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Traveling Waves in Degenerate Diffusion Equations

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Abstract. In this survey, we present a literature review on the study of traveling waves in degenerate diffusion equations by illustrating the interesting and singular wave behavior caused by degeneracy. The main results on wave existence and stability are presented for the typical degenerate equations, including porous medium equations, flux limited diffusion equations, delayed degenerate diffusion equations, and other strong degenerate diffusion equations.

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1 Introduction

Degenerate diffusion models appear in the theory of many biological and physical applications, such as population spreading, tumor invasions, fluid dynamics, and many practical problems [39, 41, 50, 51, 54]. To be precise, flow of an ideal gas through a porous medium can be described by the porous medium equation [54]; flux-saturated equations were introduced in the inertial confinement fusion [20]. In this review, we will present some of the recent developments of traveling waves related to nonlinear diffusion equations.

The mathematical interest in degenerate diffusion equations is now steadily growing. Fundamental features of degenerate diffusion equations account for the

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propagation properties of the solutions such as the finite propagation property and the waiting time phenomenon. Finite propagation means the persistence of compact supports of solutions, while waiting time phenomenon represents that the support of the solution remains motionless for some time before starting to move. Another fundamental property of this type of equations is the existence of discontinuous traveling waves in strong degenerate diffusion equations.

Traveling waves, i.e., solutions of the type $u(t,x)=u(x-ct)$, starts from nonlinear reaction-diffusion equation (with linear diffusion) of the type

$$\frac{\partial u}{\partial t}=D\Delta u+f(u),$$

where D is the diffusion coefficient and $f(u)$ is the nonlinear reaction function. Since the pioneering works of Fisher [24] and Kolmogorov *et al.* [32] in the 30s, traveling waves describe various dynamical behaviors in propagation processes. Existence and stability of traveling waves for nonlinear reaction-diffusion equations (with linear diffusion) have been widely studied in the literature, see for example, [16,22,23,29,36,42,43] and the references therein. In particular, the degenerate or singular diffusion equations have also been investigated in a series of works [4,5,11,12].

In the following sections, we will first discuss the traveling waves for the several typical degenerate diffusion equations, including the classical porous medium equations, flux limited diffusion equations, delayed degenerate diffusion equations. Finally, we will also present traveling waves for some other strong degenerate diffusion equations, especially the corresponding sharp discontinuous traveling waves.

2 Traveling waves in porous medium equations

Porous medium equations arising in many biological and physical systems are one of the typical degenerate diffusion equations. In population dynamics model, positive density-dependent dispersal implies that the population migrates to regions of lower density more rapidly as the population gets more crowded. The population dispersal can be influenced by many biological, chemical and physical factors, such as overcrowding, social behavior, mating, lights, food, chemical substances, etc [53]. The derivation of porous medium type density-dependent diffusion implies that the population jump probability is a density-dependent function due to the conspecific repelling or self-organization [49,55].

For porous media equation, there is a well-established theory on the traveling waves. To appreciate this analysis, consider the following reaction diffusion equation with porous medium diffusion:

$$u_t=\Delta u^m+f(u), \quad x\in Q_T:=\mathbb{R}^N\times(0,T)$$

with $N\geq 1$, $0<T\leq\infty$, and $m>1$. When $m>1$, the diffusion coefficient $D(u):=mu^{m-1}$ vanishes for $u=0$. Both the diffusion and reaction terms have conflicting effects on the behaviour of the traveling waves. When $f(u)=0$, the classical porous medium equations exhibit finite propagation phenomenon. At the same time, the singular reaction coefficient also influence the wave patterns.

In ecological models, the first density-dependent diffusion model was derived by Gurney and Nisbet [26, 27]. Using random walk method, they arrived at the model

$$u_t=-\nabla\cdot\vec{J}+g(u),$$

where u is the population density, g represents the growth function, the flux $\vec{J}$ involves the movement mechanism. For positive density-dependent diffusion $\vec{J}=-ku\nabla u$. A more general form of degenerate diffusion model was deduced by Gurtin and MacCamy [28] through a continuum mechanics approach

$$u_t=\phi'(u)\Delta u+g(u,\nabla u),$$

where $\phi(0)=\phi'(0)=0$, and $\phi'(u)>0$ for $u>0$.

The investigation of traveling waves for porous medium equation with Fisher-KPP source was first carried out by Aronson [2]

$$u_t=(u^m)_{xx}+u(1-u), \quad (x,t)\in\mathbb{R}\times(0,\infty).$$

Some particular cases of the porous medium equations with Fisher-KPP source were thorough studied in a series of works [3, 44, 46]. When $m=2$, $N=1$, the equations reduce to

$$u_t=[uu_x]_x+u(1-u).$$

The results showed the existence of traveling waves and critical wave speed $c^*=1/\sqrt{2}$. An interesting fact is that when the wave speed $c=c^*$, the corresponding sharp waves could be given in the explicit form in this special case.

In the simplified but nonetheless representative one dimensional case when $f(u)=\lambda u^n$, one obtain

$$u_t=(u^m)_{xx}+\lambda u^n, \quad (x,t)\in\mathbb{R}\times(0,\infty) \tag{2.1}$$

with $m>1$, $\lambda>0$, $n\in\mathbb{R}$. Pablo and Vázquez [19] proved the existence of traveling waves only if $m+n=2$ and only for wave speeds $c\geq c^*=2\sqrt{\lambda m}$. An important extension of this results is the traveling waves for porous-Fisher equations with singular reaction term

$$u_t=(u^m)_{xx}+\lambda u^n(1-u), \quad (x,t)\in\mathbb{R}\times(0,\infty) \tag{2.2}$$

with $n<1<m$. This existence result can be proved by the techniques in [4,21]. In [19], the authors proved there exist travelling waves for Eq. (2.2) only if $m+n\geq 2$, and only for wave speeds $c\geq c^*(m,n)$. The finite waves were also analyzed in [19]. The traveling wave will be briefly written as TW in the following statements.

Theorem 2.1 ([19]). *Eq.* (2.2) *with* $m>1$, $\lambda=1/m$ *admits TW solutions only if* $p=m+n-2\geq 0$. *Moreover, for* $p>0$ *there exists a critical speed* $c=c^*(p)>0$ *such that:*

(i) there exist no TWs for $0<c<c^*$;

(ii) there exists a unique TW for each $c\geq c*$.

Furthermore, the TWs with $c=c^*$ *is finite. The TWs with* $c\geq c^*$ *are finite if and only if* $n<1$. *Finally,* $c^*(p)$ *decreases as* p *increases. In particular*

$$c^*(0)=2, \quad c^*(1)=\frac{1}{\sqrt{2}}, \quad c^*(p)\to 0 \quad as \quad p\to+\infty.$$

Note that $p=1$ and $c=c^*$, the traveling waves are given by an explicit formula. A more detailed study is given by Pablo and Sánchez in [18]. They described the curve of minimal wave speed c^* in terms of m and p and studied the asymptotic behaviour of the solutions. For a more general degenerate diffusion equation

$$u_t=[D(u)u_x]_x+g(u),$$

where $D(0)=0$ with $D(u)>0$ on $[0,1]$ and $D\in C^2[0,1]$ with $D'(u)>0$ and $D''(u)\neq 0$, the source term g is the classical Fisher-KPP type function, Sánchez-Garduño and Maini [49] gave a complete analysis on the traveling waves for this nonlinear diffusion equation. Using a dynamical systems approach, they showed there exists a unique critical wave speed c^* for which the equation has a sharp traveling wave $u(x,t)=\phi(x-c^*t)$ with $\phi(-\infty)=1$, $\phi(\tau)=0$ on $\tau\geq\tau_0$. Here, the degeneracy of diffusion causes this singular dynamical behavior of wave fronts with weak regularity. In [48], the authors replaced regularity condition D with $D\in C^2[0,1]$ and $D'(0),D''(0)\neq 0$. Further, Malaguti and Marcelli [40] proved the existence of sharp-type solutions with less regularity conditions on g and D, in particular, without any sign condition on D''.

3 Traveling waves in flux limited diffusion equations

In this section, we consider the traveling waves in flux limited diffusion equations. Flux limited porous media equations include two important degenerate diffusion mechanisms: porous media diffusion and flux saturated diffusion.

The flux-saturated model was first introduced by Rosenau [47]. As a physical modification of classical diffusion equation, this model incorporates a flux that saturates as gradients become unbounded rather than a infinite flux in standard diffusion equation. This arguments indicate the fact of finite spreading of sound in a medium. Next, we will give a brief review of Rosenau's derivation. The transport of a physical quantity u in a continuum medium can be given by

$$u_t = -\mathrm{div}\,\overrightarrow{J}, \tag{3.1}$$

where $\overrightarrow{J}$ is the associated flux. For classical heat equations, the flux $\overrightarrow{J} = -\nu\nabla u$. Rosenau proposed a more physical flux

$$\overrightarrow{J} = -\frac{\nu\nabla u}{\sqrt{1+\left(\frac{\nu|\nabla u|}{cu}\right)^2}}$$

to replace the classical flux, which saturates as gradients become unbounded. By substituting the new flux into (3.1), they obtained

$$\frac{\partial u}{\partial t} = \nu\mathrm{div}\left(\frac{u\nabla u}{\sqrt{u^2+\frac{\nu^2}{c^2}|\nabla u|^2}}\right).$$

By a combination of two nonlinear diffusion mechanisms, Caselles [13] studied the following flux-limited diffusion equation:

$$\frac{\partial u}{\partial t} = \nu\mathrm{div}\left(\frac{u^m\nabla u}{\sqrt{u^2+\frac{\nu^2}{c^2}|\nabla u|^2}}\right), \tag{3.2}$$

where $m>1$, ν is a kinematic viscosity, $c>0$ is a characteristic speed. As introduced in [14, 17], we refer to this equation as flux-limited porous medium equation. When $c\to\infty$, the solutions of Eq. (3.2) converge to solutions of the porous medium equation [15]. There are a series of recent results focus on the traveling waves solutions to flux-saturated type equations in combination with reactions terms. This flux-saturated transport mechanisms could be used to describe various processes in social sciences [7] or traffic flow [10] and biology [1]. Incorporating the reaction term, one obtains the following flux limited reaction diffusion equation:

$$\frac{\partial u}{\partial t}=\nu\mathrm{div}\left(\frac{u^m\nabla u}{\sqrt{u^2+\frac{\nu^2}{c^2}|\nabla u|^2}}\right)+F(u), \tag{3.3}$$

where $F(u)$ is a sufficiently smooth function. For $m=1$ in (3.2) coupled with a reaction term of Fisher-Kolmogorov-Petrovskii-Piscounov (FKPP) type $F(u)=u(1-u)$, traveling wave solutions ware discussed in [12]. In the corresponding dynamical system, entropy wave fronts are constructed and these waves are unique and entropy piecewise smooth. When $m>1$, the situation changes to be rather complicated. The geometric characterization of entropy solutions and the Rankine-Hugoniot condition are very important in analysis on the behavior of entropy traveling waves for this strong degenerate diffusion equation [11]. Traveling wave solutions are the solutions of the form $u(x-ct)=u(\xi)$ with c being the wave speed. The entropy traveling wave solutions for flux limited diffusion equations were also analyzed in [11].

Theorem 3.1 ([11]). *Let $m>1$. The following results are verified:*

i) Existence: There exist two values $0<c_{\rm ent}<c_{\rm smooth}<mc$, depending on c, v, m and F, such that:

1) for $c>c_{\rm smooth}$ there exists a unique smooth traveling wave solution of (3.2)*,*

2) for $c=c_{\rm smooth}$ there exists a traveling wave solution of (3.2)*, which is continuous but not smooth,*

3) for $c_{\rm smooth}>c\geq c_{\rm ent}$ there exists a traveling wave solution of (3.2)*, which is discontinuous.*

ii) Uniqueness: for any fixed value of $c\in[c_{ent},+\infty)$, after normalization (modulo spatial translations) there is just one traveling wave solution in the class of piecewise smooth solutions (that is, smooth except maybe at a finite number of points) with range in $[0,1]$ and satisfying the entropy conditions.

The entropy traveling waves solutions exhibit rich dynamical behaviors. In particular, when the wave speed $c\in[c_{\rm ent},c_{\rm smooth})$, the waves are discontinuous. For critical case $c=c_{\rm ent}$, the corresponding waves are discontinuous and compact supported on the half line, which indicates the finite propagation speed of degenerate diffusion equations.

4 Traveling waves in delayed degenerate diffusion equations

In this section, we recall the main results about traveling waves for the reaction-diffusion equations with degenerate diffusion and time delay. Nonlinear dispersal and age-structured population can be modeled by the following delayed degenerate diffusion equations:

$$\frac{\partial u}{\partial t}=D\Delta u^m-d(u)+b\big(u(t-r,x)\big),\quad x\in\mathbb{R},\quad t>0, \tag{4.1}$$

where u is the population density, $b(u(t-r,x))$ is the birth function, $r\geq 0$ is the time delay, $D>0$ represents the diffusivity of the population, and $d(u)$ is the death rate function. Here, invasive phenomena in population dynamics are characterized by porous medium type diffusion to avoiding overcrowding [26,45,51].

Setting $u(x,t)=\phi(\xi)$ in Eq. (4.1), where $\xi=x+ct$ and $c>0$, we get (we write ξ as t for the sake of simplicity)

$$\begin{cases} c\phi'(t)=D\big(\phi^m(t)\big)''-d\big(\phi(t)\big)+b\big(\phi(t-cr)\big), & t\in\mathbb{R},\\ \phi(-\infty)=0, \quad \phi(+\infty)=K \end{cases} \tag{4.2}$$

with $K:=u_+$ being the positive equilibrium of (4.1).

Actually, time delay and degenerate diffusion lead to essential difficulties in proving the existence and stability results. Next, we discuss wave propagation properties of delayed degenerate diffusion equations.

For birth rate function $d(s)$ and death rate function $b(s)$, assume that

$$\begin{aligned}&\text{There exist } \ u_-=0,u_+>0 \ \text{ such that } \ d,b\in C^2([0,u_+]),\ d(0)=b(0)=0,\\ &d(u_+)=b(u_+),\ b'(0)>d'(0)\geq 0,d'(u_+)\geq b'(u_+)\geq 0,\ d'(s)\geq 0,\ b'(s)\geq 0.\end{aligned} \tag{4.3}$$

Both $u_-=0$ and $u_+>0$ are constant equilibria of (4.1). The assumption (4.3) is summarized from the classical Fisher-KPP equation [24], see also lots of evolution equations in biology, for instance, the Nicholson's blowflies equation [25] with the death rate $d_1(u)=\delta u$ or $d_2(u)=\delta u^2$, the birth rate

$$b_1(u)=pue^{-au^q},\quad p>0,\quad q>0,\quad a>0$$

and the Mackey-Glass equation [38] with the birth rate

$$b_2(u)=\frac{pu}{1+au^q},\quad p>0,\quad q>0,\quad a>0.$$

Huang *et al.* [30] gave the first fundamental result concerns the existence and stability of traveling waves for delayed degenerate diffusion equation with Nicholson's blow flies type source. They consider the following model:

$$\frac{\partial u}{\partial t} = D(u^m)_{xx} - d_1(u) + b_1(u_r), \quad t > 0, \quad x \in \mathbb{R}. \tag{4.4}$$

They proved that (4.2) admits traveling waves for some $c(r)$ if the time delay is small. Note that, these speeds are non-critical (see [30, Theorem 1.2]). The nonlinear stability of these wavefronts was also analyzed by the energy methods. Unfortunately, the perturbation method in [30] is not applicable to the large time delay and critical wave speed case. Later, Xu *et al.* [56] proved the existence of all non-critical traveling wave by upper and lower solutions method for a more general delayed case with a monotone birth rate function, where the admissible traveling wave speeds for (4.2) are greater than or equal to $c^*(m,D,r) > 0$ and time delay is small (see [56, Theorem 2.4]). Liu *et al.* [37] considered the nonlocal time-delayed reaction-diffusion equations with degenerate diffusion, and showed the stability of traveling wave with $c > c^*$, which improves the previous study of stability in [30].

The most interesting and challenging case for model (4.1) is the critical case. Weak regularity and finite propagation properties lead to a class of sharp traveling waves.

Definition 4.1 ([57]). *A profile function $\phi(t)$ is said to be a traveling wave solution (TW) of* (4.2) *if $\phi \in C^b_{unif}(\mathbb{R})$, $0 \le \phi(t) \le K := u_+$, $\phi(-\infty) = 0$, $\phi(+\infty) = K$, $\phi^m \in W^{1,2}_{\rm loc}(\mathbb{R})$, $\phi(t)$ satisfies* (4.2) *in the sense of distributions. The TW $\phi(t)$ is said to be of sharp type if the support of $\phi(t)$ is semi-compact, i.e.,* $\operatorname{supp}\phi = [t_0, +\infty)$ *for some $t_0 \in \mathbb{R}$, $\phi(t) > 0$ for $t > t_0$. On the contrary, the TW $\phi(t)$ is said to be of smooth type if $\phi(t) > 0$ for all $t \in \mathbb{R}$.*

Furthermore, for the sharp TW $\phi(t)$, if $\phi'' \notin L^1_{\rm loc}(\mathbb{R})$, we say that $\phi(t)$ is a non-C^1 type sharp TW; otherwise, if $\phi'' \in L^1_{\rm loc}(\mathbb{R})$, we say that $\phi(t)$ is a C^1 type sharp TW.

In the degenerate diffusion equations, the critical wave speed is nonlinear defined, that is, the critical wave speed can not be computed by characteristic equations directly as in most classical linear diffusion equations. The definition of critical speed need to be given. For any $m > 1$, $D > 0$, and $r \ge 0$, define the critical (or minimal) wave speed $c^*(m,D,r)$ for the degenerate diffusion equation (4.2) as follows [57]:

$$c^*(m,D,r) := \inf\{c > 0;\ \text{(4.2) admits increasing TWs with speed } c\}. \tag{4.5}$$

In Theorem 4.1 below, the authors present the existence of the critical traveling waves $\phi(x + c^* t)$ (see Fig. 1). A pioneering work by Benguria and Depassier [8,9]

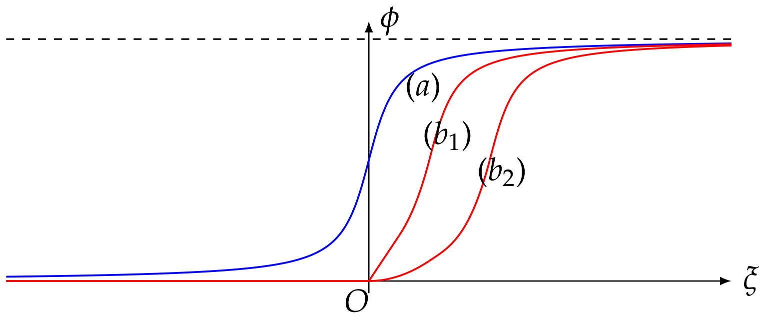

Figure 1: Traveling waves: (a) smooth type; (b_1) non-C^1 sharp type; (b_2) C^1 sharp type. Reprinted from [57].

provide an effective variational characterization to calculate the wave speed for the case $m=1$ and $r=0$,

$$c^*(1,D,0)=\max\left\{2\sqrt{D\big(b'(0)-d'(0)\big)},\ \sup_{g\in\mathscr{D}} J_1(g)\right\},$$

where J_1 is a functional defined on $\mathscr{D}$. By a new variational characterization, the reducing mechanism caused by time delay is also analysed. The critical wave speed $c^*(m,D,r)$ of (4.1) with time delay is smaller than the critical wave speed $c^*(m,D,0)$ of the case without time delay as shown in Theorem 4.1.

Theorem 4.1 (Critical Traveling Wave, [57]). *Assume that $d(s)$ and $b(s)$ satisfy* (4.3), *and $m>1$, $r\geq 0$. There exists a unique $c^*=c^*(m,D,r)>0$ defined in* (4.5) *satisfying $c^*(m,D,r)<c^*(m,D,0)$ for any time delay $r>0$, such that* (4.2) *admits a unique (up to shift) sharp traveling wave $\phi(x+c^*t)$ with speed c^*, which is the critical traveling wave of* (4.2) *and is monotonically increasing.*

Theorem 4.2 (Smooth Traveling Waves, [57]). *Assume that the conditions in Theorem* 4.1 *hold. For any $c>c^*=c^*(m,D,r)>0$ defined in* (4.5), *the traveling wave $\phi(x+ct)$ of* (4.2) *with speed c is smooth type for all $m>1$ such that $\phi\in C^2(\mathbb{R})$; while for any $c\leq c^*=c^*(m,D,r)$, there is no smooth type traveling waves of* (4.2) *with speed c.*

The degeneracy index m has essential effect on the regularity of sharp waves. The sharp traveling waves are classified into C^1 type and non-C^1 type.

Theorem 4.3 (Regularity of Sharp Wave, [57]). *Assume that the conditions in Theorem* 4.1 *hold. If $m\geq 2$, then the sharp traveling wave is of non-C^1 type; while if $1<m<2$, then the sharp traveling wave is of C^1 type.*

The results mentioned above are all related to the monotone birth rate function and monotone traveling waves. Now, we focus our attention on the non-monotone birth function for (4.1). The non-monotone nonlinearity restricted the application of monotone method and generate various singular waves, including traveling fronts, semi-waves, oscillatory waves and so on. Suppose now that the birth function $b(u)$ is non-monotone. In [58], the complex structure of traveling waves are investigated. The traveling wave solution $\phi(t)$ may be non-monotone and even non-decaying oscillating. Moreover, the degeneracy allow sharp waves with finite support. The various types of waves is defined in [58].

Definition 4.2 (Sharp wavefronts, [58])**.** *A function* $0\le\phi(t)\in W^{1,1}_{\rm loc}(\mathbb{R})\cap L^\infty(\mathbb{R})$ *with* $\phi^m(t)\in W^{1,1}_{\rm loc}(\mathbb{R})$ *is said to be a semi-wavefront of* (4.1) *if*

- *(i) the profile function* ϕ *satisfies* (4.2) *in the sense of distributions,*
- *(ii)* $\phi(-\infty)=0$, *and* $0<\liminf_{t\to+\infty}\phi(t)\le\limsup_{t\to+\infty}\phi(t)<+\infty$,
- *(iii) the leading edge of* $\phi(t)$ *near* $-\infty$ *is monotonically increasing in the sense that there exists a maximal interval* $(-\infty,t_0)$ *with* $t_0\in(-\infty,+\infty]$ *such that* $\phi(t)$ *is monotonically increasing in it and if* $t_0<+\infty$ *then* $\phi(t_0)>\kappa$. *We say that* t_0 *is the boundary of the leading edge of* ϕ.

A semi-wavefront $\phi(t)$ *is said to be a wavefront of* (4.1) *if* ϕ *converges to* κ *as* t *tends to* $+\infty$, *i.e.,* $\phi(+\infty)=\kappa$.

A semi-wavefront (including wavefront) is said to be sharp if there exists a $t_*\in\mathbb{R}$ *such that* $\phi(t)=0$ *for all* $t\le t_*$ *and* $\phi(t)>0$ *for all* $t>t_*$. *Otherwise, it is said to be a smooth semi-wavefront (or smooth wavefront) if* $\phi(t)>0$ *for all* $t\in\mathbb{R}$.

Furthermore, for the sharp semi-wavefronts (including wavefronts) $\phi(t)$, *if* $\phi''\notin L^1_{\rm loc}(\mathbb{R})$, *we say that* $\phi(t)$ *is a piecewise-*C^1 *type sharp wave; otherwise, if* $\phi''\in L^1_{\rm loc}(\mathbb{R})$, *we say that* $\phi(t)$ *is a* C^1 *type sharp wave.*

In the above definition, the possible waves are classified into nine different pattern (see Figs. 2-4). The monotonicity are related to monotone wavefronts, non-monotone wavefronts, and divergent semi-wavefronts. Regularity near $-\infty$ or t_* can be characterized by piecewise-C^1 type sharp waves, C^1 type sharp waves and smooth type waves.

An additional complication appearing in the delayed and degenerate case is the possible non-monotonicity and non-smoothness of waves. In [58], the authors discussed the existence and nonexistence results and give a classify of these waves:

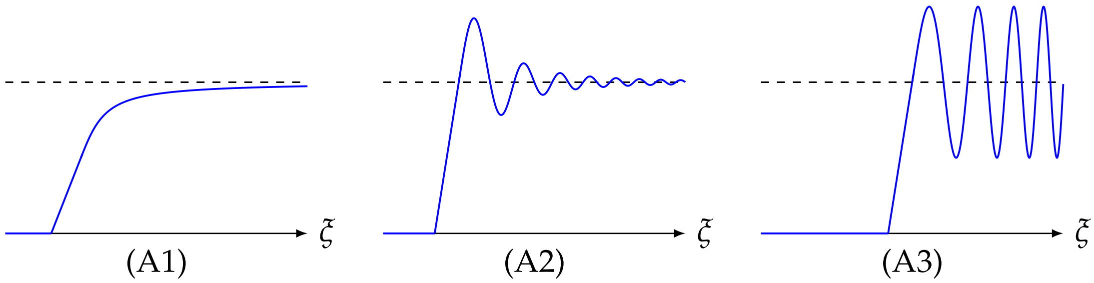

Figure 2: Sharp waves — piecewise-C^1 type: (A1) monotone wavefront; (A2) non-monotone wavefront; (A3) divergent semi-wavefront. Reprinted from [58].

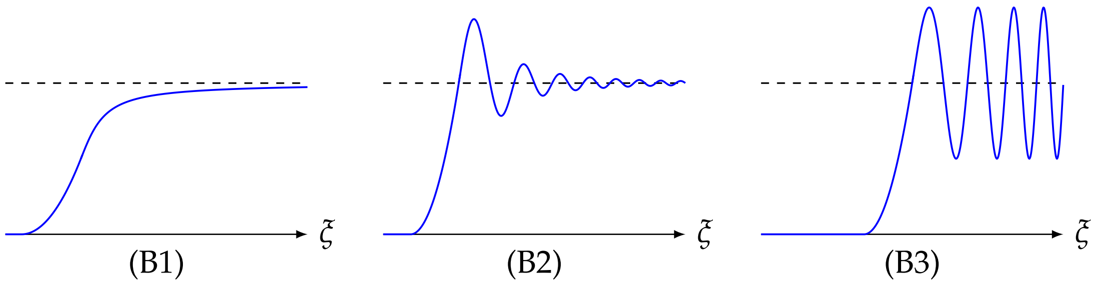

Figure 3: Sharp waves — C^1 type: (B1) monotone wavefront; (B2) non-monotone wavefront; (B3) divergent semi-wavefront. Reprinted from [58].

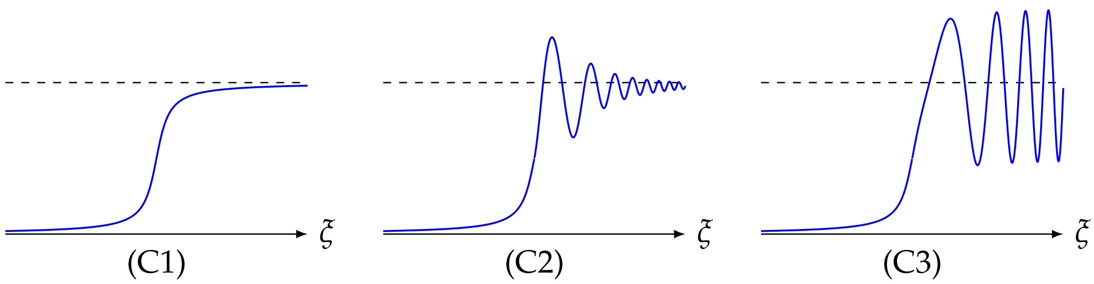

Figure 4: Smooth waves: (C1) monotone wavefront; (C2) non-monotone wavefront; (C3) divergent semi-wavefront. Reprinted from [58].

Theorem 4.4 (Existence of sharp waves, [58]). *For any $m>1$, $D>0$ and $r\geq 0$, there exists a constant $c_0(m,r,b,d)>0$ depending on m,r and the structure of $b(\cdot),d(\cdot)$ such that for $c=c_0(m,r,b,d)$,* (4.2) *admits sharp wave $\phi(t)$ (semi-wavefronts or wavefronts, piecewise-C^1 or C^1) with $\phi(t)\equiv 0$ for $t\leq 0$,*

$$0<\zeta_1\leq\liminf_{t\to+\infty}\phi(t)\leq\limsup_{t\to+\infty}\phi(t)\leq\zeta_2,$$

and

$$\left|\phi(t)-C_1t_+^{\lambda}\right|\le C_2t_+^{\Lambda}\quad \textit{for any}\quad t\in(-\infty,1),$$

where $t_+=\max\{t,0\}$, $\lambda=1/(m-1)$ *and* $\Lambda>\lambda$, $C_1,C_2>0$ *are constants.*

The regularity near the degenerate point is also investigated in [58].

Theorem 4.5 (Regularity of sharp waves, [58]). *For* $m\ge 2$, *the sharp waves in Theorem* 4.4 *are piecewise-*C^1 *(as illustrated in Fig.* 2*); while* $1<m<2$, *the sharp waves in Theorem* 4.4 *are* C^1 *(as shown in Fig.* 3*).*

A precise relationship between the oscillatory properties and the strength of the wave speed and the time delay is given in the following theorem in [58].

Theorem 4.6 (Oscillatory or divergent waves, [58]). *Assume that* $m>1$, $r>0$, $b'(\kappa)<0$, *and the birth rate function* $b(\cdot)$ *satisfies the feedback condition* (4.6). *Then there exist* $c_\kappa=c_\kappa(m,r,b'(\kappa),d'(\kappa))$ *and* $c^*=c^*(m,r,b'(\kappa),d'(\kappa))$ *with* $0<c_\kappa\le c^*\le+\infty$, *such that the waves (wavefronts or semi-wavefronts, sharp or smooth, if exist) with speed* $c>c_\kappa$ *are oscillatory and these waves with speed* $c>c^*$ *are divergent. Moreover,*

$$c_\kappa\big(m,r,b'(\kappa),d'(\kappa)\big)=\frac{\mu_\kappa\big(m,b'(\kappa),d'(\kappa)\big)+o(1)}{r},\quad r\to+\infty,$$

and if further $b'(\kappa)<-d'(\kappa)$, *then*

$$c^*\big(m,r,b'(\kappa),d'(\kappa)\big)=\frac{\mu^*\big(m,b'(\kappa),d'(\kappa)\big)+o(1)}{r},\quad r\to+\infty,$$

where

$$\mu_\kappa\big(m,b'(\kappa),d'(\kappa)\big):=\sqrt{\frac{2Dm\kappa^{m-1}\omega_\kappa}{b'(\kappa)}}e^{\frac{\omega_\kappa}{2}},\quad \omega_\kappa<-2$$

is the unique negative root of the equation $2d'(\kappa)=b'(\kappa)e^{-\omega_\kappa}(2+\omega_\kappa)$, *and*

$$\mu^*\big(m,b'(\kappa),d'(\kappa)\big):=\pi\sqrt{\frac{Dm\kappa^{m-1}}{-b'(\kappa)-d'(\kappa)}}.$$

By a new delayed iteration approach, the authors in [58] find a peculiar form of waves with sharp edges and non-decaying oscillations in the following Theorem 4.7. Suppose the following feedback condition:

$$\big(b(s)-\kappa\big)(s-\kappa)<0,\quad s\in\big[d^{-1}(\theta),d^{-1}(M)\big]\setminus\{\kappa\}. \tag{4.6}$$

Theorem 4.7 (Sharp divergent semi-wavefront, [58])**.** *Assume that the function $b(\cdot)$ satisfies the feedback condition* (4.6) *and let $\zeta_1\in(0,\kappa)$ be the constant such that $b(\zeta_1)=b(d^{-1}(M))$, where $M=\max_{s\in[0,\kappa]}b(s)$. There are positive constants ε_0,K_0,r_0, such that if $b'(s)\geq\varepsilon_0$ for $s\in(0,\zeta_1)$, $b'(\kappa)\leq -K_0$ and $r\geq r_0$, then the constants $\mu_0(m,b(\cdot),d(\cdot))>\mu^*(m,b'(\kappa),d'(\kappa))$ in Theorem* 4.6*, and the sharp type wave in Theorem* 4.4 *is a divergent semi-wavefront.*

5 Traveling waves in strong degenerate diffusion equations

In this section, we review some results about discontinuous traveling waves in the strong degenerate diffusion equations.

The theory of strong degenerate diffusion equations is more recent than that of the classical porous media equations, and not as complete. It has been observed that the fronts of travelling wave solutions in some strong degenerate diffusion equations can be shock-like. These waves, called discontinuous travelling waves, are more suitable to describe the cell migration, traffic flux, sedimentation-consolidation processes with discontinuous interfaces.

Generally speaking, discontinuous traveling waves mainly exist in the hyperbolic equations [6, 31, 33–35]. As far as we know, there are a few results on the discontinuous traveling waves in the parabolic equations with strong degeneracy. Discontinuous traveling waves can be considered in a gravitational sedimentation-consolidation model [61]

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}f(u)=\frac{\partial}{\partial x}\left(a(u)\frac{\partial u}{\partial x}\right)+h(u),\quad x\in\mathbb{R},\quad t>0, \tag{5.1}$$

where $f(s)$, $h(s)$ are appropriately smooth functions, $a(s)\geq 0$ and discontinuous. Using the discontinuous phase plane analysis method, the authors proved the existence of discontinuous traveling waves and nonexistence of continuous traveling wave entropy solution for (5.1) with discontinuous $a(s)$. Note that in this case the entropy condition guarantees the local monotonicity, which is different from the fully degenerate case [52, 60]. Discontinuous traveling waves were also studied in a modified Allen-Cahn model arising from phase transition

$$\frac{\partial u}{\partial t}=\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)+u^q f(u),\quad t\geq 0,\quad x\in\mathbb{R}, \tag{5.2}$$

where $q>0$ and $f(s)$ is a smooth and sign-changing function with the bistable structure condition

$$\begin{aligned}&f(1)=f(a)=0, \quad f'(1)<0,\\&f(s)<0 \quad \text{for} \quad s\in[0,a) \quad \text{and} \quad f(s)>0 \quad \text{for} \quad s\in(a,1),\end{aligned}$$

where $a\in(0,1)$ is a given constant. It has been proved that (5.2) has discontinuous traveling wave entropy solutions for all speed $0<c<c^*$ [59]. These solutions give a mathematical insight to the discontinuous phase transition phenomena.

Theorem 5.1 ([59])**.** *Suppose that*

$$\int_a^1 s^q f(s)ds<1 \quad \textit{and} \quad \int_0^1 s^q f(s)>0,$$

or

$$\int_a^1 s^q f(s)ds\geq 1 \quad \textit{and} \quad \int_0^a s^q|f(s)|ds<1.$$

Let c^* be the critical wave speed for smooth traveling waves. Then for any wave speed $0<c<c^*$, there exist infinitely many discontinuous traveling wave entropy solutions $u(x,t)=\varphi(\xi)$ with $\xi=x-ct$ such that $\varphi\in C^2((\xi_a,\xi_b)\backslash\{\xi^*\})$, φ has a jump discontinuity at $\xi^*\in(\xi_a,\xi_b)$, φ is strictly monotone decreasing and

$$\lim_{\xi\to-\infty}\varphi(\xi)=1, \quad \lim_{\xi\to+\infty}\varphi(\xi)=0.$$

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