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Fujita-Kato Theorem for the Inhomogeneous Incompressible Navier-Stokes Equations with Nonnegative Density

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Abstract. In this paper, we prove the global existence and uniqueness of solutions for the inhomogeneous Navier-Stokes equations with the initial data $(\rho_0,u_0)\in L^\infty\times H_0^s$, $s>\frac{1}{2}$ and $\|u_0\|_{H_0^s}\le\varepsilon_0$ in bounded domain $\Omega\subset\mathbb{R}^3$, in which the density is assumed to be nonnegative. The regularity of initial data is weaker than the previous $(\rho_0,u_0)\in(W^{1,\gamma}\cap L^\infty)\times H_0^1$ in [13] and $(\rho_0,u_0)\in L^\infty\times H_0^1$ in [7], which constitutes a positive answer to the question raised by Danchin and Mucha in [7]. The methods used in this paper are mainly the classical time weighted energy estimate and Lagrangian approach, and the continuity argument and shift of integrability method are applied to complete our proof.

AMS subject classifications: 35Q30, 76D03

Key words: Inhomogeneous Navier-Stokes equations, nonnegative density, global existence and uniqueness.

1 Introduction

In this work, we will investigate the global existence and uniqueness of solutions to the following inhomogeneous incompressible Navier-Stokes equations:

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$$\begin{cases}\rho_t+u\cdot\nabla\rho=0 & \text{in } \mathbb{R}^+\times\Omega, & (1.1a)\\ \rho u_t+\rho u\cdot\nabla u-\Delta u+\nabla P=0 & \text{in } \mathbb{R}^+\times\Omega, & (1.1b)\\ \operatorname{div} u=0 & \text{in } \mathbb{R}^+\times\Omega & (1.1c)\end{cases}$$

with the initial and the boundary conditions

$$(\rho,\rho u)|_{t=0}=(\rho_0,\rho_0 u_0),$$
$$u(t,x)=0, \quad x\in\partial\Omega,$$

and ρ,u,P standing for the density, velocity and pressure respectively. This system originated from the theory of geophysical flows, which describes incompressible fluids with different densities, and is also widely used in the research of two miscible fluids. One may check [14] for the detailed derivation of this system.

The weak solution theory of this system (see [14]) is well known. If

$$0\le\rho_0\le\rho^* \text{ for some } \rho^*>0 \text{ and } \sqrt{\rho_0}u_0\in L^2,$$

there exists global weak solutions (ρ,u) to system (1.1) such that for all $t\ge 0$

$$\left\|\sqrt{\rho(t)}u(t)\right\|_{L^2}^2+2\int_0^t\|\nabla u\|_{L^2}^2 d\tau\le\|\sqrt{\rho_0}u_0\|_{L^2}^2,$$
$$0\le\rho(t)\le\rho^* \text{ and } \int_\Omega\rho dx=\int_\Omega\rho_0 dx.$$

However, the uniqueness of such weak solutions is still an open problem. As far as we know, Ladyzhenskaya and Solonnikov [12] firstly proved the global well-posedness of system (1.1) with the small data and the density bounded away from zero. After that, many classical results appeared.

When the density is a constant, the system (1.1) is reduced to the incompressible Navier-Stokes equations

$$\begin{cases}u_t+u\cdot\nabla u-\Delta u+\nabla P=0,\\ \operatorname{div} u=0.\end{cases} \tag{1.2}$$

In the seminal paper [9], Fujita and Kato proved the global well-posedness for the initial data $u_0\in\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$ satisfying $\|u_0\|_{\dot{H}^{\frac{d}{2}-1}}\le\varepsilon_0$ with small $\varepsilon_0>0$. It is well known that the space $\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$ is critical for its important property that the norm is invariant under the scaling of the equation

$$u(t,x)\to\lambda u(\lambda^2 t,\lambda x),$$

which means that if $u(t,x)$ is a solution of (1.2), then so does $u_\lambda\triangleq\lambda u(\lambda^2 t,\lambda x)$.

In fact, the system (1.1) also has a similar scaling invariant property. And Danchin [3] proved the local well-posedness in critical Besov space with

$$\rho_0 - 1 \in \dot{B}^{\frac{d}{p}}_{p,1}(\mathbb{R}^d), \quad u_0 \in \dot{B}^{\frac{d}{p}-1}_{p,1}(\mathbb{R}^d), \quad 1 \le p \le 2d.$$

Relying on the use of Lagrangian coordinates, Danchin and Mucha [4,5] further considered the case of discontinuous density. And then Paicu and Zhang [8] established the well-posedness results with the velocity belonging to general critical Besov spaces by using the maximal regularity of heat equation. What should be mentioned is that the global results in [5] are obtained under the smallness assumption of initial density, Paicu *et al.* [15] removed the smallness condition and proved the well-posedness of solutions with the initial data $u_0 \in H^1(\mathbb{R}^3)$, which improves the results in [5] that $u_0 \in H^2(\mathbb{R}^3)$. Later, Chen *et al.* [1] established the Fujita-Kato theorem with discontinuous density. However, all of these results require that the initial density is bounded away from zero.

When the initial density is nonnegative, that is the initial vacuum is allowed, Choe and Kim [2] firstly proved the local existence and uniqueness of strong solutions. But in their work, a compatibility condition must been satisfied by initial data, that is

$$-\Delta u_0 + \nabla P_0 = \sqrt{\rho_0} g \quad \text{for} \quad g \in L^2 \quad \text{and} \quad P_0 \in H^1(\Omega). \tag{1.3}$$

Very recently, Li [13] observed that the condition (1.3) can be removed, and local existence and uniqueness are proved in bounded domain with the initial data $0 \le \rho_0 \le \rho^*$ and $\rho_0 \in W^{1,\gamma}$. After, Danchin and Mucha [7] improved their results and established global existence and uniqueness with arbitrary initial density fulfilling $0 \le \rho_0 \le \rho^* < \infty$, $\rho^* > 0$ in bounded domain.

Motivated by the above works, in this paper we are devoted to relaxing the regularity of initial velocity u_0 in the presence of vacuum, more precisely, $u_0 \in H_0^s$ and $s > \frac{1}{2}$. In fact, we are more interested in the case $s < 1$, and for convenience we always default $s \in (\frac{1}{2}, 1)$ in this article.

Notations: For the positive constant $q \in [1,\infty]$, L^q denotes the standard Lebesgue space. And for real number m, H^m and $\dot{H}^m$ denote the nonhomogeneous Sobolev space and homogeneous Sobolev space respectively, with the norms

$$\|u\|_{H^m}^2 := \int (1+|\xi|^2)^m |\hat{u}(\xi)|^2 d\xi, \quad \|u\|_{\dot{H}^m}^2 := \int |\xi|^{2m} |\hat{u}(\xi)|^2 d\xi,$$

where $\hat{u}(\xi)$ is the Fourier transform of u. We also use $H_0^m(\Omega)$ to denote the closures in H^m of the space

$$C_0^\infty(\Omega) := \{\phi \in C^\infty(\Omega) \,|\, \mathrm{supp}\phi \subset\subset \Omega\}.$$

Throughout this paper $X \hookrightarrow Y$ denotes X embedded into Y, and $X \hookrightarrow\hookrightarrow Y$ means X embedded into Y compactly. By the way, $\int f dx$ indicates the integral of f over Ω, and C stands for different positive constants in different places.

The main results of this paper are shown as follows.

Theorem 1.1. *Let Ω be a bounded subset of $\mathbb{R}^3$ with a smooth boundary. Consider any data (ρ_0, u_0) satisfying for some constant $\rho^* > 0$, $\frac{1}{2} < s < 1$ and $0 < \varepsilon_0 << 1$*

$$0 \leq \rho_0 \leq \rho^*, \quad u_0 \in H_0^s, \quad \operatorname{div} u_0 = 0, \quad \|u_0\|_{H^s} \leq \varepsilon_0,$$

where ε_0 depends only on ρ^*, s and $|\Omega|$. Then, system (1.1) *supplemented with data (ρ_0, u_0) has a unique global solution (ρ, u) satisfying $0 \leq \rho(t,x) \leq \rho^*$ and*

$$E_0(t) \leq C\|\sqrt{\rho_0}u_0\|_{L^2}^2, \quad E_1(t) \leq C\|u_0\|_{H^s}^2, \quad E_2(t) \leq C\|u_0\|_{H^s}^4 \tag{1.4}$$

for $t \in [0, \infty)$, where

$$\begin{aligned}
E_0(t) &= \int \rho |u(t,x)|^2 dx + \int_0^t \int |\nabla u|^2 dx d\tau, \\
E_1(t) &= \sigma(t)^{1-s} \int |\nabla u|^2 dx + \int_0^t \int \sigma(\tau)^{1-s} \left(\rho |u_t|^2 + |\nabla^2 u|^2 + |\nabla P|^2 \right) dx d\tau, \\
E_2(t) &= \sigma(t)^{2-s} \int \left(\rho |u_t|^2 + |\nabla^2 u|^2 + |\nabla P|^2 \right) dx + \int_0^t \int \sigma(\tau)^{2-s} |\nabla u_t|^2 dx d\tau
\end{aligned}$$

with $\sigma(t) = \min\{1, t\}$.

Remark 1.1. What we need to explain is that the power of weight $\sigma(t)$ in the energy functionals $E_i(t)$ is reasonable and natural (see [1, 15]), which plays a key role when we try to reduce the initial regularity in [7].

Remark 1.2. The initial vacuum is allowed in this paper, that is the assumption of $0 < c_0 \leq \rho_0$ in [1] is removed. Moreover, the compatibility condition in [2] is not needed.

Now let us explain our main difficulties and ideas. If the initial density is bounded away from zero, we have $0 < c_0 \leq \rho(t,x)$ and further obtain

$$\|u\|_{L^2} \leq \left\| \frac{1}{c_0} \sqrt{\rho} u \right\|_{L^2} \leq \frac{1}{c_0} \|\sqrt{\rho_0} u_0\|_{L^2},$$

while the singularity brought by vacuum makes it ineffective. Without the bound of $\|u\|_{L^2}$, the convective terms will bring us difficulties when we establish H^1

estimate (see [11, Eq. (2.5)] for details). To overcome this difficulty, we make use of continuous method and give out the bound of $\|\rho^{\frac{1}{3}}u\|_{L^3}$ instead. On the other hand, comparing that with [7], the tedious power brought by time weight is also a difficulty.

The rest of this paper is arranged as follows. In Section 2, we perform some uniform priori estimates of solutions and prove global existence of solutions. In Section 3, the uniqueness of solutions is proved. In Section A, some notations and preliminary lemmas are given for the paper self-contained and reader's convenience.

2 Regular estimate and existence of solutions

The purpose of this section is to present the proof of the existence part in Theorem 1.1. Let j_ε be the standard Friedrich's mollifier. We define

$$\rho_0^\varepsilon=\rho_0*j_\varepsilon+\varepsilon,\quad u_0^\varepsilon=u_0*j_\varepsilon,$$

then, according to the classical strong solution theory for (1.1) (see [4]), there exists a unique global smooth solution $(\rho^\varepsilon,u^\varepsilon)$ corresponding to smooth data $(\rho_0^\varepsilon,u_0^\varepsilon)$ which satisfies $\varepsilon\le\rho\le\rho^*$. And the existence part of Theorem 1.1 essentially follows from (1.4) for $(\rho^\varepsilon,u^\varepsilon)$ and a standard compactness argument.

To simplify the notations, we will omit the superscript ε in what follows. Since the standard L^2 energy estimates are well known, we start with H^1 estimates and give the following proposition.

Proposition 2.1. *Assume (ρ,u) is a smooth solution to system* (1.1) *with the data satisfying*

$$\|u_0\|_{H^s}\le\varepsilon_0,$$

where $0<\varepsilon_0<<1$ depends only on $|\Omega|$, s and ρ^*. Then there exists a constant $C>0$ such that for all $0<T<\infty$ and $t\in[0,T]$

$$\sup_{t\in[0,T]}\|\rho^{\frac{1}{3}}u\|_{L^3}^3\le 2\|u_0\|_{H^s}^2$$

and

$$\begin{aligned}&\sup_{t\in[0,T]}\|\sigma(t)^{\frac{1-s}{2}}\nabla u(t)\|_{L^2}+\|\sigma(t)^{\frac{1-s}{2}}\sqrt{\rho}u_t\|_{L^2(0,T;L^2)}+\|\sigma(t)^{\frac{1-s}{2}}\nabla^2u\|_{L^2(0,T;L^2)}\\&\quad+\|\sigma(t)^{\frac{1-s}{2}}\nabla P\|_{L^2(0,T;L^2)}\le C\|u_0\|_{H^s},\end{aligned}\tag{2.1}$$

where $\sigma(t)\triangleq\min\{1,t\}$.

Proof. We prove this proposition in three steps by continuity argument.

Step 1: We assume that there exists a $T>0$ such that

$$\sup_{t\in[0,T]} \|\rho^{\frac{1}{3}}u\|_{L^3}^3 \leq 2\|u_0\|_{H^s}^2, \tag{2.2}$$

because (ρ,u) is smooth and for $\|u_0\|_{H^s}\leq\varepsilon_0$,

$$\left\|\rho_0^{\frac{1}{3}}u_0\right\|_{L^3}^3 \leq \rho^*\|u_0\|_{L^3}^3 \leq \rho^*\|u_0\|_{H^s}^3 \leq \|u_0\|_{H^s}^2.$$

Step 2: We will prove (2.4)-(2.5) under the assumption (2.2). To do this, we take the L^2 inner product of the momentum equation in (1.1) with u_t and obtain

$$\begin{aligned}
&\int\rho|u_t|^2dx+\frac{1}{2}\frac{d}{dt}\int|\nabla u|^2dx \\
=&-\int\rho(u\cdot\nabla u)\cdot u_t dx \leq \frac{1}{2}\int\rho|u_t|^2dx+\frac{1}{2}\int\rho|u\cdot\nabla u|^2dx.
\end{aligned}$$

That is,

$$\left\|\sqrt{\rho(t)}u_t(t)\right\|_{L^2}^2+\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 \leq \int\rho|u\cdot\nabla u|^2dx.$$

In order to estimate the second derivatives of u and the gradient of pressure, we rewrite (1.1b) and (1.1c) as

$$\begin{aligned}
&-\Delta u+\nabla P=-\rho u_t-\rho u\cdot\nabla u,\\
&\operatorname{div} u=0,
\end{aligned}$$

which along with the L^2 estimate on the Stokes system ensures that

$$\begin{aligned}
&\|\nabla^2 u\|_{L^2}^2+\|\nabla P\|_{L^2}^2=\|\rho(u_t+u\cdot\nabla u)\|_{L^2}^2\\
\leq& 2\rho^*\left(\int\rho|u_t|^2dx+\int\rho|u\cdot\nabla u|^2dx\right).
\end{aligned}\tag{2.3}$$

Hence,

$$\begin{aligned}
&\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2+\frac{1}{2}\left\|\sqrt{\rho(t)}u_t(t)\right\|_{L^2}^2+\frac{1}{4\rho^*}\left(\|\nabla^2 u\|_{L^2}^2+\|\nabla P\|_{L^2}^2\right)\\
\leq&\frac{3}{2}\int\rho|u\cdot\nabla u|^2dx\leq\frac{3}{2}(\rho^*)^{\frac{1}{3}}\|\rho^{\frac{1}{3}}u\|_{L^3}^2\|\nabla u\|_{L^6}^2.
\end{aligned}$$

Thanks to the Sobolev embedding $\dot{H}^1 \hookrightarrow L^6$ and (2.2), integrating on $[0,T]$ with respect to time t, we have

$$\begin{aligned}
&\sup_{t\in[0,T]} \|\nabla u(t)\|_{L^2} + \left\|\sqrt{\rho(t)}u_t(t)\right\|_{L^2(0,T;L^2)} + \|\nabla^2 u\|_{L^2(0,T;L^2)} \\
&\qquad + \|\nabla P\|_{L^2(0,T;L^2)} \le C(\rho^*)\|\nabla u_0\|_{L^2}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left\|\sqrt{\sigma(t)}\nabla u(t)\right\|_{L^2}^2 + \frac{1}{2}\left\|\sqrt{\sigma(t)\rho(t)}u_t(t)\right\|_{L^2}^2 \\
&\qquad + \frac{1}{4\rho^*}\left(\left\|\sqrt{\sigma(t)}\nabla^2 u\right\|_{L^2}^2 + \left\|\sqrt{\sigma(t)}\nabla P\right\|_{L^2}^2\right) \\
&\le \frac{3}{2}\int \sigma(t)\rho|u\cdot\nabla u|^2 dx + \|\nabla u\|_{L^2}^2 \\
&\le \frac{3}{2}(\rho^*)^{\frac{1}{3}}\|\rho^{\frac{1}{3}}u\|_{L^3}^2\left\|\sqrt{\sigma(t)}\nabla u\right\|_{L^6}^2 + \|\nabla u\|_{L^2}^2,
\end{aligned}$$

where $\sigma(t) \triangleq \min\{1,t\}$. From which, we infer

$$\begin{aligned}
&\sup_{t\in[0,T]}\left\|\sqrt{\sigma(t)}\nabla u(t)\right\|_{L^2} + \left\|\sqrt{\sigma(t)\rho(t)}u_t(t)\right\|_{L^2(0,T;L^2)} \\
&\qquad + \left\|\sqrt{\sigma(t)}\nabla^2 u\right\|_{L^2(0,T;L^2)} + \left\|\sqrt{\sigma(t)}\nabla P\right\|_{L^2(0,T;L^2)} \\
&\le C(\rho^*)\|\nabla u\|_{L^2(0,T;L^2)} \le C(\rho^*)\|u_0\|_{L^2},
\end{aligned}$$

where we used the following energy inequality (see [2,12] for details):

$$\|\sqrt{\rho}u\|_{L^2}^2 + 2\int_0^t \|\nabla u\|_{L^2}^2 d\tau \le \|\sqrt{\rho_0}u_0\|_{L^2}^2.$$

Be similar to [15], taking advantage of Riesz-Thorin interpolation theorem and Stein interpolation theorem in [10], we obtain

$$\begin{aligned}
&\sup_{t\in[0,T]}\left\|\sigma(t)^{\frac{1-s}{2}}\nabla u(t)\right\|_{L^2} + \left\|\sigma(t)^{\frac{1-s}{2}}\sqrt{\rho(t)}u_t(t)\right\|_{L^2(0,T;L^2)} \\
&\qquad + \left\|\sigma(t)^{\frac{1-s}{2}}\nabla^2 u\right\|_{L^2(0,T;L^2)} + \left\|\sigma(t)^{\frac{1-s}{2}}\nabla P\right\|_{L^2(0,T;L^2)} \le C\|u_0\|_{H^s}.
\end{aligned} \tag{2.4}$$

That is to say, we can get (2.4) when (2.2) holds.

Next, we are going to establish

$$\sup_{t\in[0,T]} \|\rho^{\frac{1}{3}}u\|_{L^3}^3 \le \frac{3}{2}\|u_0\|_{H^s}^2, \tag{2.5}$$

under the assumption (2.2). In fact, since

$$\left\|\rho_0^{\frac{1}{3}}u_0\right\|_{L^3}^3 \le \|u_0\|_{H^s}^2,$$

there exists $0<T_1<\sigma(T)$ such that

$$\sup_{t\in[0,T_1]} \|\rho^{\frac{1}{3}}u\|_{L^3}^3 \le \frac{4}{3}\|u_0\|_{H^s}^2 < \frac{3}{2}\|u_0\|_{H^s}^2.$$

And then we only need to bound

$$\sup_{t\in[T_1,T]} \|\rho^{\frac{1}{3}}u\|_{L^3}^3 \le \frac{3}{2}\|u_0\|_{H^s}^2.$$

Multiplying the momentum equation of (1.1) with $3|u|u$ and then integrating on Ω yield

$$\begin{aligned}\frac{d}{dt}\int \rho|u|^3dx = &\int \rho_t|u|^3dx - 3\int (\rho u\cdot\nabla u)\cdot|u|udx \\ &+3\int \Delta u\cdot|u|udx - 3\int \nabla P\cdot|u|udx.\end{aligned}$$

Using the continuity equation of (1.1) and integration by parts, we obtain

$$\frac{d}{dt}\int \rho|u|^3dx = -3\int \nabla u\cdot\nabla(|u|u)dx - 3\int \nabla P|u|udx.$$

The Hölder inequality and Sobolev embedding $\dot{H}^1 \hookrightarrow L^6$ further imply

$$\begin{aligned}\frac{d}{dt}\int \rho|u|^3dx &\le 3\|u\|_{L^6}\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{L^6}^{\frac{1}{2}} + 3\|\nabla P\|_{L^2}\|u\|_{L^4}^2 \\ &\le 3\|u\|_{L^6}\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{L^6}^{\frac{1}{2}} + 3|\Omega|^{\frac{1}{6}}\|\nabla P\|_{L^2}\|u\|_{L^6}^2 \\ &\le 3\|\nabla u\|_{L^2}^{\frac{5}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}} + 3|\Omega|^{\frac{1}{6}}\|\nabla P\|_{L^2}\|\nabla u\|_{L^2}^2.\end{aligned}$$

Integrating on $[T_1,T]$ with respect to time t, we have

$$\sup_{t\in[T_1,T]} \|\rho^{\frac{1}{3}}u\|_{L^3}^3 \le \left(3+3|\Omega|^{\frac{1}{6}}\right)(\mathcal{I}_1+\mathcal{I}_2) + \frac{4}{3}\|u_0\|_{H^s}^2,$$

where

$$\mathcal{I}_1 \triangleq \int_{\sigma(T)}^{T} \|\nabla u\|_{L^2}^{\frac{5}{2}} \|\nabla^2 u\|_{L^2}^{\frac{1}{2}} + \|\nabla P\|_{L^2} \|\nabla u\|_{L^2}^2 dt,$$

$$\mathcal{I}_2 \triangleq \int_{T_1}^{\sigma(T)} \|\nabla u\|_{L^2}^{\frac{5}{2}} \|\nabla^2 u\|_{L^2}^{\frac{1}{2}} + \|\nabla P\|_{L^2} \|\nabla u\|_{L^2}^2 dt.$$

For $\mathcal{I}_1$, obviously what we care most about is the case that $T>1$, on account of that $\sigma(T)=T$, when $T\le 1$. Take advantage of Hölder inequality and Young's inequality, we get

$$\begin{aligned}
\mathcal{I}_1 &\le \sup_{t\in[1,T]} \|\nabla u\|_{L^2} \int_1^T \|\nabla u\|_{L^2}^{\frac{3}{2}} \|\nabla^2 u\|_{L^2}^{\frac{1}{2}} dt + \sup_{t\in[0,T]} \|\nabla u\|_{L^2} \int_1^T \|\nabla P\|_{L^2} \|\nabla u\|_{L^2} dt \\
&\le \sup_{t\in[1,T]} \|\nabla u\|_{L^2} \left(\int_1^T \frac{5}{4} \|\nabla u\|_{L^2}^2 + \frac{1}{4} \|\nabla^2 u\|_{L^2}^2 + \frac{1}{2} \|\nabla P\|_{L^2}^2 dt \right).
\end{aligned}$$

Notice that $\sigma(t)=1$ in (2.4) when $t\in[1,T]$, therefore

$$\mathcal{I}_1 \le C\|u_0\|_{H^s}^3 \le \frac{1}{12(3+|\Omega|^{\frac{1}{6}})} \|u_0\|_{H^s}^2.$$

When $t\in[T_1,\sigma(T)]$, it is clear that $\sigma(t)=t$, and then we obtain for $\mathcal{I}_2$

$$\begin{aligned}
\mathcal{I}_2 &= \int_{T_1}^{\sigma(T)} t^{(2\delta_0-\frac{3}{2})(1-s)} \left(t^{1-s} \|\nabla u\|_{L^2}^2 \right)^{-2\delta_0+\frac{5}{4}} \left(t^{1-s} \|\nabla^2 u\|_{L^2}^2 \right)^{\frac{1}{4}} \|\nabla u\|_{L^2}^{4\delta_0} dt \\
&\quad + \int_{T_1}^{\sigma(T)} t^{(2\delta_1-\frac{3}{2})(1-s)} \left(t^{1-s} \|\nabla u\|_{L^2}^2 \right)^{1-2\delta_1} \left(t^{1-s} \|\nabla P\|_{L^2}^2 \right)^{\frac{1}{2}} \|\nabla u\|_{L^2}^{4\delta_1} dt \\
&\le C\|u_0\|_{H^s}^{-4\delta_0+3} \left(\int_{T_1}^{\sigma(T)} \|\nabla u\|_{L^2}^2 dt \right)^{2\delta_0} \left(\int_{T_1}^{\sigma(T)} t^{-\frac{2(3-4\delta_0)(1-s)}{3-8\delta_0}} dt \right)^{\frac{3-8\delta_0}{4}} \\
&\quad + C\|u_0\|_{H^s}^{-4\delta_0+3} \left(\int_{T_1}^{\sigma(T)} \|\nabla u\|_{L^2}^2 dt \right)^{2\delta_0} \left(\int_{T_1}^{\sigma(T)} t^{-\frac{(3-4\delta_0)(1-s)}{1-4\delta_0}} dt \right)^{\frac{1-4\delta_0}{2}} \\
&\le C\|u_0\|_{H^s}^3 \le \frac{1}{12(3+|\Omega|^{\frac{1}{6}})} \|u_0\|_{H^s}^2
\end{aligned}$$

with $\delta_0 = \frac{2s-1}{4s}$. Thus, (2.5) is given when (2.2) holds.

Step 3: We will finish our argument based on the continuity method. Set

$$T^* = \sup\{T \,|\, (2.2) \text{ holds}\}, \tag{2.6}$$

we claim $T^*=\infty$. Otherwise, $T^*<\infty$ the continuity on time and (2.5) will admit us to find a $T^{**}>T^*$ such that (2.2) holds for $T=T^{**}$, which contradicts (2.6). The proof of Proposition 2.1 is finished. $\square$

Next, we turn to estimate time derivatives. More precisely, we shall bound $\sigma(t)^{\frac{2-s}{2}}\nabla u_t$ in $L^2(0,T;L^2)$.

Proposition 2.2. *Assume* (ρ,u) *is a smooth solution to system* (1.1). *Then there exists a constant* $C>0$ *such that for all* $0<T<\infty$ *and* $t\in[0,T]$

$$\sup_{t\in[0,T]}\left(\sigma(t)^{2-s}\|\sqrt{\rho}u_t\|_{L^2}^2+\sigma(t)^{2-s}\|\nabla^2 u\|_{L^2}^2+\sigma(t)^{2-s}\|\nabla P\|_{L^2}^2\right)$$
$$+\left\|\sigma(t)^{\frac{2-s}{2}}\nabla u_t\right\|_{L^2(0,T;L^2)}^2\le C\|u_0\|_{\dot{H}^s}^4.$$

Proof. Derivative the momentum equation of (1.1) with respect to t

$$\rho u_{tt}+\rho_t u_t+\rho_t u\cdot\nabla u+\rho u_t\cdot\nabla u+\rho u\cdot\nabla u_t-\Delta u_t+\nabla P_t=0,$$

and take the L^2 scalar product with $\sigma(t)^{2-s}u_t$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\rho\sigma(t)^{2-s}|u_t|^2dx+\int\sigma(t)^{2-s}|\nabla u_t|^2dx\\&=(2-s)\int\rho\sigma(t)^{1-s}|u_t|^2dx-\frac{1}{2}\int\sigma(t)^{2-s}\rho_t|u_t|^2dx\\&\quad-\int\sigma(t)^{2-s}(\rho_t u\cdot\nabla u)\cdot u_t dx-\int\sigma(t)^{2-s}(\rho u_t\cdot\nabla u)\cdot u_t dx\\&\quad-\int\sigma(t)^{2-s}(\rho u\cdot\nabla u_t)\cdot u_t dx\triangleq\sum_{i=1}^5 I_i.\end{aligned}\tag{2.7}$$

Using the continuity equation and then performing an integration by parts, we obtain

$$\begin{aligned}I_2&=-\frac{1}{2}\int\sigma(t)^{2-s}\rho_t|u_t|^2dx=\frac{1}{2}\int\sigma(t)^{2-s}\mathrm{div}(\rho u)|u_t|^2dx\\&=-\int\sigma(t)^{2-s}\rho u\cdot(u_t\cdot\nabla u_t)dx.\end{aligned}$$

Thanks to Hölder inequality, Sobolev embedding $\dot{H}^1\hookrightarrow L^6$ and Proposition 2.1, we get

$$\begin{aligned}|I_2|&\le\int\sigma(t)^{2-s}\rho|u||\nabla u_t||u_t|dx\\&\le(\rho^*)^{\frac{2}{3}}\sigma(t)^{2-s}\left\|\rho^{\frac{1}{3}}u\right\|_{L^3}\|u_t\|_{L^6}\|\nabla u_t\|_{L^2}\\&\le C(\rho^*)\varepsilon_0^{\frac{2}{3}}\sigma(t)^{2-s}\|\nabla u_t\|_{L^2}^2.\end{aligned}$$

Similarly, the term I_3 can be written as

$$\begin{aligned}I_3 &= -\int \sigma(t)^{2-s}(\rho_t u\cdot\nabla u)\cdot u_t dx\\ &= -\int \sigma(t)^{2-s}\rho u\cdot\nabla\left((u\cdot\nabla u)\cdot u_t\right)dx,\end{aligned}$$

that is,

$$|I_3| \le \int \sigma(t)^{2-s}\rho|u|\Big(|\nabla u|^2|u_t| + |u||\nabla^2 u||u_t| + |u||\nabla u||\nabla u_t|\Big)dx \triangleq \sum_{i=1}^{3} I_{3i}.$$

We bound I_{31} firstly by making use of Hölder inequality, Young's inequality and Proposition 2.1, and obtain for all $s>\frac{1}{2}$

$$\begin{aligned}I_{31} &= \sigma(t)^{2-s}\int \rho|u||\nabla u|^2|u_t|dx\\ &\le \rho^*\sigma(t)^{2-s}\|u_t\|_{L^6}\|u\|_{L^6}\|\nabla u\|_{L^6}\|\nabla u\|_{L^2}\\ &\le C(\rho^*,\epsilon)\sigma(t)^{2-s}\|\nabla^2 u\|_{L^2}^2\|\nabla u\|_{L^2}^4 + \epsilon\sigma(t)^{2-s}\|\nabla u_t\|_{L^2}^2\\ &\le C(\rho^*,\epsilon)\Big(\sup_{t\in[0,T]}\sigma(t)^{1-s}\|\nabla u\|_{L^2}^2\Big)^2\sigma(t)^{1-s}\|\nabla^2 u\|_{L^2}^2 + \epsilon\sigma(t)^{2-s}\|\nabla u_t\|_{L^2}^2\\ &\le C(\rho^*,\epsilon)\|u_0\|_{H^s}^4\sigma(t)^{1-s}\|\nabla^2 u\|_{L^2}^2 + \epsilon\sigma(t)^{2-s}\|\nabla u_t\|_{L^2}^2,\end{aligned}$$

where ϵ is a small positive constant. Next, we bound I_{32} as

$$\begin{aligned}I_{32} &= \sigma(t)^{2-s}\int \rho|u|^2|\nabla^2 u||u_t|dx\\ &\le \rho^*\sigma(t)^{2-s}\|\nabla^2 u\|_{L^2}\|u\|_{L^6}^2\|u_t\|_{L^6}\\ &\le C(\rho^*,\epsilon)\sigma(t)^{2-s}\|\nabla u\|_{L^2}^4\|\nabla^2 u\|_{L^2}^2 + \epsilon\sigma(t)^{2-s}\|\nabla u_t\|_{L^2}^2\\ &\le C(\rho^*,\epsilon)\|u_0\|_{H^s}^4\sigma(t)^{1-s}\|\nabla^2 u\|_{L^2}^2 + \epsilon\sigma(t)^{2-s}\|\nabla u_t\|_{L^2}^2.\end{aligned}$$

Finally, we deal with I_{33}

$$\begin{aligned}I_{33} &= \sigma(t)^{2-s}\int \rho|u|^2|\nabla u||\nabla u_t|dx\\ &\le \rho^*\sigma(t)^{2-s}\|u\|_{L^6}^2\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2}\\ &\le C(\rho^*,\epsilon)\sigma(t)^{2-s}\|\nabla u\|_{L^2}^4\|\nabla^2 u\|_{L^2}^2 + \epsilon\sigma(t)^{2-s}\|\nabla u_t\|_{L^2}^2\\ &\le C(\rho^*,\epsilon)\|u_0\|_{H^s}^4\sigma(t)^{1-s}\|\nabla^2 u\|_{L^2}^2 + \epsilon\sigma(t)^{2-s}\|\nabla u_t\|_{L^2}^2.\end{aligned}$$

Combining the above estimates of I_{31}-I_{33} gives

$$|I_3|\le\left(C\varepsilon_0^{\frac{2}{3}}+2\epsilon\right)\sigma(t)^{2-s}\|\nabla u_t\|_{L^2}^2+C(\rho^*,\epsilon)\|u_0\|_{H^s}^4\sigma(t)^{1-s}\|\nabla^2 u\|_{L^2}^2.$$

Next, we turn to treat I_4 as

$$\begin{aligned}
|I_4|\le&\int\sigma(t)^{2-s}\rho|u_t|^2|\nabla u|dx\\
\le&\sqrt{\rho^*}\sigma(t)^{2-s}\|\nabla u\|_{L^3}\|\sqrt{\rho}u_t\|_{L^2}\|u_t\|_{L^6}\\
\le&\sqrt{\rho^*}\sigma(t)^{2-s}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2}\|u_t\|_{L^6}\\
\le&C(\rho^*,\epsilon)\sigma(t)^{2-s}\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}^2+\epsilon\sigma(t)^{2-s}\|u_t\|_{L^6}^2\\
\le&C(\rho^*,\epsilon)\left(\sup_{t\in[0,T]}\sqrt{\sigma(t)^{1-s}}\|\nabla u\|_{L^2}\right)\left(\sqrt{\sigma(t)^{2-s}}\|\nabla^2 u\|_{L^2}\right)\sigma(t)^{1-s}\|\sqrt{\rho}u_t\|_{L^2}^2\\
&+\epsilon\sigma(t)^{2-s}\|u_t\|_{L^6}^2\\
\le&C(\rho^*,\epsilon)\left(\|u_0\|_{H^s}^2+\sigma(t)^{2-s}\|\nabla^2 u\|_{L^2}^2\right)\sigma(t)^{1-s}\|\sqrt{\rho}u_t\|_{L^2}^2+\epsilon\sigma(t)^{2-s}\|\nabla u_t\|_{L^2}^2.
\end{aligned}$$

Finally, with the similar calculation to I_2 we bound I_5 as

$$\begin{aligned}
|I_5|=&\left|-\int\sigma(t)^{2-s}(\rho u\cdot\nabla u_t)\cdot u_t dx\right|\\
\le&\int\sigma(t)^{2-s}\rho|u||\nabla u_t||u_t|dx\\
\le&C(\rho^*)\sigma(t)^{2-s}\left\|\rho^{\frac{1}{3}}u\right\|_{L^3}\|u_t\|_{L^6}\|\nabla u_t\|_{L^2}\\
\le&C(\rho^*)\varepsilon^{\frac{2}{3}}\sigma(t)^{2-s}\|\nabla u_t\|_{L^2}^2.
\end{aligned}$$

Substituting the above estimates of I_1-I_5 into (2.7), and taking ϵ suitably small, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\int\rho\sigma(t)^{2-s}|u_t|^2dx\right)+\int\sigma(t)^{2-s}|\nabla u_t|^2dx\\
\le&C(\rho^*)\|u_0\|_{H^s}^2\sigma(t)^{1-s}\|\sqrt{\rho}u_t\|_{L^2}^2+C(\rho^*)\|u_0\|_{H^s}^4\sigma(t)^{1-s}\|\nabla^2 u\|_{L^2}^2\\
&+C(\rho^*)\left(\sigma(t)^{2-s}\|\nabla^2 u\|_{L^2}^2\right)\left(\sigma(t)^{1-s}\|\sqrt{\rho}u_t\|_{L^2}^2\right).
\end{aligned}$$

Integrating with respect to t on $[0,T]$, we have by using Proposition 2.1

$$\begin{aligned}
&\sup_{t\in[0,T]}\sigma(t)^{2-s}\|\sqrt{\rho}u_t\|_{L^2}^2+\left\|\sigma(t)^{\frac{2-s}{2}}\nabla u_t\right\|_{L^2(0,T;L^2)}^2\\
\le&C(\rho^*)\|u_0\|_{H^s}^4+C(\rho^*,\epsilon)\|u_0\|_{H^s}^2\left(\sup_{t\in[0,T]}\sigma(t)^{2-s}\|\nabla^2 u\|_{L^2}^2\right).
\end{aligned}\tag{2.8}$$

From (1.1a) and (1.1b), we gather that

$$\begin{cases} -\sigma(t)^{\frac{2-s}{2}}\Delta u+\sigma(t)^{\frac{2-s}{2}}\nabla P=-\sigma(t)^{\frac{2-s}{2}}\left(\sqrt{\rho}u_t-\rho u\cdot\nabla u\right) & \text{in } (0,T)\times\Omega, \\ \sigma(t)^{\frac{2-s}{2}}\operatorname{div}u=0 & \text{in } (0,T)\times\Omega. \end{cases} \tag{2.9}$$

Be similar to (2.3), we have

$$\begin{aligned} &\sigma(t)^{2-s}\|\nabla^2 u\|_{L^2}^2+\sigma(t)^{2-s}\|\nabla P\|_{L^2}^2 \\ =&\sigma(t)^{2-s}\|\rho(u_t+u\cdot\nabla u)\|_{L^2}^2 \\ \le&2\rho^*\sigma(t)^{2-s}\left(\int\rho|u_t|^2dx+\varepsilon_0\|\nabla^2 u\|_{L^2}^2\right), \end{aligned}$$

that is,

$$\begin{aligned} &\frac{1}{4\rho^*}\sup_{t\in[0,T]}\left(\sigma(t)^{2-s}\|\nabla^2 u\|_{L^2}^2+\sigma(t)^{2-s}\|\nabla P\|_{L^2}^2\right) \\ \le&\frac{1}{2}\sup_{t\in[0,T]}\sigma(t)^{2-s}\|\sqrt{\rho}u_t\|_{L^2}^2. \end{aligned} \tag{2.10}$$

Adding (2.8) and (2.10) together, we can obtain from the smallness of energy E_1

$$\begin{aligned} &\sigma(t)^{2-s}\|\sqrt{\rho}u_t\|_{L^2}^2+\sigma(t)^{2-s}\|\nabla^2 u\|_{L^2}^2+\sigma(t)^{2-s}\|\nabla P\|_{L^2}^2 \\ &+\left\|\sigma(t)^{\frac{2-s}{2}}\nabla u_t\right\|_{L^2(0,T;L^2)}^2\le C(\rho^*)\|u_0\|_{H^s}^4. \end{aligned}$$

The Proposition 2.2 is proved. □

This gives the uniform energy estimates, thus the proof of existence. Thanks to the Aubin-Lions compactness lemma (Lemma A.1) and Lemma A.2, we can infer $u\in C(0,T;L^2)$, from which we can further obtain the time continuity of momentum through a very similar argument in [7], more specifically, $\sqrt{\rho^\epsilon}u^\epsilon\to\sqrt{\rho}u$ in $C([0,+\infty);L^2)$.

Before proving the uniqueness part of Theorem 1.1, we need to present more information on the regularity of solutions obtained above. By virtue of the shift of integrability method, we can bound ∇u in $L^1(0,T;L^\infty)$, and give the following proposition.

Proposition 2.3. *Assume* (ρ,u) *is a smooth solution to system* (1.1). *Then there exists a constant* $C>0$ *such that for all* $T>0$, $p\in[2,\infty]$

$$\left\|\sigma(t)^{\frac{1-s}{2}}\nabla u\right\|_{L^p(0,T;L^r)}+\left\|\sigma(t)^{\frac{2-s}{2}}\nabla^2 u\right\|_{L^p(0,T;L^r)}+\left\|\sigma(t)^{\frac{2-s}{2}}\nabla P\right\|_{L^p(0,T;L^r)}\le C$$

and

$$\int_0^T \|\nabla u\|_{L^\infty}^k dt \le CT^\ell, \tag{2.11}$$

where $\ell > 0$, $2 \le r \le \frac{6p}{3p-4}$, $1 \le k < \frac{2}{2-s}$.

Proof. Proposition 2.2 shows that $\sigma(t)^{\frac{2-s}{2}}\sqrt{\rho}u_t$ and $\sigma(t)^{\frac{2-s}{2}}u_t$ is in $L^\infty(0,T;L^2)$ and $L^2(0,T;L^q)$ respectively for all $q \le 6$, which implies $\sigma(t)^{\frac{2-s}{2}}\sqrt{\rho}u_t \in L^\infty(0,T;L^2) \cap L^2(0,T;L^q)$ $(q \le 6)$, since ρ is bounded. Thanks to the interpolation inequality for L^p-norms, Hölder inequality and Proposition 2.2, we are led to for all $p \in [2,\infty]$, $2 \le r \le \frac{6p}{3p-4}$

$$\begin{aligned}
&\left\|\sigma(t)^{\frac{2-s}{2}}\rho u_t\right\|_{L^p(0,T;L^r)} \\
&\le C(\rho^*)\left(\int_0^T \left\|\sigma(t)^{\frac{2-s}{2}}u_t\right\|^2_{L^{\frac{4r}{2p+(2-p)r}}} \left\|\sigma(t)^{\frac{2-s}{2}}\sqrt{\rho}u_t\right\|_{L^2}^{(1-\frac{2}{p})p} dt\right)^{\frac{1}{p}} \\
&\le C(\rho^*)\left\|\sigma(t)^{\frac{2-s}{2}}u_t\right\|^{\frac{2}{p}}_{L^2(0,T;L^{\frac{4r}{2p+(2-p)r}})} \left\|\sigma(t)^{\frac{2-s}{2}}\sqrt{\rho}u_t\right\|^{(1-\frac{2}{p})}_{L^\infty(0,T;L^2)} \\
&\le C(\rho^*)\|u_0\|_{H^s}.
\end{aligned} \tag{2.12}$$

Similarly, the bound for $\sigma(t)^{\frac{1-s}{2}}\nabla u$ in $L^\infty(0,T;L^2)$ and $\sigma(t)^{\frac{1-s}{2}}\nabla u$ in $L^2(0,T;H^1)$ imply that for all $p \in [2,\infty]$, $2 \le r \le \frac{6p}{3p-4}$

$$\begin{aligned}
&\left\|\sigma(t)^{\frac{1-s}{2}}\nabla u\right\|_{L^p(0,T;L^r)} \\
&\le \left(\int_0^T \left\|\sigma(t)^{\frac{1-s}{2}}\nabla u\right\|^{p(1-\frac{2}{p})}_{L^2} \left\|\sigma(t)^{\frac{1-s}{2}}\nabla u\right\|^2_{L^{\frac{4r}{2p+(2-p)r}}} dt\right)^{\frac{1}{p}} \\
&\le \left(\int_0^T \left\|\sigma(t)^{\frac{1-s}{2}}\nabla u\right\|^{p(1-\frac{2}{p})}_{L^2} \left\|\sigma(t)^{\frac{1-s}{2}}\nabla u\right\|^2_{H^1} dt\right)^{\frac{1}{p}} \\
&\le \left\|\sigma(t)^{\frac{1-s}{2}}\nabla u\right\|^{\frac{2}{p}}_{L^2(0,T;H^1)} \left\|\sigma(t)^{\frac{1-s}{2}}\nabla u\right\|^{(1-\frac{2}{p})}_{L^\infty(0,T;L^2)} \\
&\le C(\rho^*)\|u_0\|_{H^s},
\end{aligned} \tag{2.13}$$

by virtue of interpolation inequality for L^p-norms and embedding $H^1 \hookrightarrow L^m$ with $1 \le m \le 6$.

When $p \in [2,\infty]$ (that is, $\frac{3}{2} \le \frac{3p}{2p-2} \le 3$), we can obtain by the classical regularity results on the Stokes equations and Hölder inequality

$$\left\|\sigma(t)^{\frac{2-s}{2}}\nabla^2 u\right\|_{L^p(0,T;L^{\frac{3p}{2p-2}})} + \left\|\sigma(t)^{\frac{2-s}{2}}\nabla P\right\|_{L^p(0,T;L^{\frac{3p}{2p-2}})}$$

$$
\begin{aligned}
&\leq \left\|\sigma(t)^{\frac{2-s}{2}}\rho u\cdot\nabla u\right\|_{L^p(0,T;L^{\frac{3p}{2p-2}})}+\left\|\sigma(t)^{\frac{2-s}{2}}\rho u_t\right\|_{L^p(0,T;L^{\frac{3p}{2p-2}})}\\
&\leq \left\|\sqrt{\sigma(t)}\rho u\right\|_{L^\infty(0,T;L^6)}\left\|\sigma(t)^{\frac{1-s}{2}}\nabla u\right\|_{L^p(0,T;L^{\frac{6p}{3p-4}})}+\left\|\sigma(t)^{\frac{2-s}{2}}\rho u_t\right\|_{L^p(0,T;L^{\frac{3p}{2p-2}})}\\
&\leq C(\rho^*)\left\|\sigma(t)^{\frac{1-s}{2}}\nabla u\right\|_{L^\infty(0,T;L^2)}\left\|\sigma(t)^{\frac{1-s}{2}}\nabla u\right\|_{L^p(0,T;L^{\frac{6p}{3p-4}})}+\left\|\sigma(t)^{\frac{2-s}{2}}\rho u_t\right\|_{L^p(0,T;L^{\frac{3p}{2p-2}})}.
\end{aligned}
$$

From (2.1), (2.12) and (2.13) we can get

$$
\left\|\sigma(t)^{\frac{2-s}{2}}\nabla^2 u\right\|_{L^p(0,T;L^{\frac{3p}{2p-2}})}+\left\|\sigma(t)^{\frac{2-s}{2}}\nabla P\right\|_{L^p(0,T;L^{\frac{3p}{2p-2}})}\leq C(\rho^*)\|u_0\|_{H^s}.
$$

Combining the above inequality with (2.13), we have $\sigma(t)^{\frac{2-s}{2}}\nabla u\in L^p(0,T;W^{1,\frac{3p}{2p-2}})$ for all $p\in[0,+\infty)$. Taking advantage of $W^{1,\frac{3p}{2p-2}}\hookrightarrow L^{\frac{3p}{p-2}}$, we obtain

$$
\begin{aligned}
&\left\|\sigma(t)^{\frac{2-s}{2}}\rho u\cdot\nabla u\right\|_{L^p(0,T;L^{\frac{6p}{3p-4}})}\\
\leq& C\left\|\sigma(t)^{\frac{1}{2}}\rho u\right\|_{L^\infty(0,T;L^6)}\left\|\sigma(t)^{\frac{1-s}{2}}\nabla u\right\|_{L^p(0,T;L^{\frac{3p}{p-2}})}\\
\leq& C(\rho^*)\left\|\sigma(t)^{\frac{1-s}{2}}\nabla u\right\|_{L^\infty(0,T;L^2)}\left\|\sigma(t)^{\frac{1-s}{2}}\nabla u\right\|_{L^p(0,T;W^{1,\frac{3p}{2p-2}})}.
\end{aligned}
$$

Hence, it holds

$$
\begin{aligned}
&\left\|\sigma(t)^{\frac{2-s}{2}}\nabla^2 u\right\|_{L^p(0,T;L^{\frac{6p}{3p-4}})}+\left\|\sigma(t)^{\frac{2-s}{2}}\nabla P\right\|_{L^p(0,T;L^{\frac{6p}{3p-4}})}\\
\leq& C(\rho^*)\left\|\sigma(t)^{\frac{1-s}{2}}\nabla u\right\|_{L^\infty(0,T;L^6)}\left\|\sigma(t)^{\frac{1-s}{2}}\nabla u\right\|_{L^p(0,T;W^{1,\frac{3p}{2p-2}})}+\left\|\sigma(t)^{\frac{2-s}{2}}\rho u_t\right\|_{L^p(0,T;L^{\frac{6p}{3p-4}})}\\
\leq& C(\rho^*)\|u_0\|_{H^s}.
\end{aligned}
$$

Finally, we bound ∇u in $L^1(0,T;L^\infty)$. When $2\leq p<4$, it is obviously $W^{1,\frac{6p}{3p-4}}\hookrightarrow L^\infty$ due to $r_1\triangleq\frac{6p}{3p-4}>3$, thus

$$
\begin{aligned}
&\|\nabla u\|_{L^k(0,T;L^\infty)}\\
\leq& C\left\|\sigma(t)^{\frac{2-s}{2}}\nabla u\right\|_{L^p(0,T;W^{1,r_1})}\left(\int_0^T\sigma(t)^{\frac{(s-2)pk}{2(p-k)}}dt\right)^{\frac{1}{k}-\frac{1}{p}}\\
\leq& C\left\|\sigma(t)^{\frac{2-s}{2}}\nabla u\right\|_{L^p(0,T;W^{1,r_1})}\left(\int_0^{\sigma(T)}\sigma(t)^{\frac{(s-2)pk}{2(p-k)}}dt+\int_{\sigma(T)}^T\sigma(t)^{\frac{(s-2)pk}{2(p-k)}}dt\right)^{\frac{1}{k}-\frac{1}{p}}\\
\leq& C\left(\sigma(T)^{1+\frac{(s-2)pk}{2(p-k)}}+(T-\sigma(T))^{\frac{1}{k}-\frac{1}{p}}\right),
\end{aligned}
$$

since there exists a $p\in[2,4)$ such that $(2-s)pk<2(p-k)$ for arbitrary $1\leq k<\frac{2}{2-s}$. The proof of Proposition 2.3 is complete. $\square$

3 Uniqueness of solutions

In this section, we will prove the uniqueness of solutions in Lagrangian coordinates, owing to the lack of regularity for density.

3.1 Lagrangian formulation

We firstly transform (1.1) into the formulation in the Lagrangian coordinates. Thanks to Proposition 2.3, we can give the definition of the trajectory $X(t,y)$ of $u(t,x)$

$$\frac{dX}{dt}=u(t,X), \quad X|_{t=0}=y, \quad y\in\Omega,$$

which leads to the following relation between the Eulerian coordinates x and the Lagrangian coordinates y:

$$X(t,y)=y+\int_0^t u\big(\tau,X(\tau,y)\big)d\tau.$$

Moreover, we deduce from (2.11) that T can be taken small enough such that

$$\int_0^T \|\nabla v\|_{L^\infty}dt\leq\frac{1}{2}.$$

Of course, if that condition is fulfilled, then one may write that

$$A=\big(Id+(\nabla_y X-Id)\big)^{-1}=\sum_{k=0}^{+\infty}(-1)^k\left(\int_0^t\nabla_y v(\tau,\cdot)d\tau\right)^k. \tag{3.1}$$

For any $t\leq T$, $X(t,y)$ is invertible with respect to y variable, and we denote by $X(t,\cdot)^{-1}$ its inverse mapping. Let

$$v(t,y)\triangleq u\big(t,X(t,y)\big), \quad A(t)\triangleq\big(\nabla_y X(t,\cdot)\big)^{-1}.$$

The operators ∇, div, and Δ translate into

$$\nabla_v\triangleq A^t\cdot\nabla_y, \quad \mathrm{div}_v\triangleq A^t:\nabla_y=\mathrm{div}_y(A\,\cdot\,), \quad \Delta_v\triangleq\mathrm{div}_v\nabla_v=\mathrm{div}_y\big(A\cdot A^t\nabla_y\,\cdot\big),$$

in which A^t denotes the transpose matrix of A.

In Lagrangian coordinates the solution (ρ,u,P) of (1.1) is recast as

$$\eta(t,y)\triangleq\rho\big(t,X(t,y)\big),\quad v(t,y)\triangleq u\big(t,X(t,y)\big),\quad Q(t,y)\triangleq P\big(t,X(t,y)\big),$$

therefore the triplet (η,v,Q) satisfies

$$\begin{cases} \eta_t=0 & \text{in } (0,T)\times\Omega, \\ \eta v_t-\Delta_v v+\nabla_v Q=0 & \text{in } (0,T)\times\Omega, \\ \mathrm{div}_v v=0 & \text{in } (0,T)\times\Omega. \end{cases}$$

Now we tackle the proof of uniqueness. Assuming (ρ^1,u^1,P^1) and (ρ^2,u^2,P^2) are two solutions of (1.1), fulfilling the properties of Theorem 1.1 and emanating from the same initial data. Notations (η^1,v^1,Q^1) and (η^2,v^2,Q^2) are used to represent the corresponding triplets in Lagrangian coordinates. Denoting $\delta v=v^2-v^1$ and $\delta Q=Q^2-Q^1$, we have

$$\begin{cases} \rho_0\delta v_t-\Delta_{v^1}\delta v+\nabla_{v^1}\delta Q=(\Delta_{v^2}-\Delta_{v^1})v^2-(\nabla_{v^2}-\nabla_{v^1})Q^2, \\ \mathrm{div}_{v^1}\delta v=\big(\mathrm{div}_{v^1}-\mathrm{div}_{v^2}\big)v^2, \\ \delta v|_{t=0}=0. \end{cases} \tag{3.2}$$

Be very similar to that in [7] (see also [1,15]), we can transform the regularity information of the solution in the Eulerian coordinates into those in the Lagrangian coordinates. Here we omit the details, but shall repeatedly use the fact that for $i=1,2$

$$\begin{aligned} &\nabla v^i\in L^1(0,T;L^\infty)\cap L^2(0,T;L^2), \\ &\sigma(t)^{\frac{1-s}{2}}\nabla v^i L^\infty(0,T;L^2), && \sigma(t)^{\frac{1-s}{2}}\nabla^2 v^i\in L^2(0,T;L^2), \\ &\sigma(t)^{\frac{2-s}{2}}\nabla v^i_t\in L^2(0,T;L^2), && \sigma(t)^{\frac{1-s}{2}}\sqrt{\rho_0}v^i_t\in L^2(0,T;L^2), \\ &\sigma(t)^{\frac{1-s}{2}}\nabla Q^i\in L^2(0,T;L^2), && \sigma(t)^{\frac{2-s}{2}}\nabla Q^i\in L^2(0,T;L^6). \end{aligned}$$

3.2 Uniqueness of solutions

We claim that for sufficiently small $T>0$

$$\sup_{t\in[0,T]}\int(\rho_0|\delta v|^2)dy+\int_0^T\int_\Omega|\nabla\delta v|^2dydt=0.$$

To prove our claim, decompose δv into

$$\delta v=(v^2-2v^1)+v^1\triangleq w+z,$$

such that z satisfies $\operatorname{div}_{v^1} z=0$ and w is a solution given by Lemma A.3 to the following problem:

$$\operatorname{div}_{v^1} w=(\operatorname{div}_{v^1}-\operatorname{div}_{v^2})v^2=\operatorname{div}(-\delta A v^2)=(-\delta A)^t:\nabla v^2 \tag{3.3}$$

with $\delta A=A^2-A^1$ and $A^i\triangleq A(u^i)$.

As a first step in proving our claim, let us establish the following lemma.

Lemma 3.1. *The solution w to* (3.3) *which is given by Lemma A.*3 *satisfies*

$$\begin{aligned}
&\|w\|_{L^{\frac{8}{2s-1}}(0,T;L^2)}+\|\nabla w\|_{L^2(0,T;L^2)}+\left\|\sigma(t)^{\frac{1-s}{2}}w_t\right\|_{L^{\frac{8}{3+2s}}(0,T;L^{\frac{3}{2}})}\\
&\le c(T)\|\nabla\delta v\|_{L^2(0,T;L^2)}
\end{aligned}\tag{3.4}$$

with $c(T)$ going to 0 *when T tends to* 0.

Proof. Lemma A.3 and identity (3.1) ensure that there exist two universal positive constants c and C such that if

$$\|\nabla v^1\|_{L^1(0,T;L^\infty)}+\left\|\sigma(t)^{\frac{1-s}{2}}\nabla v^1\right\|_{L^2(0,T;L^6)}\le c,$$

then the following inequalities hold true:

$$\begin{aligned}
&\|w\|_{L^{\frac{8}{2s-1}}(0,T;L^2)}\le C\|\delta A v^2\|_{L^{\frac{8}{2s-1}}(0,T;L^2)},\\
&\|\nabla w\|_{L^2(0,T;L^2)}\le C\left\|(\delta A)^t:\nabla v^2\right\|_{L^2(0,T;L^2)}
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\sigma(t)^{\frac{1-s}{2}}w_t\right\|_{L^{\frac{8}{3+2s}}(0,T;L^{\frac{3}{2}})}\\
&\le C\left\|\delta A v^2\right\|_{L^{\frac{8}{2s-1}}(0,T;L^2)}+C\left\|\sigma(t)^{\frac{1-s}{2}}(\delta A v^2)_t\right\|_{L^{\frac{8}{3+2s}}(0,T;L^{\frac{3}{2}})}.
\end{aligned}$$

In all that follows, $c(T)$ designates a nonnegative, continuous, increasing function of T with $c(0)=0$. Now, we are going to bound $(\delta A)^t:\nabla v^2$, $\delta A v^2$ and $(\delta A v^2)_t$. Stemming from Hölder inequality and the following identity:

$$\delta A(t)=\left(\int_0^t\nabla\delta v d\tau\right)\cdot\left(\sum_{k\ge 1}\sum_{0\le j<k}C_1^jC_2^{k-1-j}\right)\quad\text{with}\quad C_i(t)\triangleq\int_0^t\nabla v^i d\tau, \tag{3.5}$$

it is not difficult to obtain

$$\sup_{t\in[0,T]}\left\|\sigma(t)^{-\frac{1}{2}}\delta A\right\|_{L^2}\le c(T)\sup_{t\in[0,T]}\left\|\sigma(t)^{-\frac{1}{2}}\int_0^t\nabla\delta v d\tau\right\|_{L^2}$$

$$\le c(T)\left(\sup_{t\in[0,T]}\sigma(t)^{-\frac{1}{2}}\|\nabla\delta v\|_{L^2(0,t;L^2)}\right)\left(\sup_{t\in[0,T]}\left(\int_0^t 1d\tau\right)^{\frac{1}{2}}\right)$$
$$\le c(T)\|\nabla\delta v\|_{L^2(0,T;L^2)}.$$

Notice that $\nabla v^i\in L^1(0,T;L^\infty)$, which implies $\sigma(t)\nabla v^i\in L^\infty(0,T;L^\infty)$ and then we have

$$\begin{aligned}
&\|(\delta A)^t:\nabla v^2\|_{L^2(0,T;L^2)}\\
&\le\|\sigma(t)^{-\frac{1}{2}}\delta A(t)\|_{L^\infty(0,T;L^2)}\|\sigma(t)^{\frac{1}{2}}\nabla v^2\|_{L^2(0,T;L^\infty)}\\
&\le C\|\sigma(t)^{-\frac{1}{2}}\delta A(t)\|_{L^\infty(0,T;L^2)}\|\sigma(t)\nabla v^2\|^{\frac{1}{2}}_{L^\infty(0,T;L^\infty)}\|\nabla v^2\|^{\frac{1}{2}}_{L^1(0,T;L^\infty)}\\
&\le c(T)\|\nabla\delta v\|_{L^2(0,T;L^2)}.
\end{aligned}$$

Similarly, for δAv^2 we get

$$\begin{aligned}
&\|\delta Av^2\|_{L^{\frac{8}{2s-1}}(0,T;L^2)}\\
&\le\|\sigma(t)^{-\frac{1}{2}}\delta A\|_{L^\infty(0,T;L^2)}\|\sigma(t)^{\frac{1}{2}}v^2\|_{L^{\frac{8}{2s-1}}(0,T;L^\infty)}\\
&\le C\|\sigma(t)^{-\frac{1}{2}}\delta A\|_{L^\infty(0,T;L^2)}\left(\|\sigma(t)^{\frac{1-s}{2}}\nabla v^2\|_{L^{\frac{8}{2s-1}}(0,T;L^2)}+\|\sigma(t)^{\frac{1+s}{2}}\nabla^2 v^2\|_{L^{\frac{8}{2s-1}}(0,T;L^2)}\right)\\
&\le C\|\sigma(t)^{-\frac{1}{2}}\delta A\|_{L^\infty(0,T;L^2)}\left(\|\sigma(t)^{\frac{1-s}{2}}\nabla v^2\|_{L^\infty(0,T;L^2)}+\|\sigma(t)^{\frac{2-s}{2}}\nabla^2 v^2\|_{L^\infty(0,T;L^2)}\right)\\
&\le c(T)\|\nabla\delta v\|_{L^2(0,T;L^2)}.
\end{aligned}$$

Next, we turn to bound $(\delta Av^2)_t$. Derivative (3.5) with respect to t yields

$$\begin{aligned}
&\|(\delta A)_t\|_{L^2}\\
&\le C\left(\|\nabla\delta v\|_{L^2}+\left\|\sigma(t)^{-\frac{1}{2}}\int_0^t\nabla\delta v d\tau\right\|_{L^2}\left(\|\sigma(t)^{\frac{1}{2}}\nabla v^1\|_{L^\infty}+\|\sigma(t)^{\frac{1}{2}}\nabla v^2\|_{L^\infty}\right)\right).
\end{aligned}$$

Based on the fact that

$$\|\sigma(t)^{\frac{1}{2}}\nabla v^i\|_{L^2(0,T;L^\infty)}\le C,\quad i=1,2,$$

we are led to by taking L^2 norm with respect to time t

$$\|(\delta A)_t\|_{L^2(0,T;L^2)}\le C\|\nabla\delta v\|_{L^2(0,T;L^2)}.$$

Finally, we can deduce the following inequality:

$$
\begin{aligned}
&\left\|\sigma(t)^{\frac{1-s}{2}}(\delta A v^2)_t\right\|_{L^{\frac{8}{3+2s}}(0,T;L^{\frac{3}{2}})} \\
\le&\left\|\sigma(t)^{\frac{1-s}{2}}\delta A v_t^2\right\|_{L^{\frac{8}{3+2s}}(0,T;L^{\frac{3}{2}})}+\left\|\sigma(t)^{\frac{1-s}{2}}(\delta A)_t v^2\right\|_{L^{\frac{8}{3+2s}}(0,T;L^{\frac{3}{2}})} \\
\le&\left\|\sigma(t)^{-\frac{1}{2}}\delta A\right\|_{L^\infty(0,T;L^2)}\left\|\sigma(t)^{\frac{2-s}{2}}v_t^2\right\|_{L^{\frac{8}{3+2s}}(0,T;L^6)} \\
&\quad+\|(\delta A)_t\|_{L^2(0,T;L^2)}\left\|\sigma(t)^{\frac{1-s}{2}}v^2\right\|_{L^{\frac{8}{2s-1}}(0,T;L^6)} \\
\le& c(T)\Big(\left\|\sigma(t)^{-\frac{1}{2}}\delta A\right\|_{L^\infty(0,T;L^2)}\left\|\sigma(t)^{\frac{2-s}{2}}\nabla v_t^2\right\|_{L^2(0,T;L^2)} \\
&\qquad+\|\nabla\delta v\|_{L^2(0,T;L^2)}\left\|\sigma(t)^{\frac{1-s}{2}}\nabla v^2\right\|_{L^\infty(0,T;L^2)}\Big) \\
\le& c(T)\|\nabla\delta v\|_{L^2(0,T;L^2)}.
\end{aligned}
$$

This ends the proof of Lemma 3.1. □

Now we give the estimation of z as the following lemma.

Lemma 3.2. *Assume that δv satisfies* (3.2) *and $z\triangleq\delta v-w$, then*

$$
\sup_{t\in[0,T]}\|\sqrt{\rho_0}z(t)\|_{L^2}^2+\|\nabla z\|_{L^2(0,T;L^2)}^2\le c(T)\|\nabla\delta v\|_{L^2(0,T;L^2)}^2.
$$

Proof. Firstly, let us restate the equations for $(\delta v,\delta Q)$ as the following system for $(z,\delta Q)$:

$$
\begin{aligned}
&\rho_0 z_t-\Delta_{v^1}z+\nabla_{v^1}\delta Q \\
&=(\Delta_{v^2}-\Delta_{v^1})v^2+(\nabla_{v^1}-\nabla_{v^2})Q^2-\rho_0 w_t+\Delta_{v^1}w, \\
&\mathrm{div}_{v^1}z=0.
\end{aligned}
$$

Testing the equation by z and noticing that

$$
\int(\nabla_{v^1}\delta Q)\cdot z dx=-\int \mathrm{div}_{v^1}z\delta Q dx=0,
$$

we have

$$
\frac{1}{2}\frac{d}{dt}\int\rho_0|z|^2dx+\int|\nabla_{v^1}z|^2dx\triangleq\sum_{k=1}^4 J_k, \tag{3.6}
$$

where

$$
J_1\triangleq\int\Big((\Delta_{v^2}-\Delta_{v^1})v^2\Big)\cdot z dx,\qquad J_3\triangleq-\int\rho_0 w_t\cdot z dx,
$$

$$J_2 \triangleq \int \Big((\nabla_{v^2} - \nabla_{v^1}) Q^2 \Big) \cdot z dx, \quad J_4 \triangleq \int (\Delta_{v^1} w) \cdot z dx.$$

For J_1, we obtain

$$\begin{aligned}
\int_0^T J_1 dt &= \int_0^T \int \operatorname{div} \Big(\big(\delta A A_2^t + A_1 (\delta A)^t \big) \nabla v^2 \Big) \cdot z dx dt \\
&\leq \int_0^T \int \big| \delta A A_2^t + A_1 (\delta A)^t \big| |\nabla v^2| |\nabla z| dx dt \\
&\leq C \int_0^T \big\| \sigma(t)^{-\frac{1}{2}} \delta A \big\|_{L^2} \big\| \sigma(t)^{\frac{1}{2}} \nabla v^2 \big\|_{L^\infty} \| \nabla z \|_{L^2} dt \\
&\leq C \big\| \sigma(t)^{-\frac{1}{2}} \delta A \big\|_{L^\infty(0,T;L^2)} \big\| \sigma(t)^{\frac{1}{2}} \nabla v^2 \big\|_{L^2(0,T;L^\infty)} \| \nabla z \|_{L^2(0,T;L^2)} \\
&\leq \epsilon \| \nabla \delta v \|^2_{L^2(0,T;L^2)} + c(T) \| \nabla z \|^2_{L^2(0,T;L^2)}.
\end{aligned} \tag{3.7}$$

For J_2, it is obviously that

$$\begin{aligned}
\int_0^T J_2 dt &\leq \int_0^T \left| \int \delta A \nabla Q^2 \cdot z dx \right| dt \\
&\leq C \big\| \sigma(t)^{-\frac{1}{2}} \delta A \big\|_{L^\infty(0,T;L^2)} \Big\| \sqrt{\sigma(t)} \nabla Q^2 \Big\|_{L^2(0,T;L^3)} \| z \|_{L^2(0,T;L^6)}
\end{aligned}$$

and

$$\Big\| \sqrt{\sigma(t)} \nabla Q^2 \Big\|_{L^2(0,T;L^3)} \leq \big\| \sigma(t)^{\frac{1-s}{2}} \nabla Q^2 \big\|_{L^2(0,T;L^2)} \big\| \sigma(t)^{\frac{2-s}{2}} \nabla Q^2 \big\|_{L^2(0,T;L^6)}.$$

Hence,

$$\int_0^T J_2 dt \leq \epsilon \| \nabla \delta v \|^2_{L^2(0,T;L^2)} + c(T) \| \nabla z \|^2_{L^2(0,T;L^2)}. \tag{3.8}$$

To estimate J_3, we should bound w_t firstly

$$\begin{aligned}
\| w_t \|_{L^{\frac{4}{3}}(0,T;L^{\frac{3}{2}})} &\leq \left(\int_0^T \Big(t^{\frac{s-1}{2}} \Big)^{\frac{8}{3-2s}} dt \right)^{\frac{3-2s}{8}} \big\| \sigma(t)^{\frac{1-s}{2}} w_t \big\|_{L^{\frac{8}{3+2s}}(0,T;L^{\frac{3}{2}})} \\
&\leq C T^{\frac{2s-1}{8}} \| \nabla \delta v \|_{L^2(0,T;L^2)}.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
\int_0^T J_3(t) dt &\leq \| \rho_0 \|^{\frac{1}{2}}_{L^\infty} \| w_t \|_{L^{\frac{4}{3}}(0,T;L^{\frac{3}{2}})} \big\| \rho_0^{\frac{1}{2}} z \big\|_{L^4(0,T;L^3)} \\
&\leq C T^{\frac{2s-1}{8}} \| \rho_0 \|_{L^\infty} \| \sqrt{\rho_0} z \|^{\frac{1}{2}}_{L^\infty(0,T;L^2)} \| z \|^{\frac{1}{2}}_{L^2(0,T;L^6)} \| \nabla \delta v \|_{L^2(0,T;L^2)}
\end{aligned}$$

$$
\begin{aligned}
&\le c(T)\left(\|\sqrt{\rho_0}z\|_{L^\infty(0,T;L^2)}+\|z\|_{L^2(0,T;H^1)}\right)\|\nabla\delta v\|_{L^2(0,T;L^2)}\\
&\le \epsilon\left(\|\sqrt{\rho_0}z\|^2_{L^\infty(0,T;L^2)}+\|z\|^2_{L^2(0,T;H^1)}\right)+c(T)\|\nabla\delta v\|^2_{L^2(0,T;L^2)}. \qquad (3.9)
\end{aligned}
$$

Finally, we get for J_4

$$
\begin{aligned}
\int_0^T J_4(t)dt &\le \int_0^T\int|\nabla_{v^1}w||\nabla_{v^1}z|dxdt\\
&\le \epsilon\int_0^T\|\nabla_{v^1}z\|^2_{L^2}dt+C(\epsilon)\int_0^T\|\nabla_{v^1}w\|^2_{L^2}dt\\
&\le \epsilon\|\nabla_{v^1}z\|^2_{L^2(0,T;L^2)}+c(T)\|\nabla\delta v\|^2_{L^2(0,T;L^2)}. \qquad (3.10)
\end{aligned}
$$

Integrating (3.6) with respect to time t, and taking ϵ suitably small, (3.7)-(3.10) imply

$$
\sup_{t\in[0,T]}\|\sqrt{\rho_0}z(t)\|^2_{L^2}+\|\nabla z\|^2_{L^2(0,T;L^2)}\le c(T)\|\nabla\delta v\|^2_{L^2(0,T;L^2)}. \qquad (3.11)
$$

The Lemma 3.2 is proved. □

Now we show the proof of uniqueness of solutions.

The proof of uniqueness: Notice that $\delta v=w+z$, (3.4) and (3.11), which implies

$$
\int_0^T\|\nabla\delta v\|^2_{L^2}dt\le c(T)\int_0^T\|\nabla\delta v\|^2_{L^2}dt.
$$

Hence, $\nabla\delta u\equiv 0$ on $[0,T]\times\Omega$ when T is small enough such that $c(T)<1$.

Then, plugging $\nabla\delta u\equiv 0$ into (3.11) yields

$$
\sup_{t\in[0,T]}\|\sqrt{\rho_0}z(t)\|^2_{L^2}+\|\nabla z\|^2_{L^2(0,T;L^2)}=0.
$$

Combining the fact that

$$
\|z(t)\|_{L^2(0,T;L^p)}\le\|\nabla z(t)\|_{L^2(0,T;L^2)}
$$

for $p\in[1,6]$, we can finally obtain $z\equiv 0$ on $[0,T]\times\Omega$. And (3.4) clearly yields $w\equiv 0$ on $[0,T]\times\Omega$. Therefore we give for small enough $T>0$,

$$
(v^1,\nabla Q^1)\equiv(v^2,\nabla Q^2)\quad\text{on }\ [0,T]\times\Omega.
$$

Reverting to Eulerian coordinates, we conclude that the two solutions of (1.1) coincide on $[0,T]\times\Omega$. Then standard connectivity arguments yield uniqueness of solutions on the whole $\mathbb{R}^+$.

Appendix A

In this section, we introduce some notations that appear throughout this article and state several preliminary lemmas which are used in this paper.

As we shall mostly consider non-smooth solutions, system (1.1) has to be understood in the distributional sense, that is, for all $t \in [0,T)$ and for all functions in $\phi \in C_0^\infty([0,T)\times\Omega;\mathbb{R})$

$$\begin{aligned}
&\langle \rho(t),\phi(t)\rangle - \langle \rho_0,\phi_0\rangle - \int_0^t \langle \rho,\phi_t\rangle d\tau - \int_0^t \langle \rho u,\nabla\phi\rangle d\tau = 0,\\
&\int_0^t \langle u,\nabla\phi\rangle d\tau = 0,
\end{aligned}$$

and that

$$\begin{aligned}
&\langle \rho u,\varphi\rangle - \langle \rho_0 u_0,\varphi\rangle - \int_0^t \langle \rho u,\varphi_t\rangle d\tau \\
&\quad - \int_0^t \langle \rho u\otimes u,\nabla\varphi\rangle d\tau + \int_0^t \langle \nabla u,\nabla\varphi\rangle d\tau = 0
\end{aligned}$$

for all divergence-free functions $\varphi \in C_0^\infty([0,T)\times\Omega;\mathbb{R}^d)$, where $\langle\cdot,\cdot\rangle$ stands for the distribution bracket in Ω (that is, $\langle f,g\rangle = \int_\Omega fg dx$ in the case of smoothness).

It is well-known that sufficiently smooth solutions to (1.1) fulfill for all $t \geq 0$:

- The energy balance

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|u|^2 dx + \int_\Omega |\nabla u|^2 dx = 0.$$

- The conservation of total momentum (in the case $\Omega = \mathbb{T}^3$)

$$\int_\Omega \rho u dx = \int_\Omega \rho_0 u_0 dx.$$

- The conservation of total mass

$$\int_\Omega \rho dx = \int_\Omega \rho_0 dx.$$

- Any Lebesgue norm of ρ that is preserved through the evolution and

$$\inf_{x\in\Omega}\rho(t,x) = \inf_{x\in\Omega}\rho_0, \quad \sup_{x\in\Omega}\rho(t,x) = \sup_{x\in\Omega}\rho_0(x).$$

Now, we state several preliminary lemmas which will be used in previous subsections.

Lemma A.1 (Aubin-Lions compactness lemma, [16, Corollaries 4 and 6])**.** *Assume* $X\hookrightarrow\hookrightarrow B\hookrightarrow Y$.

(I) Let F *be bounded in* $L^q(0,T;B)\cap L^1_{loc}(0,T;X)$ *with* $1<q\le\infty$, *and* $\partial_t F$ *be bounded in* $L^1_{loc}(0,T;Y)$, *then* F *is relatively compact in* $L^p(0,T;B)$, $\forall\ p<q$.

(II) Let F *be bounded in* $L^\infty(0,T;X)$ *and* $\partial_t F$ *be bounded in* $L^r(0,T;Y)$ *with* $r>1$, *then* F *is relatively compact in* $C(0,T;B)$.

Lemma A.2. *Let* $p\in[1,\infty]$, $0<s\le 1$ *and* $u:(0,T)\times\mathbb{R}^3\to\mathbb{R}$ *satisfy* $u\in L^2(0,T;L^p)$ *and* $\sqrt{\sigma(t)^{2-s}}u_t\in L^2(0,T;L^p)$. *Then* u *is in* $H^{\frac12-\alpha}(0,T;L^p)$ *for all* $\alpha\in(\frac{1-s}{2},\frac12)$ *and*

$$\|u\|^2_{H^{\frac12-\alpha}(0,T;L^p)}\le\|u\|^2_{L^2(0,T;L^p)}+C\left\|\sqrt{\sigma(t)^{2-s}}u_t\right\|^2_{L^2(0,T:L^p)},$$

where $C>0$ *depends only on* α,T.

Proof. The special case of $s=1$ has been proved in [7], similarly we can prove the general case. By the definition of Sobolev norms in terms of finite differences, we obtain

$$\|u\|^2_{H^{\frac12-\alpha}(0,T;L^p)}=\|u\|^2_{L^2(0,T;L^p)}+\int_0^T\left(\int_0^{T-h}\frac{\|u(t+h)-u(t)\|^2_{L^p}}{h^{2-2\alpha}}dt\right)dh.$$

By direct calculations we have

$$\begin{aligned}&\int_0^{T-h}\|u(t+h)-u(t)\|^2_{L^p}dt\\&\le\int_0^{T-h}\left\|\int_t^{t+h}\sqrt{\tau^{2-s}}u_t(\tau)\frac{d\tau}{\sqrt{\tau^{2-s}}}\right\|^2_{L^p}dt\\&\le\int_0^{T-h}\left(\int_t^{t+h}\frac{1}{\tau^{2-s}}d\tau\right)\left(\int_t^{t+h}\tau^{2-s}\|u_t\|^2_{L^p}d\tau\right)dt\\&\le\left\|\sqrt{\sigma(t)^{2-s}}u_t\right\|^2_{L^2(0,T;L^p)}\int_0^{T-h}\left(\int_t^{t+h}\frac{1}{\tau^{2-s}}d\tau\right)dt.\end{aligned}$$

Making use of Fubini theorem, we obtain for $\alpha\in(\frac{1-s}{2},\frac12)$

$$\begin{aligned}&\int_0^T h^{2\alpha-2}\left(\int_0^{T-h}\left(\int_t^{t+h}\frac{1}{\tau^{2-s}}d\tau\right)dt\right)dh\\&=\frac{1}{1-2\alpha}\int_0^T\left(\int_t^T\left((\tau-t)^{2\alpha-1}-T^{2\alpha-1}\right)\frac{d\tau}{\tau^{2-s}}\right)dt.\end{aligned}$$

It is obviously that $\frac{1}{\tau^{1-s}} \le \frac{1}{(\tau-t)^{1-s}}$ for all $0<t<\tau$, and $-1<2\alpha+s-2<s-1$ when $\frac{1-s}{2}<\alpha<\frac{1}{2}$, $0<s<1$. Thus,

$$
\begin{aligned}
&\int_0^T h^{2\alpha-2}\left(\int_0^{T-h}\left(\int_t^{t+h}\frac{1}{\tau^{2-s}}d\tau\right)dt\right)dh\\
\le&\frac{1}{1-2\alpha}\left(\int_0^T\int_t^T(\tau-t)^{2\alpha+s-2}\frac{d\tau}{\tau}dt-\int_0^T\int_t^T T^{2\alpha-1}\frac{d\tau}{\tau^{2-s}}dt\right)\le C.
\end{aligned}
$$

The proof of Lemma A.2 is finished. □

Lemma A.3. *Let Ω be a C^2 bounded domain of $\mathbb{R}^3$, and A be a matrix-valued function on $[0,T]\times\Omega$ satisfying $\det A=1$. If for all $s\in(\frac{1}{2},1)$*

$$
\|Id-A\|_{L^\infty(0,T;L^\infty)}+\left\|\sigma(t)^{\frac{1-s}{2}}A_t\right\|_{L^2(0,T;L^6)}\le c \tag{A.1}
$$

is fulfilled with small $c>0$. Then for all functions $R:[0,T]\times\Omega\to\mathbb{R}^3$ satisfying $\operatorname{div}R\in L^2(0,T\times\Omega)$, $R\in L^{\frac{8}{2s-1}}(0,T;L^2)$, $\sigma(t)^{\frac{1-s}{2}}R_t\in L^{\frac{8}{3+2s}}(0,T;L^{\frac{3}{2}})$ and $R\cdot n\equiv 0$ on $(0,T)\times\partial\Omega$, the equation

$$
\operatorname{div}(Aw)=\operatorname{div}R\triangleq g \quad in \ \ [0,T]\times\Omega
$$

admits a solution w in the space

$$
X_T\triangleq\left\{w\middle|w\in L^2(0,T;H_0^1),\ w\in L^{\frac{8}{2s-1}}(0,T;L^2),\ \sigma(t)^{\frac{1-s}{2}}w_t\in L^{\frac{8}{3+2s}}(0,T;L^{\frac{3}{2}})\right\}
$$

satisfying the following inequalities for some constant:

$$
\|w\|_{L^{\frac{8}{2s-1}}(0,T;L^2)}\le C\|R\|_{L^{\frac{8}{2s-1}}(0,T;L^2)},\quad \|\nabla w\|_{L^2(0,T;L^2)}\le C\|g\|_{L^2(0,T;L^2)}
$$

and

$$
\left\|\sigma(t)^{\frac{1-s}{2}}w_t\right\|_{L^{\frac{8}{3+2s}}(0,T;L^{\frac{3}{2}})}\le C\|R\|_{L^{\frac{8}{2s-1}}(0,T;L^2)}+C\left\|\sigma(t)^{\frac{1-s}{2}}R_t\right\|_{L^{\frac{8}{3+2s}}(0,T;L^{\frac{3}{2}})}.
$$

Proof. The case that

$$
\begin{aligned}
&\|Id-A\|_{L^\infty(0,T;L^\infty)}+\|A_t\|_{L^2(0,T;L^6)}\le c,\\
&\operatorname{div}R\in L^2(0,T\times\Omega),\quad R\in L^4(0,T;L^2),\quad R_t\in L^{\frac{4}{3}}(0,T;L^{\frac{3}{2}})
\end{aligned}
$$

has been proved by Danchin and Mucha [7]. Here, the proof of this lemma is very similar. Recalling the results of [6], there exists a linear operator $\mathcal{B}:k\to u$ which is continuous on $L^p(\Omega;\mathbb{R}^3)$ (for all $1<p<\infty$) such that for all k in $L^p(\Omega;\mathbb{R}^3)$ the vector field u satisfies

$$\int_\Omega u\cdot\nabla\phi dx=\int_\Omega k\cdot\nabla\phi dx \quad \text{for all} \quad \phi\in\mathcal{C}^\infty(\bar{\Omega};\mathbb{R}),$$

if in addition $\operatorname{div}k$ is in $L^p(\Omega)$ and $k\cdot n|_{\partial\Omega}=0$, then u is in $W_0^{1,p}(\Omega)$ with

$$\|u\|_{W_0^{1,p}(\Omega)}\le C\|\operatorname{div}k\|_{L^p(\Omega)}.$$

Define Φ on the set X_T by

$$\Phi(v)\triangleq\mathcal{B}\big((Id-A)v+R\big). \tag{A.2}$$

The above result ensures that

$$\begin{aligned}
&\|\nabla\Phi(v)\|_{L^2(0,T;L^2)}\\
&\le C\big\|((Id-A)^t:\nabla v+g)\big\|_{L^2(0,T;L^2)}\\
&\le C\Big(\|g\|_{L^2(0,T;L^2)}+\|(Id-A)\|_{L^\infty(0,T;L^\infty)}\|\nabla v\|_{L^2(0,T;L^2)}\Big)
\end{aligned}$$

as well as

$$\begin{aligned}
&\|\Phi(v)\|_{L^{\frac{8}{2s-1}}(0,T;L^2)}\\
&\le C\|(Id-A)v+R\|_{L^{\frac{8}{2s-1}}(0,T;L^2)}\\
&\le C\Big(\|Id-A\|_{L^\infty(0,T;L^\infty)}\|v\|_{L^{\frac{8}{2s-1}}(0,T;L^2)}+\|R\|_{L^{\frac{8}{2s-1}}(0,T;L^2)}\Big).
\end{aligned}$$

Moreover, derivativing (A.2) with respect to time gives

$$\big(\Phi(v)\big)_t=\mathcal{B}\big((Id-A)v_t-A_tv+R_t\big),$$

whence, using the continuity property of $\mathcal{B}$ on $L^{\frac{3}{2}}(\Omega)$ and then performing a time integration, we get

$$\begin{aligned}
&\big\|\sigma(t)^{\frac{1-s}{2}}(\Phi(v))_t\big\|_{L^{\frac{8}{3+2s}}(0,T;L^{\frac{3}{2}})}\\
&\le C\left(\big\|\sigma(t)^{\frac{1-s}{2}}(Id-A)v_t-\sigma(t)^{\frac{1-s}{2}}A_tv\big\|_{L^{\frac{8}{3+2s}}(0,T;L^{\frac{3}{2}})}+\big\|\sigma(t)^{\frac{1-s}{2}}R_t\big\|_{L^{\frac{8}{3+2s}}(0,T;L^{\frac{3}{2}})}\right)\\
&\le C\bigg(\|Id-A\|_{L^\infty(0,T;L^\infty)}\big\|\sigma(t)^{\frac{1-s}{2}}v_t\big\|_{L^{\frac{8}{3+2s}}(0,T;L^{\frac{3}{2}})}+\big\|\sigma(t)^{\frac{1-s}{2}}R_t\big\|_{L^{\frac{8}{3+2s}}(0,T;L^{\frac{3}{2}})}
\end{aligned}$$

$$+\left\|\sigma(t)^{\frac{1-s}{2}}A_t\right\|_{L^2(0,T;L^6)}\|v\|_{L^{\frac{8}{2s-1}}(0,T;L^2)}\Big).$$

This gives that Φ maps X_T to X_T. If c in (A.1) is small enough, then obvious variations on above computations give for any couple (v^1,v^2) in X_T^2

$$\begin{aligned}\left\|\Phi(v^2)-\Phi(v^1)\right\|_{X_T}&\le C\left(\|Id-A\|_{L^\infty(0,T;L^\infty)}+\left\|\sigma(t)^{\frac{1-s}{2}}A_t\right\|_{L^2(0,T;L^6)}\right)\|v^2-v^1\|_{X_T}\\&\le\frac{1}{2}\|v^2-v^1\|_{X_T}.\end{aligned}$$

Hence, applying the standard Banach fixed-point theorem in X_T provides a solution to the equation $\Phi(v)=v$. Then looking back at the above computations in the case $\Phi(v)=v$ gives the desired inequalities. □

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