Ann. of Appl. Math.
33:1(2017), 32-49

GLOBAL HOPF BIFURCATION IN A DELAYED PHYTOPLANKTON-ZOOPLANKTON MODEL WITH COMPETITION*†

Renxiang Shi, Jiang Yu‡
(*Dept. of Math., Shanghai Jiaotong University, Shanghai 200240, PR China*)

Abstract

In this paper, the dynamics of a delayed phytoplankton-zooplankton model is considered. Taking the delay due to the gestation of zooplankton as parameter, we describe the local Hopf bifurcation by center manifold theorem and normal form, then we discuss the global existence of periodic solution. At last, some simulations are given to support our result.

Keywords global Hopf bifurcation; normal form; periodic solution; competition; phytoplankton-zooplankton

2000 Mathematics Subject Classification 35B32

1 Introduction

Recently, there have been many works about the phytoplankton-zooplankton model [1-5,7]. The model is important for aquatic environment. The phytoplankton could produce much oxygen and absorb much carbon dioxide, they benefit our environment very much. As we know, some phytoplankton could be harmful for zooplankton, they could create toxin substance which could kill the aquatic animals. From [1,4,5], we know that the delay caused by the maturity of toxic-phytoplankton plays an important role on the dynamic of phytoplankton-zooplankton system, which seems that delay could cause rich dynamics. In [4], the author considered two harmful phytoplankton-zooplankton model with two delays

$$\begin{cases} \dfrac{\mathrm{d}P_1}{\mathrm{d}t} = r_1P_1\Big(1-\dfrac{P_1}{K}\Big) - \alpha_1P_1P_2 - \rho_1P_1Z, \\ \dfrac{\mathrm{d}P_2}{\mathrm{d}t} = r_2P_2\Big(1-\dfrac{P_2}{K}\Big) - \alpha_2P_1P_2 - \rho_2P_2Z, \\ \dfrac{\mathrm{d}Z}{\mathrm{d}t} = (r_1P_1+r_2P_2)Z - dZ - \theta_1P_1(t-\tau_1)Z - \theta_2P_2(t-\tau_2)Z, \end{cases} \tag{1.1}$$

*This research was supported by the National Natural Science Foundations of China (No.11431008), NSF of Shanghai grant (No.15ZR1423700).
†Manuscript received September 22, 2016; Revised November 4, 2016
‡Corresponding author. E-mail: jiangyu@sjtu.edu.cn

where local Hopf bifurcation was discussed with two different delays. In the delayed two zooplankton-phytoplankton model with competition [1]

$$\begin{cases} \dfrac{\mathrm{d}P}{\mathrm{d}t} = rP\Big(1-\dfrac{P}{K}\Big) - \dfrac{\mu_1 P Z_1}{\alpha_1 + P} - \dfrac{\mu_2 P Z_1}{\alpha_2 + P}, \\ \dfrac{\mathrm{d}Z_1}{\mathrm{d}t} = \dfrac{\beta_1 P Z_1}{\alpha_1 + P} - \dfrac{\rho_1 P(t-\tau) Z_1}{\alpha_1 + P(t-\tau)} - d_1 Z_1 - g_1 Z_1^2, \\ \dfrac{\mathrm{d}Z_2}{\mathrm{d}t} = \dfrac{\beta_2 P Z_1}{\alpha_2 + P} - \dfrac{\rho_2 P(t-\tau) Z_2}{\alpha_2 + P(t-\tau)} - d_2 Z_2 - g_1 Z_2^2. \end{cases} \tag{1.2}$$

The authors discussed the local Hopf bifurcation under taking delay τ as the parameter. As we know, when the delay τ is located in a sufficiently small neighborhood of the critical value, local hopf bifurcation occurs. But it is difficult to show the global existence of periodic solution. The works about the global Hopf bifurcation of phytoplankton-zooplankton system have been obtained in recent years, such as [2,3,7]. In [2], the authors assumed the delay of gestation equals the delay required for maturity of toxic phytoplankton. The global Hopf bifurcation of the following system was discussed

$$\begin{cases} \dfrac{\mathrm{d}P}{\mathrm{d}t} = rP(t)\Big(1-\dfrac{P(t)}{K}\Big) - \dfrac{\beta P(t) Z(t)}{1+\gamma_1 P(t)}, \\ \dfrac{\mathrm{d}Z}{\mathrm{d}t} = \dfrac{\mathrm{e}^{-\delta\tau_1}\beta_1 P(t-\tau_1) Z(t-\tau_1)}{1+\gamma_1 P(t-\tau_1)} - \delta Z(t) - \dfrac{\mathrm{e}^{-\delta\tau_2}\rho P(t-\tau_2) Z(t-\tau_2)}{1+\gamma_2 P(t-\tau_2)}. \end{cases} \tag{1.3}$$

In [7], the authors only considered the delay caused by the gestation of zooplankton, and the global Hopf bifurcation was discussed,

$$\begin{cases} \dfrac{\mathrm{d}P}{\mathrm{d}t} = rP\Big(1-\dfrac{P}{K}\Big) - \alpha P Z, \\ \dfrac{\mathrm{d}Z}{\mathrm{d}t} = \beta P(t-\tau) Z(t-\tau) - \mu Z - \dfrac{\theta P Z}{\gamma + P}. \end{cases} \tag{1.4}$$

Besides, there have been other works about the global Hopf bifurcation [8,9]. In our opinion, competition is a common phenomena in nature, so we take competition into the zooplankton, model (1.4) becomes

$$\begin{cases} \dfrac{\mathrm{d}P}{\mathrm{d}t} = rP\Big(1-\dfrac{P}{K}\Big) - \dfrac{\mu P Z}{\alpha + P}, \\ \dfrac{\mathrm{d}Z}{\mathrm{d}t} = \dfrac{\beta P(t-\tau) Z(t-\tau)}{\alpha + P(t-\tau)} - \dfrac{\rho P(t) Z(t)}{\alpha + P(t)} - dZ - gZ^2, \end{cases} \tag{1.5}$$

where P and Z denote the densities of the phytoplankton and zooplankton respectively, r denotes the intrinsic growth rate, and K is the environmental carrying capacity. We select a Holling type Π functional response, d denotes the death rate of zooplankton, g is the intraspecific competition coefficient, μ is the maximum uptake rate of zooplankton, ρ is the rate of toxic substance produced by per unit biomass of phytoplankton, β denotes the ratio of biomass conversion. We assume $\mu > \beta > \rho$, inspired by the above works, we shall describe the local Hopf bifurcation and global existence of periodic solution in this paper, also we assume that $P(\theta) \geq 0$, $Z(\theta) \geq 0$, $\theta \in [-\tau, 0]$. Similar to [1], if there is no delay and $\beta - \rho - d < 0$, then $\dot{Z} \leq 0$, we reject this situation by assuming $\beta - \rho - d > 0$.

In the following, the stability and direction of local Hopf bifurcation will be discussed in Section 1. In Section 2, we discuss the global Hopf bifurcation, at last, some numerical simulations for supporting the result are given in Section 3.

2 Stability and Direction of the Hopf Bifurcation

Before discussion, we should give the condition for the existence and uniqueness of the equilibrium (P^*, Z^*). If the equilibrium (P^*, Z^*) exists, the following equations should hold:

$$\begin{cases} rP^*\Big(1 - \dfrac{P^*}{K}\Big) = \dfrac{\mu P^* Z^*}{\alpha + P^*}, \\ \dfrac{(\beta - \rho - d)P^* - d\alpha}{g(\alpha + P^*)} = Z^*. \end{cases}$$

From these we obtain $\frac{\mathrm{d}\alpha}{\beta-\rho-d} < P^* < K$. Substituting Z^* into the first equation, we define a function:

$$h(P) = r\Big(1 - \frac{P}{K}\Big)(\alpha + P)^2 - \frac{\mu}{g}\big((\beta - \rho - d)P - d\alpha\big),$$

We know $h(K) < 0$, $h(\frac{\mathrm{d}\alpha}{\beta-\rho-d}) > 0$, so there exists at least one root P^* satisfying $h(P^*) = 0$ when $P^* \in (\frac{\mathrm{d}\alpha}{\beta-\rho-d}, K)$. If $h(P)$ is a monotone function, (P^*, Z^*) is unique, so we get following result:

Lemma 2.1 *If $\frac{\mathrm{d}\alpha}{\beta-\rho-d} < P^* < K$ and the function $h(P)$ is monotone on $(\frac{\mathrm{d}\alpha}{\beta-\rho-d}, K)$, then the unique equilibrium (P^*, Z^*) exists.*

We assume $h(P)$ is decrease on $(\frac{\mathrm{d}\alpha}{\beta-\rho-d}, K)$, so the equilibrium point (P^*, Z^*) is unique. We make translation $u(t) = P(t) - P^*$ and $v(t) = Z(t) - Z^*$, then (1.5) becomes

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = a_{11}u(t) + a_{12}v(t) + hot, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = a_{21}u(t) + a_{22}v(t) + a_{23}u(t-\tau) + a_{24}v(t-\tau) + hot, \end{cases} \tag{2.1}$$

where

$$a_{11}=-\frac{rP^*}{K}+\frac{\mu P^*Z^*}{(\alpha+P^*)^2},\quad a_{12}=-\frac{\mu P^*}{\alpha+P^*},\quad a_{21}=-\frac{\rho\alpha Z^*}{(\alpha+P^*)^2},$$
$$a_{22}=-\frac{\rho P^*}{\alpha+P^*}-d-2gZ^*,\quad a_{23}=\frac{\beta Z^*}{\alpha+P^*}-\frac{\rho P^*Z^*}{(\alpha+P^*)^2},\quad a_{24}=\frac{\beta P^*}{\alpha+P^*}.$$

The associated characteristic equation:

$$\lambda^2+A_{21}\lambda+A_{20}+(B_{21}\lambda+B_{20})\mathrm{e}^{-\lambda\tau}=0, \tag{2.2}$$

where $A_{21}=-(a_{11}+a_{22})$, $A_{20}=a_{11}a_{22}-a_{12}a_{21}$, $B_{21}=-a_{24}$, $B_{20}=a_{11}a_{24}-a_{12}a_{23}$, when $\tau=0$, (2.2) becomes $\lambda^2+(A_{21}+B_{21})\lambda+A_{20}+B_{20}=0$, then all the roots of (2.2) have negative real parts if and only if

$$A_{21}+B_{21}>0,\quad A_{20}+B_{20}>0, \tag{H$_1$}$$

so the equilibrium (P^*,Z^*) is locally asymptotically stable when (H$_1$) holds.

When $\tau>0$, from [6], if and only if when the roots pass through the imaginary axis, the stability switch occurs. Let $\lambda=\omega i\,(\omega>0)$ be a root of equation (2.2), and we obtain that

$$\begin{cases} B_{21}\omega\cos(\omega\tau)-B_{20}\sin(\omega\tau)=-A_{21}\omega, \\ B_{21}\omega\sin(\omega\tau)+B_{20}\cos(\omega\tau)=-A_{20}+\omega^2. \end{cases} \tag{2.3}$$

Squaring and adding both of the equations,

$$\omega^4+(A_{21}^2-2A_{20}-B_{21}^2)\omega+A_{20}^2-B_{20}^2=0. \tag{2.4}$$

Let $v=\omega^2$, then we get

$$v^2+(A_{21}^2-2A_{20}-B_{21}^2)v+A_{20}^2-B_{20}^2=0.$$

For the existence of positive root and transversality, we assume:

$$A_{20}^2-B_{20}^2<0,\quad A_{21}^2-2A_{20}-B_{21}^2>0. \tag{H$_2$}$$

Denote $h(z)=z^2+(A_{21}^2-2A_{20}-B_{21}^2)z+A_{20}^2-B_{20}^2$, then $h(z)$ has a positive root z under (H$_2$), so equation (2.4) has a root $\omega=\sqrt{z}$. From (2.3) we obtain

$$\tau_j=\frac{1}{\omega}\arccos\Big(\frac{B_{20}\omega^2-B_{21}\omega^2A_{21}}{B_{20}^2+B_{21}^2\omega^2}\Big)+\frac{2j\pi}{\omega},\quad j=0,1,2\cdots, \tag{2.5}$$

where $\pm i\omega$ are pairs of pure imaginary roots of (2.2). From the above discussion and (H$_1$),(H$_2$), we have:

Lemma 2.2 *Suppose λ is a root of equation* (2.2), *when $\tau=\tau_j$, $j=0,1,2,\cdots$, then*

$$\frac{\mathrm{dRe}\lambda(\tau)}{\mathrm{d}\tau}\Big|_{\tau=\tau_j}>0. \tag{2.6}$$

Proof Taking the derivative of λ with τ in equation (2.2), we get

$$\Big(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\Big)^{-1}=\frac{(2\lambda+A_{21})\mathrm{e}^{\lambda\tau}+B_{21}}{B_{21}\lambda^2+B_{20}\lambda}-\frac{\tau}{\lambda},$$
$$\mathrm{Re}\Big(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\Big)^{-1}_{\tau=\tau_j}=\mathrm{Re}\Big\{\frac{(2\lambda+A_{21})\mathrm{e}^{\lambda\tau}+B_{21}}{B_{21}\lambda^2+B_{20}\lambda}\Big\}_{\tau=\tau_j}=\frac{A_{21}^2-2A_{20}-B_{21}^2}{\omega^2B_{21}^2+B_{20}^2}.$$

From (2.3) and (H_2) we konw

$$\mathrm{sign}\frac{\mathrm{dRe}\lambda}{\mathrm{d}\tau}_{\tau=\tau_j}=\mathrm{signRe}\Big(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\Big)^{-1}_{\tau=\tau_j}>0.$$

Applying Lemmas 2.1 and 2.2, and assumption (H_1),(H_2), we obtain:

Theorem 2.1 *The positive equilibrium point E^* of system* (1.5) *is locally asymptotically stable for $\tau\in[0,\tau_0]$ and Hopf bifurcation occurs when $\tau=\tau_j$, $j=0,1,2,\cdots$.*

Let $i\omega$ be the purely imaginary roots of (2.2) when $\tau=\tau_j$. Following the method of [13], we could obtain the formulaes which determine the direction, stability and period of bifurcated periodic solution.

Theorem 2.2 (1) *The direction of Hopf bifurcation is decided by parameter μ_2, which is supercritical if $\mu_2>0$ and subcritical if $\mu_2<0$.* (2) *The periodic solution about the Hopf bifurcation is stable if $\beta_2<0$ and unstable if $\beta_2>0$.* (3) *T_2 is the period of bifurcated periodic solution, where when $T_2>0$, the period increases; when $T_2<0$, the period decreases. The calculations of μ_2, β_2, T_2 are given in Appendix.*

3 Global Hopf Bifurcation

Now we discuss the global continuation of periodic solution. Let $X=C([-\tau,0],R^2)$, $u_t(\theta)=(P(t+\theta),Z(t+\theta))$, $t\geq 0$, $\theta\in[-\tau,0]$, then system (1.5) is equivalent to

$$\dot{u_t}=F(u_t,\tau,T), \tag{3.1}$$

where

$$F(\Phi,\tau,T)=\begin{pmatrix} r\phi_1(0)\Big(1-\dfrac{\phi_1(0)}{K}\Big)-\dfrac{\mu\phi_1(0)\phi_2(0)}{\alpha+\phi_1(0)}\\ \dfrac{\beta\phi_1(-\tau)\phi_2(-\tau)}{\alpha+\phi_1(-\tau)}-\dfrac{\rho\phi_1(0)\phi_2(0)}{\alpha+\phi_1(0)}-d\phi_2(0)-g\phi_2^2(0)\end{pmatrix}$$

with $\Phi=(\phi_1,\phi_2)$, and $F:X\times R_+\times R_+\longrightarrow R_+^2$ is a mapping, we have a mapping $\widehat{F}=F|_{R_+^2}\times R_+\times R_+$ by restrict F to the subspace of constant function in X. If $\widehat{F}(\widehat{u},\widehat{\tau},\widehat{T})=0$, then the point $(\widehat{u},\widehat{\tau},\widehat{T})$ is called a stationary point, where

$$\widehat{F}(u,\tau,T)=\begin{pmatrix} rP\Big(1-\dfrac{P}{K}\Big)-\dfrac{\mu PZ}{\alpha+P} \\ \dfrac{\beta PZ}{\alpha+P}-\dfrac{\rho PZ}{\alpha+P}-dZ-gZ^2 \end{pmatrix}$$

and

$$D_u\widehat{F}(u,\tau,T)=\begin{pmatrix} r-\dfrac{2rP}{K}-\dfrac{\mu Z}{\alpha+P}+\dfrac{\mu PZ}{(\alpha+P)^2} & -\dfrac{\mu P}{\alpha+P} \\ \dfrac{(\beta-\rho)Z}{\alpha+P}-\dfrac{(\beta-\rho)PZ}{(\alpha+P)^2} & \dfrac{(\beta-\rho)P}{\alpha+P}-d-2gZ \end{pmatrix}.$$

F and $\widehat{F}$ satisfy the following conditions:

(A_1) $\widehat{F}\in C^2(R_+^2\times R_+\times R_+,R_+^2)$;

(A_2) $D_{\widehat{u}}F(\widehat{u},\tau,T)$ is an isomorphism at the equilibrium $(\widehat{u},\tau,T)$;

(A_3) $F(\Phi,\tau,T)$ is differentiable with respect to Φ.

For any stationary solution $(\widehat{u},\tau,T)$, the characteristic matrix of system (3.1) is $\Delta(\widehat{u},\tau,T)(\lambda)=\lambda I-DF(\widehat{u},\tau,T)(\mathrm{e}^{\lambda\cdot}I)$, that is

$$\triangle(\widehat{u},\tau,T)(\lambda)$$

$$=\begin{pmatrix} \lambda-r+\dfrac{2r\widehat{P}}{K}+\dfrac{\mu\widehat{Z}}{\alpha+\widehat{P}}-\dfrac{\mu\widehat{P}\widehat{Z}}{(\alpha+\widehat{P})^2} & \dfrac{\mu\widehat{P}}{\alpha+\widehat{P}} \\ \dfrac{\rho\widehat{Z}}{\alpha+\widehat{P}}-\dfrac{-\rho\widehat{P}\widehat{Z}}{(\alpha+\widehat{P})^2}-\Big(\dfrac{\beta\widehat{Z}}{\alpha+\widehat{P}}-\dfrac{\rho\widehat{P}\widehat{Z}}{(\alpha+\widehat{P})^2}\Big)\mathrm{e}^{-\lambda\tau} & \lambda+\dfrac{\rho\widehat{P}}{\alpha+\widehat{P}}+d+2g\widehat{Z}-\dfrac{\beta\widehat{P}}{\alpha+\widehat{P}}\mathrm{e}^{-\lambda\tau} \end{pmatrix}.$$

The zeros of $\det(\Delta(\widehat{u},\tau,T))(\lambda)=0$ are called characteristic roots. From (A_2) we conclude that $\lambda=0$ is not a character root.

If $(\widehat{u},\widehat{\tau},\widehat{T})$ satisfies $\det(\Delta_{\widehat{u}}(im\frac{2\pi}{\widehat{T}}))=0$, we call it is a center. A center $(\widehat{u},\widehat{\tau},\widehat{T})$ is said to be isolated if it is the only center in some neighborhood of $(\widehat{u},\widehat{\tau},\widehat{T})$ and it has finite characteristic value of form $im\frac{2\pi}{\widehat{T}}$.

From the local Hopf bifurcation, we conclude that $(u^*,\tau_j,\frac{2\pi}{\omega})$, $j=0,1,2,\cdots$ is a isolated center, then there exists a smooth curve $\lambda:(\tau_j-\delta,\tau_j+\delta)\to C$ such that $\det(\Delta(u^*,\tau_j,\frac{2\pi}{\omega})(\lambda(\tau))=0$, $|\lambda(\tau)-i\omega|<\varepsilon$ for $\tau\in[\tau_j-\delta,\tau_j+\delta]$, $\lambda(\tau_j)=i\omega$, and $\mathrm{Re}\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}|_{\tau=\tau_j}>0$. Define $\Omega_{\varepsilon,\frac{2\pi}{\omega}}=(v,T):0<v<\varepsilon,|T-\frac{2\pi}{\omega}|<\varepsilon$, and we conclude that for $\tau\in[\tau_j-\delta,\tau_j+\delta]\times\partial\Omega_{\varepsilon,\frac{2\pi}{\omega}}$,

(A_4) if and only if $v=0$, $\tau=\tau^j$, $T=\frac{2\pi}{\omega}$, $j=0,1,2,\cdots$, $\det(\Delta(u^*,\tau,T)(v+i\frac{2\pi}{T}))=0$.

The hypotheses (A_1)-(A_4) in [11] are satisfied. We put $H^{\pm}(u^*,\tau_j,\frac{2\pi}{\omega})(v,T)=\det(\Delta(u^*,\tau_j\pm\delta),\frac{2\pi}{\omega})(v+\frac{2\pi i}{T})$.

The crossing number $\gamma(u^*,\tau_j,\frac{2\pi}{\omega})$ of center $(u^*,\tau_j,\frac{2\pi}{\omega})$ can be defined as

$$\begin{aligned}\gamma\Big(u^*,\tau_j,\frac{2\pi}{\omega}\Big)&=\deg_B\Big(H^-\Big(u^*,\tau_j,\frac{2\pi}{\omega}\Big),\Omega\Big(\varepsilon,\frac{2\pi}{\omega}\Big)\Big)\\&\quad-\deg_B\Big(H^+\Big(u^*,\tau_j,\frac{2\pi}{\omega}\Big),\Omega\Big(\varepsilon,\frac{2\pi}{\omega}\Big)\Big)=-1.\end{aligned}$$

For the periodic solution of system (3.1), we define

$\Sigma(F)=Cl((u,\tau,T)\in X\times R_+\times R_+|\ u$ is a T-periodic solution),

$N(F)=((\widehat{u},\widehat{\tau},\widehat{T})\in R_+^4|\ F(\widehat{u},\widehat{\tau},\widehat{T})=0)$.

Let $l(u^*,\tau_j,\frac{2\pi}{\omega})$ denote the connected component of $(u^*,\tau_j,\frac{2\pi}{\omega})$ in $\Sigma(F)$. For $(\widehat{u},\widehat{\tau},\widehat{T})=(u^*,\tau_j,\frac{2\pi}{\omega})$, we get

$$\Sigma_{(\widehat{u},\widehat{\tau},\widehat{T})\in l(u^*,\tau_j,\frac{2\pi}{\omega})\cap N(F)}\gamma(\widehat{u},\widehat{\tau},\widehat{T})<0.$$

The connected component $l(u^*,\tau_j,\frac{2\pi}{\omega})$ through $(u^*,\tau_j,\frac{2\pi}{\omega})$ in $\Sigma(F)$ is nonempty, since the first crossing number of each center is always -1. By Theorem 3.3 [11], we conclude $l(u^*,\tau_j,\frac{2\pi}{\omega})$ is unbounded, which is presented as follows.

Lemma 3.1 *$l(u^*,\tau_j,\frac{2\pi}{\omega})$ is unbounded for each center $(u^*,\tau_j,\frac{2\pi}{\omega})$.*

By the fundamental theory in [12], system (1.5) admits the existence and uniqueness of the solutions with the initial condition:

$$P(\theta)=\phi(\theta)\geq 0,\quad Z(\theta)=\psi(\theta)\geq 0,\quad \theta\in[-\tau,0],\ \phi(0)>0,\ \psi(0)>0. \tag{3.2}$$

Lemma 3.2 *All the nonconstant periodic solutions of system* (1.5) *are positive and bounded with the initial condition* (3.2) *when τ is bounded.*

Proof Let $(P(t),Z(t))$ be a solution of system (1.5), and consider $Z(t)$ for $t\in[0,\tau]$,

$$\begin{aligned}\frac{\mathrm{d}Z}{\mathrm{d}t}&=\frac{\beta P(t-\tau)Z(t-\tau)}{\alpha+P(t-\tau)}-\frac{\rho P(t)Z(t)}{\alpha+P(t)}-dZ-gZ^2\\&\geq-\frac{\rho P(t)Z(t)}{\alpha+P(t)}-dZ-gZ^2.\end{aligned}$$

Since $\phi(\theta)\geq 0,\ \psi(\theta)\geq 0$ for $\theta\in[-\tau,0]$, we get

$$Z(t)\geq\psi(0)\exp\Big[\int_0^t\Big(-\frac{\rho P(s)}{\alpha+P(s)}-d-gZ(s)\Big)\mathrm{d}s\Big]>0,\quad t\in[0,\tau].$$

Thus, $Z(t)$ is positive for $t\in[0,\tau]$, similarly

$$P(t)=\phi(0)\exp\Big[\int_0^t\Big(r\Big(1-\frac{P(s)}{K}\Big)-\frac{\mu Z(s)}{\alpha+P(s)}\Big)\mathrm{d}s\Big]>0,\quad t\in[0,\tau],$$

so we could expand the result to $[\tau,2\tau],\cdots,[n\tau,(n+1)\tau]$, $n\in N$. Thus $P(t)>0$, $Z(t)>0$ for $t\geq 0$. Now we consider the uniformly bounded, from the first equation of (1.5), $\dot{P}\leq rP(1-\frac{P}{K})$, then we obtain $\limsup\limits_{t\to\infty}P(t)\leq K$, so for sufficiently small $\varepsilon>0$, there exists a $T_1>0$ sufficiently large, such that $P(t)<K+\varepsilon$ for all $t\geq T_1$. For the boundedness of $Z(t)$, we define $W(t)=P(t-\tau)+\frac{\mu}{\beta}Z(t)$ for $t\geq 0$. Then

$$
\begin{aligned}
\dot{W} &= \frac{\mathrm{d}P(t-\tau)}{\mathrm{d}t} + \frac{\mu}{\beta}\frac{\mathrm{d}Z}{\mathrm{d}t} \\
&= rP(t-\tau)\Big(1 - \frac{P(t-\tau)}{K}\Big) - \frac{\rho\mu}{\beta}\frac{PZ}{\alpha+P} - \frac{\mu d}{\beta}Z - \frac{\mu g}{\beta}Z^2 \\
&\le -d\Big(P(t-\tau) + \frac{\mu}{\beta}Z\Big) + P(t-\tau)\Big(d + r - \frac{rP(t-\tau)}{K}\Big) \\
&= -dW(t) + P(t-\tau)\Big(d + r - \frac{rP(t-\tau)}{K}\Big) \\
&= -dW + \frac{K}{4r}(d+r)^2,
\end{aligned}
$$

so by the comparison theory [14], we obtain $W(t) \le W(0) + \frac{K(d+r)^2}{4dr}$.

Thus we complete the proof.

Lemma 3.3 *If assumption* (H_1) *and the condition of Lemma* 2.1 *hold, system* (1.5) *has no nontrivial periodic solution with period* τ.

Proof For (1.5), suppose there is a nontrivial period solution with period τ. Then for the following system, there exists has a nontrivial periodic solution

$$
\begin{cases}
\dfrac{\mathrm{d}P}{\mathrm{d}t} = rP\Big(1 - \dfrac{P}{K}\Big) - \dfrac{\mu PZ}{\alpha + P}, \\
\dfrac{\mathrm{d}Z}{\mathrm{d}t} = \dfrac{\beta P(t)Z(t)}{\alpha + P(t)} - \dfrac{\rho P(t)Z(t)}{\alpha + P(t)} - dZ - gZ^2.
\end{cases}
\tag{3.3}
$$

We know that P-axis and Z-axis are invariant manifold of system (1.5), and there is no orbit cross the coordinate axis for the orbit do not intersect each other, so all the orbit must in the first quadrant, and the equilibrium E^* should in the interior of the periodic orbit. But from Lemma 2.1 and (H_1), the equilibrium E^* is unique and stable, so it is globally stable in the first quadrant. The periodic orbit does not exist. Thus there is no period orbit in the first quadrant.

Theorem 3.1 *If* (H_1) *and Lemma* 2.1 *hold, for each* $\tau > \tau_j$, $j = 1, 2, \cdots$, *system* (1.5) *has at least* $j+1$ *period solutions.*

Proof It is sufficient to prove the connected component $l(u^*, \tau_j, \frac{2\pi}{\omega})$ onto τ space is $[\overline{\tau}, +\infty)$, where $\overline{\tau} \le \tau_j$, $j = 0, 1, 2, \cdots$. From Lemma 3.3, we know system (3.3) have not nontrivial period solution, so the projection of $l(u^*, \tau_j, \frac{2\pi}{\omega})$ onto τ space is always from zero. From expression (2.5), we obtain $\frac{2\pi}{\omega} < \tau_j$ for $j > 0$. Suppose the projection of $l(u^*, \tau_j, \frac{2\pi}{\omega})$ onto τ space is unbounded, there exist $\tau^* > 0$, so the projection of $l(u^*, \tau_j, \frac{2\pi}{\omega})$ onto τ space is in the interval $(0, \tau^*)$, from $\frac{2\pi}{\omega} < \tau_j$, $j \ge 1$ and Lemma 3.3, $0 < T < \tau^*$ for $(u(t), \tau, T) \in l(u^*, \tau_j, \frac{2\pi}{\omega})$, which mean the projection of $l(u^*, \tau_j, \frac{2\pi}{\omega})$ onto T space is also bounded, by Lemma 3.2 we obtain that the connected component $l(u^*, \tau_j, \frac{2\pi}{\omega})$ is bounded, which contradict Lemma 3.1, so the proof is completed.

4 Numerical Simulations

Now we give simulations of (1.5), except taking $g = 0.0008$. Take the same parameter as the first and second equations in [1], system (1.5) takes:

$$\begin{cases} \dfrac{\mathrm{d}P}{\mathrm{d}t} = 5P\Big(1 - \dfrac{P}{4}\Big) - \dfrac{0.6PZ}{2+P}, \\ \dfrac{\mathrm{d}Z}{\mathrm{d}t} = \dfrac{0.43P(t-\tau)Z(t-\tau)}{2+P(t-\tau)} - \dfrac{0.1PZ}{2+P} - 0.11Z - 0.0008Z^2. \end{cases} \tag{4.1}$$

By computation, conditions of Theorem 2.1 are satisfied, then there exists a unique equilibrium point $(1.218265, 18.650750)$, such that (H_1) and (H_2) hold. We have the critical value $\tau_0 = 0.41226$, so when $\tau \in (0, \tau_0)$, E^* is asymptotically stable, and a periodic orbit turns up when τ passes through the critical value, which is shown in Figures 1 and 2, respectively.

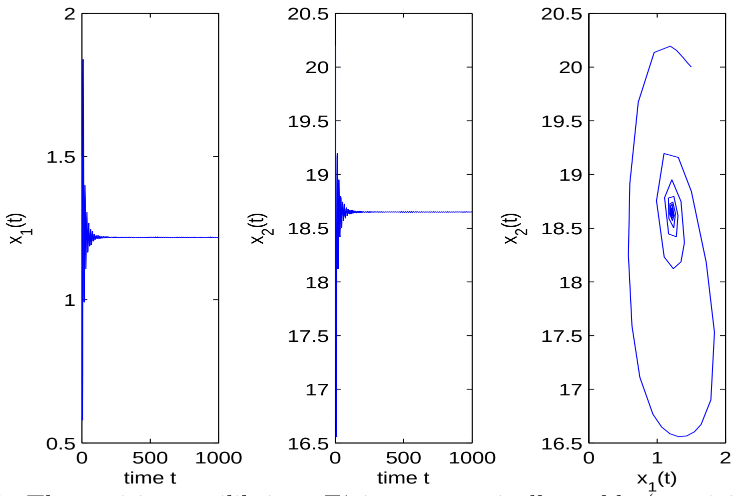


Figure 1: The positive equilibrium E^* is asymptotically stable ($\tau = 0.2637 < \tau_0$)

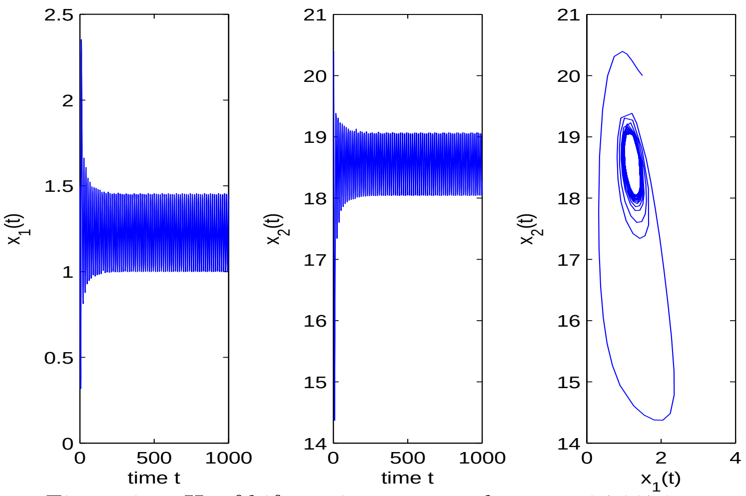
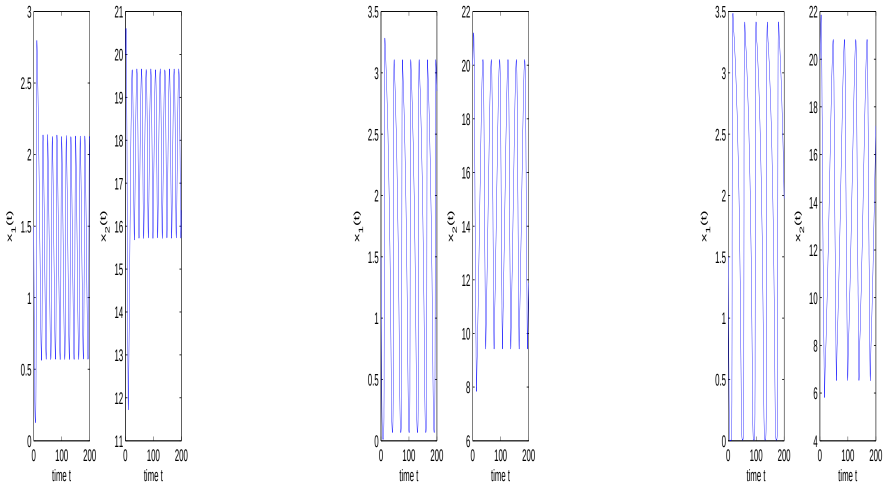


Figure 2: a Hopf bifurcation occurs when $\tau = 0.73376 > \tau_0$

By computation, we have $C_1(0) = 7.435905197 * 10^{-3} - 1.438814764 * 10^{-2}i$, $\mu_2 = -6.47369616 * 10^{-2}$, $\beta_2 = 1.487181039 * 10^{-2}$ and $T_2 = 3.770080024 * 10^{-2}$. From Theorem 2.2 we get the Hopf bifurcation which is subcritical ($\mu_2 < 0$) at τ_0, and the periodic orbit bifurcated from equilibrium point is unstable ($\beta_2 < 0$).

We know system (1.5) has a periodic solution for large τ from global Hopf bifurcation. Figure 3 shows the period and the amplitude of the period solution for different value of τ. We conclude that as τ increases from 1.2, 2.2 to 3.1, the period and the amplitude of period solution increase.



(a) (b) (c)

Figure 3: period and amplitudes of periodic solutions ($\tau = 1.2, 2.2, 3.1$, respectively)

5 Proof of Theorem 2.2

Now we take the similar method in [10] to compute the explicit formula about μ_2, β_2 and T_2. We assume system (1.5) undergoes Hopf bifurcation at $E^*(P^*, Z^*)$ when $\tau = \tau_j$.

Letting $\overline{u}(t) = u(\tau t)$, $\overline{v}(t) = v(\tau t)$, dropping the bars for simplification, system (2.1) becomes FDE in $C = C([-1, 0], R^2)$ as

$$\dot{x}(t) = L_\mu(x_t) + f(\mu, x_t), \tag{5.1}$$

where $x(t) = (u(t), v(t))^\top \in R^2$, and $L_\mu : C \to R$ and f are given respectively by

$$L_\mu(\phi) = (\tau_j + \mu)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + (\tau_j + \mu)\begin{pmatrix} 0 & 0 \\ a_{23} & a_{24} \end{pmatrix}\begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \end{pmatrix} \tag{5.2}$$

and

$$f(\tau, \phi) = (\tau_j + \mu)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{5.3}$$

where

$$f_1 = g_1\phi_1^2(0) + g_2\phi_1(0)\phi_2(0) + g_3\phi_1^3(0) + g_4\phi_1^2(0)\phi_2(0),$$
$$f_2 = h_1\phi_1^2(0) + h_2\phi_2^2(0) + h_3\phi_1(0)\phi_2(0) + h_4\phi_1(-1) + h_5\phi_1(-1)\phi_2(-1)$$
$$+h_6\phi_1^3(0) + h_7\phi_1^3(-1) + h_8\phi_1^2(0)\phi_2(0) + h_9\phi_1^2(-1)\phi_2(-1).$$
$$g_1 = -\frac{r}{K} + \frac{\mu Z^*\alpha}{(\alpha+P^*)^3}, \quad g_2 = -\frac{\mu\alpha}{(\alpha+P^*)^2}, \quad g_3 = -\frac{\mu\alpha Z^*}{(\alpha+P^*)^4}, \quad g_4 = \frac{\mu\alpha}{(\alpha+P^*)^3},$$
$$h_1 = \frac{\rho\alpha Z^*}{(\alpha+P^*)^3}, \quad h_2 = -g, \quad h_3 = -\frac{\rho\alpha}{(\alpha+P^*)^2}, \quad h_4 = -\frac{\beta\alpha Z^*}{(\alpha+P^*)^3}, \quad h_5 = \frac{\beta\alpha}{(\alpha+P^*)^2},$$
$$h_6 = -\frac{\rho\alpha Z^*}{(\alpha+P^*)^4}, \quad h_7 = \frac{\beta\alpha Z^*}{(\alpha+P^*)^3}, \quad h_8 = \frac{\rho\alpha}{(\alpha+P^*)^3}, \quad h_9 = -\frac{\beta\alpha}{(\alpha+P^*)^3}.$$

By the Riesz representation theorem, there exists a function $\eta(\theta,\mu)$ of bounded variation for $\theta \in [-1,0]$ such that

$$L_\mu = \int_{-1}^{0} \mathrm{d}\eta(\theta,\mu)\phi(\theta), \tag{5.4}$$

for $\theta \in C$. In fact, we can choose

$$\eta(\theta,\mu) = (\tau_j+\mu)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\delta(\theta) + (\tau_j+\mu)\begin{pmatrix} 0 & 0 \\ a_{23} & a_{24} \end{pmatrix}\delta(\theta+1), \tag{5.5}$$

where δ is the dirac function.

For $\phi \in C^1([-1,0],R^2)$, define

$$A(\mu)\phi = \begin{cases} \dfrac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & \theta \in [-1,0), \\ \displaystyle\int_{-1}^{0} \mathrm{d}\eta(\mu,s)\phi(s), & \theta = 0 \end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-1,0), \\ f(\mu,\phi), & \theta = 0, \end{cases}$$

then system (5.1) becomes

$$\dot{x_t} = A(\mu) + R(\mu)x_t. \tag{5.6}$$

For $\theta \in [-1,0]$, for $\psi \in C^1([0,1],(R^2)^*)$, define

$$A^*\psi(s) = \begin{cases} -\dfrac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & s \in (0,1], \\ \displaystyle\int_{-1}^{0} \mathrm{d}\eta^\top(t,0)\psi(-t), & s = 0 \end{cases}$$

and a bilinear inner product

$$\langle\psi(s),\phi(\theta)\rangle=\overline{\psi}(0)\phi(0)-\int_{-1}^{0}\int_{\xi=0}^{\theta}\overline{\psi}(\xi-\theta)\mathrm{d}\eta(\theta)\phi(\xi)\mathrm{d}\xi, \tag{5.7}$$

where $\eta(\theta)=\eta(\theta,0)$, then $A(0)$ and A^* are adjoint operators. By the discussion of Section 2, we know $\pm i\omega\tau_j$ are eigenvalues of $A(0)$, thus they are also eigenvalues of A^*, so we computer the eigenvector of $A(0)$ and A^* corresponding to $i\omega\tau_j$ and $-i\omega\tau_j$ respectively.

Suppose $q(\theta)=(1,q)^\top\mathrm{e}^{i\theta\omega\tau_j}$ is the eigenvector of $A(0)$ corresponding to $i\omega\tau_j$, then $A(0)q(\theta)=i\tau_j\omega q(\theta)$. From the definition of A, (5.4) and (5.5), we have

$$\tau_j\begin{pmatrix} i\omega+\dfrac{rP^*}{K}-\dfrac{\mu P^*Z^*}{(\alpha+P^*)^2} & \dfrac{\mu P^*}{\alpha+P^*} \\ \dfrac{\rho\alpha Z^*}{(\alpha+P^*)^2}-\Big(\dfrac{\beta Z^*}{\alpha+P^*}-\dfrac{\rho P^*Z^*}{(\alpha+P^*)^2}\Big)\mathrm{e}^{-i\omega\tau_j} & i\omega+\dfrac{\rho P^*}{\alpha+P^*}+d+2gZ^*-\dfrac{\beta P^*}{\alpha+P^*}\mathrm{e}^{-i\omega\tau_j}\end{pmatrix}\begin{pmatrix}1\\ q\end{pmatrix}$$
$$=\begin{pmatrix}0\\0\end{pmatrix},$$

then we obtain

$$q(0)=(1,q)^\top=\Big(1,\frac{Z^*}{\alpha+P^*}-\frac{r(\alpha+P^*)}{\mu K}-\frac{i\omega(\alpha+P^*)}{\mu P^*}\Big)^\top.$$

On the other hand, suppose $q(s)=D(1,q^*)\mathrm{e}^{is\omega\tau_j}$, by the definition of A^*, (5.4) and (5.5), we get

$$\tau_j\begin{pmatrix} -i\omega+\dfrac{rP^*}{K}-\dfrac{\mu P^*Z^*}{(\alpha+P^*)^2} & \dfrac{\rho\alpha Z^*}{(\alpha+P^*)^2}-\Big(\dfrac{\beta Z^*}{\alpha+P^*}-\dfrac{\rho P^*Z^*}{(\alpha+P)^2}\Big)\mathrm{e}^{i\omega\tau_j} \\ \dfrac{\mu P^*}{\alpha+P^*} & -i\omega+\dfrac{\rho P^*}{\alpha+P^*}+d+2gZ^*-\dfrac{\beta P^*}{\alpha+P^*}\mathrm{e}^{i\omega\tau_j}\end{pmatrix}\begin{pmatrix}1\\ q^*\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

Thus we obtain

$$q^*(0)=D(1,q^*)=D\Big(1,\frac{\mu P^*Z^*+(\alpha+P^*)^2(-\frac{rP^*}{K}+i\omega)}{Z^*(\rho\alpha-(\beta(\alpha+P^*)-\rho P^*)\mathrm{e}^{i\omega\tau_j})}\Big)^\top.$$

In order to assure $\langle q^*(s),q(\theta)\rangle=1$, we computer value D, from (5.7)

$$\begin{aligned}\langle q^*(s),q(\theta)\rangle&=\overline{D}(1,\overline{q^*})(1,q)^\top-\int_{-1}^{0}\int_{\xi=0}^{\theta}\overline{D}(1,\overline{q^*})\mathrm{e}^{-i(\xi-\theta)\omega\tau_j}\mathrm{d}\eta(\theta)(1,q)^\top\mathrm{e}^{i\xi\omega\tau_j}\mathrm{d}\xi\\&=\overline{D}\left(1+\overline{q^*}q+\tau_j\mathrm{e}^{-i\omega\tau_j}\Big(\frac{\beta(P^*+Z^*)}{\alpha+P^*}-\frac{\rho P^*Z^*}{(\alpha+P^*)^2}\Big)\overline{q^*}q\right),\end{aligned}$$

thus we get

$$D=\frac{1}{1+\overline{q}q^*+\tau_j\mathrm{e}^{i\omega\tau_j}\big(\frac{\beta(P^*+Z^*)}{\alpha+P^*}-\frac{\rho P^*Z^*}{(\alpha+P^*)^2}\big)q^*\overline{q}}.$$

We computer the center manifold C_0 at $\mu=0$. Let x_t be the solution of (5.1) defined by

$$z(t)=\langle q^*,x_t\rangle,\quad W(t,\theta)=x_t(\theta)-2\mathrm{Re}(z(t)q(\theta)).\tag{5.8}$$

On the center manifold C_0, we have

$$W(t,\theta)=W(z,\overline{z},\theta)=W_{20}\frac{z^2}{2}+W_{11}z\overline{z}+W_{02}\frac{\overline{z^2}}{2}+\cdots,$$

where z and $\overline{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\overline{q^*}$ respectively. Note $W(t,\theta)$ is real if x_t is real, then we only consider the real solution. For the solution $x_t\in C_0$ of (5.1), since $\mu=0$,

$$\begin{aligned}\dot z(t)&=i\omega\tau_jz+\langle\overline{q^*}(\theta),f(0,W(z,\overline{z},\theta)+2\mathrm{Re}(zq(\theta)))\rangle\\&=i\omega\tau_jz+\overline{q^*}(0)f(0,W(z,\overline{z},\theta)+2\mathrm{Re}(zq(0)))\\&\equiv i\omega\tau_jz+\overline{q^*}(0)f_0(z,\overline{z}).\end{aligned}$$

We write this equation as

$$\dot z(t)=i\omega\tau_jz(t)+g(z,\overline{z}),$$

where

$$g(z,\overline{z})=\overline{q^*}(0)f_0(z,\overline{z})=g_{20}\frac{z^2}{2}+g_{11}z\overline{z}+g_{02}\frac{\overline{z^2}}{2}+g_{21}\frac{z^2\overline{z}}{2!}+\cdots.\tag{5.9}$$

Since $x_t(\theta)=(x_{1t}(\theta),x_{2t}(\theta))=W(t,\theta)+zq(\theta)+\overline{zq}(\theta)$, $q(\theta)=(1,q)^\top e^{i\theta\omega\tau_j}$, we have

$$\begin{aligned}x_{1t}(0)&=z+\overline{z}+W_{20}^{(1)}(0)\frac{z^2}{2}+W_{11}^{(1)}(0)z\overline{z}+W_{02}^{(1)}(0)\frac{\overline{z}^2}{2}+O(|(z,\overline{z})|^3),\\x_{2t}(0)&=qz+\overline{qz}+W_{20}^{(2)}(0)\frac{z^2}{2}+W_{11}^{(2)}(0)z\overline{z}+W_{02}^{(2)}(0)\frac{\overline{z}^2}{2}+O(|(z,\overline{z})|^3),\\x_{1t}(-1)&=ze^{-i\omega\tau_j}+\overline{z}e^{i\omega\tau_j}+W_{20}^{(1)}(-1)\frac{z^2}{2}+W_{11}^{(1)}(-1)z\overline{z}\\&\quad+W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2}+O(|(z,\overline{z})|^3),\\x_{2t}(-1)&=zqe^{-i\omega\tau_j}+\overline{zq}e^{i\omega\tau_j}+W_{20}^{(2)}(-1)\frac{z^2}{2}+W_{11}^{(2)}(-1)z\overline{z}\\&\quad+W_{02}^{(2)}(-1)\frac{\overline{z}^2}{2}+O(|(z,\overline{z})|^3).\end{aligned}$$

From (5.9), we have

$$g(z,\overline{z})=\overline{q}^*(0)f_0(z,\overline{z})=\tau_j\overline{D}(1,\overline{q}^*)\begin{pmatrix}f_1\\f_1\end{pmatrix}$$

$$
\begin{aligned}
=&\tau_j\overline{D}\Big[g_1\Big(z+\overline{z}+W_{20}^{(1)}(0)\frac{z^2}{2}+W_{11}^{(1)}(0)z\overline{z}+W_{02}^{(1)}(0)\frac{\overline{z}^2}{2}+O(|(z,\overline{z})|^3)\Big)^2\\
&+g_2\Big(z+\overline{z}+W_{20}^{(1)}(0)\frac{z^2}{2}+W_{11}^{(1)}(0)z\overline{z}+W_{02}^{(1)}(0)\frac{\overline{z}^2}{2}+O(|(z,\overline{z})|^3)\Big)\\
&\cdot\Big(qz+\overline{qz}+W_{20}^{(2)}(0)\frac{z^2}{2}+W_{11}^{(2)}(0)z\overline{z}+W_{02}^{(2)}(0)\frac{\overline{z}^2}{2}+O(|(z,\overline{z})|^3)\Big)\\
&+g_3\Big(z+\overline{z}+W_{20}^{(1)}(0)\frac{z^2}{2}+W_{11}^{(1)}(0)z\overline{z}+W_{02}^{(1)}(0)\frac{\overline{z}^2}{2}+O(|(z,\overline{z})|^3)\Big)^3\\
&+g_4\Big(z+\overline{z}+W_{20}^{(1)}(0)\frac{z^2}{2}+W_{11}^{(1)}(0)z\overline{z}+W_{02}^{(1)}(0)\frac{\overline{z}^2}{2}+O(|(z,\overline{z})|^3)\Big)^2\\
&\cdot\Big(qz+\overline{qz}+W_{20}^{(2)}(0)\frac{z^2}{2}+W_{11}^{(2)}(0)z\overline{z}+W_{02}^{(2)}(0)\frac{\overline{z}^2}{2}+O(|(z,\overline{z})|^3)\Big)\Big]\\
&+\tau_j\overline{D}\overline{q}^*\Big[h_1\Big(z+\overline{z}+W_{20}^{(1)}(0)+\frac{z^2}{2}+W_{11}^{(1)}z\overline{z}+W_{02}^{(1)}(0)\frac{\overline{z}^2}{2}+O(|(z,\overline{z})|^3)\Big)^2\\
&+h_2\Big(qz+\overline{qz}+W_{20}^{(2)}(0)\frac{z^2}{2}+W_{11}^{(2)}(0)z\overline{z}+W_{02}^{(2)}(0)\frac{\overline{z}^2}{2}+O(|(z,\overline{z})|^3)\Big)^2\\
&+h_3\Big(z+\overline{z}+W_{20}^{(1)}(0)\frac{z^2}{2}+W_{11}^{(1)}(0)z\overline{z}+W_{02}^{(1)}(0)\frac{\overline{z}^2}{0}+O(|(z,\overline{z})|^3)\Big)\\
&\cdot\Big(qz+\overline{qz}+W_{20}^{(2)}(0)\frac{z^2}{2}+W_{11}^{(2)}(0)z\overline{z}+W_{02}^{(2)}(0)\frac{\overline{z}^2}{2}+O(|(z,\overline{z})|^3)\Big)\\
&+h_4\Big(z\mathrm{e}^{-i\omega\tau_j}+\overline{z}\mathrm{e}^{i\omega\tau_j}+W_{20}^{(1)}(-1)\frac{z^2}{2}+W_{11}^{(1)}(-1)z\overline{z}+W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2}+O(|(z,\overline{z})|^3)\Big)^2\\
&+h_5\Big(z\mathrm{e}^{-i\omega\tau_j}+\overline{z}\mathrm{e}^{i\omega\tau_j}+W_{20}^{(1)}(-1)\frac{z^2}{2}+W_{11}^{(1)}(-1)z\overline{z}+W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2}+O(|(z,\overline{z})|^3)\Big)\\
&\cdot\Big(zq\mathrm{e}^{-i\omega\tau_j}+\overline{zq}\mathrm{e}^{i\omega\tau_j}+W_{20}^{(2)}(-1)\frac{z^2}{2}+W_{11}^{(2)}(-1)z\overline{z}+W_{02}^{(2)}(-1)\frac{\overline{z}^2}{2}+O(|(z,\overline{z})|^3)\Big)\\
&+h_6\Big(z+\overline{z}+W_{20}^{(1)}(0)\frac{z^2}{2}+W_{11}^{(1)}(0)z\overline{z}+W_{02}^{(1)}(0)\frac{\overline{z}^2}{2}+O(|(z,\overline{z})|^3)\Big)^3\\
&+h_7\Big(z\mathrm{e}^{-i\omega\tau_j}+\overline{z}\mathrm{e}^{i\omega\tau_j}+W_{20}^{(1)}(-1)\frac{z^2}{2}+W_{11}^{(1)}(-1)z\overline{z}+W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2}+O(|(z,\overline{z})|^3)\Big)^3\\
&+h_8\Big(z+\overline{z}+W_{20}^{(1)}(0)\frac{z^2}{2}+W_{11}^{(1)}(0)z\overline{z}+W_{02}^{(1)}(0)\frac{\overline{z}^2}{2}+O(|(z,\overline{z})|^3)\Big)^2\\
&\cdot\Big(qz+\overline{qz}+W_{20}^{(2)}(0)\frac{z^2}{2}+W_{11}^{(2)}(0)z\overline{z}+W_{02}^{(2)}(0)\frac{\overline{z}^2}{2}+O(|(z,\overline{z})|^3)\Big)\\
&+h_9\Big(z\mathrm{e}^{-i\omega\tau_j}+\overline{z}\mathrm{e}^{i\omega\tau_j}+W_{20}^{(1)}(-1)\frac{z^2}{2}+W_{11}^{(1)}(-1)z\overline{z}+W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2}+O(|(z,\overline{z})|^3)\Big)^2\\
&\cdot\Big(zq\mathrm{e}^{-i\omega\tau_j}+\overline{zq}\mathrm{e}^{i\omega\tau_j}+W_{20}^{(2)}(-1)\frac{z^2}{2}+W_{11}^{(2)}(-1)z\overline{z}+W_{02}^{(2)}(-1)\frac{\overline{z}^2}{2}+O(|(z,\overline{z})|^3)\Big)\Big].
\end{aligned}
$$

Comparing the coefficients with (5.9), we get

$$
\begin{aligned}
g_{20} &= 2\tau_j\overline{D}(g_1+g_2q)+2\tau_j\overline{q}^*\overline{D}(h_1+h_2q^2+h_3q+(h_4+h_5q)\mathrm{e}^{-2i\omega\tau_j}),\\
g_{11} &= \tau_j\overline{D}(2g_1+g_2(\overline{q}+q))+\tau_j\overline{q}^*\overline{D}(2(h_1+h_4)+2h_2q\overline{q}+(h_3+h_5)(q+\overline{q})),\\
g_{02} &= 2\tau_j\overline{D}(g_1+g_2\overline{q})+2\tau_j\overline{q}^*\overline{D}(h_2\overline{q}^2+h_3\overline{q}+h_4\mathrm{e}^{2i\omega\tau_j}+h_5\overline{q}\mathrm{e}^{2i\omega\tau_j}),\\
g_{21} &= 2\tau_j\overline{D}\Big[g_1(W_{20}^{(1)}(0)+2W_{11}^{(1)}(0))+g_2\Big(\frac{1}{2}\overline{q}W_{20}^{(1)}(0)+\frac{1}{2}W_{20}^{(2)}(0)\\
&\quad+qW_{11}^{(1)}(0)+W_{11}^{(2)}(0)\Big)+3g_3+g_4(2q+\overline{q})\Big]\\
&\quad+2\tau_j\overline{q}^*\overline{D}\Big[h_1(W_{20}^{(1)}(0)+2W_{11}^{(1)}(0))+h_2(\overline{q}W_{20}^{(2)}(0)+2qW_{11}^{(2)}(0))\\
&\quad+h_3\Big(\frac{1}{2}\overline{q}W_{20}^{(1)}(0)+qW_{11}^{(1)}(0)+\frac{1}{2}W_{20}^{(2)}(0)+W_{11}^{(2)}(0)\Big)\\
&\quad+h_4(W_{20}^{(1)}(-1)\mathrm{e}^{i\omega\tau_j}+2W_{11}^{(1)}(-1)\mathrm{e}^{-i\omega\tau_j})\\
&\quad+h_5\Big(\Big(\frac{1}{2}\overline{q}W_{20}^{(1)}(-1)+\frac{1}{2}W_{20}^{(2)}(-1)\Big)\mathrm{e}^{i\omega\tau_j}+(qW_{11}^{(1)}(-1)+W_{11}^{(2)}(-1))\mathrm{e}^{-i\omega\tau_j}\Big)\\
&\quad+3h_6+3h_7\mathrm{e}^{-i\omega\tau_j}+h_8(2q+\overline{q})+h_9((2q+\overline{q})\mathrm{e}^{-i\omega\tau_j})\Big].
\end{aligned}\tag{5.10}
$$

To computer $W_{20}(\theta)$ and $W_{11}(\theta)$ in g_{21}, from (5.6) and (5.8), we have

$$
\begin{aligned}
\dot{W} &= \dot{x}_t-\dot{z}q-\dot{\overline{z}}\overline{q}\\
&= \begin{cases} AW-2\mathrm{Re}\overline{q}^*(0)f_0q(\theta), & \theta\in[-1,0],\\ AW-2\mathrm{Re}\overline{q}^*(0)f_0q(0)+f_0, & \theta=0\end{cases}\\
&\equiv AW+H(z,\overline{z},\theta),
\end{aligned}\tag{5.11}
$$

where

$$
H(z,\overline{z},\theta)=H_{20}(\theta)\frac{z^2}{2}+H_{11}(\theta)z\overline{z}+H_{02}\frac{\overline{z}^2}{2}+\cdots. \tag{5.12}
$$

Expanding the above series and comparing the corresponding coefficients, we obtain

$$
(A-2i\omega\tau_j)W_{20}(\theta)=-H_{20}(\theta),\quad AW_{11}(\theta)=-H_{11}(\theta),\ \cdots. \tag{5.13}
$$

From (5.11), we know for $\theta\in[-1,0)$,

$$
H(z,\overline{z},\theta)=-\overline{q}^*(0)f_0q(\theta)-q^*(0)\overline{f_0}\overline{q}(\theta)=-gq(\theta)-\overline{gq}(\theta). \tag{5.14}
$$

Comparing the coefficients with (5.12), we get

$$
H_{20}=-g_{20}q(\theta)-\overline{g}_{02}\overline{q}(\theta), \tag{5.15}
$$

$$
H_{11}=-g_{11}q(\theta)-\overline{g}_{11}\overline{q}(\theta). \tag{5.16}
$$

From (5.13), (5.15) and the definition of A, we get

$$
\dot{W}_{20}(\theta)=2i\tau_j\omega W_{20}(\theta)+g_{20}q\theta+\overline{g}_{02}\overline{q}(\theta).
$$

Note that $q(\theta)=(1,q)^\top \mathrm{e}^{i\theta\omega\tau_j}$, hence

$$W_{20}(\theta)=\frac{ig_{20}}{\omega\tau_j}q(0)\mathrm{e}^{i\theta\omega\tau_j}+\frac{i\overline{g}_{02}}{3\omega\tau_j}\overline{q}(0)\mathrm{e}^{-i\theta\omega\tau_j}+E_1\mathrm{e}^{2i\theta\omega\tau_j}, \tag{5.17}$$

where $E_1=(E_1^{(1)},E_1^{(2)})\in R^2$ is a constant vector, similarly from (5.13) and (5.16), we get

$$W_{11}(\theta)=-\frac{ig_{11}}{\omega\tau_j}q(0)\mathrm{e}^{i\theta\omega\tau_j}+\frac{i\overline{g}_{11}}{\omega\tau_j}\overline{q}(0)\mathrm{e}^{-i\theta\omega\tau_j}+E_2, \tag{5.18}$$

where $E_2=(E_2^{(1)},E_2^{(2)})\in R^2$ is a constant vector. In the following, we seek appropriate E_1,E_2, from the definition A and (5.13), we obtain

$$\int_{-1}^{0}\mathrm{d}\eta(\theta)W_{20}(\theta)=2i\tau_j\omega W_{20}(0)-H_{20}(0) \tag{5.19}$$

and

$$\int_{-1}^{0}\mathrm{d}\eta(\theta)W_{11}(\theta)=-H_{11}(0), \tag{5.20}$$

where $\eta(\theta)=\eta(\theta,0)$. From (5.11), we have

$$H_{20}=-g_{20}q(0)-\overline{g}_{02}\overline{q}(0)+2\tau_j\begin{pmatrix} g_1+g_2q \\ h_1+h_2q^2+h_3q+(h_4+h_5q)\mathrm{e}^{-2i\omega\tau_j}\end{pmatrix} \tag{5.21}$$

and

$$H_{11}(0)=-g_{11}q(0)-\overline{g}_{11}\overline{q}(0)+2\tau_j\begin{pmatrix} 2g_1+g_2(q+\overline{q}) \\ 2h_1+2h_2q\overline{q}+h_3(q+\overline{q})+2h_4+h_5(q+\overline{q})\end{pmatrix}. \tag{5.22}$$

Substituting (5.17) and (5.21) into (5.19) and noting

$$\Big(i\tau_j\omega I-\int_{-1}^{0}\mathrm{e}^{i\theta\omega\tau_j}\mathrm{d}\eta(\theta)\Big)q(0)=0$$

and

$$\Big(-i\tau_j\omega I-\int_{-1}^{0}\mathrm{e}^{-i\theta\omega\tau_j}\mathrm{d}\eta(\theta)\Big)\overline{q}(0)=0,$$

we obtain

$$\Big(2i\tau_j\omega I-\int_{-1}^{0}\mathrm{e}^{2i\theta\omega\tau_j}\mathrm{d}\eta(\theta)\Big)E_1=2\tau_j\begin{pmatrix} g_1+g_2q \\ h_1+h_2q^2+h_3q+(h_4+h_5q)\mathrm{e}^{-2i\omega\tau_j}\end{pmatrix},$$

which leads to

$$\begin{pmatrix} 2i\omega - a_{11} & -a_{12} \\ -a_{21} - a_{23}\mathrm{e}^{-2i\omega\tau_j} & 2i\omega - a_{22} - a_{24}\mathrm{e}^{-2i\omega\tau_j} \end{pmatrix} E_1 = 2\begin{pmatrix} g_1 + g_2 q \\ h_1 + h_2 q^2 + h_3 q + (h_4 + h_5 q)\mathrm{e}^{-2i\omega\tau_j} \end{pmatrix}. \tag{5.23}$$

Similarly, substisting (5.18) and (5.22) into (5.20), we get

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} + a_{23} & a_{22} + a_{24} \end{pmatrix} E_2 = -\begin{pmatrix} 2g_1 + g_2(q + \overline{q}) \\ 2h_1 + 2h_2 q\overline{q} + h_3(q + \overline{q}) + 2h_4 + h_5(q + \overline{q}) \end{pmatrix}. \tag{5.24}$$

From (5.23) and (5.24), we could obtain the expression of E_1 and E_2. Then we calculate the following values according to the above analysis and the expression of g_{20}, g_{11}, g_{02} and g_{21}:

$$\begin{aligned} C_1(0) &= \frac{i}{2\tau_j\omega}\Big(g_{11}g_{20} - 2|\, g_{11}\,|^2 - \frac{|\, g_{02}\,|^2}{3}\Big) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\mathrm{Re}(C_1(0))}{\mathrm{Re}(\lambda'(\tau_j))}, \quad \beta_2 = 2\mathrm{Re}(C_1(0)), \\ T_2 &= -\frac{\mathrm{Im}C_1(0) + \mu_2 \mathrm{Im}\lambda'(\tau_j)}{\tau_j\omega}. \end{aligned}$$

References

[1] Y. Lv, J. Cao, J. Song, R. Yuan and P. Yu, Global stability and Hopf-bifurcation in a zooplankton-phytoplankton model, *Nonlinear. Dyn.*, **76**(2014),345-366.

[2] Z. Jiang, W. Ma and D. Li, Dynamical behavior of a delay differential equation system on toxic producing phytoplankton and zooplankton interaction, *Japan J. Indust. Appl. Math.*, **31**(2014),583-609.

[3] Y. Wang, W. Jiang and H. Wang, Stability and global Hopf bifurcation in toxic phytoplankton-zooplankton model with delay and selective harvesting, *Nonlinear. Dyn.*, **73**(2013),881-896.

[4] J. Zhao, J. Wei, Stability and bifurcation in a two harmful phytoplankton-zooplankton system, *Chao. Solitons. Fractals*, **39**(2009),1395-1409.

[5] T. Saha, M. Bandyopadhyay, Dynamical analysis of toxic producing Phytoplankton-Zooplankton interactions, *Nonlinear Anal. Real. World. Appl.*, **10**(2009),314-332.

[6] S. Ruan, J. Wei, On the zeros of transcendental functions with applicaions to stability of delay differential equations with two delays, *Dyna. Conti. Discr. Impulsive. Systems*, **10**(2003),863-874.

[7] S. Gakkhar, A. Singh, Effects of delay and seasonality on toxin producing phytoplankton-zooplankton system, *Inter. Jour. Biomath.*, (2012), DOI: 10.1142/ S1793524511001891.

[8] X. Meng, H. Huo and X. Zhang, Stability and global hopf bifurcation in a delayed food web consisting of a prey and two predators, *Commun. Nonlin. Sci. Numer. Simulat.*, **16**(2011),4335-4348.

[9] J. Xu, Y. Zhou, Hopf bifurcation and its stability for a vector-borne disease model with delay and reinfection, *Appl. Mathe. Modell.*, **40**(2016),1685-1702.

[10] Y. Song, S. Yan, Bifurcation analysis in a predator-prey system with time delay, *Nonlinear Anal. Real. World. Appl.*, **7**(2006),265-284.

[11] J. Wu, Symmetric functional differential equations and neural networks with memory, *Trans. Amer. Math. Soc.*, **350**(1998),4799-4838.

[12] J. Hale, S. Lunel, Introduction to Functional Differential Equation, Springer, New York, 1993.

[13] B. Hassard, N. Kazarinoff and Y. Wan, Theory and Application of Hopf Bifurcation, Cambridge: Cambridge University Press, 1981.

[14] V. Lakshmikantham, S. Leela, Differential and Integral Inequalities (Theory and Application): Ordinary Differential Equation I, Academic Press, New York, 1969.

(*edited by Mengxin He*)