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EVANS FUNCTIONS AND INSTABILITY OF A STANDING PULSE SOLUTION OF A NONLINEAR SYSTEM OF REACTION DIFFUSION EQUATIONS*†

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Abstract

In this paper, we consider a nonlinear system of reaction diffusion equations arising from mathematical neuroscience and two nonlinear scalar reaction diffusion equations under some assumptions on their coefficients.

The main purpose is to couple together linearized stability criterion (the equivalence of the nonlinear stability, the linear stability and the spectral stability of the standing pulse solutions) and Evans functions to accomplish the existence and instability of standing pulse solutions of the nonlinear system of reaction diffusion equations and the nonlinear scalar reaction diffusion equations. The Evans functions for the standing pulse solutions are constructed explicitly.

Keywords nonlinear system of reaction diffusion equations; standing pulse solutions; existence; instability; linearized stability criterion; Evans functions

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1 Introduction

1.1 Mathematical Model Equations

Consider the following nonlinear system of reaction diffusion equations arising from mathematical neuroscience

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha[\beta H(u-\theta) - u] - w, \tag{1.1}$$

$$\frac{\partial w}{\partial t} = \varepsilon(u - \gamma w). \tag{1.2}$$

Also consider the following nonlinear scalar reaction diffusion equations

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$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\alpha[\beta H(u-\theta)-u], \tag{1.3}$$

and

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\alpha[\beta H(u-\theta)-u]-w_0, \tag{1.4}$$

where $w_0=\alpha(\beta-2\theta)>0$ is a positive constant. In these model equations, $u=u(x,t)$ represents the membrane potential of a neuron at position x and time t, $w=w(x,t)$ represents the leaking current, a slow process that controls the excitation of neuron membrane. The positive constants $\alpha>0$, $\beta>0$, $\gamma>0$, $\varepsilon>0$ and $\theta>0$ represent neurobiological mechanisms. The positive constant $\theta>0$ represents the threshold for excitation. The function $H=H(u-\theta)$ represents the Heaviside step function, which is defined by $H(u-\theta)=0$ for all $u<\theta$, $H(0)=1/2$ and $H(u-\theta)=1$ for all $u>\theta$. When an action potential is generated across a neuron membrane, Na^+ ion activation is considerably faster than K^+ ion activation. The positive constant ε represents the ratio of the activation of Na^+ ion channels over the activation of K^+ ion channels. The two nonlinear scalar reaction diffusion equations may be obtained by setting $\varepsilon=0$, $w=0$ and $\varepsilon=0$, $w=w_0$, respectively, in system (1.1)-(1.2). See Feroe [5-7], McKean [8-10], McKean and Moll [11], Rinzel and Keller [12], Rinzel and Terman [13], Terman [14], Wang [15] and [16] for more neurobiological backgrounds of the model system.

1.2 Main Difficulty, Main Purposes and Main Strategy

Note that there exists neither maximum principle nor conservation laws to the nonlinear system of reaction diffusion equations. The existence and instability of standing pulse solutions of the system are very important and interesting topics in applied mathematics, but they have been open for a long time, except for some numerical simulations and some claimed results without rigorous mathematical analysis. The strategy to overcome the main difficulty: coupling together linearized stability criterion and Evans functions seem to be the best way to approach the instability of the standing pulse solutions.

The main purpose of this paper is to accomplish the existence and instability of standing pulse solutions of the nonlinear system of reaction diffusion equations (1.1)-(1.2) and the nonlinear scalar reaction diffusion equation (1.3). The existence of the standing pulse solutions of both (1.1)-(1.2) and (1.3) follows from standard ideas, methods and techniques in dynamical systems. The instability of the standing pulse solutions will be accomplished by coupling together linearized stability criterion and Evans functions. The interesting and difficult points are that the eigenvalue problems obtained by using linearization technique and the method of separation of variables involve the Dirac delta impulse functions. This makes it very difficult to establish the equivalence of the nonlinear stability, the linear stability and the spectral stability of the standing pulse solutions. Another very interesting point is that the parameter ε plays no role in the existence of the standing pulse solutions, but it plays a very important role in the instability of the standing pulse solutions.

The construction and application of Evans functions to stability analysis of s-tanding pulse solutions of the nonlinear system of reaction diffusion equations (1.1)-(1.2) have been open for a long time. This paper aims to provide positive solutions to the open problems. The introduction of the Evans function and the study of the instability of the standing pulse solutions have strong impacts on stability of fast multiple traveling pulse solutions. Mathematically and biologically, these are very important/interesting problems. We believe that the same ideas also work for the existence and stability of fast multiple traveling pulse solutions and the existence and instability of slow multiple traveling pulse solutions of system (1.1)-(1.2).

1.3 Main Results

Theorem 1 (I) *Suppose that the positive constants* $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\varepsilon > 0$ *and* $\theta > 0$ *satisfy the conditions* $0 < 2(1+\alpha\gamma)\theta < \alpha\beta\gamma$. *Then there exists a unique standing pulse solution* $(U, W) \in C^1(\mathrm{R}) \cap C^2(\mathrm{R} - \{0\})$ *to the nonlinear system of reaction diffusion equations* (1.1)-(1.2). *The standing pulse solution is given explicitly by*

$$\begin{aligned}
U(x) &= \theta \exp\left(\sqrt{\alpha + \frac{1}{\gamma}}x\right), \quad \textit{on } (-\infty, 0),\\
U(x) &= \frac{\alpha\beta\gamma}{1+\alpha\gamma} + \left(\theta - \frac{\alpha\beta\gamma}{2(1+\alpha\gamma)}\right)\exp\left(\sqrt{\alpha+\frac{1}{\gamma}}x\right)\\
&\quad -\frac{\alpha\beta\gamma}{2(1+\alpha\gamma)}\exp\left(-\sqrt{\alpha+\frac{1}{\gamma}}x\right), \qquad \textit{on } (0, Z_0),\\
U(x) &= \theta \exp\left[-\sqrt{\alpha+\frac{1}{\gamma}}(x - Z_0)\right] \quad \textit{on } (Z_0, \infty),\\
Z_0 &= \frac{1}{\sqrt{\alpha + \frac{1}{\gamma}}} \ln \frac{\alpha\beta\gamma}{\alpha\beta\gamma - 2(1+\alpha\gamma)\theta}.
\end{aligned}$$

(II) *The standing pulse solution is unstable.*

(III) *Suppose that the positive constants* $\alpha > 0$, $\beta > 0$ *and* $\theta > 0$ *satisfy the conditions* $0 < 2\theta < \beta$. *Then there exists a unique unstable standing pulse solution* $U \in C^1(\mathrm{R}) \cap C^2(\mathrm{R} - \{0\})$ *to the nonlinear scalar reaction diffusion equation* (1.3). *The standing pulse solution is given explicitly by*

$$U(x) = \theta \exp(\sqrt{\alpha} x), \quad on\ (-\infty, 0),$$
$$U(x) = \beta + \left(\theta - \frac{\beta}{2}\right) \exp(\sqrt{\alpha} x) - \frac{\beta}{2} \exp(-\sqrt{\alpha} x), \quad on\ (0, Z_0),$$
$$U(x) = \theta \exp[-\sqrt{\alpha}(x - Z_0)], \quad on\ (Z_0, \infty),$$
$$Z_0 = \frac{1}{\sqrt{\alpha}} \ln \frac{\beta}{\beta - 2\theta}.$$

1.4 Previous Related Results

In early papers, the existence and instability of the standing pulse solutions had been announced to be true. For the existence and instability of the standing pulse solutions, the previous analysis missed a few key points. One must make reasonable assumptions on the constants in the model equations to establish the existence. This was missing in the old papers. One must establish the existence of finitely many points where the pulse crosses the threshold and find the intervals where the pulse lies above or below the threshold. One also has to prove that the pulse is continuously differentiable everywhere on R, particularly at the points where the pulse crosses the threshold. However, they did not do these things. That is why the previous existence analysis is not rigorously correct. The existence of the standing pulse solutions may be proved by applying standard ideas, methods and techniques in dynamical systems to guarantee that all of the above requirements are valid. For the instability, there have been no rigorous mathematical analysis on the essential spectrum or the eigenvalues of some associated linear differential operator obtained from the linearization of the nonlinear system of reaction diffusion equations about the standing pulse solution.

2 Mathematical Analysis and Proofs of the Main Results

The main purpose of this section is to accomplish the existence and instability of the standing pulse solutions. We will couple together linearized stability criterion and Evans functions (complex analytic functions) to accomplish the nonlinear instability of the standing pulse solutions.

Let $0 < (1 + \alpha\gamma)\theta < \alpha\beta\gamma$. There exist two stable constant solutions $(U, W) = (0, 0)$ and $(U, W) = (\frac{\alpha\beta\gamma}{1+\alpha\gamma}, \frac{\alpha\beta}{1+\alpha\gamma})$.

1. The Existence. First of all, let us establish the existence of the standing pulse solution to the nonlinear system of reaction diffusion equations (1.1)-(1.2). A standing pulse solution to system (1.1)-(1.2) satisfies the following differential equations

$$U'' + \alpha[\beta H(U-\theta) - U] - W = 0,$$
$$\varepsilon(U - \gamma W) = 0.$$

That is

$$U'' - \left(\alpha + \frac{1}{\gamma}\right) U + \alpha\beta H(U-\theta) = 0.$$

Suppose that there exists a positive constant $Z > 0$, such that

$$U(0) = \theta, \quad U'(0) > 0, \quad U(Z) = \theta, \quad U'(Z) < 0,$$
$$U > \theta \quad \text{on } (0, Z), \quad U < \theta \quad \text{on } (-\infty, 0) \cup (Z, \infty).$$

Then the above differential equation reduces to

$$U'' - \left(\alpha + \frac{1}{\gamma}\right) U + \alpha\beta = 0, \quad \text{on } (0, Z),$$
$$U'' - \left(\alpha + \frac{1}{\gamma}\right) U = 0, \quad \text{on } (-\infty, 0) \cup (Z, \infty),$$

subject to the boundary conditions $U(0) = \theta$ and $U(Z) = \theta$. Solving these differential equations and using the boundary conditions lead to the solution representation

$$U(x) = \theta \exp\left(\sqrt{\alpha + \frac{1}{\gamma}} x\right), \quad \text{on } (-\infty, 0),$$
$$U(x) = \frac{\alpha\beta\gamma}{1+\alpha\gamma} + C_1 \exp\left(\sqrt{\alpha + \frac{1}{\gamma}} x\right) + C_2 \exp\left(-\sqrt{\alpha + \frac{1}{\gamma}} x\right), \quad \text{on } (0, Z),$$
$$U(x) = \theta \exp\left[-\sqrt{\alpha + \frac{1}{\gamma}} (x - Z)\right], \quad \text{on } (Z, \infty),$$

where C_1, C_2 and Z are real unknown constants. Now let us find the constants. First of all, $U(0) = \theta$ and $U(Z) = \theta$, that is

$$\frac{\alpha\beta\gamma}{1+\alpha\gamma} + C_1 + C_2 = \theta,$$
$$\frac{\alpha\beta\gamma}{1+\alpha\gamma} + C_1 \exp\left(\sqrt{\alpha + \frac{1}{\gamma}} Z\right) + C_2 \exp\left(-\sqrt{\alpha + \frac{1}{\gamma}} Z\right) = \theta.$$

Second, the standing pulse solution is continuously differentiable everywhere, particularly, at $x = 0$ and $x = Z$. Hence

$$\lim_{x \to 0^-} U'(x) = \lim_{x \to 0^+} U'(x), \quad \lim_{x \to Z^-} U'(x) = \lim_{x \to Z^+} U'(x).$$

That is

$$\sqrt{\alpha+\frac{1}{\gamma}}\theta=C_1\sqrt{\alpha+\frac{1}{\gamma}}-C_2\sqrt{\alpha+\frac{1}{\gamma}},$$
$$-\sqrt{\alpha+\frac{1}{\gamma}}\theta=C_1\sqrt{\alpha+\frac{1}{\gamma}}\exp\left(\sqrt{\alpha+\frac{1}{\gamma}}Z\right)-C_2\sqrt{\alpha+\frac{1}{\gamma}}\exp\left(-\sqrt{\alpha+\frac{1}{\gamma}}Z\right).$$

Coupling all of these equations together yields

$$C_1=\theta-\frac{\alpha\beta\gamma}{2(1+\alpha\gamma)},\quad C_2=-\frac{\alpha\beta\gamma}{2(1+\alpha\gamma)},$$
$$Z=\frac{1}{\sqrt{\alpha+\frac{1}{\gamma}}}\ln\frac{\alpha\beta\gamma}{\alpha\beta\gamma-2(1+\alpha\gamma)\theta}.$$

This completes the proof of the existence and uniqueness of the single standing pulse solution to system (1.1)-(1.2). A standing pulse solution to the nonlinear scalar reaction diffusion equation (1.3) satisfies the following differential equation

$$U''+\alpha[\beta H(U-\theta)-U]=0.$$

The existence and uniqueness of the standing pulse solution to the nonlinear scalar reaction diffusion equation (1.3) may be established very similarly.

Next let us study the instability of the standing pulse solution to the nonlinear system of reaction diffusion equations (1.1)-(1.2).

2. The eigenvalue problems. Let $(P(x,t),Q(x,t))=(u(x,t),w(x,t))$. Then (1.1)-(1.2) becomes

$$\frac{\partial P}{\partial t}=\frac{\partial^2 P}{\partial x^2}+\alpha[\beta H(P-\theta)-P]-Q,$$
$$\frac{\partial Q}{\partial t}=\varepsilon(P-\gamma Q).$$

The standing pulse solution $(U,W)=(U(x),W(x))$ is a stationary solution to this system. Linearizing the nonlinear system about the standing pulse solution to get

$$\frac{\partial p}{\partial t}=\frac{\partial^2 p}{\partial x^2}+\alpha[\beta\delta(U-\theta)p-p]-q,$$
$$\frac{\partial q}{\partial t}=\varepsilon(p-\gamma q).$$

Suppose that $(p(x,t),q(x,t))=\exp(\lambda t)(\psi_1(x),\psi_2(x))$ is a complex solution to this linear system of differential equations, where λ is a complex number, ψ_1 and ψ_2 are complex, bounded, continuous functions defined on R. This leads to the following eigenvalue problem

$$\lambda\psi_1 = \psi_1'' - \alpha\psi_1 - \psi_2 + \alpha\beta\delta(U-\theta)\psi_1,$$
$$\lambda\psi_2 = \varepsilon(\psi_1 - \gamma\psi_2).$$

Define a linear differential operator $\mathcal{L}$ by

$$\mathcal{L}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}\psi_1'' - \alpha\psi_1 - \psi_2 + \alpha\beta\delta(U-\theta)\psi_1\\ \varepsilon(\psi_1 - \gamma\psi_2)\end{pmatrix}.$$

Definition 1 If there exists a complex number λ_0 and there exists a complex, vector valued, bounded, continuous function $\psi_0(\lambda_0, x) = \begin{pmatrix}\psi_{01}(\lambda_0, x)\\ \psi_{02}(\lambda_0, x)\end{pmatrix}$ on R, such that $\mathcal{L}\psi_0 = \lambda\psi_0$, then λ_0 is called an eigenvalue and $\psi_0 = \begin{pmatrix}\psi_{01}\\ \psi_{02}\end{pmatrix}$ is called an eigenfunction of the eigenvalue problem.

To see that $\lambda_0 = 0$ is an eigenvalue and $(\psi_1(x), \psi_2(x)) = (U'(x), W'(x))$ is an eigenfunction of the eigenvalue problem, let us differentiate the standing pulse equations

$$U'' + \alpha[\beta H(U-\theta) - U] - W = 0,$$
$$\varepsilon(U - \gamma W) = 0,$$

with respect to x to get

$$U''' - \alpha U' - W' + \alpha\beta\delta(U-\theta)U' = 0,$$
$$\varepsilon(U' - \gamma W') = 0.$$

Definition 2 (I) The standing pulse solution to the nonlinear system of reaction diffusion equations (1.1)-(1.2) is stable, if $\max\{\mathrm{Re}\lambda : \lambda \in \sigma(\mathcal{L}), \lambda \neq 0\} \leq -C_0$ and $\lambda_0 = 0$ is an algebraically simple eigenvalue, where $\sigma(\mathcal{L})$ represents the spectrum of the linear differential operator $\mathcal{L}$ and $C_0 > 0$ is a positive constant.

(II) The standing pulse solution to the nonlinear system of reaction diffusion equations (1.1)-(1.2) is unstable, if there exists an eigenvalue λ_0 with positive real part or if the neutral eigenvalue $\lambda_0 = 0$ is not simple.

Following John Evans' idea in [2], the essential spectrum of the linear differential operator $\mathcal{L}$ is easy to find and it is given by

$$\sigma_{\text{essential}}(\mathcal{L}) = \{\lambda \in \mathrm{C} : \lambda = \lambda_1(r) \text{ or } \lambda = \lambda_2(r), r \in \mathrm{R}\},$$

where

$$\lambda_1(r) = -\frac{1}{2}\big[\alpha + r^2 + \gamma\varepsilon + \sqrt{(\alpha + r^2 - \gamma\varepsilon)^2 - 4\varepsilon}\big],$$
$$\lambda_2(r) = -\frac{1}{2}\big[\alpha + r^2 + \gamma\varepsilon - \sqrt{(\alpha + r^2 - \gamma\varepsilon)^2 - 4\varepsilon}\big].$$

It is easy to find that the essential spectrum of $\mathcal{L}$ causes no problem to the stability of the standing pulse solution to the nonlinear system of reaction diffusion equations (1.1)-(1.2).

Let us find a simpler equation which is equivalent to the eigenvalue problem $\lambda\psi_1 = \psi_1'' - \alpha\psi_1 - \psi_2 + \alpha\beta\delta(U-\theta)\psi_1$, $\lambda\psi_2 = \varepsilon(\psi_1 - \gamma\psi_2)$, so that we can solve it to find a solution to the eigenvalue problem. Recall that the standing pulse solution satisfies the conditions: $U(0) = \theta$, $U'(0) > 0$, $U(Z_0) = \theta$, $U'(Z_0) < 0$, $U > \theta$ on $(0, Z_0)$ and $U < \theta$ on $(-\infty, 0) \cup (Z_0, \infty)$. Thus

$$H(U(x) - \theta) = H(x) - H(x - Z_0),$$

everywhere on R. Differentiating this equation with respect to x yields

$$\delta(U(x) - \theta)U'(x) = \delta(x) - \delta(x - Z_0).$$

That is

$$\delta(U(x) - \theta) = \frac{\delta(x) - \delta(x - Z_0)}{U'(x)},$$

for all x such that $U'(x) \neq 0$. Moreover, we have

$$\delta(U(x) - \theta)\psi(x) = \frac{\psi(x)}{U'(x)}[\delta(x) - \delta(x - Z_0)] = \frac{\psi(0)}{U'(0)}\delta(x) - \frac{\psi(Z_0)}{U'(Z_0)}\delta(x - Z_0),$$

for all complex valued functions ψ defined on R, in the sense of tempered distributions. Therefore, if we write $\psi_1 = \psi$ and use the relationship $\psi_2 = \frac{\varepsilon}{\lambda+\gamma\varepsilon}\psi_1$, then the eigenvalue problem $\lambda\psi_1 = \psi_1'' - \alpha\psi_1 - \psi_2 + \alpha\beta\delta(U(x) - \theta)\psi_1$, $\lambda\psi_2 = \varepsilon(\psi_1 - \gamma\psi_2)$ becomes

$$\lambda\psi = \psi'' - \alpha\psi - \frac{\varepsilon}{\lambda + \gamma\varepsilon}\psi + \alpha\beta\left[\frac{\psi(0)}{U'(0)}\delta(x) - \frac{\psi(Z_0)}{U'(Z_0)}\delta(x - Z_0)\right].$$

3. The solutions of the eigenvalue problems. Define $\Omega = \{\lambda \in \mathrm{C} : \mathrm{Re}\lambda > -\gamma\varepsilon\}$. The eigenvalue problem may be written as a nonhomogeneous, first order, linear system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\psi\\ \psi'\end{pmatrix} = \begin{pmatrix}0 & 1\\ \alpha + \lambda + \frac{\varepsilon}{\lambda+\gamma\varepsilon} & 0\end{pmatrix}\begin{pmatrix}\psi\\ \psi'\end{pmatrix} - \alpha\beta\left[\frac{\psi(0)}{U'(0)}\delta(x) - \frac{\psi(Z_0)}{U'(Z_0)}\delta(x - Z_0)\right]\begin{pmatrix}0\\ 1\end{pmatrix}.$$

For all complex numbers λ with $\mathrm{Re}\lambda > -\gamma\varepsilon$, the solution of the homogeneous system

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\psi\\ \psi'\end{pmatrix} = \begin{pmatrix}0 & 1\\ \alpha + \lambda + \frac{\varepsilon}{\lambda+\gamma\varepsilon} & 0\end{pmatrix}\begin{pmatrix}\psi\\ \psi'\end{pmatrix}$$

is given by

$$\begin{pmatrix}\psi(\lambda,\varepsilon,x)\\ \psi_x(\lambda,\varepsilon,x)\end{pmatrix} = C_1\exp\left(\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)\begin{pmatrix}1\\ \sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\end{pmatrix}$$
$$+C_2\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)\begin{pmatrix}1\\ -\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\end{pmatrix},$$

where C_1 and C_2 are constants.

Let us diagonalize the coefficient matrix. Define

$$T(\lambda,\varepsilon) = \begin{pmatrix}1 & 1\\ \sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}} & -\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\end{pmatrix}.$$

Then the inverse matrix exists and it is given by

$$[T(\lambda,\varepsilon)]^{-1} = \frac{1}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}}\begin{pmatrix}\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}} & 1\\ \sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}} & -1\end{pmatrix}.$$

Therefore

$$[T(\lambda,\varepsilon)]^{-1}\begin{pmatrix}0 & 1\\ \alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon} & 0\end{pmatrix}T(\lambda,\varepsilon) = \begin{pmatrix}\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}} & 0\\ 0 & -\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\end{pmatrix}.$$

Clearly

$$X_1(\lambda,\varepsilon,x) =$$
$$\begin{pmatrix}\exp\left(\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right) & \exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)\\ \sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\exp\left(\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right) & -\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)\end{pmatrix},$$
$$X_2(\lambda,\varepsilon,x) =$$
$$\begin{pmatrix}\exp\left[\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}(x-Z_0)\right] & \exp\left[-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}(x-Z_0)\right]\\ \sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\exp\left[\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}(x-Z_0)\right] & -\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\exp\left[-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}(x-Z_0)\right]\end{pmatrix}$$

are two fundamental matrices of the homogeneous system.

Let us use the method of variation of parameters and the two fundamental matrices to find a bounded particular solution. The particular solution to the non-homogeneous system is given by

$$\frac{\alpha\beta\psi(\lambda,\varepsilon,0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)\begin{pmatrix}1\\ \sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\end{pmatrix}[1-H(x)]$$
$$+\frac{\alpha\beta\psi(\lambda,\varepsilon,0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)\begin{pmatrix}1\\ -\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\end{pmatrix}H(x)$$
$$-\frac{\alpha\beta\psi(\lambda,\varepsilon,Z_0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(Z_0)}\exp\left[\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}(x-Z_0)\right]\begin{pmatrix}1\\ \sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\end{pmatrix}[1-H(x-Z_0)]$$
$$-\frac{\alpha\beta\psi(\lambda,\varepsilon,Z_0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(Z_0)}\exp\left[-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}(x-Z_0)\right]\begin{pmatrix}1\\ -\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\end{pmatrix}H(x-Z_0).$$

Therefore, the general solution to the eigenvalue problem is given by

$$\begin{aligned}\begin{pmatrix}\psi(\lambda,\varepsilon,x)\\ \psi_x(\lambda,\varepsilon,x)\end{pmatrix}=&C_1(\lambda,\varepsilon)\exp\left(\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)\begin{pmatrix}1\\ \sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\end{pmatrix}\\
&+C_2(\lambda,\varepsilon)\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)\begin{pmatrix}1\\ -\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\end{pmatrix}\\
&+\frac{\alpha\beta\psi(\lambda,\varepsilon,0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)\begin{pmatrix}1\\ \sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\end{pmatrix}[1-H(x)]\\
&+\frac{\alpha\beta\psi(\lambda,\varepsilon,0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)\begin{pmatrix}1\\ -\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\end{pmatrix}H(x)\\
&-\frac{\alpha\beta\psi(\lambda,\varepsilon,Z_0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(Z_0)}\exp\left[\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}(x-Z_0)\right]\\
&\cdot\begin{pmatrix}1\\ \sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\end{pmatrix}[1-H(x-Z_0)]\\
&-\frac{\alpha\beta\psi(\lambda,\varepsilon,Z_0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(Z_0)}\exp\left[-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}(x-Z_0)\right]\\
&\cdot\begin{pmatrix}1\\ -\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\end{pmatrix}H(x-Z_0),\end{aligned}$$

where $C_1(\lambda,\varepsilon)$ and $C_2(\lambda,\varepsilon)$ are independent of x, but depend on the parameters λ and ε. The general solution to the eigenvalue problem is bounded on R if and only if

$$\begin{pmatrix} C_1(\lambda,\varepsilon) \\ C_2(\lambda,\varepsilon) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The first component of the general solution to the eigenvalue problem is given by

$$\begin{aligned}
\psi(\lambda,\varepsilon,x) = {} & C_1(\lambda,\varepsilon)\exp\left(\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)+C_2(\lambda,\varepsilon)\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right) \\
& +\frac{\alpha\beta\psi(\lambda,\varepsilon,0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)[1-H(x)] \\
& +\frac{\alpha\beta\psi(\lambda,\varepsilon,0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)H(x) \\
& -\frac{\alpha\beta\psi(\lambda,\varepsilon,Z_0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(Z_0)}\exp\left[\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}(x-Z_0)\right][1-H(x-Z_0)] \\
& -\frac{\alpha\beta\psi(\lambda,\varepsilon,Z_0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(Z_0)}\exp\left[-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}(x-Z_0)\right]H(x-Z_0).
\end{aligned}$$

Let us find the relationship between $\begin{pmatrix} C_1(\lambda,\varepsilon) \\ C_2(\lambda,\varepsilon) \end{pmatrix}$ and $\begin{pmatrix} \psi(\lambda,\varepsilon,0) \\ \psi(\lambda,\varepsilon,Z_0) \end{pmatrix}$. Letting $x=0$ and $x=Z_0$, respectively, in the first component of the general solution leads to

$$\begin{aligned}
\psi(\lambda,\varepsilon,0) = {} & C_1(\lambda,\varepsilon)+C_2(\lambda,\varepsilon)+\frac{\alpha\beta\psi(\lambda,\varepsilon,0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)} \\
& -\frac{\alpha\beta\psi(\lambda,\varepsilon,Z_0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(Z_0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right), \\
\psi(\lambda,\varepsilon,Z_0) = {} & C_1(\lambda,\varepsilon)\exp\left(\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right)+C_2(\lambda,\varepsilon)\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right) \\
& +\frac{\alpha\beta\psi(\lambda,\varepsilon,0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right)-\frac{\alpha\beta\psi(\lambda,\varepsilon,Z_0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(Z_0)}.
\end{aligned}$$

These equations may be written in another way

$$\begin{aligned}
C_1(\lambda,\varepsilon)+C_2(\lambda,\varepsilon) = {} & \left[1-\frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\right]\psi(\lambda,\varepsilon,0) \\
& +\left[\frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(Z_0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right)\right]\psi(\lambda,\varepsilon,Z_0),
\end{aligned}$$

$$
\begin{aligned}
&C_1(\lambda,\varepsilon)\exp\left(\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right)+C_2(\lambda,\varepsilon)\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right)\\
=&\left[-\frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right)\right]\psi(\lambda,\varepsilon,0)\\
&+\left[1+\frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(Z_0)}\right]\psi(\lambda,\varepsilon,Z_0).
\end{aligned}
$$

Let us investigate how $\begin{pmatrix}C_1(\lambda,\varepsilon)\\ C_2(\lambda,\varepsilon)\end{pmatrix}$ and $\begin{pmatrix}\psi(\lambda,\varepsilon,0)\\ \psi(\lambda,\varepsilon,Z_0)\end{pmatrix}$ depend on each other. Rewriting the system in a nice way by using matrices and vectors yields the following relationship between $\begin{pmatrix}C_1(\lambda,\varepsilon)\\ C_2(\lambda,\varepsilon)\end{pmatrix}$ and $\begin{pmatrix}\psi(\lambda,\varepsilon,0)\\ \psi(\lambda,\varepsilon,Z_0)\end{pmatrix}$:

$$
\begin{aligned}
&\begin{pmatrix}1 & 1\\ \exp\left(\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right) & \exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right)\end{pmatrix}\begin{pmatrix}C_1(\lambda,\varepsilon)\\ C_2(\lambda,\varepsilon)\end{pmatrix}=\\
&\begin{pmatrix}1-\frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)} & \frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(Z_0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right)\\ -\frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right) & 1+\frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(Z_0)}\end{pmatrix}\\
&\cdot\begin{pmatrix}\psi(\lambda,\varepsilon,0)\\ \psi(\lambda,\varepsilon,Z_0)\end{pmatrix}.
\end{aligned}
$$

Note that $\begin{pmatrix}\psi(\lambda,\varepsilon,0)\\ \psi(\lambda,\varepsilon,Z_0)\end{pmatrix}\neq\begin{pmatrix}0\\0\end{pmatrix}$. Therefore $\begin{pmatrix}C_1(\lambda,\varepsilon)\\ C_2(\lambda,\varepsilon)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ if and only if the system of linear equations

$$
\begin{aligned}
&\begin{pmatrix}1-\frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)} & \frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(Z_0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right)\\ -\frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right) & 1+\frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(Z_0)}\end{pmatrix}\\
&\cdot\begin{pmatrix}\psi(\lambda,\varepsilon,0)\\ \psi(\lambda,\varepsilon,Z_0)\end{pmatrix}\\
=&\begin{pmatrix}1 & 1\\ \exp\left(\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right) & \exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right)\end{pmatrix}\begin{pmatrix}C_1(\lambda,\varepsilon)\\ C_2(\lambda,\varepsilon)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}
\end{aligned}
$$

has a nontrivial solution $\begin{pmatrix}\psi(\lambda,\varepsilon,0)\\ \psi(\lambda,\varepsilon,Z_0)\end{pmatrix}\neq\begin{pmatrix}0\\0\end{pmatrix}$.

4. The Evans function and its representation. Define the Evans function for the standing pulse solution to the nonlinear system of reaction diffusion equations

(1.1)-(1.2) by

$$\begin{aligned}\mathcal{E}_{\text{pulse}}(\lambda,\varepsilon) &= \left[1-\frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\right]\left[1+\frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(Z_0)}\right]\\ &\quad+\left[\frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right)\right]\\ &\quad\cdot\left[\frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(Z_0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right)\right]\\ &= \left[1-\frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\right]^2\\ &\quad-\left[\frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right)\right]^2,\end{aligned}$$

for all complex numbers $\lambda\in\Omega=\{\lambda\in \mathrm{C}\colon \operatorname{Re}\lambda>-\gamma\varepsilon\}$, where we have used the facts $U'(0)=-U'(Z_0)=\sqrt{\alpha+\frac{1}{\gamma}}\theta$.

Define the following auxiliary functions

$$F(\lambda,\varepsilon)=\frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}+\frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right),$$

$$G(\lambda,\varepsilon)=\frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}-\frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right).$$

Then

$$\mathcal{E}_{\text{pulse}}(\lambda,\varepsilon)=[1-F(\lambda,\varepsilon)][1-G(\lambda,\varepsilon)].$$

Recall that

$$U'(0)=\sqrt{\alpha+\frac{1}{\gamma}}\theta,\qquad Z_0=\frac{1}{\sqrt{\alpha+\frac{1}{\gamma}}}\ln\frac{\alpha\beta\gamma}{\alpha\beta\gamma-2(1+\alpha\gamma)\theta}.$$

First of all, we have

$$\begin{aligned}F(0,\varepsilon)&=\frac{\alpha\beta}{2\sqrt{\alpha+\frac{1}{\gamma}}U'(0)}+\frac{\alpha\beta}{2\sqrt{\alpha+\frac{1}{\gamma}}U'(0)}\exp\left(-\sqrt{\alpha+\frac{1}{\gamma}}Z_0\right)\\ &=\frac{\alpha\beta\gamma}{2(1+\alpha\gamma)\theta}+\frac{\alpha\beta\gamma-2(1+\alpha\gamma)\theta}{2(1+\alpha\gamma)\theta}=\frac{\alpha\beta\gamma}{(1+\alpha\gamma)\theta}-1>1,\end{aligned}$$

$$\begin{aligned}G(0,\varepsilon)&=\frac{\alpha\beta}{2\sqrt{\alpha+\frac{1}{\gamma}}U'(0)}-\frac{\alpha\beta}{2\sqrt{\alpha+\frac{1}{\gamma}}U'(0)}\exp\left(-\sqrt{\alpha+\frac{1}{\gamma}}Z_0\right)\\&=\frac{\alpha\beta\gamma}{2(1+\alpha\gamma)\theta}-\frac{\alpha\beta\gamma-2(1+\alpha\gamma)\theta}{2(1+\alpha\gamma)\theta}=1.\end{aligned}$$

Then we have the derivatives

$$\begin{aligned}\frac{\partial}{\partial\lambda}F(\lambda,\varepsilon)=&-\frac{\alpha\beta}{4\left(\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}\right)\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\left[1-\frac{\varepsilon}{(\lambda+\gamma\varepsilon)^2}\right]\\&-\frac{\alpha\beta}{4\left(\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}\right)\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right)\left[1-\frac{\varepsilon}{(\lambda+\gamma\varepsilon)^2}\right]\\&-\frac{\alpha\beta Z_0}{4\left(\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}\right)U'(0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right)\left[1-\frac{\varepsilon}{(\lambda+\gamma\varepsilon)^2}\right]\\=&-\frac{\alpha\beta}{4\left(\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}\right)\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\\&\cdot\left\{1+\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right)+\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right.\\&\left.\cdot\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right)\right\}\left[1-\frac{\varepsilon}{(\lambda+\gamma\varepsilon)^2}\right],\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial\lambda}G(\lambda,\varepsilon)=&-\frac{\alpha\beta}{4\left(\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}\right)\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\left[1-\frac{\varepsilon}{(\lambda+\gamma\varepsilon)^2}\right]\\&+\frac{\alpha\beta}{4\left(\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}\right)\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right)\left[1-\frac{\varepsilon}{(\lambda+\gamma\varepsilon)^2}\right]\\&+\frac{\alpha\beta Z_0}{4\left(\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}\right)U'(0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right)\left[1-\frac{\varepsilon}{(\lambda+\gamma\varepsilon)^2}\right]\\=&-\frac{\alpha\beta}{4\left(\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}\right)\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\\&\cdot\left\{1+\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\right)-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right.\\&\left.\cdot\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right)\right\}\left[1-\frac{\varepsilon}{(\lambda+\gamma\varepsilon)^2}\right].\end{aligned}$$

$$\frac{\partial}{\partial \lambda}\left(\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}\right)=1-\frac{\varepsilon}{(\lambda+\gamma\varepsilon)^2}.$$

There is a critical number $\lambda_c=-\gamma\varepsilon+\sqrt{\varepsilon}$. Obviously, $\lambda_c<0$ if $\gamma^2\varepsilon>1$ and $\lambda_c>0$ if $0<\gamma^2\varepsilon<1$.

Let us study the zeros of the Evans function. It is easy to find that

$$\mathcal{E}_{\rm pulse}(\lambda,\varepsilon)=0,$$

if and only if $F(\lambda,\varepsilon)=1$ or $G(\lambda,\varepsilon)=1$, that is $\lambda=0$ or $\lambda=\lambda_0(\varepsilon)$, where $\lambda_0(\varepsilon)$ is the unique solution of the equation

$$\frac{\alpha\beta}{2\sqrt{\alpha+\lambda_0(\varepsilon)+\frac{\varepsilon}{\lambda_0(\varepsilon)+\gamma\varepsilon}}U'(0)}+\frac{\alpha\beta}{2\sqrt{\alpha+\lambda_0(\varepsilon)+\frac{\varepsilon}{\lambda_0(\varepsilon)+\gamma\varepsilon}}U'(0)}$$
$$\cdot\exp\left[-\sqrt{\alpha+\lambda_0(\varepsilon)+\frac{\varepsilon}{\lambda_0(\varepsilon)+\gamma\varepsilon}}Z_0\right]=1.$$

Note that

$$\frac{\alpha\beta}{2\sqrt{\alpha+\frac{1}{\gamma}}U'(0)}+\frac{\alpha\beta}{2\sqrt{\alpha+\frac{1}{\gamma}}U'(0)}\exp\left(-\sqrt{\alpha+\frac{1}{\gamma}}Z_0\right)=\frac{\alpha\beta}{(\alpha+\frac{1}{\gamma})\theta}-1>1.$$

The existence and uniqueness of such a real solution $\lambda_0(\varepsilon)$ may be proved by using intermediate value theorem and mean value theorem. Let $\gamma^2\varepsilon>1$. Then $-\gamma\varepsilon<\lambda_0(\varepsilon)<0$. Let $0<\gamma^2\varepsilon<1$. Then $\lambda_0(\varepsilon)>0$.

5. The stability/instability of the standing pulse solutions. The linearized stability criterion: The nonlinear stability of the standing pulse solution to the nonlinear system of reaction diffusion equations (1.1)-(1.2) is equivalent to its linear stability, which is equivalent to the spectral stability.

By using the definitions of the stability and instability of the standing pulse solution to the nonlinear system of reaction diffusion equations (1.1)-(1.2) and by using the linearized stability criterion, we find that the standing pulse solution is unstable.

6. The instability of the standing pulse solution to the nonlinear scalar reaction diffusion equation (1.3). For the standing pulse solution to the nonlinear scalar reaction diffusion equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\alpha[\beta H(u-\theta)-u].$$

the eigenvalue problem is

$$\lambda\psi=\psi''-\alpha\psi+\alpha\beta\left[\frac{\psi(0)}{U'(0)}\delta(x)-\frac{\psi(Z_0)}{U'(Z_0)}\delta(x-Z_0)\right].$$

The linear differential operator $\mathcal{L}_0$ is defined by

$$\mathcal{L}_0\psi = \psi'' - \alpha\psi + \alpha\beta\left[\frac{\psi(0)}{U'(0)}\delta(x) - \frac{\psi(Z_0)}{U'(Z_0)}\delta(x - Z_0)\right].$$

The essential spectrum of the operator $\mathcal{L}_0$ is

$$\sigma_{\text{essential}}(\mathcal{L}_0) = \{\lambda = -\alpha - r^2 : r \in \mathrm{R}\}.$$

The eigenvalue problem may be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\psi\\ \psi'\end{pmatrix} = \begin{pmatrix}0 & 1\\ \alpha+\lambda & 0\end{pmatrix}\begin{pmatrix}\psi\\ \psi'\end{pmatrix} - \alpha\beta\left[\frac{\psi(0)}{U'(0)}\delta(x) - \frac{\psi(Z_0)}{U'(Z_0)}\delta(x - Z_0)\right]\begin{pmatrix}0\\ 1\end{pmatrix}.$$

The general solution is given by

$$\begin{aligned}\begin{pmatrix}\psi(\lambda,x)\\ \psi_x(\lambda,x)\end{pmatrix} &= C_1(\lambda)\exp(\sqrt{\alpha+\lambda}x)\begin{pmatrix}1\\ \sqrt{\alpha+\lambda}\end{pmatrix} + C_2(\lambda)\exp(-\sqrt{\alpha+\lambda}x)\begin{pmatrix}1\\ -\sqrt{\alpha+\lambda}\end{pmatrix}\\ &\quad+\frac{\alpha\beta\psi(\lambda,0)}{2\sqrt{\alpha+\lambda}U'(0)}\exp(\sqrt{\alpha+\lambda}x)\begin{pmatrix}1\\ \exp(\sqrt{\alpha+\lambda}\end{pmatrix}[1-H(x)]\\ &\quad+\frac{\alpha\beta\psi(\lambda,0)}{2\sqrt{\alpha+\lambda}U'(0)}\exp(-\sqrt{\alpha+\lambda}x)\begin{pmatrix}1\\ -\sqrt{\alpha+\lambda}\end{pmatrix}H(x)\\ &\quad-\frac{\alpha\beta\psi(\lambda,Z_0)}{2\sqrt{\alpha+\lambda}U'(Z_0)}\exp[\sqrt{\alpha+\lambda}(x-Z_0)]\begin{pmatrix}1\\ \sqrt{\alpha+\lambda}\end{pmatrix}[1-H(x-Z_0)]\\ &\quad-\frac{\alpha\beta\psi(\lambda,Z_0)}{2\sqrt{\alpha+\lambda}U'(Z_0)}\exp[-\sqrt{\alpha+\lambda}(x-Z_0)]\begin{pmatrix}1\\ -\sqrt{\alpha+\lambda}\end{pmatrix}H(x-Z_0).\end{aligned}$$

The first component of the general solution is given by

$$\begin{aligned}\psi(\lambda,x) &= C_1(\lambda)\exp(\sqrt{\alpha+\lambda}x) + C_2(\lambda)\exp(-\sqrt{\alpha+\lambda}x)\\ &\quad+\frac{\alpha\beta\psi(\lambda,0)}{2\sqrt{\alpha+\lambda}U'(0)}\exp(\sqrt{\alpha+\lambda}x)[1-H(x)]+\frac{\alpha\beta\psi(\lambda,0)}{2\sqrt{\alpha+\lambda}U'(0)}\exp(-\sqrt{\alpha+\lambda}x)H(x)\\ &\quad-\frac{\alpha\beta\psi(\lambda,Z_0)}{2\sqrt{\alpha+\lambda}U'(Z_0)}\exp[\sqrt{\alpha+\lambda}(x-Z_0)][1-H(x-Z_0)]\\ &\quad-\frac{\alpha\beta\psi(\lambda,Z_0)}{2\sqrt{\alpha+\lambda}U'(Z_0)}\exp[-\sqrt{\alpha+\lambda}(x-Z_0)]H(x-Z_0).\end{aligned}$$

The Evans function is defined by

$$\mathcal{E}_{\text{pulse}}(\lambda) = \left[1 - \frac{\alpha\beta}{2\sqrt{\alpha+\lambda}U'(0)}\right]^2 - \left[\frac{\alpha\beta}{2\sqrt{\alpha+\lambda}U'(0)}\exp(-\sqrt{\alpha+\lambda}Z_0)\right]^2.$$

There exists a positive eigenvalue $\lambda_0 > 0$. The instability of the standing pulse solution to the nonlinear scalar reaction diffusion equation (1.3) is established. The proof of Theorem 1 is finished.

3 Conclusion and Remarks

3.1 Summary

Consider the following nonlinear system of reaction diffusion equations arising from mathematical neuroscience

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \alpha[\beta H(u-\theta) - u] - w,\\ \frac{\partial w}{\partial t} &= \varepsilon(u - \gamma w),\end{aligned}$$

and the nonlinear scalar reaction diffusion equations

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha[\beta H(u-\theta) - u],$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha[\beta H(u-\theta) - u] - w_0.$$

The main purpose is to couple together the linearized stability criterion (the equivalence of the nonlinear stability, the linear stability and the spectral stability of the standing pulse solutions) and Evans functions to accomplish the nonlinear instability of the standing pulse solutions. We constructe Evans functions (complex analytic functions) to accomplish the instability of the standing pulse solutions. We study the eigenvalues and eigenfunctions of some eigenvalue problems. It turns out that a complex number λ_0 is an eigenvalue of the eigenvalue problem if and only if λ_0 is a zero of the Evans function. The introduction and application of the Evans functions to standing pulse solutions to the model equations will have great impacts on how to construct and apply Evans functions for stability of fast multiple traveling pulse solutions.

The scalar equations may be obtained by setting $\varepsilon = 0$ and $w = 0$, $\varepsilon = 0$ and $w_0 = \alpha(\beta - 2\theta)$, respectively, in the system.

Summary of the eigenvalue problem

$$\lambda\psi = \psi'' - \alpha\psi - \frac{\varepsilon}{\lambda+\gamma\varepsilon}\psi + \alpha\beta\left[\frac{\psi(\lambda,\varepsilon,0)}{U'(0)}\delta(x) - \frac{\psi(\lambda,\varepsilon,Z_0)}{U'(Z_0)}\delta(x-Z_0)\right],$$

and the Evans function

$$\begin{aligned}\mathcal{E}_{\text{pulse}}(\lambda,\varepsilon) = {}& \left[1 - \frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\right]^2\\ & - \left[\frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}Z_0\right)\right]^2,\end{aligned}$$

for the nonlinear system of reaction diffusion equations

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha[\beta H(u-\theta) - u] - w,$$
$$\frac{\partial w}{\partial t} = \varepsilon(u - \gamma w).$$

(I) The Evans function $\mathcal{E} = \mathcal{E}_{\text{pulse}}(\lambda, \varepsilon)$ is a complex analytic function of λ and ε, it is real-valued if λ is real.

(II) The complex number λ_0 is an eigenvalue of the eigenvalue problem if and only if λ_0 is a zero of the Evans function, that is, $\mathcal{E}_{\text{pulse}}(\lambda_0, \varepsilon) = 0$. In particular, $\mathcal{E}_{\text{pulse}}(0, \varepsilon) = 0$.

(III) The imaginary part of the Evans function $\mathcal{E}_{\text{pulse}}(\lambda, \varepsilon)$ is equal to zero if and only if the imaginary part of λ is equal to zero. In another word, all eigenvalues to the eigenvalue problem are real.

(IV) The algebraic multiplicity of any eigenvalue λ_0 of the eigenvalue problem is equal to the order of the zero λ_0 of the Evans function $\mathcal{E} = \mathcal{E}(\lambda, \varepsilon)$.

(V) The Evans function enjoys the following limit

$$\lim_{|\lambda| \to \infty} |\mathcal{E}_{\text{pulse}}(\lambda, \varepsilon)| = 1,$$

in the right half plane $\{\lambda \in \mathrm{C}\colon \operatorname{Re}\lambda > -\gamma\varepsilon\}$.

3.2 Future Directions and Open Problems

Consider the following nonlinear singularly perturbed system of reaction diffusion equations (1.1)-(1.2), that is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha[\beta H(u-\theta) - u] - w,$$
$$\frac{\partial w}{\partial t} = \varepsilon(u - \gamma w).$$

In the future, we would like to accomplish the existence and stability of fast multiple traveling pulse solutions

$$(U, W) = (U(\varepsilon, x + \nu_{\text{fast}}(\varepsilon)t), W(\varepsilon, x + \nu_{\text{fast}}(\varepsilon)t))$$

with fast moving coordinates $z = x + \nu_{\text{fast}}(\varepsilon)t$ and fast wave speeds $\nu_{\text{fast}}(\varepsilon)$, and to accomplish the existence and instability of slow multiple traveling pulse solutions

$$(U, W) = (U(\varepsilon, x + \nu_{\text{slow}}(\varepsilon)t), W(\varepsilon, x + \nu_{\text{slow}}(\varepsilon)t))$$

with slow moving coordinates $z = x + \nu_{\text{slow}}(\varepsilon)t$ and slow wave speeds $\nu_{\text{slow}}(\varepsilon)$. These problems are very interesting and important in applied mathematics but they have been open for a long time. We will introduce Evans functions and establish a global strong maximum principle for Evans functions, couple together linearized stability criterion, Hopf lemma and many other important ideas, methods and techniques in dynamical systems to accomplish the existence and stability of the fast multiple traveling pulse solutions.

Consider the nonlinear scalar reaction diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha[\beta H(u-\theta) - u].$$

There exists a unique stable traveling wave front $u(x,t) = U_{\text{front}}(x + \nu_0 t)$, such that $U \in C^1(\mathrm{R}) \cap C^2(\mathrm{R} - \{0\})$; $U < \theta$ on $(-\infty, 0)$, $U(0) = \theta$, $U'(0) > 0$ and $U > \theta$ on $(0, \infty)$. The wave speed ν_0 is given by

$$\nu_0 = \sqrt{\frac{\alpha(\beta - 2\theta)^2}{\theta(\beta - \theta)}}.$$

Let $z = x + \nu_0 t$. The traveling wave front is given explicitly by

$$U(z) = \theta \exp\left[\frac{1}{2}\big(\nu_0 + \sqrt{\nu_0^2 + 4\alpha}\big) z\right], \qquad \text{for all } z < 0;$$
$$U(z) = \beta + (\theta - \beta) \exp\left[\frac{1}{2}\big(\nu_0 - \sqrt{\nu_0^2 + 4\alpha}\big) z\right], \qquad \text{for all } z > 0.$$

The traveling wave front enjoys the following boundary conditions

$$\lim_{z \to -\infty} U(z) = 0, \qquad \lim_{z \to \infty} U(z) = \beta, \qquad \lim_{z \to \pm\infty} U'(z) = 0.$$

Consider the nonlinear scalar reaction diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha[\beta H(u-\theta) - u] - w_0.$$

There exists a unique stable traveling wave back $u(x,t) = U_{\text{back}}(x + \nu_0 t)$, such that $U \in C^1(\mathrm{R}) \cap C^2(\mathrm{R} - \{0\})$; $U > \theta$ on $(-\infty, 0)$, $U(0) = \theta$, $U'(0) < 0$ and $U < \theta$ on $(0, \infty)$. Let $z = x + \nu_0 t$. The traveling wave back is given explicitly by

$$U(z) = 2\theta - \theta \exp\left[\frac{1}{2}\big(\nu_0 + \sqrt{\nu_0^2 + 4\alpha}\big) z\right], \qquad \text{for all } z < 0;$$
$$U(z) = 2\theta - \beta - (\theta - \beta) \exp\left[\frac{1}{2}\big(\nu_0 - \sqrt{\nu_0^2 + 4\alpha}\big) z\right], \qquad \text{for all } z > 0.$$

The traveling wave back enjoys the following boundary conditions

$$\lim_{z \to -\infty} U(z) = 2\theta, \qquad \lim_{z \to \infty} U(z) = 2\theta - \beta, \qquad \lim_{z \to \pm\infty} U'(z) = 0.$$

Under the same assumptions $\alpha > 0$ and $0 < 2\theta < \beta$, the nonlinear scalar reaction diffusion equation (1.3) supports both a traveling wave front and a standing pulse solution. When the parameter ε is perturbed from $\varepsilon = 0$ to $\varepsilon > 0$, the traveling wave front becomes part of the fast traveling pulse solution and the standing pulse solution becomes part of the slow traveling pulse solutions.

3.3 Remarks

The fast single traveling pulse solution may be viewed as the perturbation of the traveling wave front and the traveling wave back. The fast multiple traveling pulse solution may be viewed as m copies of the fast single traveling pulse solution appropriately placed together. Therefore, the Evans function for the fast single traveling pulse solution is equal to the product of the Evans function for the traveling wave front and the Evans function for the traveling wave back plus a small function due to perturbation. The Evans function for the fast multiple traveling pulse solution is equal to the product of m Evans functions for the fast single traveling pulse solution plus a small function due to perturbation. More precisely

$$\begin{aligned}\mathcal{E}_{\text{fast}-\text{single}-\text{pulse}}(\lambda,\varepsilon) &= \mathcal{E}_{\text{front}}(\lambda)\mathcal{E}_{\text{back}}(\lambda) + \mathcal{E}_{\text{singular}-\text{perturbation}-1}(\lambda,\varepsilon),\\ \mathcal{E}_{\text{fast}-\text{multiple}-\text{pulse}}(\lambda,\varepsilon) &= [\mathcal{E}_{\text{front}}(\lambda)\mathcal{E}_{\text{back}}(\lambda) + \mathcal{E}_{\text{singular}-\text{perturbation}-1}(\lambda,\varepsilon)]^m\\ &\quad +\mathcal{E}_{\text{singular}-\text{perturbation}-2}(\lambda,\varepsilon),\end{aligned}$$

for all $\lambda \in \Omega$ and $\varepsilon > 0$, where $\Omega = \{\lambda \in \mathrm{C} : \mathrm{Re}\lambda > -\gamma\varepsilon\}$ is a right half complex plane.

Very similarly, for slow single traveling pulse solution and slow multiple traveling pulse solutions, there hold the following representations for the Evans functions

$$\begin{aligned}\mathcal{E}_{\text{slow}-\text{single}-\text{pulse}}(\lambda,\varepsilon) &= \mathcal{E}_{\text{standing}-\text{single}-\text{pulse}}(\lambda,\varepsilon) + \mathcal{E}_{\text{singular}-\text{perturbation}-1}(\lambda,\varepsilon),\\ \mathcal{E}_{\text{slow}-\text{multiple}-\text{pulse}}(\lambda,\varepsilon) &= [\mathcal{E}_{\text{standing}-\text{single}-\text{pulse}}(\lambda) + \mathcal{E}_{\text{singular}-\text{perturbation}-3}(\lambda,\varepsilon)]^m\\ &\quad +\mathcal{E}_{\text{singular}-\text{perturbation}-4}(\lambda,\varepsilon),\end{aligned}$$

for all $\lambda \in \mathrm{C}$ with $\mathrm{Re}\lambda > -\gamma\varepsilon$ and for all $\varepsilon > 0$, where

$$\begin{aligned}\mathcal{E}_{\text{singular}-\text{perturbation}-1}(\lambda,\varepsilon) &= \lambda \exp\left[-\frac{1}{\varepsilon}O(1)\right] O(1),\\ \mathcal{E}_{\text{singular}-\text{perturbation}-2}(\lambda,\varepsilon) &= \lambda \exp\left[-\frac{1}{\varepsilon}O(1)\right] O(1),\\ \mathcal{E}_{\text{singular}-\text{perturbation}-3}(\lambda,\varepsilon) &= \lambda \exp\left[-\frac{1}{\varepsilon}O(1)\right] O(1),\\ \mathcal{E}_{\text{singular}-\text{perturbation}-4}(\lambda,\varepsilon) &= \lambda \exp\left[-\frac{1}{\varepsilon}O(1)\right] O(1).\end{aligned}$$

To see if the neutral eigenvalue is algebraically simple, let us differentiate the representations with respect to λ. We have

$$\begin{aligned}
&\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{fast-single-pulse}}(\lambda,\varepsilon)\\
&=\mathcal{E}'_{\text{front}}(\lambda)\mathcal{E}_{\text{back}}(\lambda)+\mathcal{E}_{\text{front}}(\lambda)\mathcal{E}'_{\text{back}}(\lambda)+\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{singular-perturbation-1}}(\lambda,\varepsilon),\\
&\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{fast-multiple-pulse}}(\lambda,\varepsilon)\\
&=m[\mathcal{E}_{\text{front}}(\lambda)\mathcal{E}_{\text{back}}(\lambda)+\mathcal{E}_{\text{singular-perturbation-1}}(\lambda,\varepsilon)]^{m-1}\\
&\quad\cdot[\mathcal{E}'_{\text{front}}(\lambda)\mathcal{E}_{\text{back}}(\lambda)+\mathcal{E}_{\text{front}}(\lambda)\mathcal{E}'_{\text{back}}(\lambda)+\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{singular-perturbation-1}}(\lambda,\varepsilon)]\\
&\quad+\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{singular-perturbation-2}}(\lambda,\varepsilon).
\end{aligned}$$

Let $\lambda=0$, then we have

$$\begin{aligned}
&\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{fast-single-pulse}}(0,\varepsilon)\\
&=\mathcal{E}'_{\text{front}}(0)\mathcal{E}_{\text{back}}(0)+\mathcal{E}_{\text{front}}(0)\mathcal{E}'_{\text{back}}(0)+\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{singular-perturbation-1}}(0,\varepsilon)\\
&=\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{singular-perturbation-1}}(0,\varepsilon)>0,\\
&\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{fast-multiple-pulse}}(0,\varepsilon)\\
&=m[\mathcal{E}_{\text{front}}(0)\mathcal{E}_{\text{back}}(0)+\mathcal{E}_{\text{singular-perturbation-1}}(0,\varepsilon)]^{m-1}\\
&\quad\cdot[\mathcal{E}'_{\text{front}}(0)\mathcal{E}_{\text{back}}(0)+\mathcal{E}_{\text{front}}(0)\mathcal{E}'_{\text{back}}(0)+\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{singular-perturbation-1}}(0,\varepsilon)]\\
&\quad+\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{singular-perturbation-2}}(0,\varepsilon)\\
&=\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{singular-perturbation-2}}(0,\varepsilon)>0.
\end{aligned}$$

Overall, the neutral eigenvalue $\lambda=0$ is simple.

3.4 A Technical Lemma

Lemma *Let A be an $(2m)\times(2m)$ complex constant matrix. Suppose that $A\eta_k=\mu_k\eta_k$, that is, μ_1, μ_2, $\cdots$, μ_{2m} are the eigenvalues and η_1, η_2, $\cdots$, η_{2m} are the corresponding linearly independent eigenvectors of A. The real parts of the eigenvalues satisfy $Re\mu_k>0$, for all $k=1,2,\cdots,p$ and $\mathrm{Re}\mu_k<0$, for all $k=p+1,\cdots,2m$, for some positive integer $p\geq 1$. Let δ represent the Dirac delta impulse function. Let x_1, x_2, x_3, $\cdots$, x_{2m} be real constants and ξ_1, ξ_2, ξ_3, $\cdots$, ξ_{2m} be complex constant vectors. Consider the following linear system of differential* *equations*

$$\psi' = A\psi - \sum_{k=1}^{2m} \delta(x - x_k)\xi_k.$$

There exists a bounded explicit particular solution to the system

$$\psi(x) = \sum_{k=1}^{2m} T \begin{pmatrix} f_{1,k}(x) & 0 & 0 & \cdots & 0 \\ 0 & f_{2,k}(x) & 0 & \cdots & 0 \\ 0 & 0 & f_{3,k}(x) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & f_{2m,k}(x) \end{pmatrix} T^{-1}\xi_k,$$

where $T = (\eta_1, \eta_2, \cdots, \eta_{2m})$ *is the matrix consisting of the linearly independent eigenvectors, and*

$$f_{k,l}(x) = \exp[\mu_k(x - x_k)][1 - H(x - x_k)],$$

for all $k = 1, 2, \cdots, p$ *and*

$$f_{k,l}(x) = -\exp[\mu_k(x - x_k)]H(x - x_k),$$

for all $k = p+1, \cdots, 2m$, *where* $H = H(x-x_k)$ *is a Heaviside step function, defined by* $H(x - x_k) = 0$ *on* $(-\infty, x_k)$, $H(0) = \frac{1}{2}$, $H(x - x_k) = 1$ *on* (x_k, ∞).

Proof Note that

$$(\exp[\mu_1(x - x_k)]\eta_1, \exp[\mu_2(x - x_k)]\eta_2, \cdots, \exp[\mu_n(x - x_k)]\eta_{2m})$$

is a fundamental matrix of the homogeneous linear system $\psi' = A\psi$, for all $k = 1, 2, 3, \cdots, 2m$. By using the method of variation of parameters and the fundamental matrices, the proof is very easy and it is omitted.

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