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PERSISTENCE AND EXTINCTION OF A STOCHASTIC SIS EPIDEMIC MODEL WITH DOUBLE EPIDEMIC HYPOTHESIS*

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Abstract

In this paper, we aim at dynamical behaviors of a stochastic SIS epidemic model with double epidemic hypothesis. Sufficient conditions for the extinction and persistence in mean are derived via constructing suitable functions. We obtain a threshold of stochastic SIS epidemic model, which determines how the diseases spread when the white noises are small. Numerical simulations are used to illustrate the efficiency of the main results of this article.

Keywords double epidemic hypothesis; Brownian motion; extinction; persistence

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1 Introduction

Epidemiology is the science of studying the spread of infectious diseases, which is to investigate and to trace the dynamics and stabilities of infectious diseases. The modified models and recent contributions always assume that the population is separated by three subclasses: the susceptible, infective and the recovered, denoting them as S, I and R respectively.

The classical SIS model turns into SIR model or SIRS model when the recovered individuals are taken into account. Related research and modified models can be found in [1-5]. When the exposed individuals are considered into population level and participate into the spread process of disease, the classical SIS model becomes a new version, often mentioned as SEIR model or SEIRS model if the recovered individuals return into the susceptible again, for instance, see the recent literatures [6-9].

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Meng *et al.* [10] discussed an SIS epidemic model with double epidemic hypothesis of the following form:

$$\begin{cases} \dot{S}(t) = A - \mu S(t) - \dfrac{\beta_1 S(t) I_1(t)}{a_1 + I_1(t)} - \dfrac{\beta_2 S(t) I_2(t)}{a_2 + I_2(t)} + r_1 I_1(t) + r_2 I_2(t), \\ \dot{I}_1(t) = \dfrac{\beta_1 S(t) I_1(t)}{a_1 + I_1(t)} - (\mu + \alpha_1 + r_1) I_1(t), \\ \dot{I}_2(t) = \dfrac{\beta_2 S(t) I_2(t)}{a_2 + I_2(t)} - (\mu + \alpha_2 + r_2) I_2(t), \end{cases} \tag{1.1}$$

where A is the total input susceptible population size, β_1 and β_2 are the contact rates, μ is the natural mortality, α_1 and α_2 are the rates of disease-related death, r_1 and r_2 are the treatment cure rates of two diseases, respectively. Functions $\frac{\beta_1 S(t) I_1(t)}{a_1 + I_1(t)}$ and $\frac{\beta_2 S(t) I_2(t)}{a_2 + I_2(t)}$ respectively represent saturated incidence rates for two epidemic diseases. Model (1.1) admits the following equilibria:

$$E_0 : \Big(\frac{A}{\mu}, 0, 0\Big),$$

$$E_1 : (S_1^*, I_1^*, 0) \text{ with } S_1^* = \frac{(\mu+\alpha_1+r_1)(\alpha_1+I_1^*)}{\beta_1}, \quad I_1^* = \frac{\beta_1 A - a_1\mu(\mu+\alpha_1+r_1)}{\mu(\mu+\alpha_1+r_1)+\beta_1(\mu+a_1)},$$

$$E_2 : (S_2^*, 0, I_2^*) \text{ with } S_2^* = \frac{(\mu+\alpha_2+r_2)(\alpha_2+I_2^*)}{\beta_2}, \quad I_2^* = \frac{\beta_2 A - a_2\mu(\mu+\alpha_2+r_2)}{\mu(\mu+\alpha_2+r_2)+\beta_2(\mu+a_2)},$$

$$E^* : (S^*, I_1^*, I_2^*) \text{ with } S^* = \frac{A + a_1(\mu+\alpha_1) + a_2(\mu+\alpha_2)}{\mu + \frac{\beta_1(\mu+\alpha_1)}{\mu+\alpha_1+r_1} + \frac{\beta_2(\mu+\alpha_2)}{\mu+\alpha_2+r_2}},$$

$$I_1^* = \frac{\beta_1 A - a_1\mu(\mu+\alpha_1+r_1) + (\mu+\alpha_2)(\beta_1 a_2 - \beta_2 a_1)}{\mu(\mu+\alpha_1+r_1) + \beta_1(\mu+\alpha_1) + \beta_2(\mu+\alpha_2)},$$

$$I_2^* = \frac{\beta_2 A - a_2\mu(\mu+\alpha_2+r_2) + (\mu+\alpha_1)(\beta_2 a_1 - \beta_1 a_2)}{\mu(\mu+\alpha_2+r_2) + \beta_2(\mu+\alpha_2) + \beta_1(\mu+\alpha_1)}.$$

Let

$$\mathcal{R}_1 = \frac{\beta_1 A}{a_1\mu(\mu+\alpha_1+r_1)}, \quad \mathcal{R}_2 = \frac{\beta_2 A}{a_2\mu(\mu+\alpha_2+r_2)}$$

be the thresholds of model (1.1). Meng *et al.* [10] derived that: (i) If $\mathcal{R}_1 < 1$ and $\mathcal{R}_2 < 1$, then two diseases go extinct and model (1.1) has a unique stable diseases-extinction equilibrium point E_0. (ii) If $\mathcal{R}_1 > 1$ and $\mathcal{R}_2 < 1$, then the disease I_2 is extinct and model (1.1) has a unique stable equilibrium E_1. (iii) If $\mathcal{R}_1 < 1$ and $\mathcal{R}_2 > 1$, then the disease I_1 is extinct and model (1.1) has a unique stable equilibrium E_2. (iv) When (1.1) has a positive equilibrium E^*, if $\mathcal{R}_1 > 1$ and $\mathcal{R}_2 > 1$, then E^* is a unique stable equilibrium, which implies two diseases of model (1.1) are persistent.

The main aim of this article is to investigate how the dynamics behaviors when the environmental noise is considered in deterministic model (1.1). Let $B_i(t)$ $(i =$

$1,2,3)$ be independent Brownian motions and σ_i $(i=1,2,3)$ be the intensities of white noises. Let $(\Omega,\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Based on the model that has been studied in [10], we propose a stochastic model with double epidemic hypothesis as follows:

$$\begin{cases} \mathrm{d}S(t)=\Big[A-\mu S(t)-\dfrac{\beta_1 S(t)I_1(t)}{a_1+I_1(t)}-\dfrac{\beta_2 S(t)I_2(t)}{a_2+I_2(t)}+r_1I_1(t)+r_2I_2(t)\Big]\mathrm{d}t+\sigma_3S(t)\mathrm{d}B_3(t),\\ \mathrm{d}I_1(t)=\Big[\dfrac{\beta_1 S(t)I_1(t)}{a_1+I_1(t)}-(\mu+\alpha_1+r_1)I_1(t)\Big]\mathrm{d}t+\sigma_1I_1(t)\mathrm{d}B_1(t),\\ \mathrm{d}I_2(t)=\Big[\dfrac{\beta_2 S(t)I_2(t)}{a_2+I_2(t)}-(\mu+\alpha_2+r_2)I_2(t)\Big]\mathrm{d}t+\sigma_2I_2(t)\mathrm{d}B_2(t), \end{cases} \tag{1.2}$$

where all parameters have the same biological meanings with those of model (1.1).

In this paper, we prove that there is a unique positive solution of model (1.2) in Section 2. The conditions that guarantee the extinction of the diseases is derived in Section 3. And, the conditions ensuring the persistent of disease are given in Section 4. Several examples and their numerical simulations are carried out to support the main results of this article in the last section.

2 Existence and Uniqueness of Positive Solution

To investigate the dynamical behavior, the first concern is whether the solution has a global existence. Moreover, for a population dynamics model, whether the value is nonnegative is also considered. Hence in this section, we first show that the solution of model (1.2) is global and nonnegative.

Theorem 2.1 *There is a unique solution of model* (1.2) *on* $t\geq 0$ *for any initial value* $(S(0),I_1(0),I_2(0))\in\mathbb{R}^3_+$, *and the solution will remain in* $\mathbb{R}^3_+$ *with probability* 1, *namely,* $(S(t),I_1(t),I_2(t))\in\mathbb{R}^3_+$ *for all* $t\geq 0$ *almost surely.*

Proof Since the coefficients of (1.2) are locally Lipschitz continuous for any given initial value $(S(0),I_1(0),I_2(0))\in\mathbb{R}^3_+$, there is a unique local solution $(S(t),I_1(t),I_2(t))$ on $t\in[0,\tau_e)$, where τ_e is the explosion time (see [11]). To show that this solution is global, we need to show that $\tau_e=\infty$ *a.s.* Let $k_0>0$ be sufficiently large so that each component of $(S(0),I_1(0),I_2(0))$ all lies within the interval $[\frac{1}{k_0},k_0]$. For each integer $k\geq k_0$, we define the stopping time:

$$\tau_k=\inf\Big\{t\in[0,\tau_e):\min\{S(t),I_1(t),I_2(t)\}\leq\frac{1}{k}\ \text{or}\ \max\{S(t),I_1(t),I_2(t)\}\geq k\Big\}. \tag{2.1}$$

Throughout this paper, we set $\inf\emptyset=\infty$ (as usual $\emptyset$ denotes the empty set). According to the definition, τ_k is an increasing function as $k\to\infty$. We set $\tau_\infty=\lim\limits_{k\to\infty}\tau_k$, where $\tau_\infty\leq\tau_e$ a.s. If we can show that $\tau_\infty=\infty$ a.s., then $\tau_e=\infty$ and $(S(t),I_1(t),$

$I_2(t)) \in \mathbb{R}^3_+$ for all $t \geq 0$. In other words, to complete the proof, what we need to show is that $\tau_\infty = \infty$ a.s. If this statement is false, then there exists a pair of constants $T > 0$ and $\varepsilon \in (0,1)$ such that

$$P\{\tau_\infty \leq T\} > \varepsilon. \tag{2.2}$$

Hence there is an integer $k_1 \geq k_0$ such that

$$P\{\tau_k \leq T\} \geq \varepsilon, \quad k \geq k_1. \tag{2.3}$$

Define a C^2-function $W : \mathbb{R}^3_+ \to \overline{\mathbb{R}}$ as follows:

$$W(S, I_1, I_2) = (S - 1 - \log S) + (I_1 - 1 - \log I_1) + (I_2 - 1 - \log I_2), \tag{2.4}$$

where

$$\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : x_i > 0,\ i = 1,2,3\}, \quad \overline{\mathbb{R}} = \{x \in \mathbb{R}^3 : x_i \geq 0,\ i = 1,2,3\}.$$

The nonnegativity of function W is clear since $u - 1 - \log u \geq 0$ on $u > 0$. The generalized Itô's formula gives that

$$\begin{aligned}
\mathrm{d}W(S, I_1, I_2) = {} & \Big(1 - \frac{1}{S}\Big)\Big[A - \mu S - \frac{\beta_1 S I_1}{a_1 + I_1} - \frac{\beta_2 S I_2}{a_2 + I_2} + r_1 I_1 + r_2 I_2\Big]\mathrm{d}t \\
& + \frac{\sigma_3^2}{2}\mathrm{d}t + \Big(1 - \frac{1}{I_1}\Big)\Big[\frac{\beta_1 S I_1}{a_1 + I_1} - (\mu + \alpha_1 + r_1)I_1\Big]\mathrm{d}t + \frac{\sigma_1^2}{2}\mathrm{d}t \\
& + \Big(1 - \frac{1}{I_2}\Big)\Big[\frac{\beta_2 S I_2}{a_2 + I_2} - (\mu + \alpha_2 + r_2)I_2\Big]\mathrm{d}t + \frac{\sigma_2^2}{2}\mathrm{d}t \\
& + (S-1)\sigma_3 \mathrm{d}B_3(t) + (I_1 - 1)\sigma_1 \mathrm{d}B_1(t) + (I_2 - 1)\sigma_1 \mathrm{d}B_2(t),
\end{aligned}$$

which can be written as

$$\begin{aligned}
\mathrm{d}W(S, I_1, I_2) = {} & \mathcal{L}W(S, I_1, I_2)\mathrm{d}t + (S-1)\sigma_3 \mathrm{d}B_3(t) \\
& + (I_1 - 1)\sigma_1 \mathrm{d}B_1(t) + (I_2 - 1)\sigma_1 \mathrm{d}B_2(t),
\end{aligned} \tag{2.5}$$

where $\mathcal{L}$ maps from $\mathbb{R}^3_+$ to $\overline{\mathbb{R}}$ and is expressed as

$$\begin{aligned}
\mathcal{L}W(S, I_1, I_2) = {} & A - \mu(S + I_1 + I_2) - \alpha_1 I_1 - \alpha_2 I_2 - \frac{A}{S} + \mu + \frac{\beta_1 I_1}{a_1 + I_1} \\
& + \frac{\beta_2 I_2}{a_2 + I_2} - \frac{r_1 I_1 + r_2 I_2}{S} - \frac{\beta_1 S}{a_1 + I_1} + \mu + \alpha_1 + r_1 - \frac{\beta_2 S}{a_2 + I_2} \\
& + \mu + \alpha_2 + r_2 + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} \\
\leq {} & A + \mu + \beta_1 + \beta_2 + \mu + \alpha_1 + r_1 + \mu + \alpha_2 + r_2 + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} \\
:= {} & K.
\end{aligned} \tag{2.6}$$

The remainder of the proof follows from that of Mao *et al.* [12]. The proof is complete.

3 Extinction

Let us prepare two useful lemmas before proving Theorem 3.1. According to the similar mechanism in [13] and same arguments, we can obtain Lemmas 3.1 and 3.2. The proofs of these two lemmas are omitted here.

Lemma 3.1 *Let $(S(t), I_1(t), I_2(t))$ be a solution of model (1.2) with any initial value $(S(0), I_1(0), I_2(0)) \in \mathbb{R}^3_+$, then*

$$\lim_{t\to\infty} \frac{S(t)+I_1(t)+I_2(t)}{t} = 0 \ \ a.s.$$

Moreover

$$\lim_{t\to\infty} \frac{S(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{I_1(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{I_2(t)}{t} = 0 \quad a.s.$$

Lemma 3.2 *Assume that $\mu > (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)/2$. Let $(S(t), I_1(t), I_2(t))$ be a solution of model (1.2) with any initial value $(S(0), I_1(0), I_2(0)) \in \mathbb{R}^3_+$, then*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t S(r)\mathrm{d}B_3(r)=0, \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t I_1(r)\mathrm{d}B_1(r)=0, \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t I_2(r)\mathrm{d}B_2(r)=0, \ \ a.s. \tag{3.1}$$

Theorem 3.1 *Assume that $\mu > (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)/2$. Let $(S(t), I_1(t), I_2(t))$ be a solution of model (1.2) with any initial value $(S(0), I_1(0), I_2(0)) \in \mathbb{R}^3_+$. If $\mathcal{R}_1^* < 1$ and $\mathcal{R}_2^* < 1$ hold, then the densities of two infective individuals of model (1.2) go to extinction almost surely, that is*

$$\lim_{t\to\infty} I_1(t) = 0, \quad \lim_{t\to\infty} I_2(t) = 0.$$

Moreover,

$$\lim_{t\to\infty} S(t) = 0,$$

where

$$\mathcal{R}_i^* = \frac{\beta_i A}{\mu a_i(\mu+\alpha_i+r_i+\frac{\sigma_i^2}{2})}, \quad i = 1, 2. \tag{3.2}$$

Proof The integration of model (1.2) yields

$$\begin{aligned}
&\frac{S(t)-S(0)}{t} + \frac{I_1(t)-I_1(0)}{t} + \frac{I_2(t)-I_2(0)}{t} \\
&= A - \mu\langle S(t)\rangle - (\mu+\alpha_1)\langle I_1(t)\rangle - (\mu+\alpha_2)\langle I_2(t)\rangle \\
&\quad + \frac{\sigma_3}{t}\int_0^t S(\tau)\mathrm{d}B_3(\tau) + \frac{\sigma_1}{t}\int_0^t I_1(\tau)\mathrm{d}B_1(\tau) + \frac{\sigma_2}{t}\int_0^t I_2(\tau)\mathrm{d}B_2(\tau).
\end{aligned}$$

We simplify the above expression as follows:

$$\langle S(t)\rangle = \frac{A}{\mu} - \frac{\mu+\alpha_1}{\mu}\langle I_1(t)\rangle - \frac{\mu+\alpha_2}{\mu}\langle I_2(t)\rangle + \frac{1}{\mu}\varphi(t), \tag{3.3}$$

where

$$\varphi(t)=\frac{\sigma_3}{t}\int_0^t S(\tau)\mathrm{d}B_3(\tau)+\frac{\sigma_1}{t}\int_0^t I_1(\tau)\mathrm{d}B_1(\tau)+\frac{\sigma_2}{t}\int_0^t I_2(\tau)\mathrm{d}B_2(\tau)$$
$$-\frac{S(t)-S(0)}{t}-\frac{I_1(t)-I_1(0)}{t}-\frac{I_2(t)-I_2(0)}{t}.$$

By (3.1) of Lemma 3.2, we have

$$\lim_{t\to\infty}\varphi(t)=0 \ \ a.s. \tag{3.4}$$

Applying Itô's formula to model (1.2) leads to

$$\mathrm{d}\log I_i=\left[\frac{\beta_i S}{a_i+I_i}-\left(\mu+\alpha_i+r_i+\frac{\sigma_i^2}{2}\right)\right]\mathrm{d}t+\sigma_i\mathrm{d}B_i(t),\quad i=1,2. \tag{3.5}$$

Integrating both sides of (3.5) from 0 to t and dividing t on both sides gives that

$$\begin{aligned}\frac{\log I_i(t)-\log I_i(0)}{t}&=\left\langle\frac{\beta_i S(t)}{a_i+I_i}\right\rangle-\left(\mu+\alpha_i+r_i+\frac{\sigma_i^2}{2}\right)+\frac{\sigma_i B_i(t)}{t}\\&\le\frac{\beta_i}{a_i}\langle S(t)\rangle-\left(\mu+\alpha_i+r_i+\frac{\sigma_i^2}{2}\right)+\frac{\sigma_i B_i(t)}{t}.\end{aligned} \tag{3.6}$$

Substituting (3.3) into (3.6) yields

$$\begin{aligned}\frac{\log I_i(t)}{t}&\le\frac{A\beta_i}{\mu a_i}-\frac{\beta_i(\mu+\alpha_1)}{a_i\mu}\langle I_1(t)\rangle-\frac{\beta_i(\mu+\alpha_2)}{a_i\mu}\langle I_2(t)\rangle-\left(\mu+\alpha_i+r_i+\frac{\sigma_i^2}{2}\right)\\&\quad+\frac{\beta_i}{a_i\mu}\varphi(t)+\frac{\sigma_i B_i(t)}{t}+\frac{\log I_i(0)}{t}\\&\le\frac{A\beta_i}{\mu a_i}-\left(\mu+\alpha_i+r_i+\frac{\sigma_i^2}{2}\right)+M_i(t)\\&=\left(\mu+\alpha_i+r_i+\frac{\sigma_i^2}{2}\right)(\mathcal{R}_i^*-1)+M_i(t),\end{aligned} \tag{3.7}$$

where

$$M_i(t)=\frac{\beta_i}{a_i\mu}\varphi(t)+\frac{\sigma_i B_i(t)}{t}+\frac{\log I_i(0)}{t},\quad i=1,2.$$

Obviously,

$$\lim_{t\to\infty}M_i(t)=0,\ a.s.$$

Since $\mathcal{R}_i^*<1$ for $i=1,2$, taking superior limit on both sides of (3.7) gives

$$\limsup_{t\to\infty}\frac{\log I_i(t)}{t}\le\left(\mu+\alpha_i+r_i+\frac{\sigma_i^2}{2}\right)(\mathcal{R}_i^*-1)<0,$$

which implies

$$\lim_{t\to\infty}\frac{I_i(t)}{t}=0. \tag{3.8}$$

From (3.3) and (3.8), we have

$$\lim_{t\to\infty}\frac{S(t)}{t}=\frac{A}{\mu}.$$

The proof is complete.

Remark 3.1 Note that the expressions of $\mathcal{R}_i^*$ $(i=1,2)$ carry the same information with thresholds of model (1.1), say $\mathcal{R}_i$ $(i=1,2)$. These conditions that guarantee the extinction of $I_i(t)$ $(i=1,2)$ in the deterministic model (1.1) are stronger than those in the corresponding stochastic model (1.2).

4 Persistence in Mean

Theorem 4.1 *Assume that* $\mu>(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)/2$. *Let* $(S(t),I_1(t),I_2(t))$ *be a solution of model* (1.2) *with any initial value* $(S(0),I_1(0),I_2(0))\in\mathbb{R}_+^3$.

(i) *If* $\mathcal{R}_1^*>1$, $\mathcal{R}_2^*<1$, *then the disease* I_2 *is extinct and the disease* I_1 *is persistent in mean, moreover,* I_1 *satisfies*

$$\liminf_{t\to\infty}\langle I_1(t)\rangle\geq\frac{\mu a_1(\mu+\alpha_1+r_1+\frac{\sigma_1^2}{2})}{\beta_1(\mu+\alpha_1)+\mu(\mu+\alpha_1+r_1)}(\mathcal{R}_1^*-1).$$

(ii) *If* $\mathcal{R}_1^*<1$, $\mathcal{R}_2^*>1$, *then the disease* I_1 *is extinct and the disease* I_2 *is persistent in mean, moreover,* I_2 *satisfies*

$$\liminf_{t\to\infty}\langle I_2(t)\rangle\geq\frac{\mu a_2(\mu+\alpha_2+r_2+\frac{\sigma_2^2}{2})}{\beta_2(\mu+\alpha_2)+\mu(\mu+\alpha_2+r_2)}(\mathcal{R}_2^*-1).$$

(iii) *If* $\mathcal{R}_1^*>1$, $\mathcal{R}_2^*>1$, *then two infectious diseases* I_1 *and* I_2 *are persistent in mean, moreover,* I_1 *and* I_2 *satisfy*

$$\liminf_{t\to\infty}\langle I_1(t)+I_2(t)\rangle\geq\frac{1}{\Delta_{\max}}\sum_{i=1}^{2}a_i\Big(\mu+\alpha_i+r_i+\frac{\sigma_i^2}{2}\Big)(\mathcal{R}_i^*-1),$$

where

$$\Delta_{\max}=\max\left\{\frac{\beta_1+\beta_2}{\mu}(\mu+\alpha_1)+\mu+\alpha_1+r_1,\ \frac{\beta_1+\beta_2}{\mu}(\mu+\alpha_2)+\mu+\alpha_2+r_2\right\}.$$

Proof Case (i). By Theorem 3.1, since $\mathcal{R}_2^*<1$, $\lim\limits_{t\to\infty}I_2(t)=0$. Since $\mathcal{R}_1^*>1$, for ε small enough, such that $0<I_2(t)<\varepsilon$ for all t large enough, we obtain

$$\frac{\beta_1(A-(\mu+\alpha_2)\varepsilon)}{\mu a_1(\mu+\alpha_1+r_1+\frac{\sigma_1^2}{2})}>1.$$

According to (3.3), we can derive that

$$\begin{aligned}\langle S(t)\rangle&=\frac{A}{\mu}-\frac{\mu+\alpha_1}{\mu}\langle I_1(t)\rangle-\frac{\mu+\alpha_2}{\mu}\langle I_2(t)\rangle+\frac{1}{\mu}\varphi(t)\\&\geq\frac{A-(\mu+\alpha_2)\varepsilon}{\mu}-\frac{\mu+\alpha_1}{\mu}\langle I_1(t)\rangle+\frac{1}{\mu}\varphi(t).\end{aligned}\tag{4.1}$$

Generalized Itô's formula gives

$$\mathrm{d}(a_1\log I_1(t)+I_1(t))=\Big[\beta_1 S(t)-a_1(\mu+\alpha_1+r_1)-(\mu+\alpha_1+r_1)I_1(t)-\frac{a_1\sigma_1^2}{2}\Big]\mathrm{d}t$$
$$+(a_1\sigma_1+\sigma_1 I_1(t))\mathrm{d}B_1(t). \tag{4.2}$$

Integrating this from 0 to t and dividing it by t on both sides of (4.2) yields

$$\begin{aligned}&\frac{a_1(\log I_1(t)-\log I_1(0))}{t}+\frac{I_1(t)-I_1(0)}{t}\\
&=\beta_1\langle S(t)\rangle-a_1\Big(\mu+\alpha_1+r_1+\frac{\sigma_1^2}{2}\Big)-(\mu+\alpha_1+r_1)\langle I_1(t)\rangle+\frac{\sigma_1}{t}\int_0^t(a_1+I_1(\tau))\mathrm{d}B_1(\tau)\\
&\geq\frac{\beta_1(A-(\mu+\alpha_2)\varepsilon)}{\mu}-a_1\Big(\mu+\alpha_1+r_1+\frac{\sigma_1^2}{2}\Big)-\Big[\frac{\beta_1(\mu+\alpha_1)}{\mu}+(\mu+\alpha_1+r_1)\Big]\langle I_1(t)\rangle\\
&\quad+\frac{\beta_1}{\mu}\varphi(t)+\frac{\sigma_1}{t}\int_0^t(a_1+I_1(\tau))\mathrm{d}B_1(\tau)\\
&=a_1\Big(\mu+\alpha_1+r_1+\frac{a_1\sigma_1^2}{2}\Big)\Big[\frac{\beta_1(A-(\mu+\alpha_2)\varepsilon)}{\mu a_1(\mu+\alpha_1+r_1+\frac{\sigma_1^2}{2})}-1\Big]\\
&\quad-\Big[\frac{\beta_1(\mu+\alpha_1)}{\mu}+(\mu+\alpha_1+r_1)\Big]\langle I_1(t)\rangle+\frac{\beta_1}{\mu}\varphi(t)+\frac{\sigma_1}{t}\int_0^t(a_1+I_1(\tau))\mathrm{d}B_1(\tau).\end{aligned}\tag{4.3}$$

Inequality (4.3) can be rewritten as

$$\begin{aligned}\langle I_1(t)\rangle\geq&\frac{1}{\Delta}\Big[a_1\Big(\mu+\alpha_1+r_1+\frac{\sigma_1^2}{2}\Big)\Big(\frac{\beta_1(A-(\mu+\alpha_2)\varepsilon)}{\mu a_1(\mu+\alpha_1+r_1+\frac{\sigma_1^2}{2})}-1\Big)\\
&+\frac{\beta_1}{\mu}\varphi(t)+\frac{\sigma_1}{t}\int_0^t(a_1+I_1(\tau))\mathrm{d}B_1(\tau)-\frac{a_1(\log I_1(t)-\log I_1(0))}{t}-\frac{I_1(t)-I_1(0)}{t}\Big]\\
\geq&\begin{cases}\frac{1}{\Delta}\Big[a_1\Big(\mu+\alpha_1+r_1+\frac{\sigma_1^2}{2}\Big)\Big(\frac{\beta_1(A-(\mu+\alpha_2)\varepsilon)}{\mu a_1(\mu+\alpha_1+r_1+\frac{\sigma_1^2}{2})}-1\Big)+\frac{\beta_1}{\mu}\varphi(t)\\
+\frac{\sigma_1}{t}\int_0^t(a_1+I_1(\tau))\mathrm{d}B_1(\tau)+\frac{a_1\log I_1(0)}{t}-\frac{I_1(t)-I_1(0)}{t}\Big],\quad 0<I_1(t)<1;\\
\frac{1}{\Delta}\Big[a_1\Big(\mu+\alpha_1+r_1+\frac{\sigma_1^2}{2}\Big)\Big(\frac{\beta_1(A-(\mu+\alpha_2)\varepsilon)}{\mu a_1(\mu+\alpha_1+r_1+\frac{\sigma_1^2}{2})}-1\Big)+\frac{\beta_1}{\mu}\varphi(t)\\
+\frac{\sigma_1}{t}\int_0^t(a_1+I_1(\tau))\mathrm{d}B_1(\tau)-\frac{a_1(\log I_1(t)-\log I_1(0))}{t}-\frac{I_1(t)-I_1(0)}{t}\Big],\quad 1\leq I_1(t),\end{cases}\end{aligned}\tag{4.4}$$

where $\Delta=\frac{\beta_1(\mu+\alpha_1)}{\mu}+(\mu+\alpha_1+r_1)$. By Lemma 3.2, we get

$$\lim_{t\to\infty}\varphi(t)=0,\quad \lim_{t\to\infty}\frac{\sigma_1}{t}\int_0^t(a_1+I_1(\tau))\mathrm{d}B_1(\tau)=0.$$

According to Lemma 3.1, one has

$$\lim_{t\to\infty}\frac{I_1(t)}{t}=0,\quad \lim_{t\to\infty}\frac{\log I_1(t)}{t}=0 \ \text{ as } I_1(t)\geq 1.$$

Taking inferior limit on both sides of (4.4) yields

$$\liminf_{t\to\infty}\langle I_1(t)\rangle \geq \frac{a_1(\mu+\alpha_1+r_1+\frac{\sigma_1^2}{2})}{\Delta}\left(\frac{\beta_1(A-(\mu+\alpha_2)\varepsilon)}{\mu a_1(\mu+\alpha_1+r_1+\frac{\sigma_1^2}{2})}-1\right)>0.$$

Letting $\varepsilon\to 0$ gives that

$$\liminf_{t\to\infty}\langle I_1(t)\rangle \geq \frac{a_1\mu(\mu+\alpha_1+r_1+\frac{\sigma_1^2}{2})}{\beta_1(\mu+\alpha_1)+\mu(\mu+\alpha_1+r_1)}\big(\mathcal{R}_1^*-1\big).$$

By the similar argument, we can prove Case (ii), so we omit it here.

To prove Case (iii), we define

$$V(t)=\log I_1^{a_1}(t)+\log I_2^{a_2}(t)+I_1(t)+I_2(t),$$

then we have

$$\begin{aligned}\mathrm{d}V(t)=&\left[(\beta_1+\beta_2)S(t)-\sum_{i=1}^{2}a_i\Big(\mu+\alpha_i+r_i+\frac{\sigma_i^2}{2}\Big)-\sum_{i=1}^{2}(\mu+\alpha_i+r_i)I_i(t)\right]\mathrm{d}t\\&+\sum_{i=1}^{2}\sigma_i(a_i+I_i(t))\mathrm{d}B_i(t).\end{aligned}\tag{4.5}$$

Integrating this from 0 to t and dividing it by t on both sides of (4.5), together with (3.3), yields that

$$\begin{aligned}\frac{V(t)}{t}-\frac{V(0)}{t}=&(\beta_1+\beta_2)\langle S(t)\rangle-\sum_{i=1}^{2}a_i\Big(\mu+\alpha_i+r_i+\frac{\sigma_i^2}{2}\Big)\\&-\sum_{i=1}^{2}(\mu+\alpha_i+r_i)\langle I_i(t)\rangle+\sum_{i=1}^{2}\Big(\frac{\sigma_i}{t}\int_0^t(a_i+I_i(\tau))\mathrm{d}B_i(\tau)\Big)\\=&(\beta_1+\beta_2)\frac{A}{\mu}-\sum_{i=1}^{2}a_i\Big(\mu+\alpha_i+r_i+\frac{\sigma_i^2}{2}\Big)\\&-\sum_{i=1}^{2}\Big(\frac{\beta_1+\beta_2}{\mu}(\mu+\alpha_i)+\mu+\alpha_i+r_i\Big)\langle I_i(t)\rangle+\frac{\beta_1+\beta_2}{\mu}\varphi(t)\\&+\sum_{i=1}^{2}\Big(\frac{\sigma_i}{t}\int_0^t(a_i+I_i(\tau))\mathrm{d}B_i(\tau)\Big)\\\geq&(\beta_1+\beta_2)\frac{A}{\mu}-\sum_{i=1}^{2}a_i\Big(\mu+\alpha_i+r_i+\frac{\sigma_i^2}{2}\Big)-\Delta_{\max}\big[\langle I_1(t)+I_2(t)\rangle\big]\\&+\frac{\beta_1+\beta_2}{\mu}\varphi(t)+\sum_{i=1}^{2}\Big(\frac{\sigma_i}{t}\int_0^t(a_i+I_i(\tau))\mathrm{d}B_i(\tau)\Big).\end{aligned}\tag{4.6}$$

Inequality (4.6) can be rewritten as

$$\langle I_1(t)+I_2(t)\rangle \ge \frac{1}{\Delta_{\max}}\Big[(\beta_1+\beta_2)\frac{A}{\mu}-\sum_{i=1}^{2}a_i\Big(\mu+\alpha_i+r_i+\frac{\sigma_i^2}{2}\Big)+\frac{\beta_1+\beta_2}{\mu}\varphi(t)$$
$$+\sum_{i=1}^{2}\Big(\frac{\sigma_i}{t}\int_0^t(a_i+I_i(\tau))\mathrm{d}B_i(\tau)\Big)-\frac{V(t)}{t}+\frac{V(0)}{t}\Big], \tag{4.7}$$

where

$$-\frac{V(t)}{t} = -\frac{a_1\log I_1(t)+a_2\log I_2(t)+I_1(t)+I_2(t)}{t}$$
$$\ge \begin{cases} -\dfrac{a_1\log I_1(t)+a_2\log I_2(t)+I_1(t)+I_2(t)}{t}, & 1\le I_1(t),\ 1\le I_2(t);\\ -\dfrac{a_1\log I_1(t)+I_1(t)+I_2(t)}{t}, & 1\le I_1(t),\ 0<I_2(t)<1;\\ -\dfrac{a_2\log I_2(t)+I_1(t)+I_2(t)}{t}, & 0<I_1(t)<1,\ 1\le I_2(t);\\ -\dfrac{I_1(t)+I_2(t)}{t}, & 0<I_1(t)<1,\ 0<I_2(t)<1. \end{cases}$$

By Lemma 3.2, we get that

$$\lim_{t\to\infty}\varphi(t)=0, \quad \lim_{t\to\infty}\frac{\sigma_i}{t}\int_0^t(a_i+I_i(\tau))\mathrm{d}B_i(\tau)=0.$$

According to Lemma 3.1 and taking inferior limit on both sides of (4.7), implies that

$$\liminf_{t\to\infty}\langle I_1(t)+I_2(t)\rangle \ge \frac{1}{\Delta_{\max}}\sum_{i=1}^{2}a_i\Big(\mu+\alpha_i+r_i+\frac{\sigma_i^2}{2}\Big)(\mathcal{R}_i^*-1).$$

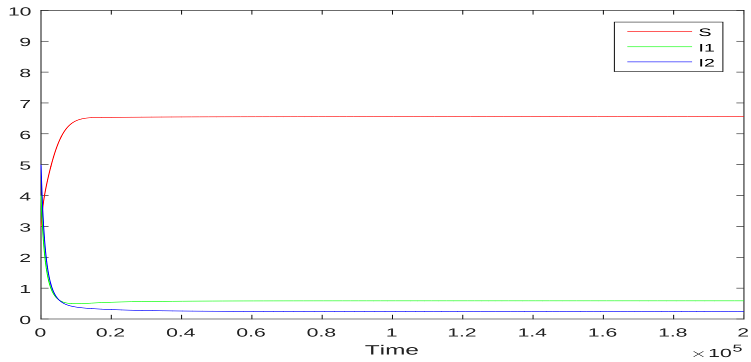
The proof is complete.

Remark 4.1 Theorem 4.1 shows that two diseases prevail if the white noises are small enough and $\mathcal{R}_i^*>1$. On the contrary, if the white noises are large enough, then two diseases become extinct. This implies that random perturbations may cause epidemic diseases to die out.

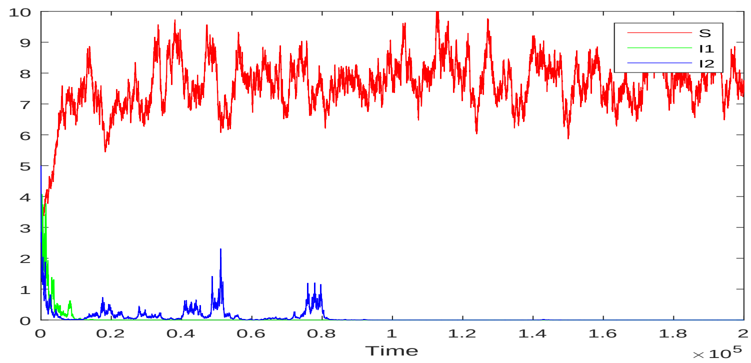
5 Conclusion and Simulations

In this paper, we investigate the dynamics of an SIS epidemic model with nonlinear growth rate and double epidemic hypothesis. The thresholds of stochastic model which guarantee the extinction and permanence of two epidemic diseases are derived in Theorems 3.1 and 4.1. Compared with the known results given by Meng *et al.*, from Theorem 3.1, we can see that conditions that guarantee the extinction of stochastic model (1.2) are weaker than that of deterministic model (1.1), and the conditions of Theorem 3.1 depend on the intensity of white noise.

By using of Euler Maruyama (EM) method [14, 15], we present several simulations to support main results of this article. We show the property of deterministic model (1.1) in Figure 1 (a). And, we demonstrate the extinction and persistence of diseases in Figure 1 (b) and Figure 2 (a)(b)(c).



(a) deterministic model (1.1) with $\mathcal{R}_1 = 1.4000 > 1$, $\mathcal{R}_2 = 1.2800 > 1$



(b) stochastic model (1.2) with $\mathcal{R}_1^* = 0.9964 < 1$, $\mathcal{R}_2^* = 0.9110 < 1$

Figure 1: Numerical simulation of paths $S(t), I_1(t), I_2(t)$ for SIS epidemic model with double epidemic hypothesis

We choose initial values $(S(0), I_1(0), I_2(0)) = (3, 4, 5)$ and the parameters in models (1.1) and (1.2) as follows:

$A{=}4,\ a_1{=}4,\ a_2{=}5,\ \beta_1{=}0.7,\ \beta_2{=}0.8,\ r_1{=}0.1,\ r_2{=}0.2,\ \mu{=}0.5,\ \alpha_1{=}0.4,\ \alpha_2{=}0.3.$

Notice that in Figure 1:

(a) $\sigma_1 = 0$, $\sigma_2 = 0$, $\sigma_3 = 0$, $\mathcal{R}_1 = 1.4000 > 1$, $\mathcal{R}_2 = 1.2800 > 1$,

(b) $\sigma_1 = 0.9$, $\sigma_2 = 0.9$, $\sigma_3 = 0.1$, $\mathcal{R}_1^* = 0.9964 < 1$, $\mathcal{R}_2^* = 0.9110 < 1$,

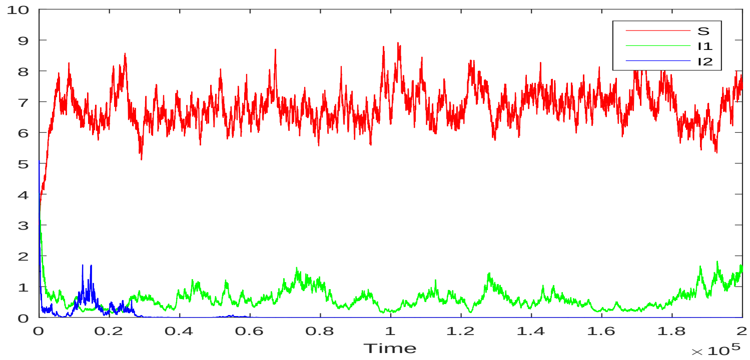
and in Figure 2:

(a) $\sigma_1 = 0.3$, $\sigma_2 = 0.9$, $\sigma_3 = 0.1$, $\mathcal{R}_1^* = 1.3397 > 1$, $\mathcal{R}_2^* = 0.9110 < 1$,

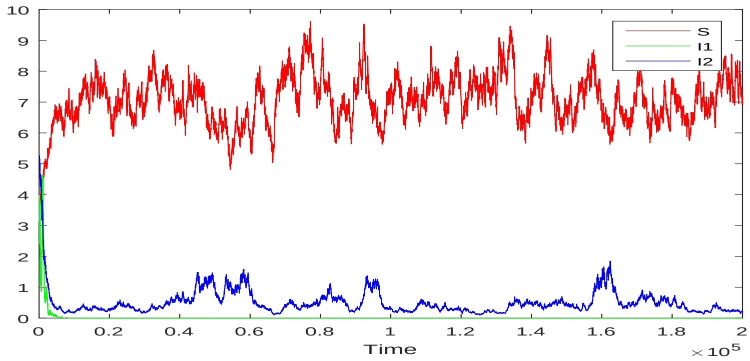
(b) $\sigma_1 = 0.9$, $\sigma_2 = 0.3$, $\sigma_3 = 0.1$, $\mathcal{R}_1^* = 0.9964 < 1$, $\mathcal{R}_2^* = 1.2249 > 1$,

(c) $\sigma_1 = 0.3$, $\sigma_2 = 0.3$, $\sigma_3 = 0.1$, $\mathcal{R}_1^* = 1.3397 > 1$, $\mathcal{R}_2^* = 1.2249 > 1$,

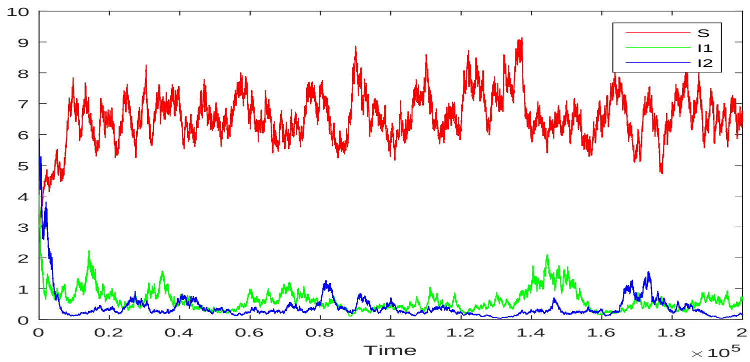
and
$$\mu > (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)/2 = 0.405.$$



(a) $\mathcal{R}_1^* = 1.3397 > 1$, $\mathcal{R}_2^* = 0.9110 < 1$



(b) $\mathcal{R}_1^* = 0.9964 < 1$, $\mathcal{R}_2^* = 1.2249 > 1$



(c) $\mathcal{R}_1^* = 1.3397 > 1$, $\mathcal{R}_2^* = 1.2249 > 1$.

Figure 2: Numerical simulations of stochastic model (1.2) with double epidemic hypothesis

Figure 1 shows that two diseases are persistent in a deterministic model (see Figure 1 (a)), and they die out due to taking intensities of white noises into account in a stochastic model with $\sigma_1 = \sigma_2 = 0.9$ (see Figure 1 (b)). When $\mathcal{R}_i^* < 1 < \mathcal{R}_i$, the persistence of deterministic model changes into the extinction of the corresponding stochastic model due to random perturbation. Therefore, the intensities of white noises can be referred as to control parameters when stochastic epidemic models are considered.

Now, we keep the most parameters the same as shown in Figure 1, but σ_1 and σ_2 take different values. When σ_1 is smaller, and σ_2 is larger ($\sigma_1 = 0.3$, $\sigma_2 = 0.9$), here $\mathcal{R}_1^* = 1.3397 > 1$, $\mathcal{R}_2^* = 0.9110 < 1$, thus I_2 goes to extinction and I_1 is persistent (see Figure 2 (a)). When σ_1 is larger and σ_2 is smaller ($\sigma_1 = 0.9$, $\sigma_2 = 0.3$), here $\mathcal{R}_1^* = 0.9964 < 1$, $\mathcal{R}_2^* = 1.2249 > 1$, Figure 2 (b) shows that I_1 goes to extinction and I_2 is persistent. Furthermore, let σ_1 and σ_2 take small values, then I_1 and I_2 are persistent (see Figure 2 (c)). That is, two diseases will prevail in a long run, which strongly supports the theoretical results derived in Theorem 4.1.

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