Ann. of Appl. Math.
33:1(2017), 102-110

ON THE BOUNDEDNESS OF A CLASS OF NONLINEAR DYNAMIC EQUATIONS OF THE THIRD ORDER*

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Abstract

In this paper, a modified nonlinear dynamic inequality on time scales is used to study the boundedness of a class of nonlinear third-order dynamic equations on time scales. These theorems contain as special cases results for dynamic differential equations, difference equations and q-difference equations.

Keywords time scales; dynamic equation; integral inequality; boundedness; third-order

2000 Mathematics Subject Classification 34C11; 34N05

1 Introduction

To unify and extend continuous and discrete analyses, the theory of time scales was introduced by Hilger [1] in his Ph.D.Thesis in 1988. Since then, the theory has been evolving, and it has been applied to various fields of mathematics; for example, see [2,3] and the references therein. It is well known that Gronwall-type integral inequalities and their discrete analogues play a dominant role in the study of quantitative properties of solutions of differential, integral and difference equations.

During the last few years, some Gronwall-type integral inequalities on time scales and their applications have been investigated by many authors. For example, we refer readers to [5-11]. In this paper, motivated by the paper [4], we obtain the bounds of the solutions of a class of nonlinear dynamic equations of the third order on time scales, which generalizes the main result of [4]. For all the detailed definitions, notation and theorems on time scales, we refer the readers to the excellent monographs [2,3] and references given therein. We also present some preliminary

*This research was partially supported by the NSF of China (Grant.11271225) and Program for Scientific Research Innovation Team in Colleges and Universities of Shandong Province.

†Manuscript received July 9, 2016

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results that are needed in the remainder of this paper as useful lemmas for the discussion of our proof.

In what follows, R denotes the set of real number, $R_+ = [0, +\infty)$; $C(M, S)$ denotes the class of all continuous functions defined on a set M with range in a set S; T is an arbitrary time scale and C_{rd} denotes the set of rd-continuous functions. Throughout this paper, we always assume that $t_0 \in T$, $T_0 = [t_0, +\infty) \cap T$.

2 Preliminary

Lemma 2.1 *Suppose $u(t), a(t) \in C_{rd}(T_0, R_+)$, and a is nondecreasing, $f(t,s)$, $f_t^\Delta(t,s) \in C_{rd}(T_0 \times T_0, R_+)$, $\omega \in C(R_+, R_+)$ is nondecreasing. If for $t \in T_0$, $u(t)$ satisfies the following inequality*

$$u(t) \leq a(t) + \int_{t_0}^{t} f(t,s)\omega(u(s))\Delta s, \quad t \in T_0, \tag{2.1}$$

then

$$u(t) \leq G^{-1}\Big[G(a(t)) + \int_{t_0}^{t} f(t,s)\Delta s\Big], \quad t \in T_0, \tag{2.2}$$

where

$$G(v) = \int_{v_0}^{v} \frac{1}{\omega(r)}\mathrm{d}r, \quad v \geq v_0 > 0,\ G(+\infty) = +\infty. \tag{2.3}$$

Proof For arbitrarily fixed $\widetilde{t_0} > t_0$, by the condition, we have

$$u(t) \leq a(\widetilde{t_0}) + \int_{t_0}^{t} f(\widetilde{t_0}, s)\omega(u(s))\Delta s, \quad t \in [t_0, \widetilde{t_0}].$$

Let $z(t) = a(\widetilde{t_0}) + \int_{t_0}^{t} f(\widetilde{t_0}, s)\omega(u(s))\Delta s$, then we get $z(t_0) = a(\widetilde{t_0})$ and $u(t) \leq z(t)$. Since

$$z^\Delta(t) = f(\widetilde{t_0}, t)\omega(u(t)) \leq f(\widetilde{t_0}, t)\omega(z(t)),$$

we have

$$\frac{z^\Delta(t)}{\omega(z(t))} \leq f(\widetilde{t_0}, t).$$

Furthermore, for $t \in [t_0, \widetilde{t_0}]$, if $\sigma(t) > t$, then

$$\begin{aligned}[G(z(t))]^\Delta &= \frac{G(z(\sigma(t))) - G(z(t))}{\sigma(t) - t} = \frac{1}{\sigma(t) - t}\int_{z(t)}^{z(\sigma(t))} \frac{1}{\omega(r)}\mathrm{d}r \\ &\leq \frac{z(\sigma(t)) - z(t)}{\sigma(t) - t}\frac{1}{\omega(z(t))} = \frac{z^\Delta(t)}{\omega(z(t))}.\end{aligned}$$

If $\sigma(t) = t$, then

$$[G(z(t))]^{\Delta} = \lim_{s\to t}\frac{G(z(t))-G(z(s))}{t-s} = \lim_{s\to t}\frac{1}{t-s}\int_{z(s)}^{z(t)}\frac{1}{\omega(r)}\mathrm{d}r$$
$$= \lim_{s\to t}\frac{z(t)-z(s)}{t-s}\frac{1}{\omega(\xi)} = \frac{z^{\Delta}(t)}{\omega(z(t))},$$

where G is defined in (2.3) and ξ lies between $z(s)$ and $z(t)$. So we always have

$$[G(z(t))]^{\Delta} \le \frac{z^{\Delta}(t)}{\omega(z(t))}.$$

Using the above statements, we deduce that

$$[G(z(t))]^{\Delta} \le \frac{z^{\Delta}(t)}{\omega(z(t))} \le f(\widetilde{t_0}, t). \tag{2.4}$$

Replacing t with s in (2.4), and an integration with respect to s from t_0 to t yields

$$G(z(t)) - G(z(t_0)) \le \int_{t_0}^{t} f(\widetilde{t_0}, s)\Delta s.$$

Hence,

$$G(z(t)) \le G(z(t_0)) + \int_{t_0}^{t} f(\widetilde{t_0}, s)\Delta s = G(a(\widetilde{t_0})) + \int_{t_0}^{t} f(\widetilde{t_0}, s)\Delta s,$$

and we have

$$z(t) \le G^{-1}\Big[G(a(\widetilde{t_0})) + \int_{t_0}^{t} f(\widetilde{t_0}, s)\Delta s\Big], \quad t \in [t_0, \widetilde{t_0}], \tag{2.5}$$

where G^{-1} is the inverse function of G.

Let $t = \widetilde{t_0}$ in (2.5), then we get

$$z(\widetilde{t_0}) \le G^{-1}\Big[G(a(\widetilde{t_0})) + \int_{t_0}^{\widetilde{t_0}} f(\widetilde{t_0}, s)\Delta s\Big]. \tag{2.6}$$

Since $\widetilde{t_0}$ is chosen arbitrarily, from (2.6) we can obtain

$$z(t) \le G^{-1}\Big[G(a(t)) + \int_{t_0}^{t} f(t, s)\Delta s\Big].$$

And then we get

$$u(t) \le G^{-1}\Big[G(a(t)) + \int_{t_0}^{t} f(t, s)\Delta s\Big], \quad t \in T_0.$$

The proof of Lemma 2.1 is completed.

Remark 2.1 Lemma 2.1 is similar to Theorem 3.1 in [10], but by defining a new function of G, which has more extensive applications.

Definition[12] A function $g \in C(R_+, R_+)$ is said to belong to the class of $\Re$ if

(1) g is nondecreasing,

(2) $\frac{g(u)}{v} \le g(\frac{u}{v})$ for $u \ge 0,\ v \ge 1$.

It is easy to see that $g(u) \in \Re$ implies $\int_1^{+\infty} \frac{1}{g(s)} \mathrm{d}s = +\infty$.

Lemma 2.2 *Suppose*

(1) $u(t)$ *and* $a(t) \in C_{rd}(T_0, R_+)$, $a(t) \ge 1$ *is nondecreasing on* T_0;

(2) $f_i(t,s), f_i^{\Delta}(t,s) \in C_{rd}(T_0 \times T_0, R_+)$;

(3) $h_i \in \Re$ $(i = 1, 2, \cdots, m)$.

If for $t \in T_0$, $u(t)$ *satisfies the following inequality*

$$u(t) \le a(t) + \sum_{i=1}^{m} \int_{t_0}^{t} f_i(t,s) h_i[u(s)] \Delta s, \quad t \in T_0, \tag{2.7}$$

then

$$u(t) \le a(t) \prod_{i=1}^{m} L_i(t), \quad t \in T_0, \tag{2.8}$$

where

$$\begin{cases} L_i(t) = G_i^{-1}\Big[G_i(1) + \displaystyle\int_{t_0}^{t} f_i(t,s)\Big(\prod_{k=1}^{i-1} L_k(s)\Big)\Delta s\Big], \quad i = 1, 2, \cdots, m, \\ G_i(v) = \displaystyle\int_{v_0}^{v} \frac{1}{h_i(r)} \mathrm{d}r, \quad v \ge v_0 > 0, \\ \displaystyle\prod_{k=1}^{0} L_k(t) = 1. \end{cases} \tag{2.9}$$

Proof The proof is completely similar to that of Lemma 2.2 in [4], and we omit the details here.

Lemma 2.3[13] *Assume* $a < b \in T$ *and* $F(\tau, s)$ *is a real-valued function on* $T \times T$. *Then*

$$\int_a^b \int_a^{\tau} F(\tau,s) \Delta s \Delta \tau = \int_a^b \int_{\sigma(s)}^{b} F(\tau,s) \Delta \tau \Delta s, \tag{2.10}$$

where $\sigma(s)$ *is the forward jump operator at* s.

3 Main Results

Consider the following equation

$$(p_2(t)(p_1(t)x^{\Delta})^{\Delta})^{\Delta} + f(t, x(t)) = 0, \tag{3.1}$$

and assume that the following hypotheses (denoted by (H)) are satisfied:

(1) $p_1(t), p_2(t) \in C_{rd}$ are positive for all $t \geq t_0$;

(2) $f : T \times R \to R$ satisfies

$$|f(t,x)| \leq \sum_{i=1}^{m} b_i(t) h_i(|x|) + b_{m+1}(t),$$

where $h_i \in \Re$, $b_i \in C_{rd}$ are nonnegative $(i = 1, 2, \cdots, m+1)$;

(3) the uniqueness and the local existence of the solution of (3.1) are valid.

For convenience, for any function $d_i \in C_{rd}$, we define

$$\begin{cases} W_1(t,s;d_1) = \displaystyle\int_s^t \frac{1}{d_1(u)} \Delta u, \\ W_2(t,s;d_1,d_2) = \displaystyle\int_s^t \frac{1}{d_1(u)} W_1(u,s;d_2) \Delta u, \end{cases} \tag{3.2}$$

and from Lemma 2.3, we can conclude another form of $W_2(t,s;d_1,d_2)$:

$$\begin{aligned} W_2(t,s;d_1,d_2) &= \int_s^t \frac{1}{d_1(u)} W_1(u,s;d_2) \Delta u = \int_s^t \int_{\sigma(\tau)}^t \frac{1}{d_2(\tau)} \frac{1}{d_1(u)} \Delta u \Delta \tau \\ &= \int_s^t \frac{1}{d_2(\tau)} W_1(t,\sigma(\tau);d_1) \Delta \tau. \end{aligned}$$

Lemma 3.1 *If $W_2(t,t_0;d_1,d_2)$ is bounded on T_0, then $W_1(t,t_0;d_1)$ is also bounded on T_0; if $\lim\limits_{t\to+\infty} W_1(t,t_0;d_1) = +\infty$, then $\lim\limits_{t\to+\infty} W_2(t,t_0;d_1,d_2) = +\infty$.*

Proof Let $t_1 \in T_0$ be fixed and $t_1 > t_0$, then by (3.2), we can get

$$\begin{aligned} W_2(t,t_0;d_1,d_2) &= \int_{t_0}^{t_1} \frac{1}{d_1(u)} W_1(u,t_0;d_2) \Delta u + \int_{t_1}^{t} \frac{1}{d_1(u)} W_1(u,t_0;d_2) \Delta u \\ &\geq \int_{t_1}^{t} \frac{1}{d_1(u)} W_1(t_1,t_0;d_2) \Delta u = W_1(t_1,t_0;d_2) W_1(t,t_1;d_1), \end{aligned}$$

which implies the validity of Lemma 3.1.

Theorem 3.1 *Suppose that hypotheses* (H) *hold and the following conditions are satisfied:*

(1) $\int_{t_0}^t b_i(s) W_2(t,\sigma(s);p_1,p_2) \Delta s$ *is bounded on* T_0 *for* $1 \leq i \leq m+1$;

(2) $W_2(t,t_0;p_1,p_2)$ *is bounded on* T_0.

Then (i) *every solution* $x(t)$ *of* (3.1) *is bounded on* T_0; (ii) *if* $b_i(t) \in L_1(t_0,+\infty)$ *for* $1 \leq i \leq m+1$, *then* $p_2(t)(p_1(t)x^{\Delta})^{\Delta}$ *is also bounded on* T_0.

Proof (i) Let $x(t)$ be any solution of (3.1) with the initial time $t = t_0$, existing on some maximal interval $I_0 = [t_0, L)$, here $t_0 < L \leq +\infty$. By conditions (H), we can easily see $L = +\infty$.

Integrating (3.1) from t_0 to t, we get

$$(p_1(t)x^{\Delta}(t))^{\Delta} = \frac{p_2(t_0)[p_1(t_0)x^{\Delta}(t_0)]^{\Delta}}{p_2(t)} - \frac{\displaystyle\int_{t_0}^{t} f(s,x(s))\Delta s}{p_2(t)}. \tag{3.3}$$

Integrating (3.3) from t_0 to t, we get

$$x^{\Delta}(t)=\frac{p_1(t_0)x^{\Delta}(t_0)}{p_1(t)}+\frac{p_2(t_0)[p_1(t_0)x^{\Delta}(t_0)]^{\Delta}\displaystyle\int_{t_0}^{t}\frac{1}{p_2(s)}\Delta s}{p_1(t)}-\frac{\displaystyle\int_{t_0}^{t}\frac{1}{p_2(\tau)}\int_{t_0}^{\tau}f(s,x(s))\Delta s\Delta\tau}{p_1(t)}.$$

Integrating this equation again from t_0 to t, and then using Lemma 2.3 we obtain

$$\begin{aligned}
x(t) &= x(t_0)+p_1(t_0)x^{\Delta}(t_0)\int_{t_0}^{t}\frac{1}{p_1(u)}\Delta u+p_2(t_0)[p_1(t_0)x^{\Delta}(t_0)]^{\Delta}\int_{t_0}^{t}\frac{1}{p_1(u)}\int_{t_0}^{u}\frac{1}{p_2(s)}\Delta s\Delta u\\
&\quad -\int_{t_0}^{t}\frac{1}{p_1(u)}\int_{t_0}^{u}\frac{1}{p_2(\tau)}\int_{t_0}^{\tau}f(s,x(s))\Delta s\Delta\tau\Delta u\\
&= x(t_0)+p_1(t_0)x^{\Delta}(t_0)W_1(t,t_0;p_1)+p_2(t_0)[p_1(t_0)x^{\Delta}(t_0)]^{\Delta}W_2(t,t_0;p_1,p_2)\\
&\quad -\int_{t_0}^{t}\int_{\sigma(s)}^{t}\frac{1}{p_1(u)}W_1(u,\sigma(s);p_2)f(s,x(s))\Delta u\Delta s\\
&= x(t_0)+p_1(t_0)x^{\Delta}(t_0)W_1(t,t_0;p_1)+p_2(t_0)[p_1(t_0)x^{\Delta}(t_0)]^{\Delta}W_2(t,t_0;p_1,p_2)\\
&\quad -\int_{t_0}^{t}W_2(t,\sigma(s);p_1,p_2)f(s,x(s))\Delta s,
\end{aligned} \tag{3.4}$$

where W_1 and W_2 are defined as in (3.2).

Now by conditions (H) and (3.4), we can get

$$|x(t)| \le N(t)+\sum_{i=1}^{m}\int_{t_0}^{t}W_2(t,\sigma(s);p_1,p_2)b_i(s)h_i(|x(s)|)\Delta s, \quad t\in T_0, \tag{3.5}$$

where

$$\begin{aligned}
N(t) &= 1+|x(t_0)|+p_1(t_0)W_1(t,t_0;p_1)|x^{\Delta}(t_0)|+p_2(t_0)W_2(t,t_0;p_1,p_2)|[p_1(t_0)x^{\Delta}(t_0)]^{\Delta}|\\
&\quad +\int_{t_0}^{t}W_2(t,\sigma(s);p_1,p_2)b_{m+1}(s)\Delta s.
\end{aligned}$$

Since $W_2(t,t_0;p_1,p_2)$ is bounded on T_0, by Lemma 3.1, $W_1(t,t_0;p_1)$ is bounded on T_0.

By Lemma 2.2 and the last inequality, we conclude that

$$|x(t)| \le N(t)\prod_{i=1}^{m}U_i(t), \quad t\in T_0, \tag{3.6}$$

where

$$U_i(t) = G_i^{-1}\Big[G_i(1) + \int_{t_0}^{t} W_2(t,\sigma(s);p_1,p_2)b_i(s)\Big(\prod_{k=1}^{i-1} U_k(s)\Big)\Delta s\Big]$$
$$\le G_i^{-1}\Big[G_i(1) + \Big(\prod_{k=1}^{i-1} U_k(t)\Big)\int_{t_0}^{t} W_2(t,\sigma(s);p_1,p_2)b_i(s)\Delta s\Big],$$

where G_i is defined in (2.9).

By (3.6) we can easily observe that $x(t)$ is bounded on T_0.

(ii) Moreover, we easily obtain from (3.3) that

$$p_2(t)|(p_1(t)x^{\Delta}(t))^{\Delta}| \le p_2(t_0)|(p_1(t_0)x^{\Delta}(t_0))^{\Delta}| + \int_{t_0}^{t} b_{m+1}(s)\Delta s + \sum_{i=1}^{m} h_i(C)\int_{t_0}^{t} b_i(s)\Delta s,$$

where $|x(t)| \le C$ holds for all $t \in T_0$ by (i), here C is a constant. Hence, if also $b_i(t) \in L_1(t_0,+\infty)$ for $1 \le i \le m+1$, then the boundedness of $p_2(t)(p_1(t)x^{\Delta}(t))^{\Delta}$ follows from the above inequality immediately. The proof of Theorem 3.1 is completed.

Theorem 3.2 *Suppose that hypotheses* (H) *hold and the following conditions are satisfied:*

(1) $\int_{t_0}^{t} W_2(t,\sigma(s);p_1,p_2)b_i(s)\Delta s$ *is bounded on* T_0, $i = 1,2,\cdots,m$;

(2) $\lim\limits_{t\to+\infty} W_1(t,t_0;p_1) = +\infty$, $\lim\limits_{t\to+\infty} W_1(t,t_0;p_2) = +\infty$;

(3) $\int_{t_0}^{+\infty} b_{m+1}(s)\Delta s < +\infty$, $\int_{t_0}^{+\infty} W_2(s,t_0;p_1,p_2)b_i(s)\Delta s < +\infty$, $i = 1,2,\cdots,m$.

Then for any solution of (3.1), *we have* (i) $|x(t)| = O(W_2(t,t_0;p_1,p_2))$ *as* $t \to +\infty$; (ii) $|p_2(t)(p_1(t)x^{\Delta}(t))^{\Delta}| = O(1)$ *as* $t \to +\infty$.

Proof (i) By Theorem 3.1, the solution of (3.1) exists on T_0. Since

$$\lim_{t\to+\infty} W_1(t,t_0;p_1) = +\infty, \qquad \lim_{t\to+\infty} W_1(t,t_0;p_2) = +\infty,$$

we can easily get

$$\lim_{t\to+\infty}\frac{W_1(t,t_0;p_1)}{W_2(t,t_0;p_1,p_2)} = \lim_{t\to+\infty}\frac{\displaystyle\int_{t_0}^{t}\frac{1}{p_1(s)}\Delta s}{\displaystyle\int_{t_0}^{t}\frac{1}{p_1(s)}W_1(s,t_0;p_2)\Delta s} = 0.$$

And by Lemma 3.1, we have $\lim\limits_{t\to+\infty} W_2(t,t_0;p_1,p_2) = +\infty$.

By the definition of $W_2(t,s;p_1,p_2)$, we easily observe that $W_2(t,\sigma(s);p_1,p_2) \le W_2(t,t_0;p_1,p_2)$ when $t_0 \le s \le t$.

From (3.4) in the proof of Theorem 3.1 and conditions (H) we have

$$\begin{aligned}
&\frac{|x(t)|}{W_2(t,t_0;p_1,p_2)}\\
&\le \frac{|x(t_0)|}{W_2(t,t_0;p_1,p_2)}+p_1(t_0)|x^\Delta(t_0)|\frac{W_1(t,t_0;p_1)}{W_2(t,t_0;p_1,p_2)}+p_2(t_0)|[p_1(t_0)x^\Delta(t_0)]^\Delta|\\
&\quad+\int_{t_0}^t\frac{W_2(t,\sigma(s);p_1,p_2)}{W_2(t,t_0;p_1,p_2)}f(s,x(s))\Delta s\\
&\le \frac{|x(t_0)|}{W_2(t,t_0;p_1,p_2)}+p_1(t_0)|x^\Delta(t_0)|\frac{W_1(t,t_0;p_1)}{W_2(t,t_0;p_1,p_2)}+p_2(t_0)|[p_1(t_0)x^\Delta(t_0)]^\Delta|\\
&\quad+\sum_{i=1}^m\int_{t_0}^t\frac{W_2(t,\sigma(s);p_1,p_2)}{W_2(t,t_0;p_1,p_2)}b_i(s)h_i(|x(s)|)\Delta s+\int_{t_0}^t\frac{W_2(t,\sigma(s);p_1,p_2)}{W_2(t,t_0;p_1,p_2)}b_{m+1}(s)\Delta s\\
&\le H(t)+\sum_{i=1}^m\int_{t_0}^t W_2(t,\sigma(s);p_1,p_2)b_i(s)h_i\Big(\frac{|x(s)|}{W_2(s,t_0;p_1,p_2)}\Big)\Delta s, \qquad (3.7)
\end{aligned}$$

where

$$\begin{aligned}
H(t)=1+\frac{|x(t_0)|}{W_2(t,t_0;p_1,p_2)}&+p_1(t_0)|x^\Delta(t_0)|\frac{W_1(t,t_0;p_1)}{W_2(t,t_0;p_1,p_2)}\\
&+p_2(t_0)|[p_1(t_0)x^\Delta(t_0)]^\Delta|+\int_{t_0}^t b_{m+1}(s)\Delta s.
\end{aligned}$$

Now using Lemma 2.2 to the last inequality, we find

$$|x(t)|\le W_2(t,t_0;p_1,p_2)H(t)\prod_{i=1}^m V_i(t),\quad t\in T_0, \qquad (3.8)$$

where

$$\begin{aligned}
V_i(t)&=G_i^{-1}\Big[G_i(1)+\int_{t_0}^t W_2(t,\sigma(s);p_1,p_2)b_i(s)\Big(\prod_{k=1}^{i-1}V_k(s)\Big)\Delta s\Big]\\
&\le G_i^{-1}\Big[G_i(1)+\Big(\prod_{k=1}^{i-1}V_k(t)\Big)\int_{t_0}^t W_2(t,\sigma(s);p_1,p_2)b_i(s)\Delta s\Big],
\end{aligned}$$

where G_i is defined in (2.9).

By conditions of Theorem 3.2 and letting $t\to+\infty$ in (3.8), we obtain the desired relation in (i).

(ii) By (3.8) we derive from (3.3) that

$$\begin{aligned}
&p_2(t)|(p_1(t)x^\Delta(t))^\Delta|\\
&\le p_2(t_0)|(p_1(t_0)x^\Delta(t_0))^\Delta|+\int_{t_0}^t b_{m+1}(s)\Delta s+\sum_{i=1}^m\int_{t_0}^t b_i(s)h_i(|x(s)|)\Delta s
\end{aligned}$$

$$\begin{aligned}
&\le p_2(t_0)|(p_1(t_0)x^{\Delta}(t_0))^{\Delta}| + \int_{t_0}^{t} b_{m+1}(s)\Delta s \\
&\quad + \sum_{i=1}^{m}\int_{t_0}^{t} b_i(s)W_2(s,t_0;p_1,p_2)h_i\Big(\frac{|x(s)|}{W_2(s,t_0;p_1,p_2)}\Big)\Delta s \\
&\le p_2(t_0)|(p_1(t_0)x^{\Delta}(t_0))^{\Delta}| + \int_{t_0}^{t} b_{m+1}(s)\Delta s + \sum_{i=1}^{m} h_i(M)\int_{t_0}^{t} b_i(s)W_2(s,t_0;p_1,p_2)\Delta s,
\end{aligned}$$

where the number $M > 0$ is the upper bound of $H(t)\prod_{i=1}^{m} V_i(t)$ on T_0. Thus the proof of the Theorem 3.2 is now completed.

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(*edited by Liangwei Huang*)