - m) + \frac{d_1 r\beta}{eK} > 0$ . The unique endemic-coexistence equilibrium  $E^*(x^*, S^*, I^*)$  is locally asymptotically stable if  $0 < 1 - m < \min\{\frac{r\beta}{cd_2}, 1\}$  and  $cd_2(1 - m)^2 - r\beta(1 - m) + \frac{d_1 r\beta}{eK} < 0$ .*

By the definition of  $x^*$ , we have  $x^* < K$ . It follows from the second equation of (2.2) that  $1 - m > \frac{d_1}{ex^*} > \frac{d_1}{eK}$ , that is, the existence of  $E^*(x^*, S^*, I^*)$  implies that  $1 - m > \frac{d_1}{eK}$  holds.

Define

$$H(z) = cd_2 z^2 - r\beta z + \frac{d_1 r\beta}{eK}, \quad \frac{d_1}{eK} < z < 1. \quad (2.5)$$

Let  $\Delta_1 = r\beta(r\beta - \frac{4cd_1 d_2}{eK})$ .

If  $r\beta < \frac{4cd_1 d_2}{eK}$ , then  $H(z) > 0$  for all  $\frac{d_1}{eK} < z < 1$ .

If  $r\beta = \frac{4cd_1 d_2}{eK}$  and  $0 < \frac{d_1}{eK} < \frac{1}{2}$ , then  $H(z) > 0$  for  $\frac{d_1}{eK} < z < \frac{d_1}{2eK}$  or  $\frac{d_1}{2eK} < z < 1$ . When  $\frac{1}{2} \leq \frac{d_1}{eK} < 1$ , then  $H(z) > 0$  for all  $\frac{d_1}{eK} < z < 1$ .

If  $r\beta > \frac{4cd_1 d_2}{eK}$ , equation (2.5) has two positive roots  $z_1 = \frac{r\beta - \sqrt{\Delta_1}}{2c_2}$  and  $z_2 = \frac{r\beta + \sqrt{\Delta_1}}{2c_2}$ . By calculation, we have  $H(\frac{d_1}{eK}) = cd_2(\frac{d_1}{eK})^2 > 0$ ,  $H(\frac{r\beta}{2cd_2}) = r\beta(\frac{d_1}{eK} - \frac{3r\beta}{4cd_2}) < 0$  and  $H(\frac{r\beta}{cd_2}) = \frac{d_1 r\beta}{eK} > 0$ . Hence  $\frac{d_1}{eK} < z_1 < \frac{r\beta}{2cd_2} < z_2 < \frac{r\beta}{cd_2}$ . Then, we have:

(i) If  $z_2 < 1$ , then  $H(z) > 0$  for  $\frac{d_1}{eK} < z < z_1$  and  $z_2 < z < 1$ ,  $H(z) < 0$  for  $z_1 < z < z_2$ .

(ii) If  $z_1 < 1 \leq z_2$ , then  $H(z) > 0$  for  $\frac{d_1}{eK} < z < z_1$ ,  $H(z) < 0$  for  $z_1 < z < 1$ .

(iii) If  $z_1 \geq 1$ , then  $H(z) > 0$  for  $\frac{d_1}{eK} < z < 1$ .

By analyse, we have the following results.

**Proposition 2.1** *If  $r\beta > \frac{4cd_1 d_2}{eK}$  and  $\frac{d_1}{eK} < 1$ , then*

(i)  $z_2 < 1$  if and only if  $\frac{r\beta}{cd_2} < 2$  and  $\frac{r\beta}{cd_2} < \frac{1}{1 - \frac{d_1}{eK}}$ .

(ii)  $z_1 < 1 \leq z_2$  if and only if one of the following conditions hold:

(ii.a)  $\frac{r\beta}{cd_2} < 2$  and  $\frac{r\beta}{cd_2} \geq \frac{1}{1 - \frac{d_1}{eK}}$ .

(ii.b)  $\frac{r\beta}{cd_2} = 2$ .

(ii.c)  $\frac{r\beta}{cd_2} > 2$  and  $\frac{r\beta}{cd_2} > \frac{1}{1 - \frac{d_1}{eK}}$ .

(iii)  $z_1 \geq 1$  if and only if  $\frac{r\beta}{cd_2} > 2$  and  $\frac{r\beta}{cd_2} \leq \frac{1}{1 - \frac{d_1}{eK}}$ .

**Proof** (i)  $z_2 < 1$  is equivalent to  $\sqrt{r\beta(r\beta - \frac{4cd_1d_2}{eK})} < 2cd_2 - r\beta$ . Hence, if  $\frac{r\beta}{cd_2} < 2$  and  $\frac{r\beta}{cd_2} < \frac{1}{1 - \frac{d_1}{eK}}$  hold, then the conclusion (i) holds.

The proofs of (ii) and (iii) are similar to that of (i) and we omit the detail here. This completes the proof of Proposition 2.1.

From Proposition 2.1, we have the following proposition.

**Proposition 2.2** *Proposition 2.1 is equivalent to the following statements.*

- (i) When  $0 < \frac{d_1}{eK} < \frac{1}{2}$ , we obtain:
  - (i.a) If  $\frac{4d_1}{eK} < \frac{r\beta}{cd_2} < \frac{1}{1 - \frac{d_1}{eK}}$ , then  $z_2 < 1$ .
  - (i.b) If  $\frac{r\beta}{cd_2} \geq \frac{1}{1 - \frac{d_1}{eK}}$ , then  $z_1 < 1 \leq z_2$ .
- (ii) When  $\frac{d_1}{eK} = \frac{1}{2}$ , if  $\frac{r\beta}{cd_2} > \frac{4d_1}{eK} = 2$ , then  $z_1 < 1 \leq z_2$ .
- (iii) When  $\frac{1}{2} < \frac{d_1}{eK} < 1$ , we obtain:
  - (iii.a) If  $\frac{4d_1}{eK} < \frac{r\beta}{cd_2} \leq \frac{1}{1 - \frac{d_1}{eK}}$ , then  $z_1 \geq 1$ .
  - (iii.b) If  $\frac{r\beta}{cd_2} > \frac{1}{1 - \frac{d_1}{eK}}$ , then  $z_1 < 1 \leq z_2$ .

**Proof** Note that  $\frac{1}{1-x} > 4x$  for  $0 < x < 1$ . It follows from Proposition 2.1 that  $\frac{r\beta}{cd_2} > \frac{4d_1}{eK}$  and  $\frac{d_1}{eK} < 1$ . Then  $\frac{1}{1 - \frac{d_1}{eK}} > \frac{4d_1}{eK}$  for  $\frac{d_1}{eK} < 1$ .

(i) It follows from  $0 < \frac{d_1}{eK} < \frac{1}{2}$  that  $1 < \frac{1}{1 - \frac{d_1}{eK}} < 2$ . It follows from (i) of Proposition 2.1 that  $\frac{r\beta}{cd_2} < \frac{1}{1 - \frac{d_1}{eK}}$ . Hence, if  $\frac{4d_1}{eK} < \frac{r\beta}{cd_2} < \frac{1}{1 - \frac{d_1}{eK}}$ , then  $z_2 < 1$ . The conditions (ii.a), (ii.b) and (ii.c) of Proposition 2.1 are equivalent to  $\frac{r\beta}{cd_2} \geq \frac{1}{1 - \frac{d_1}{eK}}$ . Therefore, if  $\frac{r\beta}{cd_2} \geq \frac{1}{1 - \frac{d_1}{eK}}$ , then  $z_1 < 1 \leq z_2$ . Note that condition (iii) of Proposition 2.1 is impossible.

The proofs of (ii) and (iii) are similar to that of (i) and we omit the detail here. This completes the proof of Proposition 2.2.

By the above discussion and Propositions 2.1 and 2.2, we have the following result.

**Proposition 2.3** *Equation (2.5) has the following result.*

- (i) When  $0 < \frac{d_1}{eK} < \frac{1}{2}$ , we obtain:
  - (i.a) If  $\frac{r\beta}{cd_2} < \frac{4d_1}{eK}$ , then  $H(z) > 0$  for  $\frac{d_1}{eK} < z < 1$ .
  - (i.b) If  $\frac{r\beta}{cd_2} = \frac{4d_1}{eK}$ , then  $H(z) > 0$  for  $\frac{d_1}{eK} < z < \frac{d_1}{2eK}$  or  $\frac{d_1}{2eK} < z < 1$ .
  - (i.c) If  $\frac{4d_1}{eK} < \frac{r\beta}{cd_2} < \frac{1}{1 - \frac{d_1}{eK}}$ , then  $H(z) > 0$  for  $\frac{d_1}{eK} < z < z_1$  or  $z_2 < z < 1$ ,  $H(z) < 0$  for  $z_1 < z < z_2$ .
  - (i.d) If  $\frac{r\beta}{cd_2} \geq \frac{1}{1 - \frac{d_1}{eK}}$ , then  $H(z) > 0$  for  $\frac{d_1}{eK} < z < z_1$ ,  $H(z) < 0$  for  $z_1 < z < 1$ .
- (ii) When  $\frac{d_1}{eK} = \frac{1}{2}$ , we obtain:
  - (ii.a) If  $\frac{r\beta}{cd_2} \leq \frac{4d_1}{eK} = 2$ , then  $H(z) > 0$  for  $\frac{d_1}{eK} < z < 1$ .

(ii.b) If  $\frac{r\beta}{cd_2} > \frac{4d_1}{eK} = 2$ , then  $H(z) > 0$  for  $\frac{d_1}{eK} < z < z_1$ ,  $H(z) < 0$  for  $z_1 < z < 1$ .

(iii) When  $\frac{1}{2} < \frac{d_1}{eK} < 1$ , we obtain:

(iii.a) If  $\frac{r\beta}{cd_2} \leq \frac{1}{1-\frac{d_1}{eK}}$ , then  $H(z) > 0$  for  $\frac{d_1}{eK} < z < 1$ .

(iii.b) If  $\frac{r\beta}{cd_2} > \frac{1}{1-\frac{d_1}{eK}}$ , then  $H(z) > 0$  for  $\frac{d_1}{eK} < z < z_1$ ,  $H(z) < 0$  for  $z_1 < z < 1$ .

Next we show the global stability of equilibrium of system (1.3), and obtain the following lemmas.

**Lemma 2.3** *If the predator-extinction equilibrium  $E_1(K, 0, 0)$  is locally asymptotically stable, then  $E_1(K, 0, 0)$  is globally asymptotically stable.*

**Proof** Consider the following Lyapunov function

$$V(t) = x - K - K \ln \frac{x}{K} + \eta_1 S + \eta_2 I,$$

where  $\eta_i = \frac{c}{e}$ ,  $i = 1, 2$ .

Calculating the derivative of  $V$  along the solution  $(x(t), S(t), I(t))$  of system (1.3), we have

$$\begin{aligned} V'(t) &= (x - K) \left( -\frac{r}{K}(x - K) - c(1 - m)S \right) \\ &\quad + \eta_1 S(e(1 - m)(x - K) + e(1 - m)K - d_1 - \beta I) + \eta_2 I(\beta S - d_2) \quad (2.6) \\ &= -\frac{r}{K}(x - K)^2 + \eta_1(e(1 - m)K - d_1)S - \eta_2 d_2 I. \end{aligned}$$

It follows from Lemma 2.2 that  $e(1 - m)K - d_1 < 0$ . Then  $V'(t) < 0$  for all  $(x, S, I) \neq (K, 0, 0)$ . By the Lyapunov-LaSalle invariance principle [15],  $E_1(K, 0, 0)$  is globally asymptotically stable. This completes the proof of Lemma 2.3.

**Lemma 2.4** *If the disease-free equilibrium  $E_2(x_1, S_1, 0)$  is locally asymptotically stable, then  $E_2(x_1, S_1, 0)$  is globally asymptotically stable.*

**Proof** Define a Lyapunov function as follows

$$V(t) = x - x_1 - x_1 \ln \frac{x}{x_1} + \eta_1 \left( S - S_1 - S_1 \ln \frac{S}{S_1} \right) + \eta_2 I,$$

where  $\eta_i = \frac{c}{e}$ ,  $i = 1, 2$ .

Calculating the derivative of  $V$  along the solution  $(x(t), S(t), I(t))$  of system (1.3), we have

$$\begin{aligned} V'(t) &= (x - x_1) \left( -\frac{r}{K}(x - x_1) - c(1 - m)(S - S_1) \right) + \eta_1 (S - S_1)(e(1 - m)(x - x_1) - \beta I) \\ &\quad + \eta_2 I(\beta(S - S_1) + \beta S_1 - d_2) \quad (2.7) \\ &= -\frac{r}{K}(x - x_1)^2 + \eta_2(\beta S_1 - d_2)I. \end{aligned}$$

It follows from Lemma 2.2 that  $\beta S_1 - d_2 < 0$ . Let  $\delta = \min\{\frac{r}{K}, |\beta S_1 - d_2|\}$ . Then it follows from (2.7) that

$$V'(t) \leq -\delta((x - x_1)^2 + I). \quad (2.8)$$

Thus  $V(t)$  is nonincreasing. From Lemma 2.1,  $(x - x_1)^2$  and  $I$  are bounded. On the other hand, it is easy to see that  $x'(t)$  and  $I'(t)$  are bounded. Therefore,  $(x - x_1)^2$  and  $I$  are uniformly continuous on  $[0, +\infty)$ . Integrating both sides of (2.8) over the interval  $[0, +\infty)$ , we have

$$V(t) + \delta \int_0^t (x(s) - x_1)^2 ds + \delta \int_0^t I(s) ds \leq V(0), \quad t \geq 0.$$

Therefore,  $V(t)$  is bounded on  $[0, +\infty)$  and satisfies

$$\int_0^t (x(s) - x_1)^2 ds < +\infty, \quad \int_0^t I(s) ds < +\infty.$$

It follows from the above inequality that  $(x(t) - x_1)^2 \in L^1[0, +\infty)$  and  $I(t) \in L^1[0, +\infty)$ . By Barbalat's lemma [16], we conclude that

$$\lim_{t \rightarrow +\infty} (x(t) - x_1)^2 = 0, \quad \lim_{t \rightarrow +\infty} I(t) = 0, \quad (2.9)$$

so

$$\lim_{t \rightarrow +\infty} x(t) = x_1. \quad (2.10)$$

From the first equation of system (1.3) and the above equality, we have

$$\lim_{t \rightarrow +\infty} x'(t) = \lim_{t \rightarrow +\infty} \left[ rx(t) \left( 1 - \frac{x(t)}{K} \right) - c(1 - m)x(t)S(t) \right] = 0.$$

Hence,

$$c(1 - m) \lim_{t \rightarrow +\infty} x(t)S(t) = rx_1 \left( 1 - \frac{x_1}{K} \right),$$

that is

$$\lim_{t \rightarrow +\infty} S(t) = \frac{r(K - x_1)}{Kc(1 - m)} = \frac{r}{c(1 - m)^2} \left( 1 - m - \frac{d_1}{Ke} \right) = S_1. \quad (2.11)$$

Then, from (2.9)-(2.11),  $E_2(x_1, S_1, 0)$  is globally asymptotically stable. This completes the proof of Lemma 2.4.

**Lemma 2.5** *If the unique endemic-coexistence equilibrium  $E^*(x^*, S^*, I^*)$  is locally asymptotically stable, then  $E^*(x^*, S^*, I^*)$  is globally asymptotically stable.*

**Proof** Define a Lyapunov function as follows

$$V(t) = x - x^* - x^* \ln \frac{x}{x^*} + \eta_1 \left( S - S^* - S^* \ln \frac{S}{S^*} \right) + \eta_2 \left( I - I^* - I^* \ln \frac{I}{I^*} \right),$$

where  $\eta_i = \frac{c}{e}$ ,  $i = 1, 2$ .

Calculating the derivative of  $V$  along the solution  $(x(t), S(t), I(t))$  of system (1.3), we obtain

$$\begin{aligned}
V'(t) &= (x - x^*) \left( -\frac{r}{K}(x - x^*) - c(1 - m)(S - S^*) \right) \\
&\quad + \eta_1(S - S^*)(e(1 - m)(x - x^*) - \beta(I - I^*)) + \eta_2\beta(I - I^*)(S - S^*) \quad (2.12) \\
&= -\frac{r}{K}(x - x^*)^2.
\end{aligned}$$

Thus  $V(t)$  is nonincreasing. By Lemma 2.1,  $(x - x^*)^2$  is bounded. On the other hand, it is easy to see that  $x'(t)$  is bounded. Therefore,  $(x - x^*)^2$  is uniformly continuous on  $[0, +\infty)$ . Integrating both sides of (2.12) over the interval  $[0, +\infty)$ , we have

$$V(t) + \frac{r}{K} \int_0^t (x(s) - x^*)^2 ds = V(0), \quad t \geq 0.$$

Hence,  $V(t)$  is bounded on  $[0, +\infty)$  and satisfies

$$\int_0^t (x(s) - x^*)^2 ds < +\infty.$$

The above inequality implies that  $(x(t) - x^*)^2 \in L^1[0, +\infty)$ . By Barbalat's lemma [16], we have

$$\lim_{t \rightarrow +\infty} (x(t) - x^*)^2 = 0, \quad (2.13)$$

so

$$\lim_{t \rightarrow +\infty} x(t) = x^*. \quad (2.14)$$

It follows from the first equation of system (1.3) and the above equality that

$$\lim_{t \rightarrow +\infty} x'(t) = \lim_{t \rightarrow +\infty} \left[ rx(t) \left( 1 - \frac{x(t)}{K} \right) - c(1 - m)x(t)S(t) \right] = 0.$$

Hence,

$$c(1 - m) \lim_{t \rightarrow +\infty} x(t)S(t) = rx^* \left( 1 - \frac{x^*}{K} \right),$$

that is

$$\lim_{t \rightarrow +\infty} S(t) = \frac{r(K - x^*)}{Kc(1 - m)}. \quad (2.15)$$

By the definition of  $x^*$ , we have  $\frac{r(K - x^*)}{Kc(1 - m)} = \frac{d_2}{\beta}$ . Then, from (2.15), we obtain

$$\lim_{t \rightarrow +\infty} S(t) = \frac{d_2}{\beta} = S_1. \quad (2.16)$$

From the second equation of system (1.3) and (2.16), we have

$$\lim_{t \rightarrow +\infty} S'(t) = \lim_{t \rightarrow +\infty} (e(1 - m)x(t)S(t) - d_1S(t) - \beta S(t)I(t)) = 0.$$

Hence, from (2.14) and (2.16), we obtain

$$\beta \lim_{t \rightarrow +\infty} S(t)I(t) = e(1 - m)x^*S^* - d_1S^*,$$

that is

$$\lim_{t \rightarrow +\infty} I(t) = \frac{e(1-m)x^* - d_1}{\beta} = I^*. \quad (2.17)$$

Therefore, by (2.14), (2.16) and (2.17),  $E^*(x^*, S^*, I^*)$  is globally asymptotically stable. This completes the proof of Lemma 2.5.

Now, we give the main result of this section. Let  $z = 1 - m$  in (2.5). According to the above analysis and summarizing Propositions 2.1-2.3 and Lemmas 2.2-2.5, we obtain the following theorem.

**Theorem 2.1**

(i) If  $\frac{d_1}{eK} \geq 1$ , then the predator-extinction equilibrium  $E_1(K, 0, 0)$  is globally asymptotically stable.

(ii) If  $\frac{1}{2} < \frac{d_1}{eK} < 1$ , we obtain:

(ii.a) If  $\frac{r\beta}{cd_2} \leq \frac{1}{1 - \frac{d_1}{eK}}$ , then the disease-free equilibrium  $E_2(x_1, S_1, 0)$  is globally asymptotically stable for all  $0 < m < 1 - \frac{d_1}{eK}$ , and the predator-extinction equilibrium  $E_1(K, 0, 0)$  is globally asymptotically stable for all  $1 - \frac{d_1}{eK} < m < 1$ .

(ii.b) If  $\frac{r\beta}{cd_2} > \frac{1}{1 - \frac{d_1}{eK}}$ , then the unique endemic-coexistence equilibrium  $E^*(x^*, S^*, I^*)$  is globally asymptotically stable for all  $0 < m < 1 - z_1$ , the disease-free equilibrium  $E_2(x_1, S_1, 0)$  is globally asymptotically stable for all  $1 - z_1 < m < 1 - \frac{d_1}{eK}$ , and the predator-extinction equilibrium  $E_1(K, 0, 0)$  is globally asymptotically stable for all  $1 - \frac{d_1}{eK} < m < 1$ .

(iii) If  $\frac{d_1}{eK} = \frac{1}{2}$ , we obtain:

(iii.a) If  $\frac{r\beta}{cd_2} \leq 2$ , then the disease-free equilibrium  $E_2(x_1, S_1, 0)$  is globally asymptotically stable for all  $0 < m < 0.5$ , and the predator-extinction equilibrium  $E_1(K, 0, 0)$  is globally asymptotically stable for all  $0.5 < m < 1$ .

(iii.b) If  $\frac{r\beta}{cd_2} > 2$ , then the unique endemic-coexistence equilibrium  $E^*(x^*, S^*, I^*)$  is globally asymptotically stable for all  $0 < m < 1 - z_1$ , the disease-free equilibrium  $E_2(x_1, S_1, 0)$  is globally asymptotically stable for all  $1 - z_1 < m < 0.5$ , and the predator-extinction equilibrium  $E_1(K, 0, 0)$  is globally asymptotically stable for all  $0.5 < m < 1$ .

(iv) If  $0 < \frac{d_1}{eK} < \frac{1}{2}$ , we obtain:

(iv.a) If  $\frac{r\beta}{cd_2} < \frac{4d_1}{eK}$ , then the disease-free equilibrium  $E_2(x_1, S_1, 0)$  is globally asymptotically stable for all  $0 < m < 1 - \frac{d_1}{eK}$ , and the predator-extinction equilibrium  $E_1(K, 0, 0)$  is globally asymptotically stable for all  $1 - \frac{d_1}{eK} < m < 1$ .

(iv.b) If  $\frac{r\beta}{cd_2} = \frac{4d_1}{eK}$ , then the disease-free equilibrium  $E_2(x_1, S_1, 0)$  is globally asymptotically stable for all  $0 < m < 1 - \frac{2d_1}{eK}$  or  $1 - \frac{2d_1}{eK} < m < 1 - \frac{d_1}{eK}$ , and the predator-extinction equilibrium  $E_1(K, 0, 0)$  is globally asymptotically stable for all  $1 - \frac{d_1}{eK} < m < 1$ .

(iv.c) If  $\frac{4d_1}{eK} < \frac{r\beta}{cd_2} < \frac{1}{1-\frac{d_1}{eK}}$ , then the disease-free equilibrium  $E_2(x_1, S_1, 0)$  is globally asymptotically stable for all  $0 < m < 1 - z_2$  or  $1 - z_1 < m < 1 - \frac{d_1}{eK}$ , the unique endemic-coexistence equilibrium  $E^*(x^*, S^*, I^*)$  is globally asymptotically stable for all  $1 - z_2 < m < 1 - z_1$ , and the predator-extinction equilibrium  $E_1(K, 0, 0)$  is globally asymptotically stable for all  $1 - \frac{d_1}{eK} < m < 1$ .

(iv.d) If  $\frac{r\beta}{cd_2} \geq \frac{1}{1-\frac{d_1}{eK}}$ , then the unique endemic-coexistence equilibrium  $E^*(x^*, S^*, I^*)$  is globally asymptotically stable for all  $0 < m < 1 - z_1$ , the disease-free equilibrium  $E_2(x_1, S_1, 0)$  is globally asymptotically stable for all  $1 - z_1 < m < 1 - \frac{d_1}{eK}$ , and the predator-extinction equilibrium  $E_1(K, 0, 0)$  is globally asymptotically stable for all  $1 - \frac{d_1}{eK} < m < 1$ .

### 3 Numerical Simulations

In this section we give some numerical simulations of systems (1.3). We consider the following system

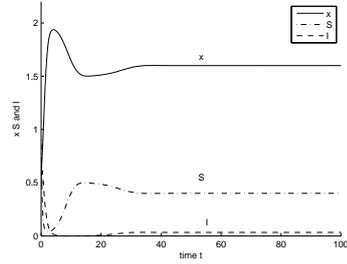
$$\begin{aligned}x' &= 1.6x\left(1 - \frac{x}{2}\right) - 0.8(1 - m)xS, \\S' &= (1 - m)xS - d_1S - \beta SI, \\I' &= \beta SI - 1.2I,\end{aligned}\tag{3.1}$$

where  $r = 1.6$ ;  $K = 2$ ;  $c = 0.8$ ;  $e = 1$ ;  $d_2 = 1.2$ .

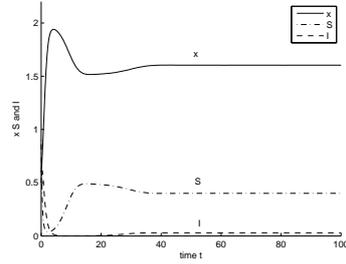
When  $d_1 = 1.5$ ,  $0.5 < \frac{d_1}{eK} = 0.75 < 1$ . Let  $\beta = 3$ , then  $\frac{r\beta}{cd_2} = 5 > \frac{1}{1-\frac{d_1}{eK}} = 4$ . It follows from Theorem 2.1 (ii.b) that the unique endemic-coexistence equilibrium  $E^*(x^*, S^*, I^*)$  is globally asymptotically stable for all  $0 < m < 0.081$  (see Figure 1(b)), the disease-free equilibrium  $E_2(x_1, S_1, 0)$  is globally asymptotically stable for all  $0.081 < m < 0.25$  (See Figure 1(c)), and the predator-extinction equilibrium  $E_1(2, 0, 0)$  is globally asymptotically stable for all  $0.25 < m < 1$  (See Figure 1(d)).

When  $d_1 = 1$ ,  $\frac{d_1}{eK} = \frac{1}{2}$ . Let  $\beta = 1.5$ , then  $\frac{r\beta}{cd_2} = 2.5 > 2$ . It follows from Theorem 2.1 (iii.b) that the unique endemic-coexistence equilibrium  $E^*(x^*, S^*, I^*)$  is globally asymptotically stable for all  $0 < m < 0.309$  (see Figure 2(b)), the disease-free equilibrium  $E_2(x_1, S_1, 0)$  is globally asymptotically stable for all  $0.309 < m < 0.5$  (see Figure 2(c)), and the predator-extinction equilibrium  $E_1(2, 0, 0)$  is globally asymptotically stable for all  $0.5 < m < 1$  (see Figure 2(d)).

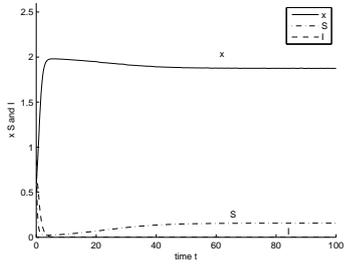
When  $d_1 = 0.4$ ,  $\frac{d_1}{eK} = 0.2 < 0.5$ . Let  $\beta = 0.6$ , then  $\frac{4d_1}{eK} = 0.8 < \frac{r\beta}{cd_2} = 1 < \frac{1}{1-\frac{d_1}{eK}} = 1.25$ . It follows from Theorem 2.1 (iv.c) that the disease-free equilibrium  $E_2(x_1, S_1, 0)$  is globally asymptotically stable for all  $0 < m < 0.276$  or  $0.724 < m < 0.8$  (see Figures 3(b) and 3(d)), the unique endemic-coexistence equilibrium  $E^*(x^*, S^*, I^*)$  is globally asymptotically stable for all  $0.276 < m < 0.724$  (see Figure 3(c)), and the predator-extinction equilibrium  $E_1(2, 0, 0)$  is globally asymptotically stable for all  $0.8 < m < 1$  (see Figure 3(e)).



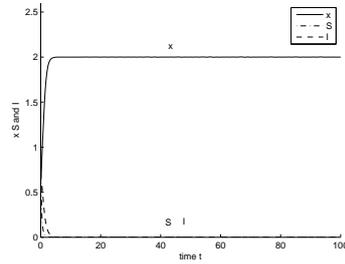
(a)  $m = 0, E^*(1.6, 0.4, 0.033)$



(b)  $m = 0.01, E^*(1.604, 0.4, 0.029)$

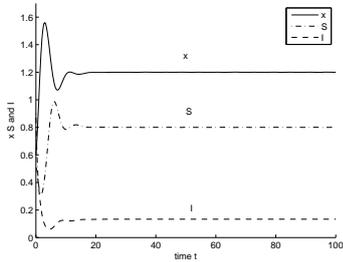


(c)  $m = 0.2, E_2(1.875, 0.156, 0)$

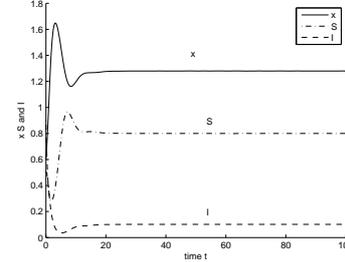


(d)  $m = 0.8, E_1(2, 0, 0)$

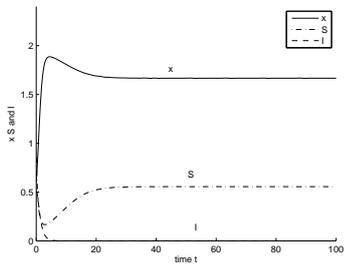
Figure 1: Dynamics behavior of system (3.1) with  $d_1 = 1.5; \beta = 3$ .



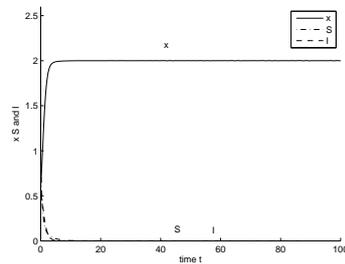
(a)  $m = 0, E^*(1.2, 0.8, 0.133)$



(b)  $m = 0.1, E^*(1.28, 0.8, 0.101)$

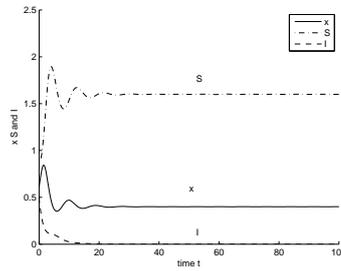


(c)  $m = 0.4, E_2(1.667, 0.556, 0)$

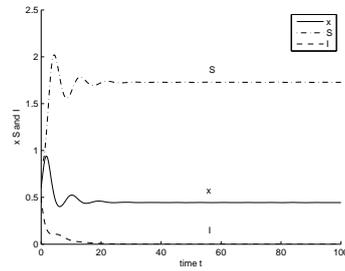


(d)  $m = 0.7, E_1(2, 0, 0)$

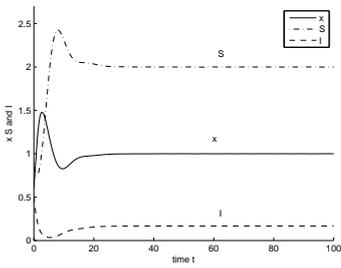
Figure 2: Dynamics behavior of system (3.1) with  $d_1 = 1; \beta = 1.5$ .



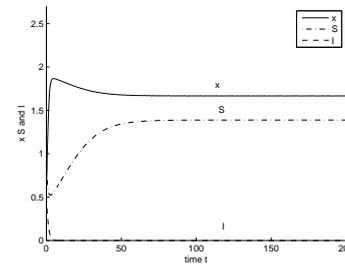
(a)  $m = 0, E_2(0.4, 1.6, 0)$



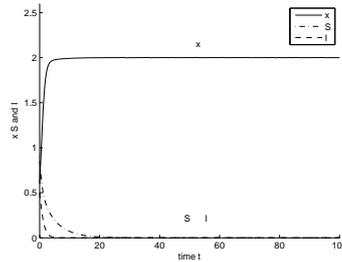
(b)  $m = 0.1, E_2(0.444, 1.728, 0)$



(c)  $m = 0.5, E^*(1, 2, 0.167)$



(d)  $m = 0.76, E_2(1.667, 1.389, 0)$



(e)  $m = 0.9, E_1(2, 0, 0)$

Figure 3: Dynamics behavior of system (3.1) with  $d_1 = 0.4; \beta = 0.6$ .

### 4 Discussion

In this paper, we study the global dynamics of a predator-prey model with prey refuge and disease in the predator. We show that prey refuge plays an important role in the dynamics of a predator-prey system (1.3). From the above results, if the refuge  $m$  is sufficiently small, then the dynamics behavior of system is accordant with the corresponding system without prey refuges. By decreasing the value of natural death rate of the susceptible predator  $d_1$ , the dynamics behavior of system becomes complicated, that is the global stable equilibrium may be changed by increasing the value of refuges  $m$ . This shows that prey refuge plays an important role in the dynamics behavior of system (1.3).

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