Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **14**, No. 6, pp. 1433-1455

DOI: 10.4208/aamm.OA-2021-0187
December 2022

Numerical Analysis of Two-Grid Block-Centered Finite Difference Method for Two-Phase Flow in Porous Medium

Jing Zhang and Hongxing Rui*

School of Mathematics, Shandong University, Jinan, Shandong 250100, China

Received 31 May 2021; Accepted (in revised version) 10 November 2021

Abstract. In this paper, a two-grid block-centered finite difference method for the incompressible miscible displacement in porous medium is introduced and analyzed, which is to solve a nonlinear equation on coarse mesh space of size H and a linear equation on fine grid of size h. We establish the full discrete two-grid block-centered finite difference scheme on a uniform grid. The error estimates for the pressure, Darcy velocity, concentration variables are derived, which show that the discrete L_2 error is $\mathcal{O}(\Delta t+h^2+H^4)$. Finally, two numerical examples are provided to demonstrate the effectiveness and accuracy of our algorithm.

AMS subject classifications: 65M06, 65M12, 65M15, 65M55

Key words: Porous media, two phase flow, block-centered finite difference, two-grid, numerical analysis.

1 Introduction

In this paper, we consider the incompressible miscible displacement in porous media [1–3]

$$\begin{cases} \nabla\cdot u=q(x,t), & x\in\Omega, \quad t\in J, \\ u=-\dfrac{\kappa(x)}{\mu(c)}\nabla p, & x\in\Omega, \quad t\in J, \\ \varphi(x)\dfrac{\partial c}{\partial t}+\nabla\cdot(uc)-\nabla\cdot(D\nabla c)=\tilde{c}q, & x\in\Omega, \quad t\in J. \end{cases}$$

We assume that Ω is a rectangular domain in R^2, $t\in J=(0,T]$, and T denotes the final time. The concentration is denoted by $c(x,t)$, $p(x,t)$ is the fluid pressure, and $u=(u_1,u_2)^T$ is Darcy velocity of the fluid, $\kappa(x)$ and $\varphi(x)$ represent the permeability and porosity of

*Corresponding author.
Emails: zhangjing169277@163.com (J. Zhang), hxrui@sdu.edu.cn (H. Rui)

the porous medium, respectively. $\mu(c)$ is the concentration dependent viscosity. The function $\tilde{c}$ is the concentration of the same component as measured by c in the injected fluid, which must be specified whenever injection is taking place, and it will be assumed that $\tilde{c}=c$, when the fluid is being produced, q is the external flow rate at wells.

For the sake of simplicity, let $a(c)=\kappa(x)/\mu(c)$, $D=\varphi d_m I=\lambda I$, and $w=(w^x,w^y)=uc-D\nabla c=uc-\lambda\nabla c$, where d_m is the molecular diffusivity, I is the second order identity matrix. Then the question may be equivalently written in the form:

$$\begin{cases} \nabla\cdot u=q(x,t), & x\in\Omega, \quad t\in J, & (1.1a)\\ u=-a(c)\nabla p, & x\in\Omega, \quad t\in J, & (1.1b)\\ \varphi(x)\dfrac{\partial c}{\partial t}+\nabla\cdot w=f(\tilde{c}), & x\in\Omega, \quad t\in J, & (1.1c)\\ w=uc-\lambda\nabla c, & x\in\Omega, \quad t\in J, & (1.1d)\end{cases}$$

where $f(\tilde{c})=\tilde{c}q$.

We consider the following boundary condition and initial condition for the problem:

$$u\cdot\mathbf{n}=0, \qquad x\in\partial\Omega, \quad t\in J, \tag{1.2a}$$

$$(\lambda\nabla c)\cdot\mathbf{n}=0, \qquad x\in\partial\Omega, \quad t\in J, \tag{1.2b}$$

$$c|_{t=0}=c_0, \qquad x\in\Omega, \tag{1.2c}$$

where $\mathbf{n}$ is the unit outward normal vextor to $\partial\Omega$, the compatibility condition and the uniqueness condition are as follows

$$\int_\Omega p(x)\mathrm{d}x=0. \tag{1.3}$$

For problem (1.1a)-(1.3), we consider the following smoothness hypotheses (H):

(1) The funtions $a(c),b(c),\lambda$ are bounded, And, there exist positive constants a_0, a_1, b_0, b_1, λ_0, λ_1, such that

$$0<a_0\le a\le a_1, \quad 0<b_0\le b\le b_1, \quad 0<\lambda_0\le\lambda\le\lambda_1.$$

(2) The second derivative of f, q are continuously bounded in $\Omega\times J$, f, a is Lipschitz-continuous corresponding to variable c. The function φ is continuous, there exist a positive constant φ_0, such that $\varphi\ge\varphi_0>0$.

(3) $p\in L_\infty(J;W^4_\infty(\Omega))$, $u\in C^1(J;W^1_\infty(\Omega))^2$, $c\in W^2_\infty(J;W^4_\infty(\Omega))$.

People have been interested in efficient oil exploitation and improving the utilization of groundwater resource for a long time. Two-phase flow and transportation of fluids in porous media play a vital role in both theoretic and applicative aspects in groundwater contamination or petroleum engineering. The incompressible miscible displacement problem in porous media is a large nonlinear system, so we want to study highly efficient and highly accurate numerical schemes to improve the numerical computational efficiency.

As we know, the block-centered finite difference method, sometimes called cell-centered finite difference method, is a simple but effective method. It can be considered as the lowest order Raviart-Thomas mixed element method in [4], with proper quadrature formulation. In [5], Wheeler presented on convergence of block-centered finite difference for elliptic problem. Moreover, the mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences was considered in [6]. In [1], Lui proofed Convergence of the block center finite difference scheme for two-phase flow in porous media. Then, Rui had done many research about the block-centered finite difference in [7–9].

The two-grid method is a quite effective method to solve the nonlinear equations. This method was introduced by Xu in [10,11]. And Layton studied the nonlinear boundary value problems by two-grid method in [12]. Nonlinear parabolic equations with this method are considered in [13]. Many people are attracted to do research about two-grid techniques with the finite element, discontinuous Galerkin methods, and mixed element, for example, in [14–17]. Moreover, the two-grid method is presented and analyzed in the work of Chen [18–21] for miscible displacement problem. The basic process of two-grid method is to solve a nonlinear equation on coarse mesh space of size H and a linear equation on fine grid of size h.

All in all, the two-grid block-centered finite difference method concludes the advantages of the block-centered finite difference method and the two-grid method, which can not only solve the nonlinear problem efficiently and accurately, but also make the numerical results of the problem reach the second-order convergence accuracy. In addition, the two-grid block-centered finite difference method is applied to many problems [22, 23], for example, the Darcy-Forchheimer model and the non-Fickian flow model. There is no two-grid block-centered finite difference methods for the incompressible miscible displacement in porous medium, therefore we propose the corresponding algorithm in this paper. By using this method, we establish the full discrete two-grid block-centered finite difference scheme on a uniform grid and prove the error estimates for the pressure, Darcy velocity, concentration variables. Some numerical examples are carried out to check the accuracy and efficiency of the method. The convergence rates for the pressure and velocity in discrete L^2 norms are second-order under the conditions $h=\mathcal{O}(H^2)$, which are consistent with the numerical analysis. Moreover, we also give the numerical examples, compared with the nonlinear implicit scheme, the efficiency and accuracy of the two-grid block-centered finite difference method are illustrated.

The paper is organized as follows. In Section 2, we give some notations and lemmas. In Section 3, we give the two-grid block-centered finite difference algorithm. Then in Section 4, we present the error estimates for the two-grid block-centered finite difference method. Finally in Section 5, Two numerical examples of the two-grid block-centered finite difference scheme are drawn.

Through out the paper we use M, with or without subscript, to denote a positive constant, which could have different values at different appearances.

2 Some notation and lemmas

Firstly, we define some notations. Let $N>0$ be a positive integer. Set $\Delta t=T/N$, $t_n=n\Delta t$, for $n\leq N$, $\Delta t^n=t_n-t_{n-1}$, $\Delta t=\max_n \Delta t^n$. For simplicity in constructing the finite difference algorithm, we suppose that the domain Ω is a rectangular, $\Omega=[d_1^x,d_2^x]\times[d_1^y,d_2^y]$.

Let $L^p(\Omega)$ be the standard Banach space with norm

$$\|v\|_{L^p(\Omega)}=\left(\int_\Omega |v|^p d\Omega\right)^{1/p}.$$

For the sake of simplicity, let $(\cdot,\cdot)$ denote the $L^2(\Omega)$ inner product. And let $W_p^k(\Omega)$ be the standard Sobolev space

$$W_p^k(\Omega)=\{g:\|g\|_{W_p^k(\Omega)}<\infty\},$$

where

$$\|g\|_{W_p^k(\Omega)}=\left(\sum_{|\alpha|\leq k}\|D^\alpha g\|^p_{L^p(\Omega)}\right)^{1/p}.$$

Let

$$S=L^2(\Omega)\quad\text{and}\quad V=H(\Omega,div)=\{v\in(L^2(\Omega))^d,\ \nabla\cdot v\in L^2(\Omega)\}.$$

And V^0 is denoted as the subspaces of V containing functions with normal traces equal to 0.

Let $\mathcal{T}_h$ be the quasi-uniform partition of Ω into rectangles in two dimensions with mesh size h. The lowest-order Raviart-Thomas-Nédélec (RTN) space on rectangles [4,24] is considered. Thus, on an element $D\in\mathcal{T}_h$, we have

$$\begin{aligned}&V_h(D)=\{(\alpha_1x+\beta_1,\ \alpha_2y+\beta_2)^T:\alpha_i,\ \beta_i\in R,\ i=1,2\},\\&S_h(D)=\{\alpha:\alpha\in R\}.\end{aligned}$$

To construct the two-grid algorithm we have to define a coarse partition and a fine partition of Ω simultaneously. First we define the fine grid Ω_h in detail. The notations are similar to those in [5]. The fine partition $\Omega_h=\delta_h^x\times\delta_h^y$ for Ω is as follows:

$$\begin{aligned}&\delta_h^x:\ d_1^x=x_{1/2}<x_{3/2}<\cdots<x_{N_x^h-1/2}<x_{N_x^h+1/2}=d_2^x,\\&\delta_h^y:\ d_1^y=y_{1/2}<y_{3/2}<\cdots<y_{N_y^h-1/2}<y_{N_y^h+1/2}=d_2^y.\end{aligned}$$

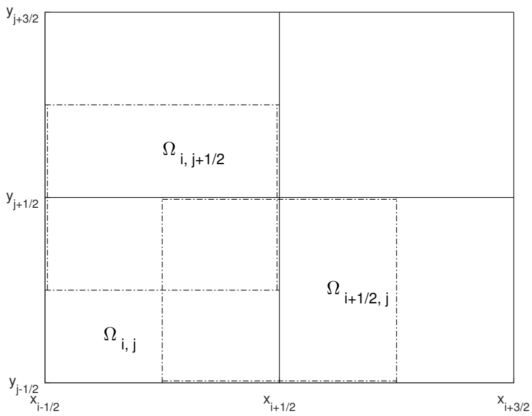


Figure 1: An example of mesh partition.

For $i=1,\cdots,N_x^h$ and $j=1,\cdots,N_y^h$ define

$$
\begin{aligned}
&x_i=(x_{i-1/2}+x_{i+1/2})/2, && y_j=(y_{j-1/2}+y_{j+1/2})/2,\\
&h_i^x=x_{i+1/2}-x_{i-1/2}, && h_j^y=y_{j+1/2}-y_{j-1/2},\\
&h_{i+1/2}^x=x_{i+1}-x_i=(h_i^x+h_{x+1}^x)/2, && h_{j+1/2}^y=y_{j+1}-y_j=(h_j^y+h_{j+1}^y)/2,\\
&h=\max_{i,j}\{h_i^x,\, h_j^y\}, && \Omega_{i,j}=(x_{i-1/2},x_{i+1/2})\times(y_{j-1/2},y_{j+1/2}),\\
&\Omega_{i+1/2,j}=(x_i,x_{i+1})\times(y_{j-1/2},y_{j+1/2}), && \Omega_{i,j+1/2}=(x_{i-1/2},x_{i+1/2})\times(y_j,y_{j+1}).
\end{aligned}
$$

Fig. 1 is the description of mesh construction and the nodes.

Let $g_{i,j}$, $g_{i+1/2,j}$, $g_{i,j+1/2}$ denote $g(x_i,y_j)$, $g(x_{i+1/2},y_j)$, $g(x_i,y_{j+1/2})$. Define the discrete inner products and norms:

$$
\begin{aligned}
&(f,g)=\sum_{i=1}^{N_x^h}\sum_{j=1}^{N_y^h}h_i^x h_j^y f_{i,j}g_{i,j}, && (f,g)_x=\sum_{i=1}^{N_x^h-1}\sum_{j=1}^{N_y^h}h_{i+1/2}^x h_j^y f_{i+1/2,j}g_{i+1/2,j},\\
&(f,g)_y=\sum_{i=1}^{N_x^h}\sum_{j=1}^{N_y^h-1}h_i^x h_{j+1/2}^y f_{i,j+1/2}g_{i,j+1/2}, && (\mathbf{v},\mathbf{r})_{TM}=(v^x,r^x)_x+(v^y,r^y)_y.
\end{aligned}
$$

Define

$$
\begin{aligned}
&[d_x g]_{i+1/2,j}=(g_{i+1,j}-g_{i,j})/h_{i+1/2}^x, && [d_y g]_{i,j+1/2}=(g_{i,j+1}-g_{i,j})/h_{j+1/2}^y,\\
&[D_x g]_{i,j}=(g_{i+1/2,j}-g_{i-1/2,j})/h_i^x, && [D_y g]_{i,j}=(g_{i,j+1/2}-g_{i,j-1/2})/h_j^y,\\
&[Dg]_{i,j}=[D_x g,D_y g]_{i,j}^T, && d_t g^n=(g^n-g^{n-1})/\Delta t^n.
\end{aligned}
$$

Similarly to Ω_h we define a coarse grid $\Omega_H = \delta_H^x \times \delta_H^y$. Replacing h^x, h^y we use H^x, H^y to denote the mesh sizes in x and y directions of the coarse grid, respectively. And similar to $[d_x g]$, $[D_x g]$, $[d_y g]$, $[D_y g]$ we use $[d_{x,H} g]$, $[D_{x,H} g]$, $[d_{y,H} g]$, $[D_{y,H} g]$ to denote the finite difference operators on the coarse grid. For the sake of simplicity, we do not give their definitions in detail. For a discrete function, its norms and seminorms can also be defined similarly on the coarse grid.

Then we present some lemmas as follow.

Lemma 2.1. *Let $p_{i,j}$, $w^x_{i+1/2,j}$ and $w^y_{i,j+1/2}$ be any values such that $w^x_{1/2,j} = w^x_{N_x+1/2,j} = w^y_{i,1/2} = w^y_{i,N_y+1/2} = 0$, then we have*

$$(p, D_x w^x) = -(d_x p, w^x)_x, \quad (p, D_y w^y) = -(d_y p, w^y)_y.$$

This lemma can be proven similar to [5].

Next the interpolant operator is defined, which is similar to that in [13]. For points (x,y), set $x \in [x_i, x_{i+1}]$, $y \in [y_j, y_{j+1}]$, then, $\Pi_h c(x,y)$ can be defined as follow. For $i = 0, \cdots, N_x$, $j = 1, \cdots, N_y$,

$$\begin{aligned}\Pi_h c(x,y) = &\left(c_{i,j} \left(\frac{x_{i+1} - x}{x_{i+1} - x_i} \right) + c_{i+1,j} \left(\frac{x - x_i}{x_{i+1} - x_i} \right) \right) \left(\frac{y_{j+1} - y}{y_{j+1} - y_j} \right) \\ &+ \left(c_{i,j+1} \left(\frac{x_{i+1} - x}{x_{i+1} - x_i} \right) + c_{i+1,j+1} \left(\frac{x - x_i}{x_{i+1} - x_i} \right) \right) \left(\frac{y - y_j}{y_{j+1} - y_j} \right).\end{aligned}$$

For $j = 1, \cdots, N_y$, the two-point extrapolation is defined

$$\Pi_h c(x_{1/2}, y_j) = \frac{(2h_1^x + h_2^x) c_{1j} - h_1^x c_{2j}}{h_1^x + h_2^x},$$

and by Taylor's theorem, we can obtain that

$$|(\Pi_h c - c)(x_{1/2}, y_j)| = \mathcal{O}(h^2).$$

For points (x,y), assuming $x \in [x_{1/2}, x_1]$, $y \in [y_j, y_{j+1}]$, then, $\Pi_h c(x,y)$ can be obtained as the bilinear interpolant between $c_{1,j}$, $c_{1,j+1}$, $\Pi_h c(x_{1/2}, y_j)$, and $\Pi_h c(x_{1/2}, y_{j+1})$. Then for these points, we can evaluate that $|\Pi_h c - c| = M h^2$ by interpolation theory. Moreover, for points (x,y), such that $x \in [x_{N_x}, x_{N_x+1/2}]$, $y \in [y_j, y_{j+1}]$ or $x \in [x_i, x_{i+1}]$, $y \in [y_{1/2}, y_1]$ or $x \in [x_i, x_{i+1}]$, $y \in [y_{N_y}, y_{N_y+1/2}]$, we can define $\Pi_h c(x,y)$ similarly. Lastly, by three-point extrapolantion, we define $\Pi_h c(x_{1/2}, y_{1/2})$:

$$\begin{aligned}\Pi_h c(x_{1/2}, y_{1/2}) &= \Pi_h c(x_1, y_{1/2}) + \Pi_h c(x_{1/2}, y_1) - p_{1,1} \\ &= c_{1,1/2} + c_{1/2,1} - c_{1,1} + M h^2.\end{aligned}$$

We can easily get that

$$|(\Pi_h c - c)(x_{1/2}, y_{1/2})| \leq M h^2$$

by Taylor's expansion. Hence, for points (x,y), assuming $x\in[x_{1/2},x_1]$, $y\in[y_{1/2},y_1]$, $\Pi_h c(x,y)$ can be obtained as the bilinear interpolant between $c_{1,1}$, $\Pi_h c_{1/2,1}$, $\Pi_h c_{1,1/2}$, and $\Pi_h c_{1/2,1/2}$. We can define, $\Pi_h c_{1/2,N_y+1/2}$, $\Pi_h c_{N_x+1/2,1/2}$, $\Pi_h c_{N_x+1/2,N_y+1/2}$ similarly. And in the other three corner regions, we can ge the same approximations. Thus the following lemma is obtained.

Lemma 2.2. *Suppose c is twice differentiable in space, then we can have the estimate that*

$$\|\Pi_h c - c\|_\infty \le Mh^2.$$

3 Two-grid block-centered finite difference algorithm

The full discrete two-grid block-centered finite difference scheme for the nonlinear problem (1.1a)-(1.2c) is as follows:

$$[D_x U_h^x]_{i,j}^n + [D_y U_h^y]_{i,j}^n = q_{i,j}^n, \tag{3.1a}$$

$$[\alpha(\Pi_h C_h^x)]_{i+1/2,j}^{n-1}[U_h^x]_{i+1/2,j}^n = -[d_x P_h]_{i+1/2,j}^n, \tag{3.1b}$$

$$[\alpha(\Pi_h C_h^y)]_{i,j+1/2}^{n-1}[U_h^y]_{i,j+1/2}^n = -[d_y P_h]_{i,j+1/2}^n, \tag{3.1c}$$

$$\varphi_{i,j}[d_t C_h]_{i,j}^n + [D_x W_h^x]_{i,j}^n + [D_y W_h^y]_{i,j}^n = [f(\tilde{C})]_{i,j}^n, \tag{3.1d}$$

$$[W_h^x]_{i+1/2,j}^n = [U_h^x]_{i+1/2,j}^n[\Pi_h C_h^x]_{i+1/2,j}^n - [\lambda d_x C_h]_{i+1/2,j}^n, \tag{3.1e}$$

$$[W_h^y]_{i,j+1/2}^n = [U_h^y]_{i,j+1/2}^n[\Pi_h C_h^y]_{i,j+1/2}^n - [\lambda d_y C_h]_{i,j+1/2}^n, \tag{3.1f}$$

$$C_{i,j}|_{t=0} = c_0, \tag{3.1g}$$

where

$$\alpha(c) = \frac{1}{a(c)}.$$

The condition (1.3) can be discretized as follows:

$$\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}[P_h^n]_{ij} = 0.$$

For the sake of simplicity, let

$$g = -\lambda\nabla c. \tag{3.2}$$

Then the boundary and initial approximations can be discretized as follows:

$$\begin{aligned}
&[U_h^x]_{1/2,j}^n = [U_h^x]_{N_x+1/2,j}^n = 0, && 1\le j\le N_y,\\
&[U_h^y]_{i,1/2}^n = [U_h^y]_{i,N_y+1/2}^n = 0, && 1\le i\le N_x,\\
&[G_h^x]_{1/2,j}^n = [G_h^x]_{N_x+1/2,j}^n = 0, && 1\le j\le N_y,\\
&[G_h^y]_{i,1/2}^n = [G_h^y]_{i,N_y+1/2}^n = 0, && 1\le i\le N_x,\\
&[C_h]_{i,j}^0 = c_{0,i,j}, && 1\le i\le N_x, \quad 1\le j\le N_y.
\end{aligned}$$

Noting Lemma 2.1, the question is equivalent to the mixed finite element problem with discrete inner product. For $U_h^n \in S_h$, $P_h^n \in V_h$, $C_h^n \in S_h$, $W_h^n \in V_h$,

$$(D_x U_h^{x,n} + D_y U_h^{y,n}, s) = (\nabla \cdot U_h^n, s) = (q_h^n, s), \qquad \forall s \in S_h, \tag{3.3a}$$
$$(\alpha(\Pi_h C_h^n) U_h^n, v)_{TM} = (P_h^n, \nabla \cdot v), \qquad \forall v \in V_h, \tag{3.3b}$$
$$(\varphi d_t C_h^n, s) + (\nabla \cdot W_h^n, s) = \big(f(\tilde{C}_h^n, s)\big), \qquad \forall s \in S_h, \tag{3.3c}$$
$$(W_h^n, v)_{TM} = (U_h^n \Pi_h C_h^n, v)_{TM} + (\lambda C_h^n, \nabla \cdot v), \qquad \forall v \in V_h. \tag{3.3d}$$

The two-grid algorithm has two steps:

Step 1. On the coarse grid Ω_H with mesh sizes H^x and H^y, we compute $\{U_H^n, P_H^n, C_H^n, W_H^n\}_{n=1}^N \in S_H \times V_H \times S_H \times V_H$ to satisfy the following nonlinear system:

$$(\nabla \cdot U_H^n, s) = (q_H^n, s), \qquad \forall s \in S_H, \tag{3.4a}$$
$$(\alpha(\Pi_H C_H^n) U_H^n, v)_{TM} = (P_H^n, \nabla \cdot v), \qquad \forall v \in V_H, \tag{3.4b}$$
$$(\varphi d_t C_H^n, s) + (\nabla \cdot W_H^n, s) = \big(f(\tilde{C}_H^n, s)\big), \qquad \forall s \in S_H, \tag{3.4c}$$
$$(W_H^n, v)_{TM} = (U_H^n \Pi_H C_H^n, v)_{TM} + (\lambda C_H^n, \nabla \cdot v), \qquad \forall v \in V_H, \tag{3.4d}$$

where the initial approximation $C_H^0 = c^0$.

Step 2. On the fine grid Ω_h with mesh sizes h^x and h^y, we compute $\{U_h^n, P_h^n, C_h^n, W_h^n\}_{n=1}^N \in S_h \times V_h \times S_h \times V_h$ to satisfy the following linear system:

$$(\nabla \cdot U_h^n, s) = (q_h^n, s), \qquad \forall s \in S_h, \tag{3.5a}$$
$$(A(\Pi_h C_h^n) U_h^n, v)_{TM} = (P_h^n, \nabla \cdot v), \qquad \forall v \in V_h, \tag{3.5b}$$
$$(\varphi d_t C_h^n, s) + (\nabla \cdot W_h^n, s) = \big(f(\tilde{C}_h^n), s\big), \qquad \forall s \in S_h, \tag{3.5c}$$
$$(W_h^n, v)_{TM} = (\overline{U_h^n \Pi_h C_h^n}, v)_{TM} + (\lambda C_h^n, \nabla \cdot v), \qquad \forall v \in V_h, \tag{3.5d}$$

where the initial approximation $C_h^0 = c^0$, and

$$A(\Pi_h C_h^n) U_h^n = \alpha(\Pi_H C_H^n) U_h^n + \alpha_c(\Pi_H C_H^n) R_H U_H^n (C_h^n - \Pi_H C_H^n), \tag{3.6a}$$
$$\overline{U_h^n \Pi_h C_h^n} = R_H U_H^n \Pi_h C_h^n + \Pi_H C_H^n (R_h U_h^n - R_H U_H^n). \tag{3.6b}$$

Here, we define the interpolant operator $R_H u = (R_H^x u^x, R_H^y u^y)$ which is similar to that in [23]. For points (x,y), assuming $x \in [x_{i-1/2}, x_{i+1/2}]$, $y \in [y_j, y_{j+1}]$, then, $R_H^x u^x(x,y)$ can be presented as the bilinear interpolant of $u^x_{i-1/2,j}$, $u^x_{i-1/2,j+1}$, $u^x_{i+1/2,j}$, $u^x_{i+1/2,j+1}$. Moreover, we define $R_H^x u^x_{i+1/2,1/2}$ by the two-point extrapolation of $u^x_{i+1/2,1}$ and $u^x_{i+1/2,2}$. For points (x,y) such that $x \in [x_{i-1/2}, x_{i+1/2}]$, $y \in [y_{1/2}, y_1]$, $R_H^x u^x(x,y)$ can be presented as the bilinear interpolant of $u^x_{i-1/2,1}$, $u^x_{i+1/2,1}$, $R_H^x u^x_{i+1/2,1/2}$, $R_H^x u^x_{i-1/2,1/2}$. And other points are similar to the above, as well as the definition of $R_H^y u^y$. By interpolation theory, we can obtain that

$$\|R_H u - u\|_\infty \le M H^2,$$

where u^x, u^y are twice differentiable in space.

The block-centered finite difference method, can be considered as the lowest order Raviart-Thomas mixed element method in [4], so the existence and uniqueness of a solution to the discrete nonlinear question are easily obtained, the proof is similar to [2].

4 Error estimates of the two-grid scheme

In this section, we consider the error estimation for the two-grid block-centered finite difference algorithm.

First, to analyze the error estimation, we consider the following question and discrete scheme firstly.

For the elliptic problem

$$\begin{cases} \nabla\cdot u = q, & x\in\Omega, \\ u = -a(c)\nabla p, & x\in\Omega, \\ u\cdot\nu = 0, & x\in\partial\Omega, \end{cases}$$

$\{\hat{U}_h^n,\hat{P}_h^n\}\in S_h\times V_h$ denote the nonlinear block-centered finite difference approximations to $\{u^n,p^n\}$, respectively. Their values are defined by the following scheme:

$$(\nabla\cdot\hat{U}_h^n,s) = (q_h^n,s), \qquad \forall s\in S_h, \tag{4.1a}$$

$$\big(\alpha(c)\hat{U}_h^n,v\big)_{TM} = (\hat{P}_h^n,\nabla\cdot v), \qquad \forall v\in V_h, \tag{4.1b}$$

where c is the exact solution to problems (1.1a)-(1.3).

For the parabolic problem

$$\begin{cases} \varphi\dfrac{\partial c}{\partial t}+\nabla\cdot g = f(x,c,t)-\nabla\cdot(uc) = F(x,t), & x\in\Omega, & t\in J, \\ g = -\lambda\nabla c, & x\in\Omega, & t\in J, \\ g\cdot\nu = 0, & x\in\partial\Omega, & t\in J, \end{cases}$$

$\{\hat{C}_h^n,\hat{G}_h^n\}\in S_h\times V_h$ denote the nonlinear block-centered finite difference approximations to $\{c^n,g^n\}$, respectively. Their values are defined by the following scheme:

$$(\varphi d_t\hat{C}_h^n,s)+(\nabla\cdot\hat{G}_h^n,s) = (F_h^n,s), \qquad \forall s\in S_h, \tag{4.2a}$$

$$(\hat{G}_h^n,v)_{TM} = (\lambda\hat{C}_h^n,\nabla\cdot v), \qquad \forall v\in V_h. \tag{4.2b}$$

By Eq. (3.2), we have

$$(G_h,v)_{TM} = (\lambda C_h,\nabla\cdot v), \quad \forall v\in V_h. \tag{4.3}$$

Now for the convenience of analysis, we set

$$\begin{cases} P_h-p = P_h-\hat{P}_h+\hat{P}_h-p = \eta_h+\gamma_h, \\ U_h-u = U_h-\hat{U}_h+\hat{U}_h-u = \xi_h+\beta_h, \\ C_h-c = C_h-\hat{C}_h+\hat{C}_h-c = \theta_h+\sigma_h, \\ W_h-\hat{G}_h = \pi_h+U_h\Pi_h C_h, \\ \pi_h = G_h-\hat{G}_h. \end{cases}$$

In [5] and [8], we can obtain that the approximate solution of the discrete scheme (4.1a)-(4.1b) and (4.2a)-(4.2b) exists uniquely. Moreover, the following results are easily got.

Lemma 4.1. *Suppose the hypotheses (H) hold, then there exists a positive constant M independent of h and Δt such that*

$$\begin{aligned}&\|\gamma_h^n\|_M \le Mh^2, \qquad \|\beta_h^n\|_{TM} \le Mh^2,\\ &\|\sigma_h^n\|_M \le M(\Delta t + h^2),\\ &\|\hat{U}_h^n\|_{L_\infty} \le M.\end{aligned}$$

Next we give the error analysis of the nonlinear scheme (3.4a)-(3.4d) on the course grid.

Subtracting (4.1a) from (3.4a), we can obtain

$$(\nabla\cdot\xi_H^n, s) = 0, \quad \forall s \in S_H. \tag{4.4}$$

And subtracting (4.1b) from (3.4b), we have that

$$\begin{aligned}&\Big(\alpha(\Pi_H C_H^{n-1})\xi_H^n, v\Big)_{TM} - (\eta_H^n, \nabla\cdot v)\\ =& -\Big(\Big(\alpha(\Pi_H C_H^{n-1}) - \alpha(c)\Big)\hat{U}_H^n, v\Big)_{TM}, \quad \forall v \in V_H.\end{aligned} \tag{4.5}$$

Subtracting (4.2a) from (3.4c), we can get

$$\begin{aligned}&(\varphi d_t \theta_H^n, s) + (\nabla\cdot\pi_H^n, s)\\ =&(f(\tilde{C}_H^n) - F_H^n, s) + (\nabla\cdot(U_H^n \Pi_H C_H^n), s), \quad \forall s \in S_H.\end{aligned} \tag{4.6}$$

And (3.4d) subtracting (4.2b), we have that

$$(\pi_H^n, v)_{TM} = (\lambda\theta_H^n, \nabla\cdot v), \quad \forall v \in V_H. \tag{4.7}$$

Selecting $s=\eta_H^n$ in (4.4) and $v=\xi_H^n$ in (4.5), then adding (4.4) to (4.5), we have the following equations.

$$(\alpha(\Pi_H C_H^n)\xi_H^n, \xi_H^n)_{TM} = -\big((\alpha(\Pi_H C_H^n) - \alpha(c))\,\hat{U}_H^n, \xi_H^n\big)_{TM}. \tag{4.8}$$

Together the assumption of $a(c)$ with Taylor expansion and noting Lemma 2.2, Lemma 4.1 and ϵ-Chauchy inequality, we have that

$$\begin{aligned}\|\xi_H^n\|_{TM}^2 &\le M\|(\Pi_H C_H^n - c^n)\xi_H^n\|_{TM}\\ &\le M\|\Pi_H C_H^n - c^n\|_{TM}\|\xi_H^n\|_{TM}\\ &\le M\|\Pi_H C_H^n - \Pi_h c_H^n + \Pi_H c_H^n - c^n\|_{TM}^2 + \epsilon\|\xi_H^n\|_{TM}^2\\ &\le M(\|\theta_H^n\|^2 + \|\sigma_H^n\|^2 + H^4) + \epsilon\|\xi_H^n\|_{TM}^2.\end{aligned}$$

Then,

$$\|\xi_H^n\|_{TM}^2 \le M(H^4 + \|\theta_H^n\|^2 + \|\sigma_H^n\|^2). \tag{4.9}$$

Selecting $v=\theta_H^n$ in (4.6) and $s=\pi_H^n$ in (4.7), then adding (4.7) to (4.6), we can obtain

$$\begin{aligned}&(\varphi d_t\theta_H^n,\theta_H^n)+\Big(\frac{1}{\lambda}\pi_H^n,\pi_H^n\Big)_{TM}\\ =&(f(\tilde{C}_H^n)-F_H^n,\theta_H^n)-(\nabla\cdot(U_H^n\Pi_H C_H^n),\theta_H^n)\\ =&(f(\tilde{C}_H^n)-f(\tilde{c}^n),\theta_H^n)+(\nabla\cdot(uc)-D(u^nc^n)+D(u^nc^n)-\nabla\cdot(U_H^n\Pi_HC_H^n),\theta_H^n)\\ =&\Big(\sum_{i=1}^3 E_i,\theta_H^n\Big).\end{aligned}\tag{4.10}$$

Next we will estimate the three terms in the right side of (4.10). Note that

$$\tilde{C}_H^n-\tilde{c}^n=\begin{cases} C_H^n-c^n, & \text{if } q^n>0,\\ 0, & \text{if } q^n<0.\end{cases}$$

Therefore together the assumption of $f(c)$ with Taylor expansion and Lemma 4.1, we can get that

$$\begin{aligned}|(E_1,\theta_H^n)|=&|(f(\tilde{C}_H^n)-f(\tilde{c}^n),\theta_H^n)|\\ \le& M|(f_c(\theta_1^n)(\tilde{C}_H^n-\tilde{c}^n),\theta_H^n)|\\ \le& M|(f_c(\theta_1^n)(C_H^n-c^n),\theta_H^n)|\\ \le& M(\|\theta_H^n\|^2+\|\sigma_H^n\|^2),\end{aligned}\tag{4.11}$$

where θ_1^n is between $\tilde{C}_H^n$ and $\tilde{c}^n$. By the smoothness assumption (H_1),

$$\begin{aligned}|(E_2,\theta_H^n)|\le& M|(\nabla\cdot(uc)-\nabla\cdot(u^nc^n),\theta_H^n)|+|(\nabla\cdot(u^nc^n)-D(u^nc^n))|\\ \le& M(H^4+\|\theta_H^n\|^2).\end{aligned}\tag{4.12}$$

Noting Lemma 2.1 and (4.9), we have that

$$\begin{aligned}|(E_3,\theta_H^n)|\le& M\Big|\Big(u^nc^n-U_H^n\Pi_HC_H^n,-\frac{1}{\lambda}\pi_H^n\Big)_{TM}\Big|\\ \le& M|(u^nc^n-u^n\Pi_HC_H^n,\pi_H^n)_{TM}|+M|(u^n\Pi_HC_H^n-U_H^n\Pi_HC_H^n,\pi_H^n)_{TM}|\\ \le& M(\|c^n-\Pi_HC_H^n\|^2+\|u^n-U_H^n\|^2)+\epsilon\|\pi\|_{TM}^2\\ \le& M(\|H^4+\theta_H^n\|^2+\|\sigma_H^n\|^2)+\epsilon\|\pi\|_{TM}^2.\end{aligned}\tag{4.13}$$

Furthermore, we have

$$\begin{aligned}(\varphi d_t\theta_H^n,\theta_H^n)=&\Big(\varphi\frac{(\theta_H^n-\theta_H^{n-1})}{\Delta t},\theta_H^n\Big)=\frac{1}{2\Delta t}[2(\varphi\theta_H^n,\theta_H^n)-2(\varphi\theta_H^{n-1},\theta_H^n)]\\ =&\frac{1}{2\Delta t}[(\varphi\theta_H^n,\theta_H^n)-(\varphi\theta_H^{n-1},\theta_H^{n-1})+(\varphi\theta_H^{n-1},\theta_H^{n-1})+(\varphi\theta_H^n,\theta_H^n)-2(\varphi\theta_H^{n-1},\theta_H^n)]\\ =&\frac{1}{2}d_t(\varphi\theta_H^n,\theta_H^n)+\frac{1}{2\Delta t}[(\varphi(\theta_H^n-\theta_H^{n-1}),\theta_H^n-\theta_H^n)]\\ =&\frac{1}{2}d_t(\varphi\theta_H^n,\theta_H^n)+\frac{\Delta t}{2}(\varphi d_t\theta_H^n,\theta_H^n).\end{aligned}\tag{4.14}$$

Analyzing the left hand side of (4.10), we have

$$\begin{aligned}(\varphi d_t\theta_H^n,\theta_H^n)+(\lambda\pi_H^n,\pi_H^n)_{TM}&\geq\frac{\varphi_0}{2}d_t\|\theta_H^n\|^2+\varphi_0\frac{\Delta t}{2}\|d_t\theta_H^n\|^2+\frac{1}{\lambda_1}\|\pi_H^n\|_{TM}^2\\&\geq\frac{\varphi_0}{2\Delta t}(\|\theta_H^n\|^2-\|\theta_H^{n-1}\|^2)+\frac{1}{\lambda_1}\|\pi_H^n\|_{TM}^2.\end{aligned}\tag{4.15}$$

Then combing (4.10) with (4.11)-(4.15), we can get that:

$$\frac{\varphi_0}{2\Delta t}(\|\theta_H^n\|^2-\|\theta_H^{n-1}\|^2)+\frac{1}{\lambda_1}\|\pi_H^n\|_{TM}^2\leq M(H^4+\|\theta_H^n\|^2+\|\sigma_H^n\|^2)+\epsilon\|\pi_H^n\|_{TM}^2.\tag{4.16}$$

Noting Lemma 4.1 and ϵ-Cauchy inequality, we give that

$$\frac{\varphi_0}{2\Delta t}(\|\theta_H^n\|^2-\|\theta_H^{n-1}\|^2)+\frac{1}{\lambda_1}\|\pi_H^n\|_{TM}^2\leq M\|\theta_H^n\|^2+M((\Delta t)^2+H^4).\tag{4.17}$$

Multiplying $2\Delta t$ in two sides, summing for n from 1 to N and noting $\tau_H^0=0$, we have that

$$\varphi_0(\|\theta_H^n\|^2-\|\theta_H^{n-1}\|^2)+\frac{1}{\lambda_1}\Delta t\sum_{n=1}^N\|\pi_H^n\|_{TM}^2\leq M\sum_{n=1}^N\|\theta_H^n\|^2\Delta t+M((\Delta t)^2+H^4).\tag{4.18}$$

Recalling $\theta_H^0=0$ and noting Gronwall's lemma, we give the result

$$\|\theta_H^N\|^2+\Delta t\sum_{n=1}^N\|\pi_H^n\|_{TM}^2\leq M((\Delta t)^2+H^4),\tag{4.19}$$

where Δt is selected sufficiently small.

Noting Lemma 4.1 and (4.19), we get (4.9),

$$\|\xi_H^n\|_{TM}\leq M(\Delta t+H^2).\tag{4.20}$$

By Liu [1], using the dual method, we get the estimate of η_H^n,

$$\|\eta_H^N\|_{TM}\leq M(\Delta t+H^2).\tag{4.21}$$

By Lemma 4.1, we obtain the following theorem.

Theorem 4.1. *Let U_H^n, P_H^n, C_H^n be obtained by Step 1 of the two-grid finite difference algorithm. Suppose the hypotheses (H) hold and that the time step Δt is sufficiently small, then for $1\leq n\leq N$, there exists a positive constant M independent of H and Δt such that*

$$\|U_H^n-u^n\|_{TM}+\|P_H^n-p^n\|+\|C_H^n-c^n\|\leq M(\Delta t+H^2).\tag{4.22}$$

Next we give the error estimates on the fine grid. Subtracting (4.1a) from (3.5a), we can obtain

$$(\nabla\cdot\xi_h^n,s)=0,\quad \forall s\in S_h. \tag{4.23}$$

And subtracting (4.1b) from (3.5b), we have that

$$(A(\pi_h C_h^n)U_h^n-\alpha(c^n))\tilde{U}_h^n,v)_{TM}=(\eta_h^n,\nabla\cdot v),\quad \forall v\in V_h.$$

By Taylor expansion, we have

$$\begin{aligned}&\alpha(c^n))\tilde{U}_h^n\\ =&\alpha(\Pi_H C_H^n)\tilde{U}_h^n+\alpha_c(\Pi_H C_H^n)\tilde{U}_h^n(c^n-\Pi_H C_H^n)+\frac{1}{2}\alpha_{cc}(c^*)(c^n-\Pi_H C_H^n)^2\tilde{U}_h^n,\end{aligned}$$

where c^* is selected between c^n and $\Pi_H C_H^n$. Thus, we get

$$\begin{aligned}&(\alpha(\Pi_H C_H^n)\xi_h^n,v)_{TM}\\ =&(\eta_h^n,\nabla\cdot v)+(\alpha_c(\Pi_H C_H^n)R_H U_H^n(c^n-\Pi_h C_h^n),v)_{TM}\\ &+(\alpha_c(\Pi_H C_H^n)(c^n-\Pi_H C_H^n)(u^n-R_H U_H^n),v)_{TM}\\ &+(\alpha_c(\Pi_H C_H^n)(c^n-\Pi_H C_H^n)(\tilde{U}_h^n-u^n),v)_{TM}\\ &+\Big(\frac{1}{2}\alpha_{cc}(c^*)(c^n-\Pi_H C_H^n)^2\tilde{U}_h^n,v\Big)_{TM}\\ =&(\eta_h^n,\nabla\cdot v)+\Big(\sum_{i=1}^4 T_i,v\Big)_{TM},\quad \forall v\in V_h.\end{aligned} \tag{4.24}$$

We can obtain the following equation by subtracting (4.2a) from (3.5c)

$$\begin{aligned}&(\varphi d_t\theta_h^n,s)+(\nabla\cdot\pi_h^n,s)\\ =&(f(\tilde{C}_h^n)-F_h^n,s)-(\nabla\cdot(U_h^n\Pi_h C_h^n),s),\quad \forall s\in S_h.\end{aligned} \tag{4.25}$$

And (3.5d) subtracting (4.2b), we can get

$$\begin{aligned}&(\pi_h^n,v)_{TM}+(U_h^n\Pi_h C_h^n,v)_{TM}\\ =&(R_H U_H^n\Pi_h C_h^n+\Pi_H C_H^n(R_h U_h^n-R_H U_H^n),v)_{TM}+(\lambda\theta_h^n,\nabla\cdot v),\quad \forall v\in V_h.\end{aligned} \tag{4.26}$$

Setting $s=\eta_h^n$, $v=\xi_h^n$, (4.23) and (4.24) can be transformed into the following:

$$(\alpha(\Pi_H C_H^n)\xi_h^n,\xi_h^n)_{TM}=\Big(\sum_{i=1}^4 T_i,\xi_h^n\Big)_{TM}. \tag{4.27}$$

We now analyze each term on the right hand side of (4.27). By Lemma 2.2, Lemma 4.1 and ϵ-Chauchy inequality, we have that

$$\begin{aligned}|(T_1,\xi_h^n)_{TM}|\le&\|\alpha_c R_H U_H^n\|_\infty\|\Pi_h C_h^n-c^n\|\|\xi_h^n\|_{TM}\\ \le& M\|\Pi_h C_h^n-\Pi_h c^n+\Pi_h c^n-c^n\|^2+\epsilon\|\xi_h^n\|_{TM}^2\\ \le& M\|C_h^n-c^n\|^2+Mh^4+\epsilon\|\xi_h^n\|_{TM}^2\\ \le& M((\Delta t)^2+h^4+\|\theta_h^n\|^2)+\epsilon\|\xi_h^n\|_{TM}^2, &\text{(4.28a)}\\ |(T_2,\xi_h^n)_{TM}|\le&\|\alpha_c\|_\infty\|(\Pi_H C_H^n-c^n)(R_H U_H^n-u^n)\|\|\xi_h^n\|_{TM}\\ \le& M\|\Pi_H C_H^n-c^n\|^2\|R_H U_H^n-u^n\|^2+\epsilon\|\xi_h^n\|_{TM}^2\\ \le& M((\Delta t)^2+H^4)((\Delta t)^2+H^4)+\epsilon\|\xi_h^n\|_{TM}^2, &\text{(4.28b)}\\ |(T_3,\xi_h^n)_{TM}|\le& M\|(\Pi_H C_H^n-c^n)(\tilde{U}_h^n-u^n)\|\|\xi_h^n\|_{TM}\\ \le& M\|\Pi_H C_H^n-c^n\|^2\|\tilde{U}_h^n-u^n\|^2+\epsilon\|\xi_h^n\|_{TM}^2\\ \le& M((\Delta t)^2+H^4)h^4+\epsilon\|\xi_h^n\|_{TM}^2, &\text{(4.28c)}\\ |(T_4,\xi_h^n)_{TM}|\le&\|\alpha_{cc}\tilde{U}_h^n\|_\infty\|\Pi_H C_H^n-c^n\|^2\|\xi_h^n\|_{TM}\\ \le& M((\Delta t)^4+H^8)+\epsilon\|\xi_h^n\|_{TM}^2. &\text{(4.28d)}\end{aligned}$$

Analyzing the left hand side of (4.27), we infer that

$$(\alpha(\Pi_H C_H^n)\xi_h^n,\xi_h^n)_{TM}\ge\frac{1}{a_1}\|\xi_h^n\|_{TM}^2. \tag{4.29}$$

Then, from the analyzing of the both sides of (4.27), we can obtain

$$\|\xi_h^n\|_{TM}^2\le M((\Delta t)^2+h^4+H^8)+M\|\theta_h^n\|^2. \tag{4.30}$$

Selecting $v=\theta_h^n$ in (4.25), and $s=\pi_h^n$ in (4.26), then adding (4.25) to (4.26), we can get

$$\begin{aligned}&(\varphi d_t\theta_h^n,\theta_h^n)+\frac{1}{\lambda}(\pi_h^n,\pi_h^n)_{TM}\\ =&(f(\tilde{C}_h^n)-F_h^n,\theta_h^n)-(\nabla\cdot(U_h^n\Pi_h C_h^n),\theta_h^n)\\ &-\frac{1}{\lambda}(R_H U_H^n\Pi_h C_h^n-U_h^n\Pi_h C_h^n+\Pi_H C_H^n(R_h U_h^n-R_H U_H^n),\pi_h^n)_{TM}\\ =&(f(\tilde{C}_h^n)-f(\tilde{c}^n),\theta_h^n)+(\nabla\cdot(uc)-\nabla\cdot(U_h^n\Pi_h C_h^n),\theta_h^n)\\ &-\frac{1}{\lambda}(R_H U_H^n\Pi_h C_h^n-U_h^n\Pi_h C_h^n+\Pi_H C_H^n(R_h U_h^n-R_H U_H^n),\pi_h^n)_{TM}\\ =&\Big(\sum_{i=1}^3 J_i,\theta_h^n\Big).\end{aligned}\tag{4.31}$$

Similar to the proof process of Lemma (4.1a), we obtain

$$\begin{aligned}
|(J_1+J_2,\theta_h^n)| &\le M(h^4+\|\theta_h^n\|^2+\|\sigma_h^n\|^2)\\
&\le M((\Delta t)^2+h^4+\|\theta_h^n\|^2), \qquad (4.32a)\\
|(J_3,\theta_h^n)| &\le M|(\Pi_H C_H^n(U_h^n-u^n),\pi_h^n)_{TM}|+M|(\Pi_h C_h^n(u^n-U_h^n),\pi_h^n)_{TM}|\\
&\quad +|((\Pi_H C_H^n-\Pi_h C_h^n)(u^n-R_H U_H^n),\pi_h^n)_{TM}|\\
&\le M(h^4+H^8+\Delta t^2)+M(H^4+h^4+\Delta^2+\|\theta_h^n\|^2)(H^4+\Delta^2)+\epsilon\|\pi_h^n\|_{TM}^2\\
&\le M((\Delta)^2+h^4+H^8)+\epsilon\|\pi_h^n\|_{TM}^2. \qquad (4.32b)
\end{aligned}$$

Next we will estimate the terms in the right side of (4.31), we have that

$$(\varphi d_t\theta_h^n,\theta_h^n)+\Big(\frac{1}{\lambda}\pi_h^n,\pi_h^n\Big)_{TM}\ge\frac{\varphi_0}{2\Delta t}(\|\theta_h^n\|^2-\|\theta_h^{n-1}\|^2)+\frac{1}{\lambda_1}\|\pi_h^n\|_{TM}^2. \tag{4.33}$$

Then, from the analysis of the both sides of (4.31), we can obtain

$$\begin{aligned}
&\frac{\varphi_0}{2\Delta t}(\|\theta_h^n\|^2-\|\theta_h^{n-1}\|^2)+\frac{1}{\lambda_1}\|\pi_h^n\|_{TM}^2\\
\le& M((\Delta t)^2+h^4+H^8)+M\|\theta_h^n\|^2+\epsilon\|\pi_h^n\|_{TM}^2.
\end{aligned} \tag{4.34}$$

Multiplying $2\Delta t$ in two sides, summing for n from 1 to N and noting $\pi_H^0=0$, we infer that

$$\begin{aligned}
&\varphi_0(\|\theta_h^N\|^2-\|\theta_h^0\|^2)+\frac{1}{\lambda_1}\Delta t\sum_{n=1}^N\|\pi_h^n\|_{TM}^2\\
\le& M\sum_{n=1}^N\|\theta_h^n\|^2\Delta t+M((\Delta t)^2+h^4+H^8).
\end{aligned} \tag{4.35}$$

Recalling $\theta_H^0=0$ and applying Gronwall's lemma, we get the result

$$\|\theta_h^N\|^2+\Delta t\sum_{n=1}^N\|\pi_h^n\|_{TM}^2\le M((\Delta t)^2+h^4+H^8), \tag{4.36}$$

where Δt is selected sufficiently small.

Noting Lemma 4.1, we have the following theorem.

Theorem 4.2. *Let U_h^n, P_h^n, C_h^n be obtained by Step 2 of the two-grid finite difference algorithm. Suppose the hypotheses (H) hold and that the time step Δt is sufficiently small, then for $1\le n\le N$, we have*

$$\|U_h^n-u^n\|_{TM}+\|P_h^n-p^n\|+\|C_h^n-c^n\|\le M(\Delta t+h^2+H^4). \tag{4.37}$$

5 Numerical tests

In this section, some numerical experiments using the two-grid block-centered finite difference method have been constructed.

For the following incompressible miscible displacement problem:

$$\begin{cases} \nabla\cdot u = q(x,t), & x\in\Omega, \quad t\in(0,T], \\ u = -a(c)\nabla p, & x\in\Omega, \quad t\in(0,T], \\ \varphi\dfrac{\partial c}{\partial t} + \nabla\cdot(uc) - \nabla\cdot(D\nabla c) = f(\tilde{c}), & x\in\Omega, \quad t\in(0,T]. \end{cases} \tag{5.1}$$

For convenient, the domain $\Omega=(0,1)^2$ is uniformly divided by the rectangle of uniform mesh size H and h, respectively. And as presented in Theorem 4.2, we set $h=H^2$. Meanwhile, we choose the time step τ sufficiently small to illustrate the space convergence rate in Examples 5.1 and 5.2. And we also set $\tau=h^2$ to illustrate the time convergence rate in Example 5.3. In addition, to verify the efficiency of the two-grid block-centered finite difference method, we apply the nonlinear implicit scheme in (3.4a)-(3.4d) with mesh size h. Tables 1, 2, 3 and 4 are presented to get the space convergence rate and the efficiency of the two-grid block-centered finite difference method, while the Tables 5 and 6 are given in order to obtain the time convergence rate and the efficiency of the two-grid block-centered finite difference method. Moreover, to present vividly the numerical solutions, Figs. 2-7 are given with $H=1/8$, $h=1/64$, $T=1e-3$, and Figs. 8-10 are given with $H=1/8$, $h=1/64$, $T=1$. And we define the norm

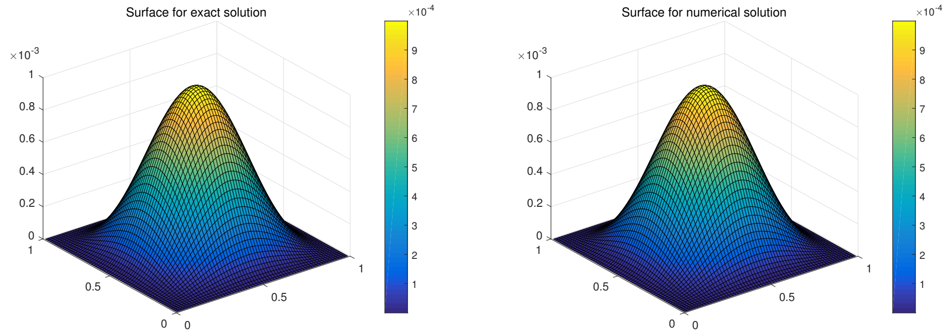
$$\|P_h-p\|_{0,\infty} = \max_{1\le n\le N}\{\|(P_h-p)^n\|\}.$$

Example 5.1. Here, $\varphi=1$, $D=0.01$, $a(c)^{-1}=(c+2)$, $\tau=1.0e-5$, $T=1.0e-3$. Meanwhile, $f(x,t,c)$ and $q(x,t)$ are suitably chosen such that the exact solution of (5.1) is

$$\begin{cases} c=\sin^2(\pi x)\sin^2(\pi y)t, \\ p=-\dfrac{1}{2}\sin^4(\pi x)\sin^4(\pi y)t^2 - 2\sin^2(\pi x)\sin^2(\pi y)t + \dfrac{9}{128}t^2 + \dfrac{1}{2}t. \end{cases}$$

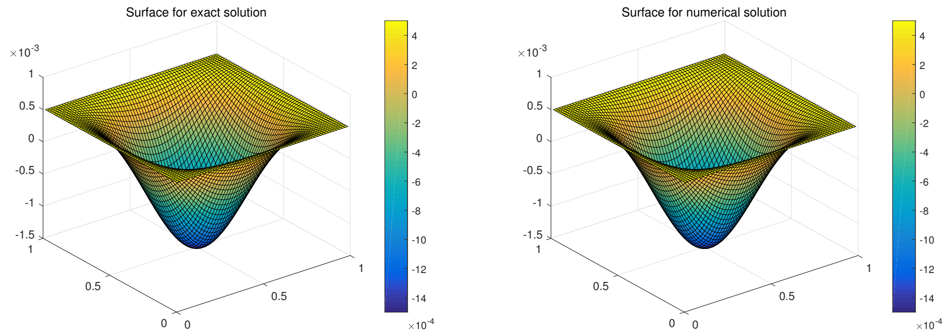
Then, we have

$$\begin{aligned} q =\, & 2t\pi^2\cos(2\pi x)\sin(\pi y)^2 + 2t\pi^2\cos(2\pi y)\sin(\pi x)^2, \\ f =\, & c^2 + \sin(\pi x)^2\sin(\pi y)^2 - \lambda(2t\pi^2\cos(2\pi x)\sin(\pi y)^2 + 2t\pi^2\cos(2\pi y)\sin(\pi x)^2) \\ & - t^2\sin(\pi x)^4\sin(\pi y)^4 + 2t^2\pi^2\cos(2\pi x)\sin(\pi x)^2\sin(\pi y)^4 \\ & + 2t^2\pi^2\cos(2\pi y)\sin(\pi x)^4\sin(\pi y)^2 + 2t^2\pi^2\cos(\pi x)\sin(\pi x)\sin(2\pi x)\sin(\pi y)^4 \\ & + 2t^2\pi^2\cos(\pi y)\sin(\pi x)^4\sin(\pi y)\sin(2\pi y). \end{aligned}$$



(a) (b)

Figure 2: The concentration figures for Example 5.1. (a): the exact solution c, (b): the numerical solution C_h.



(a) (b)

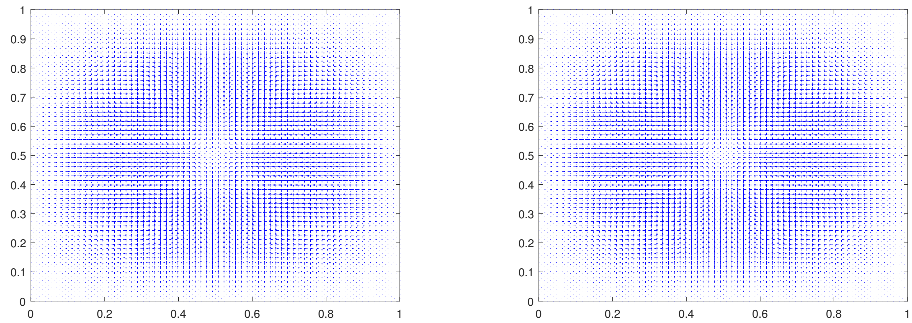
Figure 3: The pressure figures for Example 5.1. (a): the exact solution p, (b): the numerical solution P_h.

Table 1: Error and CPU time cost of two-grid method with Example 5.1.

H	h	$\|\|c-C_h\|\|_{0,\infty}$	rate	$\|\|p-P_h\|\|_{0,\infty}$	rate	$\|\|u-U_h\|\|_{0,\infty}$	rate	CPU time
2^{-1}	2^{-2}	1.3350e-08	—	1.3062e-04	—	2.1301e-04	—	0.1143 s
2^{-2}	2^{-4}	9.0352e-10	1.94	7.2375e-06	2.09	1.2417e-05	2.05	0.5750 s
2^{-3}	2^{-6}	5.6745e-11	2.00	4.4802e-07	2.01	7.7282e-07	2.00	299.5745 s

Table 2: Error and CPU time cost of nonlinear implicit scheme with Example 5.1.

h	$\|\|c-C_h\|\|_{0,\infty}$	rate	$\|\|p-P_h\|\|_{0,\infty}$	rate	$\|\|u-U_h\|\|_{0,\infty}$	rate	CPU time
2^{-2}	1.3350e-08	—	1.3060e-04	—	2.1301e-04	—	0.1249 s
2^{-4}	9.0352e-10	1.94	7.2409e-06	2.09	1.2417e-05	2.05	0.6754 s
2^{-6}	5.6745e-11	2.00	4.4930e-07	2.01	7.7282e-07	2.00	864.5798 s



(a) (b)

Figure 4: The Darcy velocity figures for Example 5.1. (a): the exact solution u, (b): the numerical solution U_h.

Table 3: Error and CPU time cost of two-grid method with Example 5.2.

H	h	$\|c-C_h\|_{0,\infty}$	rate	$\|p-P_h\|_{0,\infty}$	rate	$\|u-U_h\|_{0,\infty}$	rate	CPU time
2^{-1}	2^{-2}	1.3350e-08	—	6.5321e-05	—	2.1301e-04	—	0.1116 s
2^{-2}	2^{-4}	9.0352e-10	1.94	3.6096e-06	2.09	1.2418e-05	2.05	0.5514 s
2^{-3}	2^{-6}	5.6745e-11	2.00	2.2367e-07	2.00	7.7351e-07	2.00	306.0547 s

Table 4: Error and CPU time cost of nonlinear implicit scheme with Example 5.2.

h	$\|c-C_h\|_{0,\infty}$	rate	$\|p-P_h\|_{0,\infty}$	rate	$\|u-U_h\|_{0,\infty}$	rate	CPU time
2^{-2}	1.3350e-08	—	6.5321e-05	—	2.1301e-04	—	0.2545 s
2^{-4}	9.0352e-10	1.94	3.6198e-06	2.09	1.2417e-05	2.05	0.7521 s
2^{-6}	5.6745e-11	2.00	2.2461e-07	2.01	7.7282e-07	2.00	887.9120 s

Example 5.2. Here, $\varphi=1$, $D=0.01$, $a(c)^{-1}=(c^2+1)$, $\tau=1.0e-5$, $T=1.0e-3$. Meanwhile, $f(x,t,c)$ and $q(x,t)$ are suitably chosen, such that the exact solution of (5.1) is

$$\begin{cases} c=\cos^2(\pi x)\cos^2(\pi y)t, \\ p=-\dfrac{1}{3}\cos^6(\pi x)\cos^6(\pi y)t^3-\cos^2(\pi x)\cos^2(\pi y)t+\dfrac{252}{7741}t^3+\dfrac{1}{4}t. \end{cases}$$

Then, we have

$$\begin{aligned} q=&2tpi^2\cos(\pi x)^2*\sin(\pi y)^2-4t\pi^2\cos(\pi x)^2\cos(\pi y)^2+2t\pi^2\cos(\pi y)^2\sin(\pi x)^2, \\ f=&c^2+\lambda(2t\pi^2\cos(2\pi x)\cos(\pi y)^2+2t\pi^2\cos(\pi x)^2\cos(2\pi y))+\cos(\pi x)^2\cos(\pi y)^2 \\ &-t^2\cos(\pi x)^4\cos(\pi y)^4-t\cos(\pi x)^2\cos(\pi y)^2(2t\pi^2\cos(\pi x)^2\cos(\pi y)^2 \\ &-2t\pi^2\cos(\pi x)^2\sin(\pi y)^2)-t\cos(\pi x)^2\cos(\pi y)^2(2t\pi^2\cos(\pi x)^2\cos(\pi y)^2 \\ &-2t\pi^2\cos(\pi y)^2*\sin(\pi x)^2)+4t^2\pi^2\cos(\pi x)^2\cos(\pi y)^4\sin(\pi x)^2 \\ &+4t^2\pi^2\cos(\pi x)^4\cos(\pi y)^2*\sin(\pi y)^2. \end{aligned}$$

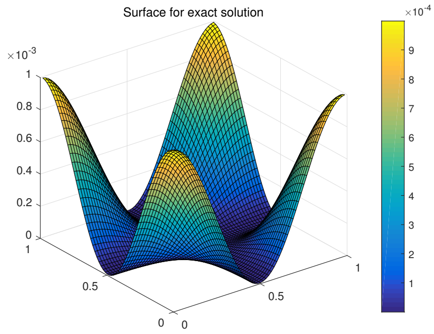


Figure 5: The concentration figures for Example 5.2. (a): the exact solution c, (b): the numerical solution C_h.

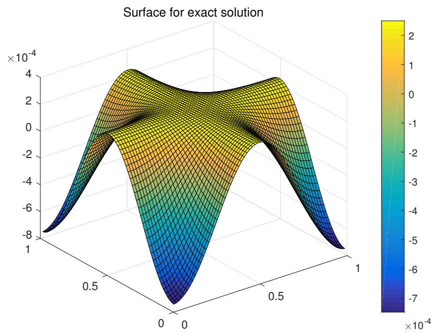


Figure 6: The pressure figures for Example 5.2. (a): the exact solution p, (b): the numerical solution P_h.

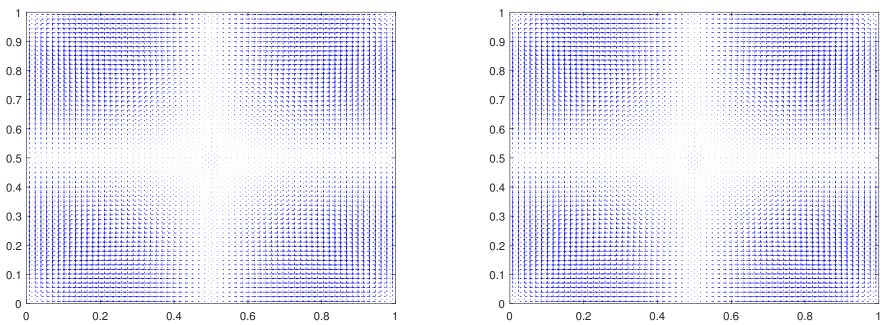


Figure 7: The Darcy velocity figures for Example 5.2. (a): the exact solution u, (b): the numerical solution U_h.

Example 5.3. Here, $\varphi=1$, $D=1$, $a(c)^{-1}=(c^2+1)$, $\tau=h^2$, $T=1.0$. Meanwhile, $f(x,t,c)$ and

Table 5: Error and CPU time cost of two-grid method with Example 5.2.

H	h	$\Vert c-C_h\Vert_{0,\infty}$	rate	$\Vert p-P_h\Vert_{0,\infty}$	rate	$\Vert u-U_h\Vert_{0,\infty}$	rate	CPU time
2^{-1}	2^{-2}	2.23e-03	—	9.74e-04	—	3.65e-03	—	0.0924 s
2^{-2}	2^{-4}	1.31e-04	2.04	6.48e-05	1.96	3.33e-04	1.73	1.5203 s
2^{-3}	2^{-6}	8.17e-06	2.00	3.97e-06	2.01	2.58e-05	1.84	1037.4227 s

$q(x,t)$ are suitably chosen, such that the exact solution of (5.1) is

$$\begin{cases} c=x^2(1-x)^2y^2(1-y)^2t, \\ p=-\dfrac{1}{3}x^6(1-x)^6y^6(1-y)^6t^3-x^2(1-x)^2y^2(1-y)^2t+\dfrac{1}{432864471}t^3+\dfrac{1}{900}t. \end{cases}$$

Then, we have

$$\begin{aligned} q=&2tx^2(x-1)^2(y-1)^2+2ty^2(x-1)^2(y-1)^2+2tx^2y^2(x-1)^2+2tx^2y^2(y-1)^2 \\ &+4txy^2(2x-2)(y-1)^2+4tx^2y(2y-2)(x-1)^2, \\ f=&c^2+(2txy^2(x-1)^2(y-1)^2+tx^2y^2(2x-2)(y-1)^2)^2+(2tx^2y(x-1)^2(y-1)^2 \\ &+tx^2y^2(2y-2)(x-1)^2)^2-\lambda(2tx^2(x-1)^2(6y^2-6y+1) \\ &+2ty^2(y-1)^2(6x^2-6x+1))+x^2y^2(x-1)^2(y-1)^2-t^2x^4y^4(x-1)^4(y-1)^4 \\ &+tx^2y^2(x-1)^2(y-1)^2(2tx^2(x-1)^2(y-1)^2+2tx^2y^2(x-1)^2 \\ &+4tx^2y(2y-2)(x-1)^2)+tx^2y^2(x-1)^2(y-1)^2(2ty^2(x-1)^2(y-1)^2 \\ &+2tx^2y^2(y-1)^2+4txy^2(2x-2)(y-1)^2). \end{aligned}$$

From Tables 1-6 and Fig. 2-10, we can obtain that the convergence order of the two-grid block-centered finite difference method is the $\mathcal{O}(\Delta t+h^2+H^4)$ accuracy in discrete L_2 norm. These results are corresponding to the error estimates in Theorem 4.2. Moreover, we also can see that the two-grid method spends less time than the nonlinear implicit

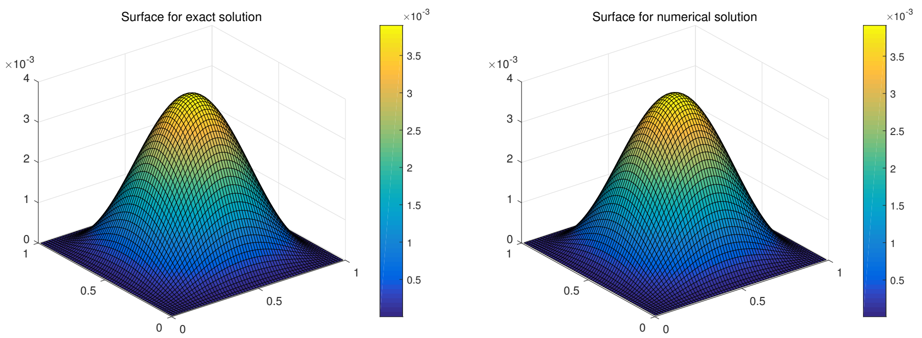


Figure 8: The concentration figures for Example 5.3. (a): the exact solution c, (b): the numerical solution C_h.

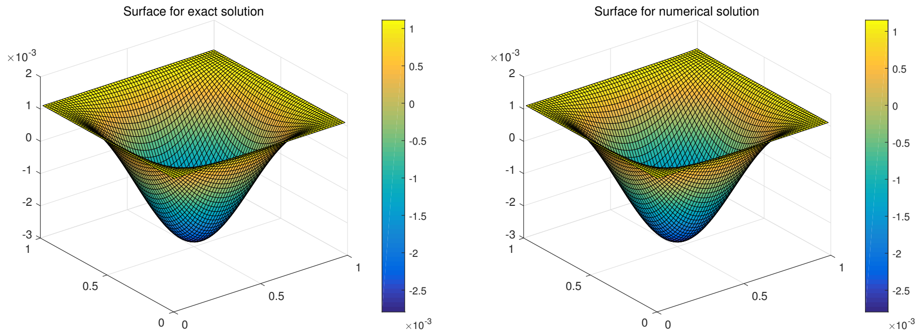


Figure 9: The pressure figures for Example 5.3. (a): the exact solution p, (b): the numerical solution P_h.

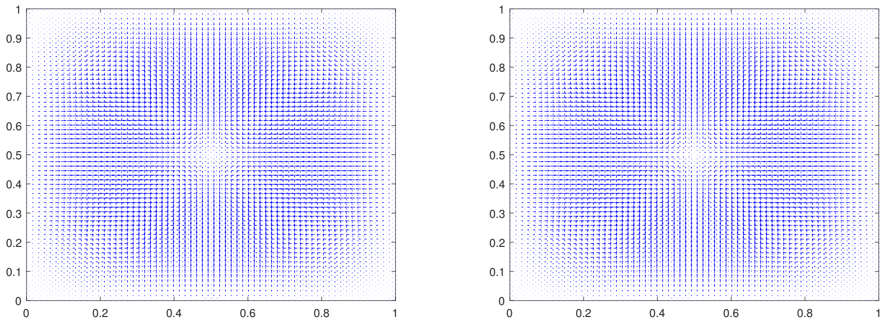


Figure 10: The Darcy velocity figures for Example 5.3. (a): the exact solution u, (b): the numerical solution U_h.

Table 6: Error and CPU time cost of nonlinear implicit scheme with Example 5.3.

h	$\|c-C_h\|_{0,\infty}$	rate	$\|p-P_h\|_{0,\infty}$	rate	$\|u-U_h\|_{0,\infty}$	rate	CPU time
2^{-2}	3.48e-09	—	9.91e-07	—	3.65e-06	—	0.1227 s
2^{-4}	3.42e-10	1.67	6.67e-08	1.95	3.33e-07	1.73	5.4492 s
2^{-6}	2.63e-11	1.85	4.20e-09	1.99	2.58e-08	1.84	3535.9231 s

scheme by comparing Tables 1, 3, 5 with Tables 2, 4, 6. This phenomenon shows that two-grid method is a highly effective method for the incompressible miscible displacement problem.

Acknowledgements

The authors thank the editor and referees for their valuable comments and suggeations which helped us to improve the results of this paper. This work is supported by National Natural Science Foundation of China No. 12131014.

References

[1] Y. LUI, *On convergence of block-centered finite difference scheme for two-phase flow in porous medium (in chinese),* Journal of Shandong University (Natural Science), 30(1) (1995), pp. 20–29.

[2] J. DOUGLAS, R. E. EWING, AND M. F. WHEELER, *The approximation of the pressure by a mixed method in the simulation of miscible displacement,* RAIRO Analyse Numérique, 17(1) (1983), pp. 17–33.

[3] J. ZHANG, J. ZHU, R. ZHANG, AND D. YANG, *A combined discontinuous Galerkin finite element method for miscible displacement problem,* J. Comput. Appl. Math., 309 (2017), pp. 44–55.

[4] P. A. RAVIART, AND J. M. THOMAS, *A mixed finite element method for 2-nd order elliptic problems,* Mathematical Aspects of Finite Element Methods, 1977.

[5] A. WEISER, AND M. W. FANETT, *On convergence of block-centered finite differences for elliptic-problems,* SIAM J. Numer. Anal., 25(2) (1988), pp. 351–375.

[6] T. ARBOGAST, M. F. WHEELER AND I. YOTOV, *Mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences,* SIAM J. Numer. Anal., (1997).

[7] H. RUI, AND H. PAN, *A block-centered fifinite difference method for the Darcy-Forchheimer model,* SIAM J. Numer. Anal., 50(5) (2012), pp. 2612–2631.

[8] H. RUI, AND H. PAN, *Block-centered finite difference methods for parabolic equation with time-dependent coefficient,* Japan J. Industr. Appl. Math., 30(3) (2013), pp. 681–699.

[9] X. LI, AND H. RUI, *Characteristic block-centred finite difference methods for nonlinear convection-dominated diffusion equation,* Int. J. Comput. Math., (2017), pp. 1–19.

[10] J. XU, *A novel two-grid method for semilinear elliptic equations,* SIAM J. Sci. Comput., 15(1) (1994), pp. 231–237.

[11] J. XU, *Two-grid discretization techniques for linear and nonlinear PDEs,* SIAM J. Sci. Comput., 33(5) (1996), pp. 1759–1777.

[12] O. A. LAYTON, *A two-level method for the discretization of nonlinear boundary value problems,* SIAM J. Numer. Anal., 33(6) (1996), pp. 2359–2374.

[13] C. N. DAWSON, M. F. WHEELER, AND C. S. WOODWARD, *A two-grid finite difference scheme for nonlinear parabolic equations,* SIAM J. Numer. Anal., 35(2) (1998), pp. 435–452.

[14] L. WU, AND M. B. A. III, *A two-grid method for mixed finite-element solution of Reaction-Diffusion equations,* Numer. Methods Partial Differential Equations, 15 (1999), pp. 317–332.

[15] C. BI, AND V. E. GINTING, *Two-grid discontinuous Galerkin method for quasi-linear elliptic problems,* J. Sci. Comput., 49(3) (2011), pp. 311–331.

[16] S. CONGREVE, P. HOUSTON, AND T. P. WIHLER, *Two-grid hp-version discontinuous galerkin finite element methods for second-order quasilinear elliptic PDEs,* J. Sci. Comput., 55(2) (2013), pp. 471–497.

[17] L. CHEN, AND Y. CHEN, *Two-grid method for nonlinear reaction-diffusion equations by mixed finite element methods,* J. Sci. Comput., 49(3) (2011), pp. 383–401.

[18] J. ZENG, Y. CHEN, AND H. HU, *Two-grid method for compressible miscible displacement problem by CFEM-MFEM,* J. Comput. Appl. Math., (2018).

[19] H. HU, Y. CHEN, AND J. ZHOU, *Two-grid method for miscible displacement problem by mixed finite element methods and finite element method of characteristics,* Comput. Math. Appl., 72(11) (2016), pp. 2694–2715.

[20] Y. CHEN, J. ZENG, AND J. ZHOU, *L^p error estimates of two-grid method for miscible displacement problem,* J. Sci. Comput., 69(1) (2016), pp. 28–51.

[21] S. LIU, Y. CHEN, Y. HUANG, AND J. ZHOU, *Two-grid methods for miscible displacement problem*

by Galerkin methods and mixed finite-element methods, Int. J. Comput. Math., (2017), pp. 1–28.
[22] H. RUI, AND W. LIU, *A two-grid block-centered finite difference method for Darcy-Forchheimer flow in porous media,* SIAM J. Numer. Anal., 53(4) (2015), pp. 1941–1962.
[23] X. LI, AND H. RUI, *A two-grid block-centered finite difference method for nonlinear non-fickian flow model,* Appl. Math. Comput., 281 (2016), pp. 300–313.
[24] J. C. NÉDÉLEC, *Mixed finite elements in* R^3*,* Numerische Mathematik, 35(3) (1980), pp. 315–341.