Commun. Math. Res.
doi: 10.4208/cmr.2021-0108

Vol. **39**, No. 1, pp. 54-78
February 2023

Asymptotic Behavior of Solutions to a Class of Semilinear Parabolic Equations with Boundary Degeneracy

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Received 30 December 2021; Accepted 15 April 2022

Abstract. This paper concerns the asymptotic behavior of solutions to one-dimensional semilinear parabolic equations with boundary degeneracy both in bounded and unbounded intervals. For the problem in a bounded interval, it is shown that there exist both nontrivial global solutions for small initial data and blowing-up solutions for large one if the degeneracy is not strong. Whereas in the case that the degeneracy is strong enough, the nontrivial solution must blow up in a finite time. For the problem in an unbounded interval, blowing-up theorems of Fujita type are established. It is shown that the critical Fujita exponent depends on the degeneracy of the equation and the asymptotic behavior of the diffusion coefficient at infinity, and it may be equal to one or infinity. Furthermore, the critical case is proved to belong to the blowing-up case.

AMS subject classifications: 35K65, 35D30, 35B33

Key words: Asymptotic behavior, boundary degeneracy, blowing-up.

1 Introduction

In this paper, we consider the following semilinear degenerate equation of the form:

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right)=f(x,t,u),\quad 0<x<1,\quad t>0, \tag{1.1}$$

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where $a\in C([0,1])\cap C^1((0,1])$ such that $a>0$ in $(0,1]$ and $a(0)=0$. As a parabolic equation with boundary degeneracy, (1.1) is degenerate at $x=0$, a portion of the lateral boundary. Such equations are used to describe some models, such as the Budyko-Sellers climate model [18], the Black-Scholes model coming from the option pricing problem [3], and a simplified Crocco-type equation coming from the study on the velocity field of a laminar flow on a flat plate [7]. The typical case of a is

$$a(x)=x^\lambda, \quad x\in[0,1], \quad \lambda>0. \tag{1.2}$$

In recent years, the null controllability of the control system governed by (1.1) was studied in [1, 8, 9, 17, 22, 25, 26]. In particular, the following control system was studied:

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(x^\lambda\frac{\partial u}{\partial x}\right)+c(x,t)u=h(x,t)\chi_\omega, \qquad (x,t)\in(0,1)\times(0,T), \tag{1.3}$$

$$\begin{cases} u(0,t)=u(1,t)=0, & \text{if} \quad 0<\lambda<1, \\ \lim\limits_{x\to0^+}x^\lambda\dfrac{\partial u}{\partial x}(x,t)=u(1,t)=0, & \text{if} \quad \lambda\ge1, \end{cases} \qquad t\in(0,T), \tag{1.4}$$

$$u(x,0)=u_0(x), \qquad x\in(0,1), \tag{1.5}$$

where $\lambda>0, c\in L^\infty((0,1)\times(0,T))$. It was shown that the system (1.3)-(1.5) is null controllable if $0<\lambda<2$, while not if $\lambda\ge2$. Although the system (1.3)-(1.5) is not null controllable for $\lambda\ge2$, it was proved in [11, 19, 21] and [4–6] that it is approximately controllable in $L^2((0,1))$ and regional null controllable for each $\lambda>0$, respectively.

In this paper, we study the asymptotic behavior of solutions to (1.1) with

$$f(x,t,u)=u^p, \quad (x,t,u)\in(0,1)\times(0,+\infty)\times\mathbb{R}, \quad p>1.$$

That is to say, we consider the following problem:

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right)=u^p, \qquad (x,t)\in(0,1)\times(0,T), \tag{1.6}$$

$$\lim_{x\to0^+}a(x)\frac{\partial u}{\partial x}(x,t)=0, \ \ u(1,t)=0, \quad t\in(0,T), \tag{1.7}$$

$$u(x,0)=u_0(x), \qquad x\in(0,1). \tag{1.8}$$

By a weighted energy estimate, it is shown that the asymptotic behavior of solutions to the problem (1.6)-(1.8) depends on the degenerate rate of a at $x=0$. Precisely, it is assumed that $a\in C([0,1])\cap C^1((0,1])$ satisfies

$$a(0)=0, \quad a(x)>0 \quad \text{for} \quad 0<x\le1. \tag{1.9}$$

Furthermore, a satisfies one of the following two asymptotic behaviors as $x \to 0^+$:

$$\frac{x}{a(x)} \text{ is integrable near } x=0 \tag{1.10}$$

or

$$\overline{\lim_{x\to 0^+}} \frac{|a'(x)|}{a^{1/2}(x)} < +\infty \text{ and } \underline{\lim_{x\to 0^+}} \frac{a(x)}{x^\gamma} > 0 \text{ for some constant } \gamma \geq 2. \tag{1.11}$$

For the typical a given by (1.2), it satisfies (1.10) if $0<\lambda<2$, while satisfies (1.11) if $\lambda \geq 2$. In this paper it is proved that any nontrivial solution to the problem (1.6)-(1.8) blows up in a finite time if a satisfies (1.9) and (1.11), while there exist both nontrivial global and blowing-up solutions if a satisfies (1.9) and (1.10).

We also study the following problem in an unbounded interval:

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right) = u^p, \quad && (x,t) \in (0,+\infty)\times(0,T), && (1.12)\\
&\lim_{x\to 0^+} a(x)\frac{\partial u}{\partial x}(x,t) = 0, && t\in(0,T), && (1.13)\\
&u(x,0) = u_0(x), && x \in (0,+\infty), && (1.14)
\end{aligned}$$

where $p>1$, and $a \in C([0,+\infty)) \cap C^1((0,+\infty))$ satisfies

$$a(0)=0, \quad a(x)>0 \quad \text{for} \quad x>0. \tag{1.15}$$

Furthermore, a satisfies the asymptotic behavior (1.10) or (1.11) as $x \to 0^+$. For the case that a satisfies (1.15) and (1.11), it is proved that any nontrivial solution to the problem (1.12)-(1.14) must blow up in a finite time for $p>1$. As to the case that a satisfies (1.15) and (1.10), the asymptotic behavior of solutions to the problem (1.12)-(1.14) is determined by the asymptotic behavior of a as $x \to +\infty$. It is assumed that a also satisfies

$$\overline{\lim_{x\to 0^+}} \frac{x^2}{a(x)} < +\infty, \quad \lim_{x\to+\infty} \frac{xa'(x)}{a(x)} = \lambda, \quad \underline{\lim_{x\to+\infty}} \frac{a(x)}{x^2} > -|\lambda-2|, \tag{1.16}$$

where $\lambda \geq 0$ is a constant. Owing to (1.15), it is noted that the third formula in (1.16) is trivial for $\lambda \neq 2$. Using weighted energy estimates and suitable self-similar supersolutions, we prove that, if a satisfies (1.15), (1.10) and (1.16), the critical Fujita exponent to the problem (1.12)-(1.14) is $\max\{3-\lambda,1\}$. That is to say, in the case that a satisfies (1.15), (1.10) and (1.16), any nontrivial solution to the problem (1.12)-(1.14) must blow up in a finite time if $1<p<\max\{3-\lambda,1\}$, while there are both nontrivial global and blowing-up solutions to the problem (1.12)-(1.14) if $p>\max\{3-\lambda,1\}$. Furthermore, the critical case $p=3-\lambda$ for $\lambda\in[0,2)$ belongs to the blowing-up case under the following additional condition:

$$\overline{\lim_{x\to+\infty}}\frac{a(x)}{x^{\lambda}}<+\infty. \tag{1.17}$$

Summing up, it is shown that the critical Fujita exponent to the problem (1.12)-(1.14) is

$$p_c=\begin{cases}3-\lambda, & \text{if } a \text{ satisfies (1.15),(1.10) and (1.16) with } 0\le\lambda<2,\\ 1, & \text{if } a \text{ satisfies (1.15),(1.10) and (1.16) with } \lambda\ge 2,\\ +\infty, & \text{if } a \text{ satisfies (1.15) and (1.11).}\end{cases}$$

In particular, $p_c=1$ if a is suitably large as $x\to+\infty$, $p_c=+\infty$ if a is suitably small as $x\to 0^+$.

In 1966, Fujita [12] proved that for the Cauchy problem of the semilinear equation

$$\frac{\partial u}{\partial t}-\Delta u=u^p,\quad x\in\mathbb{R}^n,\quad t>0,$$

any nontrivial solution must blow up in a finite time if $1<p<1+\frac{n}{2}$, whereas there exist both nontrivial global and blowing-up solutions when $p>1+\frac{n}{2}$. For this problem, $p_c=1+\frac{n}{2}$ is called the critical Fujita exponent, and the critical case $p=p_c$ was proved to belong to the blowing-up case in [13, 15]. Fujita revealed an important topic of nonlinear partial differential equations. And there have been a great number of extensions of Fujita's results in several directions since then, including similar results for numerous of quasilinear parabolic equations and systems in various of geometries with nonlinear sources or nonhomogeneous boundary conditions, see the survey papers [10, 16] and also the recent papers [2, 14, 23, 24, 27].

In this paper, we study the asymptotic behavior of solutions to the problem (1.6)-(1.8) in a bounded interval and the problem (1.12)-(1.14) in an unbounded interval. The methods used in this paper are similar to the ones in [20], where the following special case was considered:

$$a(x)=x^{\lambda},\quad x\ge 0. \tag{1.18}$$

For this special a given by (1.18), it satisfies (1.10), (1.16) and (1.17) if $0<\lambda<2$, while satisfies (1.11) if $\lambda\ge 2$. For the blowing-up of solutions to the problem (1.6)-(1.8) in a bounded interval and the problem (1.12)-(1.14) in an unbounded interval, we apply the method of weighted energy estimates to determine the interaction of the degenerate diffusions and the reactions, and the key is to choose appropriate weights. To prove the global existence of nontrivial solutions, we construct suitable self-similar supersolutions. Since the diffusion coefficients are more general functions in this paper, the weights and self-similar supersolutions are more complicated, and we have to overcome some technical difficulties. Furthermore, for the critical case $p=p_c$ when a satisfies (1.15), (1.10), (1.16) and (1.17), we need a series of elaborate energy estimates.

The paper is organized as follows. Main results are stated in Section 2. The problem (1.6)-(1.8) in a bounded interval and the problem (1.12)-(1.14) in an unbounded interval are studied in Sections 3 and 4, respectively. Finally, we state the results for the problems with inner degeneracy in Section 5.

2 Main results

Solutions to the problems (1.6)-(1.8) and (1.12)-(1.14) are defined as follows.

Definition 2.1. *Let $0<T\le+\infty$. A nonnegative function u is said to be a subsolution (supersolution, solution) to the problem* (1.6)-(1.8) *in $(0,T)$, if*

(i) For any $0<\tilde{T}<T$, $u\in L^\infty((0,1)\times(0,\tilde{T}))$, and $\dfrac{\partial u}{\partial t}$, $a^{\frac12}\dfrac{\partial u}{\partial x}\in L^2((0,1)\times(0,\tilde{T}))$.

(ii) For any $0<\tilde{T}<T$ and any nonnegative function $\varphi\in C^1([0,1]\times[0,\tilde{T}])$ vanishing at $x=1$, it holds that

$$\begin{aligned}&\int_0^{\tilde{T}}\int_0^1\left(\frac{\partial u}{\partial t}(x,t)\varphi(x,t)+a(x)\frac{\partial u}{\partial x}(x,t)\frac{\partial \varphi}{\partial x}(x,t)\right)\mathrm{d}x\mathrm{d}t\\ \le(\ge,=)&\int_0^{\tilde{T}}\int_0^1 u^p(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

(iii) $u(1,\cdot)\le(\ge,=)0$ in $(0,T)$ and $u(\cdot,0)\le(\ge,=)u_0(\cdot)$ in $(0,1)$ in the sense of trace.

Definition 2.2. *Let $0<T\le+\infty$. A nonnegative function u is said to be a subsolution (supersolution, solution) to the problem* (1.12)-(1.14) *in $(0,T)$, if*

(i) For any $0<\tilde{T}<T$ and any $R>0$, $u\in L^\infty((0,+\infty)\times(0,\tilde{T}))$, and $\dfrac{\partial u}{\partial t}$, $a^{\frac12}\dfrac{\partial u}{\partial x}\in L^2((0,R)\times(0,\tilde{T}))$.

(ii) *For any* $0<\tilde{T}<T$ *and any nonnegative function* $\varphi\in C^1([0,+\infty)\times[0,\tilde{T}])$ *vanishing when* x *is large, it holds that*

$$\begin{aligned}&\int_0^{\tilde{T}}\int_0^{+\infty}\left(\frac{\partial u}{\partial t}(x,t)\varphi(x,t)+a(x)\frac{\partial u}{\partial x}(x,t)\frac{\partial \varphi}{\partial x}(x,t)\right)\mathrm{d}x\mathrm{d}t\\ \le(\ge,=)&\int_0^{\tilde{T}}\int_0^{+\infty}u^p(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

(iii) $u(\cdot,0)\le(\ge,=)u_0(\cdot)$ *in* $(0,+\infty)$ *in the sense of trace.*

Similarly to [20], one can establish the well-posedness and the comparison principles for the problems (1.6)-(1.8) and (1.12)-(1.14).

Proposition 2.1. *Assume that* $a\in C([0,1])\cap C^1((0,1])$ *satisfies* (1.9).

(i) *For any* $0\le u_0\in L^\infty((0,1))$ *with* $a^{\frac{1}{2}}u_0'\in L^2((0,1))$*, there is a unique solution to the problem* (1.6)-(1.8) *locally in time.*

(ii) *Assume that* $\hat{u}$ *and* $\check{u}$ *are a supersolution and a subsolution to the problem* (1.6)-(1.8) *in* $(0,T)$*, respectively. Then* $\check{u}\le\hat{u}$ *in* $(0,1)\times(0,T)$.

Proposition 2.2. *Assume that* $a\in C([0,+\infty))\cap C^1((0,+\infty))$ *satisfies* (1.15).

(i) *For any* $0\le u_0\in L^\infty((0,+\infty))$ *with* $a^{\frac{1}{2}}u_0'\in L^2((0,R))$ *for each* $R>0$*, there is a unique solution to the problem* (1.12)-(1.14) *locally in time.*

(ii) *Assume that* $\hat{u}$ *and* $\check{u}$ *are a supersolution and a subsolution to the problem* (1.12)-(1.14) *in* $(0,T)$*, respectively. Then* $\check{u}\le\hat{u}$ *in* $(0,+\infty)\times(0,T)$.

If u is a solution to the problem (1.6)-(1.8) (or to the problem (1.12)-(1.14)) in $(0,+\infty)$, we say that u is a global solution in time. Otherwise, there exists $T>0$ such that u is a solution in $(0,T)$ and satisfies

$$\lim_{t\to T^-}\sup_{(0,1)}u(\cdot,t)=+\infty \quad \Big(\text{or } \lim_{t\to T^-}\sup_{(0,+\infty)}u(\cdot,t)=+\infty\Big),$$

and we say that u blows up in a finite time.

The main results of the paper are the following theorems.

Theorem 2.1. *Assume that* $a\in C([0,1])\cap C^1((0,1])$ *satisfies* (1.9) *and* (1.10)*. The solution to the problem* (1.6)-(1.8) *exists globally in time if* u_0 *is small, while blows up in a finite time if* u_0 *is large.*

Theorem 2.2. *Assume that* $a\in C([0,1])\cap C^1((0,1])$ *satisfies* (1.9) *and* (1.11). *Then any nontrivial solution to the problem* (1.6)-(1.8) *must blow up in a finite time.*

Theorem 2.3. *Assume that* $a\in C([0,+\infty))\cap C^1((0,+\infty))$ *satisfies* (1.15), (1.10) *and* (1.16) *with* $0\le\lambda<2$.

(i) If $1<p<3-\lambda$, *then any nontrivial solution to the problem* (1.12)-(1.14) *must blow up in a finite time.*

(ii) If $p>3-\lambda$, *then the solution to the problem* (1.12)-(1.14) *exists globally in time if* u_0 *is small, while blows up in a finite time if* u_0 *is large.*

Theorem 2.4. *Assume that* $a\in C([0,+\infty))\cap C^1((0,+\infty))$ *satisfies* (1.15), (1.10) *and* (1.16) *with* $\lambda\ge 2$. *For* $p>1$, *the solution to the problem* (1.12)-(1.14) *exists globally in time if* u_0 *is small, while blows up in a finite time if* u_0 *is large.*

Theorem 2.5. *Assume that* $a\in C([0,+\infty))\cap C^1((0,+\infty))$ *satisfies* (1.15), (1.10), (1.16) *and* (1.17) *with* $0\le\lambda<2$. *For* $p=3-\lambda$, *any nontrivial solution to the problem* (1.12)-(1.14) *must blow up in a finite time.*

Theorem 2.6. *Assume that* $a\in C([0,+\infty))\cap C^1((0,+\infty))$ *satisfies* (1.15) *and* (1.11). *Then any nontrivial solution to the problem* (1.12)-(1.14) *must blow up in a finite time.*

3 Problem in a bounded interval

In this section, we prove Theorems 2.1 and 2.2 for the problem (1.6)-(1.8) in a bounded interval.

Proof of Theorem 2.1. First consider the global case. Due to (1.9) and (1.10), $x/a(x)$ is integrable on $[0,1]$. We study self-similar supersolutions to (1.6) of the form

$$\hat{u}(x,t)=(t+L)^{-\frac{1}{p-1}}\left(\frac{1}{L}\int_0^1\frac{s}{a(s)}\mathrm{d}s-\frac{1}{t+L}\int_0^x\frac{s}{a(s)}\mathrm{d}s\right),\quad 0\le x\le 1,\quad t\ge 0,$$

where $L\ge 1$ is a constant to be determined later. Owing to $L\ge 1$ and $p>1$, a direct calculation shows that

$$\begin{aligned}&\frac{\partial\hat{u}}{\partial t}-\frac{\partial}{\partial x}\left(a(x)\frac{\partial\hat{u}}{\partial x}\right)-\hat{u}^p \qquad (3.1)\\&=(t+L)^{-\frac{p}{p-1}}\left(1-\frac{1}{(p-1)L}\int_0^1\frac{s}{a(s)}\mathrm{d}s+\frac{p}{(p-1)(t+L)}\int_0^x\frac{s}{a(s)}\mathrm{d}s\right.\end{aligned}$$

$$
\begin{aligned}
&-\left(\frac{1}{L}\int_0^1 \frac{s}{a(s)}\mathrm{d}s-\frac{1}{t+L}\int_0^x \frac{s}{a(s)}\mathrm{d}s\right)^p\Bigg)\\
&\geq (t+L)^{-\frac{p}{p-1}}\left(1-\frac{1}{(p-1)L}\int_0^1\frac{s}{a(s)}\mathrm{d}s-\frac{1}{L}\left(\int_0^1\frac{s}{a(s)}\mathrm{d}s\right)^p\right),\quad 0<x<1,\quad t>0.
\end{aligned}
$$

Set

$$
L_0=\frac{1}{p-1}\int_0^1\frac{s}{a(s)}\mathrm{d}s+\left(\int_0^1\frac{s}{a(s)}\mathrm{d}s\right)^p+1.
$$

For each $L\geq L_0$, one gets from (3.1) that

$$
\frac{\partial \hat{u}}{\partial t}-\frac{\partial}{\partial x}\left(a(x)\frac{\partial \hat{u}}{\partial x}\right)-\hat{u}^p\geq 0,\quad 0<x<1,\quad t>0.
$$

It is noted that

$$
\lim_{x\to 0^+} a(x)\frac{\partial \hat{u}}{\partial x}(x,t)=0,\quad \hat{u}(1,t)\geq 0,\quad t>0.
$$

Therefore, $\hat{u}$ is a supersolution to the problem (1.6)-(1.8) if

$$
u_0(x)\leq \hat{u}(x,0),\quad 0<x<1. \tag{3.2}
$$

Thanks to Proposition 2.1 (ii), there is a global solution to the problem (1.6)-(1.8) if u_0 satisfies (3.2).

Turn to the blowing-up case. Set

$$
\zeta(x)=\begin{cases}2, & 0\leq x\leq 1/2,\\ 1+\cos(2x-1)\pi, & 1/2<x\leq 1.\end{cases}
$$

It is clear that $\zeta\in C^1([0,1])$ is piecewise smooth, and satisfies $\zeta'(0)=0$ and $\zeta(1)=0$. Owing to (1.9), one gets that

$$
\begin{aligned}
\left(a(x)\zeta'(x)\right)'&=-2\pi a'(x)\sin(2x-1)\pi-4\pi^2 a(x)\cos(2x-1)\pi\\
&\geq -2\pi|a'(x)|-4\pi^2 a(x)\cos(2x-1)\pi\\
&\geq -4\pi^2\left(a(x)+\frac{|a'(x)|}{2\pi}\right)(1+\cos(2x-1)\pi)\\
&\geq -4\pi^2 M\zeta(x),\quad 1/2<x<1,
\end{aligned}
$$

where

$$
M=\sup\left\{a(x)+\frac{|a'(x)|}{2\pi}:1/2<x<1\right\}.
$$

Assume that u is a global solution to the problem (1.6)-(1.8). It follows from Definition 2.1 and the Hölder inequality that u satisfies

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u(x,t)\zeta(x)\mathrm{d}x \\
=&\int_0^1 \frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right)\zeta(x)\mathrm{d}x+\int_0^1 u^p(x,t)\zeta(x)\mathrm{d}x \\
\geq&-4\pi^2 M\int_0^1 u(x,t)\zeta(x)\mathrm{d}x+\left(\int_0^1 \zeta(x)\mathrm{d}x\right)^{1-p}\left(\int_0^1 u(x,t)\zeta(x)\mathrm{d}x\right)^p \\
\geq&-4\pi^2 M\int_0^1 u(x,t)\zeta(x)\mathrm{d}x+2^{1-p}\left(\int_0^1 u(x,t)\zeta(x)\mathrm{d}x\right)^p, \quad t>0.
\end{aligned}$$

If u_0 is sufficiently large such that

$$\int_0^1 u_0(x)\zeta(x)\mathrm{d}x\geq\left(2^{p+2}\pi^2 M\right)^{\frac{1}{p-1}},$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u(x,t)\zeta(x)\mathrm{d}x\geq 2^{-p}\left(\int_0^1 u(x,t)\zeta(x)\mathrm{d}x\right)^p, \quad t>0.$$

Therefore, there exists $T>0$ such that

$$\lim_{t\to T^-}\int_0^1 u(x,t)\zeta(x)\mathrm{d}x=+\infty,$$

which leads to

$$\lim_{t\to T^-}\sup_{(0,1)} u(\cdot,t)=+\infty.$$

That is to say, u must blow up in a finite time. □

Proof of Theorem 2.2. Thanks to (1.9) and (1.11), there exist two positive constants M_1 and M_2 such that

$$-M_1 a^{\frac{1}{2}}(x)\leq a'(x)\leq M_1 a^{\frac{1}{2}}(x), \quad a(x)\geq M_2 x^\gamma, \quad 0<x<1. \tag{3.3}$$

Set

$$\zeta_\delta(x)=\left(\int_x^1 \frac{1}{a(s)}\mathrm{d}s\right)^\delta, \quad 0<x\leq 1,$$

where $0<\delta<\frac{1}{\gamma}$ is a constant to be determined. It is clear that $\zeta_\delta\in C^2((0,1])$ satisfies

$$\zeta_\delta(1)=0, \quad \zeta_\delta(x)>0 \quad \text{for} \quad 0<x<1,$$

and

$$\big(a(x)\zeta_\delta'(x)\big)'=\frac{\delta(\delta-1)}{a(x)}\left(\int_x^1\frac{1}{a(s)}\mathrm{d}s\right)^{\delta-2},\quad 0<x<1. \tag{3.4}$$

It follows from the second formula in (3.3) that

$$\int_x^1\frac{1}{a(s)}\mathrm{d}s\le\frac{1}{M_2(\gamma-1)}\left(\frac{1}{x^{\gamma-1}}-1\right),\quad 0<x<1.$$

Hence, $\zeta_\delta\in L^1((0,1))$ and there exists a constant $M_3>0$ independent of δ such that

$$\int_0^1\zeta_\delta(x)\mathrm{d}x\le M_3. \tag{3.5}$$

It follows from (1.9) and the first formula in (3.3) that

$$\frac{1}{a(x)}\ge\frac{a'(x)}{M_1a^{3/2}(x)}=\frac{2}{M_1}\big(-a^{-\frac{1}{2}}(x)\big)',\quad 0<x<1,$$

which yields

$$\int_x^1\frac{1}{a(s)}\mathrm{d}s\ge\frac{2}{M_1}\big(a^{-\frac{1}{2}}(x)-a^{-\frac{1}{2}}(1)\big),\quad 0<x<1. \tag{3.6}$$

Due to (1.9), there exists $x_0\in(0,1)$ such that

$$a(x)<\frac{a(1)}{4},\quad 0\le x\le x_0,$$

which, together with (3.4), leads to

$$\int_x^1\frac{1}{a(s)}\mathrm{d}s\ge\frac{1}{M_1a^{1/2}(x)},\quad 0<x<x_0. \tag{3.7}$$

Hence,

$$\frac{1}{a(x)}\left(\int_x^1\frac{1}{a(s)}\mathrm{d}s\right)^{-2}\le M_1^2,\quad 0<x<x_0. \tag{3.8}$$

Thanks to (3.4), (3.8) and (1.9), there exists a constant $M_4>0$ independent of δ such that

$$\big(a(x)\zeta_\delta'(x)\big)'\ge -M_4\delta\zeta_\delta(x),\quad 0<x<1. \tag{3.9}$$

For $0<\varepsilon<\frac{1}{2}$, let $\mu_\varepsilon\in C^\infty([0,1])$ satisfy

$$\mu_\varepsilon(x)=\begin{cases}0, & 0\le x\le\varepsilon,\\ 1, & 2\varepsilon\le x\le 1,\end{cases}$$

and

$$0\le\mu_\varepsilon(x)\le 1,\quad 0\le\mu'_\varepsilon(x)\le\frac{2}{\varepsilon},\quad \left|\mu''_\varepsilon(x)\right|\le\frac{4}{\varepsilon^2},\quad 0\le x\le 1.$$

Assume that u is a global solution to the problem (1.6)-(1.8). It follows from Definition 2.1 that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u(x,t)\mu_\varepsilon(x)\zeta_\delta(x)\mathrm{d}x\\
=&\int_0^1 u(x)\big(a(x)(\mu_\varepsilon(x)\zeta_\delta(x))'\big)'\mathrm{d}x+\int_0^1 u^p(x,t)\mu_\varepsilon(x)\zeta_\delta(x)\mathrm{d}x\\
=&\int_0^1 u(x)\mu_\varepsilon(x)\big(a(x)\zeta'_\delta(x)\big)'\mathrm{d}x+\int_\varepsilon^{2\varepsilon} u(x)\mu'_\varepsilon(x)\big(2a(x)\zeta'_\delta(x)+a'(x)\zeta_\delta(x)\big)\mathrm{d}x\\
&+\int_\varepsilon^{2\varepsilon} u(x)a(x)\mu''_\varepsilon(x)\zeta_\delta(x)\mathrm{d}x+\int_0^1 u^p(x,t)\mu_\varepsilon(x)\zeta_\delta(x)\mathrm{d}x,\quad t>0.
\end{aligned}\tag{3.10}$$

Owing to (3.7) and $0<\delta<\frac{1}{\gamma}<1$, one gets that

$$\begin{aligned}
\left|a(x)\zeta'_\delta(x)\right|&=\delta\left(\int_x^1\frac{1}{a(s)}\mathrm{d}s\right)^{\delta-1}\le\delta\big(M_1a^{\frac12}(x)\big)^{1-\delta}\\
&=\delta M_1^{1-\delta}a^{\frac{1-\delta}{2}}(x),\quad 0<x<x_0.
\end{aligned}\tag{3.11}$$

It follows from the first formula in (3.3) that

$$\big(a^{\frac12}(x)\big)'\le\frac{M_1}{2},\quad 0<x<1,$$

which, together with (1.9), leads to

$$a(x)\le\frac{M_1^2}{4}x^2,\quad 0\le x\le 1.\tag{3.12}$$

Thanks to (3.3), (3.12) and $0<\delta<\frac{1}{\gamma}$, one gets that

$$\begin{aligned}
&\left|a'(x)\zeta_\delta(x)\right|=\left|a'(x)\right|\left(\int_x^1\frac{1}{a(s)}\mathrm{d}s\right)^{\delta}\\
\le&M_1a^{\frac12}(x)\left(\int_x^1\frac{1}{M_2s^\gamma}\mathrm{d}s\right)^{\delta}\le\frac{M_1^2}{2}x\left(\frac{1}{M_2(\gamma-1)}\left(\frac{1}{x^{\gamma-1}}-1\right)\right)^{\delta}\\
\le&\frac{M_1^2}{2M_2^\delta(\gamma-1)^\delta}x^{1-\delta(\gamma-1)},\quad 0<x<1.
\end{aligned}\tag{3.13}$$

Due to (3.11), (3.13), (1.9), $\gamma\geq 2$ and $0<\delta<1/\gamma$, it holds that

$$\lim_{\varepsilon\to 0^+}\sup\left\{\left|2a(x)\zeta_\delta'(x)+a'(x)\zeta_\delta(x)\right|;\, \varepsilon<x<2\varepsilon\right\}=0. \tag{3.14}$$

Letting $\varepsilon\to 0^+$ in (3.10), one can obtain from (3.14) and the Hölder inequality that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u(x,t)\zeta_\delta(x)\mathrm{d}x \\ =&\int_0^1 u(x)\big(a(x)\zeta_\delta'(x)\big)'\mathrm{d}x+\int_0^1 u^p(x,t)\zeta_\delta(x)\mathrm{d}x \\ \geq&\int_0^1 u(x)\big(a(x)\zeta_\delta'(x)\big)'\mathrm{d}x+\left(\int_0^1\zeta_\delta(x)\mathrm{d}x\right)^{1-p}\left(\int_0^1 u(x,t)\zeta_\delta(x)\mathrm{d}x\right)^p, \quad t>0.\end{aligned} \tag{3.15}$$

Substitute (3.5) and (3.9) into (3.15) to get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u(x,t)\zeta_\delta(x)\mathrm{d}x \\ \geq& -M_4\delta\int_0^1 u(x,t)\zeta_\delta(x)\mathrm{d}x+M_3^{1-p}\left(\int_0^1 u(x,t)\zeta_\delta(x)\mathrm{d}x\right)^p, \quad t>0.\end{aligned} \tag{3.16}$$

For a nontrivial u_0, it is noted that

$$\inf_{0<\delta<1/2}\int_0^1 u_0(x)\zeta_\delta(x)\mathrm{d}x>0.$$

Hence, there exists a sufficiently small $0<\delta<\frac{1}{2}$ such that

$$2M_4\delta\leq M_3^{1-p}\left(\int_0^1 u_0(x)\zeta_\delta(x)\mathrm{d}x\right)^{p-1}.$$

It follows from (3.16) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u(x,t)\zeta_\delta(x)\mathrm{d}x\geq\frac{1}{2}M_3^{1-p}\left(\int_0^1 u(x,t)\zeta_\delta(x)\mathrm{d}x\right)^p, \quad t>0.$$

Therefore, there exists $T>0$ such that

$$\lim_{t\to T^-}\int_0^1 u(x,t)\zeta_\delta(x)\mathrm{d}x=+\infty,$$

which leads to

$$\lim_{t\to T^-}\sup_{(0,1)}u(\cdot,t)=+\infty.$$

That is to say, u must blow up in a finite time. $\square$

4 Problem in an unbounded interval

In this section, we prove the theorems for the problem (1.12)-(1.14) in an unbounded interval. It is noted that Theorem 2.6 is a corollary of Theorem 2.2 and Proposition 2.2, and we need only to prove Theorems 2.3-2.5.

Proof of Theorem 2.3. First prove the Case (i). For $p<3-\lambda$, set $\eta=(3-\lambda-p)/2$. Owing to (1.16), there exists a constant $R_1>0$ depending only on a such that

$$\frac{xa'(x)}{a(x)}<\lambda+\eta, \quad x\geq R_1. \tag{4.1}$$

Hence,

$$\left(\frac{a(x)}{x^{\lambda+\eta}}\right)'<0, \quad x\geq R_1. \tag{4.2}$$

It follows from (4.2) and (1.15) that

$$a(x)\leq \frac{a(R_1)}{R_1^{\lambda+\eta}}x^{\lambda+\eta}, \quad x\geq R_1. \tag{4.3}$$

For $R>0$, set

$$\zeta_R(x)=\begin{cases}1, & 0\leq x\leq R,\\ \dfrac{1}{2}\left(1+\cos\dfrac{(x-R)\pi}{R}\right), & R<x<2R,\\ 0, & x\geq 2R.\end{cases} \tag{4.4}$$

It is clear that $\zeta_R\in C^1([0,+\infty))$ is piecewise smooth and satisfies

$$\big(a(x)\zeta_R'(x)\big)'=-\frac{\pi}{2R}a'(x)\sin\frac{(x-R)\pi}{R}-\frac{\pi^2}{2R^2}a(x)\cos\frac{(x-R)\pi}{R}, \quad R<x<2R. \tag{4.5}$$

Thanks to (4.1), (4.3) and (4.5), one gets that for $R\geq R_1$,

$$\begin{aligned}\big(a(x)\zeta_R'(x)\big)'&\geq -\frac{\pi(\lambda+\eta)}{2xR}a(x)\sin\frac{(x-R)\pi}{R}-\frac{\pi^2}{2R^2}a(x)\cos\frac{(x-R)\pi}{R}\\ &\geq -\frac{\pi^2}{2R^2}a(x)\left(\frac{\lambda+\eta}{\pi}+1\right)\left(1+\cos\frac{(x-R)\pi}{R}\right)\\ &\geq -\frac{\pi^2a(R_1)}{2R_1^{\lambda+\eta}R^2}\left(\frac{\lambda+\eta}{\pi}+1\right)x^{\lambda+\eta}\left(1+\cos\frac{(x-R)\pi}{R}\right)\\ &\geq -N_1R^{\lambda+\eta-2}\zeta_R(x), \quad R<x<2R,\end{aligned} \tag{4.6}$$

where

$$N_1=\frac{2^{\lambda+\eta}\pi^2 a(R_1)}{R_1^{\lambda+\eta}}\left(\frac{\lambda+\eta}{\pi}+1\right).$$

Assume that u is a solution to the problem (1.12)-(1.14). Definition 2.2, (4.6) and the Hölder inequality yield

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x \qquad (4.7)\\
=&\int_0^{+\infty}u(x,t)\big(a(x)\zeta_R'(x)\big)'\mathrm{d}x+\int_0^{+\infty}u^p(x,t)\zeta_R(x)\mathrm{d}x\\
\ge&-N_1R^{\lambda+\eta-2}\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x+\left(\int_0^{+\infty}\zeta_R(x)\mathrm{d}x\right)^{1-p}\left(\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x\right)^p\\
\ge&-N_1R^{\lambda+\eta-2}\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x+2^{1-p}R^{1-p}\left(\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x\right)^p,\quad t>0.
\end{aligned}$$

It follows from the choice of η that $\lambda+\eta-2<1-p$. Hence, there exists a sufficiently large $R\ge R_1$ such that

$$2N_1R^{\lambda+\eta-2}\le 2^{1-p}R^{1-p}\left(\int_0^{+\infty}u_0(x)\psi_R(x)\mathrm{d}x\right)^{p-1}.$$

It follows from (4.7) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x\ge 2^{-p}R^{1-p}\left(\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x\right)^p,\quad t>0.$$

Therefore, there exists $T>0$ such that

$$\lim_{t\to T^-}\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x=+\infty,$$

which leads to

$$\lim_{t\to T^-}\sup_{(0,+\infty)}u(\cdot,t)=+\infty,$$

i.e. u blows up in a finite time.

Turn to the Case (ii) that $p>3-\lambda$. Thanks to Theorem 2.1, Propositions 2.1 and 2.2, the solution to the problem (1.12)-(1.14) blows up in a finite time if u_0 is suitably large. Below we prove that the solution to the problem (1.12)-(1.14) exists globally if u_0 is suitably small. Set

$$\hat{u}(x,t)=\frac{\varepsilon}{(t+L)^{1/(p-1)}}\exp\left\{-\frac{\eta A(x)}{t+L}\right\},\quad x\ge 0,\quad t\ge 0,$$

where ε and L are positive constants to be determined below, η is a constant such that

$$\frac{1}{p-1}<\eta<\frac{1}{2-\lambda}, \tag{4.8}$$

and

$$A(x)=\int_0^x \frac{s}{a(s)}\mathrm{d}s, \quad x\geq 0.$$

Here η and A are well-defined owing to $p>3-\lambda$ and (1.10). Direct calculations show that

$$\begin{aligned}
&\frac{\partial \hat{u}}{\partial t}-\frac{\partial}{\partial x}\left(a(x)\frac{\partial \hat{u}}{\partial x}\right)-\hat{u}^p\\
=&\frac{\varepsilon}{(t+L)^{p/(p-1)}}\left(\eta\big(a(x)A'(x)\big)'-\frac{1}{p-1}\right)\exp\left\{-\frac{\eta A(x)}{t+L}\right\}\\
&+\frac{\varepsilon\eta}{(t+L)^{p/(p-1)+1}}\left(A(x)-\eta a(x)\big(A'(x)\big)^2\right)\exp\left\{-\frac{\eta A(x)}{t+L}\right\}\\
&+\frac{\varepsilon^p}{(t+L)^{p/(p-1)}}\exp\left\{-\frac{\eta pA(x)}{t+L}\right\}\\
=&\frac{\varepsilon}{(t+L)^{p/(p-1)}}\left(\eta-\frac{1}{p-1}\right)\exp\left\{-\frac{\eta A(x)}{t+L}\right\}\\
&+\frac{\varepsilon\eta}{(t+L)^{p/(p-1)+1}}\left(\int_0^x\frac{s}{a(s)}\mathrm{d}s-\frac{\eta x^2}{a(x)}\right)\exp\left\{-\frac{\eta A(x)}{t+L}\right\}\\
&+\frac{\varepsilon^p}{(t+L)^{p/(p-1)}}\exp\left\{-\frac{\eta pA(x)}{t+L}\right\}, \quad x>0, \quad t>0.
\end{aligned} \tag{4.9}$$

It follows from the L'Hospital rule and (1.16) that

$$\lim_{x\to+\infty}\frac{a(x)}{x^2}\int_0^x\frac{s}{a(s)}\mathrm{d}s=\lim_{x\to+\infty}\frac{a(x)}{2a(x)-xa'(x)}=\frac{1}{2-\lambda}. \tag{4.10}$$

Thanks to (1.15), (4.8) and (4.10), there exists a constant $x_0>0$ such that

$$\int_0^x\frac{s}{a(s)}\mathrm{d}s-\frac{\eta x^2}{a(x)}\geq 0, \quad x\geq x_0. \tag{4.11}$$

It follows from (1.15) and the first formula in (1.16) that $x^2/a(x)\,(x\in(0,x_0))$ is bounded. Choose suitably large $L>0$ and suitably small $\varepsilon>0$ such that

$$\frac{\eta}{L}\sup\left\{\frac{\eta x^2}{a(x)}:0<x<x_0\right\}\leq\frac{1}{2}\left(\eta-\frac{1}{p-1}\right), \quad \varepsilon^{p-1}\leq\frac{1}{2}\left(\eta-\frac{1}{p-1}\right). \tag{4.12}$$

Using (4.11) and (4.12), one gets from (4.9) that

$$\frac{\partial \hat{u}}{\partial t}-\frac{\partial}{\partial x}\left(a(x)\frac{\partial \hat{u}}{\partial x}\right)-\hat{u}^p\geq 0,\quad x>0,\quad t>0.$$

It is noted that

$$\lim_{x\to 0^+} a(x)\frac{\partial \hat{u}}{\partial x}(x,t)=0,\quad t>0.$$

Therefore, $\hat{u}$ is a supersolution to the problem (1.12)-(1.14) if

$$u_0(x)\leq \hat{u}(x,0),\quad x>0. \tag{4.13}$$

Thanks to Proposition 2.2 (ii), the solution to the problem (1.12)-(1.14) exists globally in time if u_0 satisfies (4.13). □

Proof of Theorem 2.4. Let $p>1$. It follows from Theorem 2.1, Propositions 2.1 and 2.2 that the solution to the problem (1.12)-(1.14) blows up in a finite time if u_0 is suitably large. Below we prove that the solution to the problem (1.12)-(1.14) exists globally if u_0 is suitably small. Set

$$\hat{u}(x,t)=\frac{\varepsilon}{(t+L)^{1/(p-1)}}\exp\left\{-\frac{2A(x)}{(p-1)(t+L)}\right\},\quad x\geq 0,\quad t\geq 0,$$

where ε and L are positive constants to be determined below, and

$$A(x)=\int_0^x \frac{s}{a(s)}\mathrm{d}s,\quad x\geq 0,$$

which is well-defined due to (1.10). Similar to the proof of (4.9), it holds that

$$\begin{aligned}
&\frac{\partial \hat{u}}{\partial t}-\frac{\partial}{\partial x}\left(a(x)\frac{\partial \hat{u}}{\partial x}\right)-\hat{u}^p\\
=&\frac{\varepsilon}{(p-1)(t+L)^{p/(p-1)}}\exp\left\{-\frac{2A(x)}{(p-1)(t+L)}\right\}\\
&+\frac{2\varepsilon}{(p-1)(t+L)^{p/(p-1)+1}}\left(\int_0^x\frac{s}{a(s)}\mathrm{d}s-\frac{2x^2}{(p-1)a(x)}\right)\exp\left\{-\frac{2A(x)}{(p-1)(t+L)}\right\}\\
&+\frac{\varepsilon^p}{(t+L)^{p/(p-1)}}\exp\left\{-\frac{2pA(x)}{(p-1)(t+L)}\right\}\\
\geq&\frac{\varepsilon}{(p-1)(t+L)^{p/(p-1)}}\left(1-\frac{4x^2}{(p-1)(t+L)a(x)}\right)\exp\left\{-\frac{2A(x)}{(p-1)(t+L)}\right\}\\
&+\frac{\varepsilon^p}{(t+L)^{p/(p-1)}}\exp\left\{-\frac{2pA(x)}{(p-1)(t+L)}\right\},\quad x>0,\quad t>0.
\end{aligned}\tag{4.14}$$

If $\lambda=2$, it follows from the third formula in (1.16) that there exist two constants $x_1>0$ and $S_1>0$ such that

$$a(x)\geq S_1 x^2,\quad x\geq x_1. \tag{4.15}$$

If $\lambda>2$, it follows from the second formula in (1.16) that there exists a constant $x_2>0$ such that

$$xa'(x)-2a(x)>0,\quad x\geq x_2,$$

which yields

$$a(x)\geq \frac{a(x_2)}{x_2^2}x^2,\quad x\geq x_2. \tag{4.16}$$

Thanks to (1.15), the first formula in (1.16), (4.15) and (4.16), there exists a constant $S_2>0$ such that

$$\frac{x^2}{a(x)}\leq S_2,\quad x>0. \tag{4.17}$$

Choose

$$L=\frac{8S_2}{p-1},\quad \varepsilon=\left(\frac{1}{2(p-1)}\right)^{\frac{1}{p-1}}. \tag{4.18}$$

One gets from (4.14), (4.17) and (4.18) that

$$\frac{\partial \hat{u}}{\partial t}-\frac{\partial}{\partial x}\left(a(x)\frac{\partial \hat{u}}{\partial x}\right)-\hat{u}^p\geq 0,\quad x>0,\quad t>0.$$

It is noted that

$$\lim_{x\to 0^+} a(x)\frac{\partial \hat{u}}{\partial x}(x,t)=0,\quad t>0.$$

Therefore, $\hat{u}$ is a supersolution to the problem (1.12)-(1.14) if

$$u_0(x)\leq \hat{u}(x,0),\quad x>0. \tag{4.19}$$

Thanks to Proposition 2.2 (ii), the solution to the problem (1.12)-(1.14) exists globally in time if u_0 satisfies (4.19). □

In order to prove Theorem 2.5, we need the following two lemmas.

Lemma 4.1. *Assume that $a\in C([0,+\infty))\cap C^1((0,+\infty))$ satisfies* (1.15), (1.10), (1.16) *and* (1.17). *Let $p=p_c=3-\lambda$ and u be a global solution to the problem* (1.12)-(1.14). *There exist two positive constants R_2 and N_2 depending only on λ and a, such that for any $R\geq R_2$,*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x \\ \geq & -N_2R^{\lambda-2}\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x+2^{\lambda-2}R^{\lambda-2}\left(\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x\right)^{3-\lambda}, \quad t>0,\end{aligned} \tag{4.20}$$

where ζ_R *is defined in* (4.4).

Proof. It follows from Definition 2.2 that u satisfies

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x \\ =&\int_0^{+\infty}u(x,t)\big(a(x)\zeta_R'(x)\big)'\mathrm{d}x+\int_0^{+\infty}u^{3-\lambda}(x,t)\zeta_R(x)\mathrm{d}x, \quad t>0.\end{aligned} \tag{4.21}$$

Owing to (1.16) and (1.17), there exist two constants $R_2>0$ and $L>0$, depending only on a, such that

$$\frac{xa'(x)}{a(x)}<\lambda+1, \quad a(x)<Lx^\lambda, \quad x\geq R_2. \tag{4.22}$$

Hence,

$$a'(x)<(\lambda+1)Lx^{\lambda-1}, \quad x\geq R_2. \tag{4.23}$$

Thanks to (4.22) and (4.23), one gets that for $R\geq R_2$,

$$\begin{aligned}\big(a(x)\zeta_R'(x)\big)'&=-\frac{\pi}{2R}a'(x)\sin\frac{(x-R)\pi}{R}-\frac{\pi^2}{2R^2}a(x)\cos\frac{(x-R)\pi}{R}\\ &\geq -\frac{\pi(\lambda+1)L}{2R}x^{\lambda-1}\sin\frac{(x-R)\pi}{R}-\frac{\pi^2L}{2R^2}x^\lambda\cos\frac{(x-R)\pi}{R}\\ &\geq -\frac{\pi^2L}{2R^2}\left(\frac{\lambda+1}{\pi}+1\right)x^\lambda\left(1+\cos\frac{(x-R)\pi}{R}\right)\\ &\geq -N_2R^{\lambda-2}\zeta_R(x), \quad R<x<2R,\end{aligned} \tag{4.24}$$

where

$$N_2=2^\lambda\pi^2L\left(\frac{\lambda+1}{\pi}+1\right).$$

Thanks to (4.21), (4.24) and the Hölder inequality, one gets that for $R\geq R_2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x\geq -N_2R^{\lambda-2}\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x+\left(\int_0^{+\infty}\zeta_R(x)\mathrm{d}x\right)^{\lambda-2}$$

$$\times\left(\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x\right)^{3-\lambda},\quad t>0,$$

which leads to (4.20). □

Lemma 4.2. *Assume that $a\in C([0,+\infty))\cap C^1((0,+\infty))$ satisfies* (1.15), (1.10), (1.16) *and* (1.17). *Let $p=p_c=3-\lambda$ and u be a global solution to the problem* (1.12)-(1.14). *Then for any $R\geq R_2$,*

$$\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x\leq 2^{\frac{3-\lambda}{2-\lambda}}N_2^{\frac{1}{2-\lambda}},\qquad t>0,\tag{4.25}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x\geq -2N_2^{\frac{3-\lambda}{2-\lambda}}R^{\lambda-2},\quad t>0,\tag{4.26}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x\geq{}& R^{\lambda-2}\left(\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x\right)^{\frac{1}{2}}\\&\times\left(-N_2\left(\int_R^{2R}u(x,t)\zeta_R(x)\mathrm{d}x\right)^{\frac{1}{2}}+2^{\lambda-2}\left(\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x\right)^{\frac{5}{2}-\lambda}\right),\quad t>0,\end{aligned}\tag{4.27}$$

where ζ_R is defined in (4.4), *and R_2 and N_2 are given in Lemma* 4.1.

Proof. First we prove (4.25) by a contradiction. Otherwise, there exists $t_0>0$ and $R\geq R_2$ such that

$$2N_2\leq 2^{\lambda-2}\left(\int_0^{+\infty}u(x,t_0)\zeta_R(x)\mathrm{d}x\right)^{2-\lambda}.$$

It follows from (4.20) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x\geq 2^{\lambda-3}R^{\lambda-2}\left(\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x\right)^{3-\lambda},\quad t>t_0,$$

which leads to that u must blow up in a finite time since $0\leq\lambda<2$. Hence, (4.25) is proved.

Second, from (4.20) and the Young inequality, one can get that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x\\&\geq 2^{\lambda-2}R^{\lambda-2}\left(-2^{2-\lambda}N_2\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x+\left(\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x\right)^{3-\lambda}\right)\end{aligned}$$

$$
\begin{aligned}
&\geq 2^{\lambda-2}R^{\lambda-2}\Bigg(-\frac{1}{3-\lambda}\left(\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x\right)^{3-\lambda}-\frac{2-\lambda}{3-\lambda}(2^{2-\lambda}N_2)^{\frac{3-\lambda}{2-\lambda}}\\
&\qquad\qquad +\left(\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x\right)^{3-\lambda}\Bigg)\\
&\geq -2N_2^{\frac{3-\lambda}{2-\lambda}}R^{\lambda-2},\quad t>0,
\end{aligned}
$$

which is just (4.26).

Finally, it follows from (4.21), (4.24) and the Hölder inequality that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x\\
=&\int_R^{2R}u(x,t)\big(a(x)\zeta_R'(x)\big)'\mathrm{d}x+\int_0^{+\infty}u^{3-\lambda}(x,t)\zeta_R(x)\mathrm{d}x\\
\geq&-N_2R^{\lambda-2}\int_R^{2R}u(x,t)\zeta_R(x)\mathrm{d}x+\left(\int_0^{+\infty}\zeta_R(x)\mathrm{d}x\right)^{\lambda-2}\left(\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x\right)^{3-\lambda}\\
\geq&-N_2R^{\lambda-2}\int_R^{2R}u(x,t)\zeta_R(x)\mathrm{d}x+2^{\lambda-2}R^{\lambda-2}\left(\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x\right)^{3-\lambda}\\
\geq&R^{\lambda-2}\left(\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x\right)^{\frac{1}{2}}\\
&\times\left(-N_2\left(\int_R^{2R}u(x,t)\zeta_R(x)\mathrm{d}x\right)^{\frac{1}{2}}+2^{\lambda-2}\left(\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x\right)^{\frac{5}{2}-\lambda}\right),\quad t>0,
\end{aligned}
$$

which is just (4.27). □

Proof of Theorem 2.5. Let ζ_R be defined in (4.4), and R_2 and N_2 be given in Lemma 4.1. Assume that u is a global solution to the problem (1.12)-(1.14). For any $R\geq R_2$, set

$$
w_R(t)=\int_0^{+\infty}u(x,t)\zeta_R(x)\mathrm{d}x,\quad t>0.
$$

Denote

$$
\Lambda=\sup_{R>0,t>0}w_R(t)=\sup_{t>0}\int_0^{+\infty}u(x,t)\mathrm{d}x. \tag{4.28}
$$

It follows from (4.25) and the nontriviality of u_0 that $0<\Lambda<+\infty$. For ε_0, there exists $t_1\geq 0$ and $R_0\geq R_2$ such that

$$
w_{R_0}(t_1)\geq\Lambda-\varepsilon_0, \tag{4.29}
$$

where $\varepsilon_0>0$ is a constant to be determined below. For any $t\geq t_1$, it follows from (4.26) with $R=R_0$ and (4.29) that

$$\begin{aligned} w_{R_0}(t)&\geq w_{R_0}(t_1)-2N_2^{\frac{3-\lambda}{2-\lambda}}R_0^{\lambda-2}(t-t_1)\\ &\geq\Lambda-\varepsilon_0-2N_2^{\frac{3-\lambda}{2-\lambda}}R_0^{\lambda-2}(t-t_1), \end{aligned}$$

which, together with (4.28), leads to

$$\begin{aligned} &\int_{2R_0}^{4R_0}u(x,t)\zeta_{2R_0}(x)\mathrm{d}x\\ \leq&\int_0^{+\infty}u(x,t)\mathrm{d}x-\int_0^{+\infty}u(x,t)\zeta_{R_0}(x)\mathrm{d}x\\ \leq&\,\varepsilon_0+2N_2^{\frac{3-\lambda}{2-\lambda}}R_0^{\lambda-2}(t-t_1). \end{aligned}\tag{4.30}$$

Choosing $R=2R_0$ in (4.27) yields

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}w_{2R_0}(t)\geq(2R_0)^{\lambda-2}w_{2R_0}^{\frac12}(t)\\ &\quad\times\left(-N_2\left(\int_{2R_0}^{4R_0}u(x,t)\zeta_{2R_0}(x)\mathrm{d}x\right)^{\frac12}+2^{\lambda-2}w_{2R_0}^{\frac52-\lambda}(t)\right),\quad t>t_1. \end{aligned}$$

Fix $\varepsilon_0\in(0,\Lambda)$ and $\tau>0$ such that

$$N_2(\varepsilon_0+\tau)^{\frac12}\leq2^{\lambda-3}(\Lambda-\varepsilon_0)^{\frac52-\lambda}.$$

Owing to (4.28)-(4.30), it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}w_{2R_0}(t)\geq2^{2\lambda-5}R_0^{\lambda-2}(\Lambda-\varepsilon_0)^{3-\lambda},\quad t_1<t<t_2,\tag{4.31}$$

where

$$t_2=t_1+\frac12N_2^{-\frac{3-\lambda}{2-\lambda}}\tau R_0^{2-\lambda}.$$

It follows from (4.29) and (4.31) that

$$w_{2R_0}(t_2)\geq w_{2R_0}(t_1)+2^{2\lambda-5}R_0^{\lambda-2}(\Lambda-\varepsilon_0)^{3-\lambda}(t_2-t_1)\geq\Lambda-\varepsilon_0+\gamma_0,\tag{4.32}$$

where

$$\gamma_0=2^{2\lambda-6}N_2^{-\frac{3-\lambda}{2-\lambda}}\tau(\Lambda-\varepsilon_0)^{3-\lambda}.$$

Thanks to (4.26) with $R=2R_0$ and (4.32), one gets that

$$\begin{aligned}w_{2R_0}(t)&\geq w_{2R_0}(t_2)-2N_2^{\frac{3-\lambda}{2-\lambda}}(2R_0)^{\lambda-2}(t-t_2)\\&\geq\Lambda-\varepsilon_0-2N_2^{\frac{3-\lambda}{2-\lambda}}(2R_0)^{\lambda-2}(t-t_2),\quad t\geq t_2,\end{aligned}$$

which, together with (4.28) with $R=2R_0$, leads to

$$\begin{aligned}&\int_{4R_0}^{8R_0}u(x,t)\zeta_{4R_0}(x)\mathrm{d}x\\&\leq\int_0^{+\infty}u(x,t)\mathrm{d}x-\int_0^{+\infty}u(x,t)\zeta_{2R_0}(x)\mathrm{d}x\\&\leq\varepsilon_0+2N_2^{\frac{3-\lambda}{2-\lambda}}(2R_0)^{\lambda-2}(t-t_2),\quad t\geq t_2.\end{aligned}\tag{4.33}$$

Taking $R=4R_0$ in (4.27) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}w_{4R_0}(t)\geq(4R_0)^{\lambda-2}w_{4R_0}^{\frac{1}{2}}(t)\\&\quad\times\left(-N_2\left(\int_{4R_0}^{8R_0}u(x,t)\zeta_{4R_0}(x)\mathrm{d}x\right)^{\frac{1}{2}}+2^{\lambda-2}w_{4R_0}^{\frac{5}{2}-\lambda}(t)\right),\quad t>t_2.\end{aligned}$$

Thanks to (4.31)-(4.33), one gets that

$$\frac{\mathrm{d}}{\mathrm{d}t}w_{4R_0}(t)\geq2^{2\lambda-5}(2R_0)^{\lambda-2}(\Lambda-\varepsilon_0)^{3-\lambda},\quad t_2<t<t_3,\tag{4.34}$$

where

$$t_3=t_2+\frac{1}{2}N_2^{-\frac{3-\lambda}{2-\lambda}}\tau(2R_0)^{2-\lambda}.$$

It follows from (4.31) and (4.34) that

$$\begin{aligned}w_{4R_0}(t_3)&\geq w_{4R_0}(t_2)+2^{2\lambda-5}(2R_0)^{\lambda-2}(\Lambda-\varepsilon_0)^{3-\lambda}(t_3-t_2)\\&\geq w_{2R_0}(t_2)+\gamma_0\geq\Lambda-\varepsilon_0+2\gamma_0.\end{aligned}$$

Repeating the procedure in turn, one obtains that for any positive integer i,

$$w_{2^iR_0}(t_{i+1})\geq w_{2^iR_0}(t_i)+\gamma_0\geq w_{2^{i-1}R_0}(t_i)+\gamma_0\geq\Lambda-\varepsilon_0+i\gamma_0,$$

where

$$t_{i+1}=t_i+\frac{1}{2}N_2^{-\frac{3-\lambda}{2-\lambda}}\tau(2^{i-1}R_0)^{2-\lambda}.$$

Therefore

$$\sup_{t>0}\int_0^{+\infty}u(x,t)\mathrm{d}x=+\infty,$$

which contradicts (4.28) and completes the proof of Theorem 2.5. $\square$

5 Problems with inner degeneracy

Similarly to the proof for the problems (1.6)-(1.8) and (1.12)-(1.14) in Sections 3 and 4, one can establish the similar theorems for the following problems with inner degeneracy

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(a(|x|)\frac{\partial u}{\partial x}\right)=u^p, \qquad (x,t)\in(-1,1)\times(0,T), \tag{5.1}$$

$$u(\pm 1,t)=0, \qquad t\in(0,T), \tag{5.2}$$

$$u(x,0)=u_0(x), \qquad x\in(-1,1), \tag{5.3}$$

and

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(a(|x|)\frac{\partial u}{\partial x}\right)+u^p, \qquad (x,t)\in\mathbb{R}\times(0,T), \tag{5.4}$$

$$u(x,0)=u_0(x), \qquad x\in\mathbb{R}. \tag{5.5}$$

We state the results without proof.

Theorem 5.1. *Assume that* $a\in C([0,1])\cap C^1((0,1])$ *satisfies* (1.9) *and* (1.10). *The solution to the problem* (5.1)-(5.3) *exists globally in time if* u_0 *is small, while blows up in a finite time if* u_0 *is large.*

Theorem 5.2. *Assume that* $a\in C([0,1])\cap C^1((0,1])$ *satisfies* (1.9) *and* (1.11). *Then any nontrivial solution to the problem* (5.1)-(5.3) *must blow up in a finite time.*

Theorem 5.3. *Assume that* $a\in C([0,+\infty))\cap C^1((0,+\infty))$ *satisfies* (1.15), (1.10) *and* (1.16) *with* $0\le\lambda<2$.

(i) If $1<p<3-\lambda$, *then any nontrivial solution to the problem* (5.4)-(5.5) *must blow up in a finite time.*

(ii) If $p>3-\lambda$, *then the solution to the problem* (5.4)-(5.5) *exists globally in time if* u_0 *is small, while blows up in a finite time if* u_0 *is large.*

Theorem 5.4. *Assume that* $a\in C([0,+\infty))\cap C^1((0,+\infty))$ *satisfies* (1.15), (1.10) *and* (1.16) *with* $\lambda\ge 2$. *For* $p>1$, *the solution to the problem* (5.4)-(5.5) *exists globally in time if* u_0 *is small, while blows up in a finite time if* u_0 *is large.*

Theorem 5.5. *Assume that* $a\in C([0,+\infty))\cap C^1((0,+\infty))$ *satisfies* (1.15), (1.10), (1.16) *and* (1.17) *with* $0\le\lambda<2$. *For* $p=3-\lambda$, *any nontrivial solution to the problem* (5.4)-(5.5) *must blow up in a finite time.*

Theorem 5.6. *Assume that* $a\in C([0,+\infty))\cap C^1((0,+\infty))$ *satisfies* (1.15) *and* (1.11). *Then any nontrivial solution to the problem* (5.4)-(5.5) *must blow up in a finite time.*

Acknowledgements

This work was supported by the National Key R & D Program of China (Grant No. 2020YFA0714101) and by the National Natural Science Foundation of China (Grant No. 11925105).

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