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Regularity for 3-D MHD Equations in Lorentz Space $L^{3,\infty}$

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Abstract. The regularity for 3-D MHD equations is considered in this paper, it is proved that the solutions (v,B,p) are Hölder continuous if the velocity field $v\in L^\infty(0,T;L_x^{3,\infty}(\mathbb{R}^3))$ with local small condition

$$r^{-3}\left|\left\{x\in B_r(x_0):|v(x,t_0)|>\varepsilon r^{-1}\right\}\right|\le\varepsilon$$

and the magnetic field $B\in L^\infty(0,T;VMO^{-1}(\mathbb{R}^3))$.

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1 Introduction

In this paper, we will consider the following MHD equations in $\mathbb{R}^3\times(0,T)$:

$$\begin{cases}\partial_t v+v\cdot\nabla v-\Delta v+\nabla p=B\cdot\nabla B,\\ \partial_t B-\Delta B+v\cdot\nabla B-B\cdot\nabla v=0,\\ \mathrm{div}v=0,\quad \mathrm{div}B=0,\end{cases}\tag{1.1}$$

where v is the fluid velocity field and B is the magnetic field. The function p describes the scalar pressure. The MHD equations (1.1) reduce to the incompressible Navier-Stokes equations when there is no electromagnetic field, that is, $B\equiv 0$.

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There is a very rich literatures dedicated to the mathematical study of the Navier-Stokes system by many researchers; see, for example, [1,3–5,8–10,13–15,18,20,22]. Compared with Navier-Stokes equation, the MHD equation (1.1) is a combination of the incompressible Navier-Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. Such a system depicts the motion of many conducting impressible immiscible fluids without surface tension under the action of a magnetic field.

For a point $z=(x,t)\in\mathbb{R}^3\times\mathbb{R}_+$, we denote

$$B_r(x):=\left\{y\in\mathbb{R}^3:|y-x|<r\right\},\quad Q_r(z):=B_r(x)\times\left(t-r^2,t\right),\quad Q_r=Q_r(0,0),$$

and define the space

$$L^{q,p}(Q_r)=L^q\left(t-r^2,t;L^p\left(B_r(x)\right)\right)$$

with the norm

$$\|u\|^q_{L^{p,q}(Q_r(z))}=\int_{t-r^2}^t\|u(t)\|^q_{L^p(B_r(x))}dt.$$

Let $\Omega\subset\mathbb{R}^3$. The Lorentz space $L^{(p,q)}(\Omega)$ with $p,q\in(0,\infty)$ is the set of measurable functions f on Ω such that the following quasi-norm is bounded:

$$\|f\|_{L^{(p,q)}(\Omega)}:=\begin{cases}\left(p\displaystyle\int_0^\infty\alpha^q\big|\{x\in\Omega:|f(x)|>\alpha\}\big|^{\frac{q}{p}}\frac{d\alpha}{\alpha}\right)^{\frac{1}{q}}, & \text{if}\quad q<\infty,\\ \sup_{\alpha>0}\alpha\big|\{x\in\Omega:|f(x)|>\alpha\}\big|^{\frac{1}{p}}, & \text{if}\quad q=\infty.\end{cases}$$

Lorentz spaces can be used to capture logarithmic singularities. For example, in $\mathbb{R}^3$, for any $\beta>0$ we have

$$|x|^{-1}\left(\frac{\log|x|}{2}\right)^{-\beta}\in L^{(3,q)}\left(\mathbb{R}^3\right),\quad\text{if and if}\quad q>\frac{1}{\beta}.$$

It is known that $L^{(p,p)}(\Omega)=L^p(\Omega)$ and $L^{(p,q_1)}(\Omega)\subset L^{(p,q_2)}(\Omega)$ while $q_1\le q_2$. If $|\Omega|$ is finite then $L^{(p,q)}(\Omega)\subset L^r(\Omega)$ for all $0<q\le\infty$ and $0<r<p$,

$$\|g\|_{L^r(\Omega)}\le|\Omega|^{\frac{1}{r}-\frac{1}{p}}\|g\|_{L^{(p,q)}(\Omega)}.\tag{1.2}$$

In the classical literatures, Leray [9] and Hopf [8] established the existence weak solution to the Navier Stokes system. However, the smoothness of the Leray-Hopf weak solution is still a challenging open problem. Serrin [18] proved that if the weak solution to the Navier-Stokes equation satisfies the well-know Ladyzhenskaya-Prodi-Serrin type condition

$$u\in L^q\big(0,T;L^p\big(\mathbb{R}^3\big)\big) \quad \text{with} \quad \frac{2}{q}+\frac{3}{p}=1, \quad p>1,$$

then the weak solution u is regular in $\mathbb{R}^3\times(0,T)$. Later, for $p=3$, $q=\infty$ (i.e., $u\in L^\infty(0,T;L^3(\mathbb{R}^3))$), Escauriaza-Seregin-Šverak [3] established the regularity of the Navier-Stokes equation by using backward uniqueness for the parabolic operator. In [14], Phuc established the regularity when the velocity field belongs to $L_t^\infty(L_x^{(3,q)})\,(q\neq\infty)$, it is due to that $L_x^{(3,q)}$ contains L_x^3, thus the result is an improvement of the result by Escauriaza *et al.* [3].

For mathematical questions related to the MHD equations we refer the reader to [17]. He-Xin [7] established some regularity criterions to the MHD equations (1.1). Wu [23] promoted Ladyzhenskaya-Prodi-Serrin type criterions to the MHD equations (1.1) in the term of both the velocity field v and the magnetic field B, and Mahalov *et al.* [12] established the limiting case. Recently, Wang and Zhang [21] proved that any weak solution (v,B) of MHD equations is regular if $v\in L^\infty(0,T;L^3(\mathbb{R}^3))$ and $B\in L^\infty(0,T;VMO^{-1}(\mathbb{R}^3))$. For general Lorenz space $L^{(3,q)}(\mathbb{R}^3)$, $3<q<\infty$ case, we prove in [11] if $v\in L^\infty(0,T;L^{(3,q)}(\mathbb{R}^3))$ and $B(t)\in VMO^{-1}(\mathbb{R}^3)$ for any t, then (v,B) is smooth. For $L^{(3,\infty)}(\mathbb{R}^3)$ case, Choe *et al.* [2] proved that if $u\in L^\infty(0,T;L^{(3,\infty)}(\mathbb{R}^3))$ and an additional local small condition

$$\frac{1}{r^3}\left|\left\{x\in B_r(x_0)\,\big|\,|u(\cdot,t_0)|>\frac{\epsilon}{r}\right\}\right|\le\epsilon, \tag{1.3}$$

then weak solution u of Navier-Stokes equations is smooth on $Q_{\epsilon r}(z_0)$, also see Seregin [16].

In this paper, we shall established the regularity of weak solutions of MHD equations in Lorentz space $L^{(3,\infty)}$, the technicality is different from Navier-Stokes equations.

Under the condition $v\in L^\infty(0,T;L^{(3,\infty)}(\mathbb{R}^3))$, we can get $C(v,z_0,r)\le M(\forall r\in(0,1))$. The main difficulty of this paper is to obtain $C(B,z_0,\frac{r}{2})+D(p,z_0,\frac{r}{2})\le C(M)$. The main technic is to deal with the new norms. Our main results can be stated as following.

Theorem 1.1. *Assume that the triplet functions (v,B,p) is weak solution to the MHD equations* (1.1) *in $Q_T:=\mathbb{R}^3\times(0,T)$ with*

$$v\in L^\infty\big(0,T;L^{(3,\infty)}(\mathbb{R}^3)\big)$$

and

$$B\in L^\infty\left(0,T;BMO^{-1}(\mathbb{R}^3)\right),\quad B(t)\in VMO^{-1}(\mathbb{R}^3)\quad \text{for}\quad t\in(0,T).$$

Then there exists a positive constant ε with the following property. Let $N>0$ be a constant. If $z_0=(x_0,t_0)\in Q_T$ and $R>0$ such that $Q_R(z_0)\subset Q_T$, (B,p) satisfy

$$C(B,z_0,R)+D(p,z_0,R)\le N,$$

and for some $0<r\le R/2$,

$$r^{-3}\left|\left\{x\in B_r(x_0):|v(x,t_0)|>\varepsilon r^{-1}\right\}\right|\le\varepsilon.$$

Note that here ε is dependent with N. Then (v,B) is smooth in $Q_{\varepsilon r}(z_0)$. Here $C(B,z_0,R)$ and $D(p,z_0,R)$ are dimensionless quantities in Section 2.

Remark 1.1. In Theorem 1.1, for fixed z_0 and $R>0$, naturally, the scaling quantity $C(B,z_0,R)+D(p,z_0,R)$ is finite. We add the condition $C(B,z_0,R)+D(p,z_0,R)\le N$ just to make the proof more convenient.

Theorem 1.2. *Let (v,B) be a suitable weak solution to MHD equations in $Q_T:=\mathbb{R}^3\times(0,T)$ with $v\in L^\infty(0,T;L^{(3,\infty)}(\mathbb{R}^3))$, $B\in L^\infty(0,T;BMO^{-1}(\mathbb{R}^3))$, $B(t)\in VMO^{-1}(\mathbb{R}^3)$ for $t\in(0,T)$. Then there exist at most finite number $\mathcal{N}$ of singular points at any singular time t.*

We can change the condition of B to get following theorem.

Theorem 1.3. *Assume that the triplet functions (v,B,p) is weak solution to the MHD equations* (1.1) *in $Q_T:=\mathbb{R}^3\times(0,T)$. Suppose that*

$$v,B\in L^\infty\left(0,T;L^{(3,\infty)}(\mathbb{R}^3)\right).$$

There exists a positive constant ε with following property. If $z_0=(x_0,t_0)\in Q_T$ and $R>0$ such that $Q_R(z_0)\subset Q_T$, (B,p) satisfy

$$C(B,z_0,R)+D(p,z_0,R)\le N,$$

for some $0<r\le R/2$,

$$r^{-3}\left|\left\{x\in B_r(x_0):\left|(v(x,t_0),B(x,t_0))\right|>\varepsilon r^{-1}\right\}\right|\le\varepsilon.$$

Then (v,B) is smooth in $Q_{\varepsilon r}(z_0)$. Here $C(B,z_0,R)$ and $D(p,z_0,R)$ are dimensionless quantities in Section 2.

We need the following small energy regularity proposition ([7,12]).

Theorem (A). *Let the triple* (v,B,p) *be a suitable weak solution to system* (1.1). *There exists a small constant* $\epsilon_0>0$, *such that if*

$$\int_{Q_1}|v|^3+|B|^3+|p|^{\frac{3}{2}}<\epsilon_0,$$

then v *and* B *are smooth in* $\overline{Q_{\frac{1}{2}}}$. *In particular, for any* z_0 *if*

$$\frac{1}{r^2}\int_{Q_r(z_0)}|v|^3+|B|^3<\epsilon_0 \quad \text{for all} \quad 0<r\le 1,$$

then z_0 *is a regular point.*

We can drop the pressure p by Wolf's method of local suitable weak solutions, the proof to see Section 5.

Proposition 1.1. *Let the triple* (v,B,p) *be a suitable weak solution to system* (1.1). *There exists a small constant* $\varepsilon_0>0$, *such that if*

$$\int_{Q_1}|v|^3+|B|^3<\varepsilon_0,$$

then v *and* B *are smooth in* $\overline{Q_{\frac{1}{2}}}$.

2 Preliminaries and local estimates

At present, we introduce some useful space and properties. We call a local integrable function $f\in BMO(\mathbb{R}^3)$, if f satisfies

$$\|f\|_{BMO(\mathbb{R}^3)}:=\sup_{R>0,x_0\in\mathbb{R}^3}\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}|f(x)-f_{B_R(x_0)}|dx<\infty,$$

where and in what follows we denote

$$f_\Omega=\frac{1}{|\Omega|}\int_\Omega f(x)dx.$$

A remarkable property of BMO function is

$$\sup_{R>0,x_0\in\mathbb{R}^3}\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}|f(x)-f_{B_R(x_0)}|^q dx<\infty$$

for any $1 \le q < \infty$. A function $f(x) \in VMO(\mathbb{R}^3)$ if $f(x) \in BMO(\mathbb{R}^3)$ and for any $x_0 \in \mathbb{R}^3$

$$\limsup_{R\downarrow 0} \frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} |f(x)-f_{B_R(x_0)}|dx = 0.$$

We say that a function $f \in BMO^{-1}(\mathbb{R}^3)$, if there exists $U_j \in BMO(\mathbb{R}^3)$ such that $f = \sum_{j=1}^3 \partial_j U_j$. The definition of $VMO^{-1}(\mathbb{R}^3)$ is similar.

We will use the following scaling invariant quantities:

$$\begin{aligned}
&A(v,z_0,r) = \sup_{-r^2+t_0\le t\le t_0} r^{-1}\int_{B_r(x_0)} |v(x,t)|^2 dx,\\
&A(B,z_0,r) = \sup_{-r^2+t_0\le t\le t_0} r^{-1}\int_{B_r(x_0)} |B(x,t)|^2 dx,\\
&C_1(v,z_0,r) = r^{-2}\int_{Q_r(z_0)} |v(x,t)|^3 dxdt, \qquad C_1(B,z_0,r) = r^{-2}\int_{Q_r(z_0)} |B(x,t)|^3 dxdt,\\
&C(v,z_0,r) = r^{-\frac{16}{7}}\int_{t_0-r^2}^{t_0} \|v\|^4_{L^{\frac{14}{5}}(B_r(x_0))} dt, \quad C(B,z_0,r) = r^{-\frac{16}{7}}\int_{t_0-r^2}^{t_0} \|B\|^4_{L^{\frac{14}{5}}(B_r(x_0))} dt,\\
&E(v,z_0,r) = r^{-1}\int_{Q_r(z_0)} |\nabla v(x,t)|^2 dxdt, \quad E(B,z_0,r) = r^{-1}\int_{Q_r(z_0)} |\nabla B(x,t)|^2 dxdt,\\
&D_1(p,z_0,r) = r^{-2}\int_{Q_r(z_0)} |p(x,t)|^{\frac{3}{2}} dxdt, \quad D(p,z_0,r) = r^{-\frac{16}{7}}\int_{t_0-r^2}^{t_0} \|p\|^2_{L_{\frac{7}{5}}(B_r(x_0))} dt.
\end{aligned}$$

For simplicity, we write

$$\begin{aligned}
&A(v,B,z_0,r) = A(v,z_0,r) + A(B,z_0,r),\\
&E(v,B,z_0,r) = E(v,z_0,r) + E(B,z_0,r),\\
&C(v,B,z_0,r) = C(v,z_0,r) + C(B,z_0,r),\\
&C_1(v,B,z_0,r) = C_1(v,z_0,r) + C_1(B,z_0,r).
\end{aligned}$$

If $z_0 = (0,0)$ we shall simplify to rewrite $C(v,r) = C(v,z_0,r)$, $C(B,r) = C(B,z_0,r)$, and so on.

Definition 2.1. *Let $\Omega \subset \mathbb{R}^3$ and $-\infty < a < b < \infty$. A triplet (v,B,p) is called a suitable weak solution to the MHD equations* (1.1) *in $Q = \Omega \times (a,b)$ if the following conditions are valid:*

(i) $(v,B) \in L^\infty(a,b;L^2(\Omega)) \cap L^2(a,b;W^{1,2}(\Omega))$ *and* $p \in L^{\frac{3}{2}}(\Omega \times (a,b))$;

(ii) (v,B,p) *satisfy the MHD equations in the sense of distributions in* $\mathscr{D}'(Q)$;

(iii) *For each real-valued* $0\leq\phi\in C_0^\infty(Q)$*, the following generalized energy inequality is valid:*

$$\begin{aligned}&\int_\Omega\left(|v|^2+|B|^2\right)\phi dx+2\int_a^t\int_\Omega\left(|\nabla v|^2+|\nabla B|^2\right)\phi dxds\\&\leq\int_a^t\int_\Omega\left(|v|^2+|B|^2\right)(\Delta\phi+\partial_s\phi)dxds+\int_a^t\int_\Omega v\cdot\nabla\phi\left(|v|^2+|B|^2+2p\right)dxds\\&\quad-\int_a^t\int_\Omega(B\cdot v)(B\cdot\nabla\phi)dxds,\end{aligned}$$

for a.e. $t\in[a,b]$.

A weak solution (v,B) of the MHD equation (1.1) satisfies the energy inequality, for a.a. $t\in(a,b)$

$$\begin{aligned}&\|v(t)\|_{L^2}^2+\|B(t)\|_{L^2}^2+2\int_0^t\|\nabla v(s)\|_{L^2}^2ds+2\int_0^t\|\nabla B(s)\|_{L^2}^2ds\\&\leq\|v_0\|_{L^2}^2+\|B_0\|_{L^2}^2\equiv c_0.\end{aligned}$$

First we have the following standard local estimates for MHD equations.

Lemma 2.1. *Assume that the triplet functions* (v,B,p) *is a suitable weak solution to the MHD equation in* $Q=\Omega\times(a,b)$*. Let* $z_0=(x_0,t_0)$ *and* $\rho>0$ *be such that* $Q_\rho(z_0)\subset Q$*. For every* $r\in(0,\rho]$*, we have*

$$C_1(v,z_0,r)\leq c\left(\frac{\rho}{r}\right)^3A(v,z_0,\rho)^{\frac34}E(v,z_0,\rho)^{\frac34}+c\left(\frac{r}{\rho}\right)^3A(v,z_0,\rho)^{\frac32},\tag{2.1}$$

$$C_1(B,z_0,r)\leq c\left(\frac{\rho}{r}\right)^3A(B,z_0,\rho)^{\frac34}E(B,z_0,\rho)^{\frac34}+c\left(\frac{r}{\rho}\right)^3A(B,z_0,\rho)^{\frac32}.\tag{2.2}$$

We also need the so-called decay estimate for pressure.

Lemma 2.2. *Let* (v,B,p) *be a suitable weak solution to the MHD equations* (1.1) *in* $Q=\Omega\times(a,b)$*. Let* $z_0=(x_0,t_0)$ *and let* $\rho>0$ *be such that* $Q_\rho\subset Q$*. For every* $r<(0,\frac{\rho}{4}]$*, we have*

$$\begin{aligned}D_1(p,z_0,r)\leq{}&c\left(\frac{\rho}{r}\right)^{\frac32}\left[A(v,z_0,\rho)^{\frac34}E(v,z_0,\rho)^{\frac34}+A(B,z_0,\rho)^{\frac34}E(B,z_0,\rho)^{\frac34}\right]\\&+c\frac{r}{\rho}D(p,z_0,\rho)^{\frac34},\end{aligned}\tag{2.3}$$

$$D(p,z_0,r)\leq c\left[\left(\frac{r}{\rho}\right)^2D(p,z_0,\rho)+\left(\frac{\rho}{r}\right)^{\frac{16}{7}}C(v,B,z_0,\rho)\right],\tag{2.4}$$

$$K(B,z_0,r)\leq c\left(\frac{\rho}{r}\right)^3C_1(v,z_0,\rho)^{\frac23}\left[A(B,z_0,\rho)+E(B,z_0,\rho)\right]+\left(\frac{r}{\rho}\right)^2K(B,z_0,\rho),\tag{2.5}$$

where

$$K(B,z_0,r)=\frac{1}{r^3}\int_{Q_r(z_0)}|B|^2.$$

Proof. Let $z_0=(0,0)$, we decompose p so that

$$p=p_1+p_2,$$

where p_1 satisfies in B_ρ for a.e. $t\in[-\rho^2,0]$, in the weak sense,

$$\begin{cases}\Delta p_1=-\mathrm{div}\,\mathrm{div}(v\otimes v-[v\otimes v]_{B_\rho})-\mathrm{div}\,\mathrm{div}(B\otimes B-[B\otimes B]_{B_\rho}),\\ p_1|_{\partial B_\rho}=0,\end{cases}\tag{2.6}$$

and p_2 is a harmonic function in B_ρ, i.e.

$$\Delta p_2=0.$$

Regarding p_1, by theory of Laplace operator and Calderón-Zygmund theorem, we have

$$\begin{aligned}\left(\int_{B_\rho}|p_1|^{\frac{3}{2}}dx\right)^{\frac{2}{3}}&\le c\left(\int_{B_\rho}\left(|v\otimes v-[v\otimes v]_{B_\rho}|^{\frac{3}{2}}+|B\otimes B-[B\otimes B]_{B_\rho}|^{\frac{3}{2}}\right)dx\right)^{\frac{2}{3}}\\ &\le c\int_{B_\rho}|\nabla v||v|+|\nabla B||B|\\ &\le c\left[\|\nabla v\|_{L^2(B_\rho)}\|v\|_{L^2(B_\rho)}+\|\nabla B\|_{L^2(B_\rho)}\|B\|_{L^2(B_\rho)}\right],\\ \int_{Q_\rho}|p_1|^{\frac{3}{2}}\le c\rho^2&\left[A^{\frac{3}{4}}(v,\rho)E^{\frac{3}{4}}(v,\rho)+A^{\frac{3}{4}}(B,\rho)E^{\frac{3}{4}}(B,\rho)\right].\end{aligned}$$

For $x\in B_{\frac{\rho}{2}}$,

$$|p_2(x,t)|\le c-\!\!\!\!\!\!\int_{B_\rho}|p_2|\le c\left(-\!\!\!\!\!\!\int_{B_\rho}|p_2|^l\right)^{\frac{1}{l}},\quad l>1,$$

i.e. for $r\le\frac{\rho}{2}$,

$$\begin{aligned}&\frac{1}{r^2}\int_{Q_r}|p_2|^{\frac{3}{2}}\le\frac{cr}{\rho^{\frac{45}{14}}}\int_{-\rho^2}^0\|p_2\|^{\frac{3}{2}}_{L^{\frac{7}{5}}(B_\rho)},\\ &D_1(p,r)\le\frac{1}{r^2}\int_{Q_\rho}|p_1|^{\frac{3}{2}}+\frac{1}{r^2}\int_{Q_r}|p_2|^{\frac{3}{2}}\le c\left(\frac{\rho}{r}\right)^2\left[A^{\frac{3}{4}}(v,\rho)E^{\frac{3}{4}}(v,\rho)+A^{\frac{3}{4}}(B,\rho)E^{\frac{3}{4}}(B,\rho)\right]\\ &\qquad\qquad+\frac{cr}{\rho^{\frac{45}{14}}}\int_{-\rho^2}^0\left(\|p\|^{\frac{3}{2}}_{L^{\frac{7}{5}}(B_\rho)}+\|p_1\|^{\frac{3}{2}}_{L^{\frac{7}{5}}(B_\rho)}\right).\end{aligned}$$

Now

$$\frac{r}{\rho^{\frac{45}{14}}}\int_{-\rho^2}^{0}\|p_1\|^{\frac{3}{2}}_{L^{\frac{7}{5}}(B_\rho)} \le \frac{r}{\rho}D_1(p_1,\rho), \quad \frac{r}{\rho^{\frac{45}{14}}}\int_{-\rho^2}^{0}\|p\|^{\frac{3}{2}}_{L^{\frac{7}{5}}(B_\rho)} \le \frac{r}{\rho}D(p,\rho)^{\frac{3}{4}},$$

so that

$$D_1(p,r) \le c\left(\frac{\rho}{r}\right)^2\left[A^{\frac{3}{4}}(v,\rho)E^{\frac{3}{4}}(v,\rho)+A^{\frac{3}{4}}(B,\rho)E^{\frac{3}{4}}(B,\rho)\right]+c\frac{r}{\rho}D(p,\rho)^{\frac{3}{4}}.$$

On the other hand,

$$\int_{B_r}|p_1|^{\frac{7}{5}} \le c\int_{B_r}|v|^{\frac{14}{5}}+|B|^{\frac{14}{5}},$$

we have

$$D(p_1,r) \le c\left(\frac{\rho}{r}\right)^{\frac{16}{7}}C(v,B,\rho).$$

For $x\in B_{\frac{\rho}{2}}$,

$$|p_2(x,t)| \le c\left(-\!\!\!\!\!\!\int_{B_\rho}|p_2|^{\frac{7}{5}}dx\right)^{\frac{5}{7}},$$

we have

$$\begin{aligned} D(p_2,r) &= r^{-\frac{16}{7}}\int_{-r^2}^{0}\|p_2\|^2_{L^{\frac{7}{5}}(B_r)} \le c\left(\frac{r}{\rho}\right)^2 D(p_2,\rho) \\ &\le c\left(\frac{r}{\rho}\right)^2[D(p,\rho)+D(p_1,\rho)]. \end{aligned}$$

Therefore,

$$\begin{aligned} D(p,r) &\le c[D(p_1,r)+D(p_2,r)] \\ &\le c\left(\frac{\rho}{r}\right)^{\frac{16}{7}}C(v,B,\rho)+cD(p_2,r) \\ &\le c\left[\left(\frac{r}{\rho}\right)^2 D(p,\rho)+\left(\frac{\rho}{r}\right)^{\frac{16}{7}}C(v,B,\rho)\right]. \end{aligned}$$

Let η be a smooth function such that

$$0\le\eta\le 1, \quad |\nabla\eta|\le\frac{C}{\rho}, \quad \eta\equiv 1 \quad \text{in } Q_{\frac{\rho}{2}} \quad \text{and} \quad \eta=0 \quad \text{in } Q_\rho^c.$$

We decompose the magnetic field B so that

$$B=\hat{B}+\tilde{B},$$

$\hat{B}$ satisfies in $\mathbb{R}^3\times(-\rho^2,0)$,

$$(\partial_t-\Delta)\hat{B}=\partial_j(v_jB_i-v_iB_j)\eta^2,$$

and $\tilde{B}$ satisfies in Q_ρ

$$(\partial_t-\Delta)\tilde{B}=0.$$

Therefore,

$$\hat{B}(x,t)=-\int_{-\rho^2}^{0}ds\int_{B_\rho}\partial_j\Gamma(x-y,t-s)\left[(v_jB_i-v_iB_j)\eta^2\right]dy,$$

where $\Gamma(x,t)$ is the heat kernel. By Young's inequality, we get

$$\|\hat{B}(\cdot,t)\|_{L^2(B_\rho)}\le c\int_{-\rho^2}^{0}\|\nabla\Gamma(\cdot,t-s)\|_{L^\alpha(B_{2\rho})}\|v(\cdot,s)\|_{L^3(B_\rho)}\|B(\cdot,s)\|_{L^p(B_\rho)}ds,$$

where $\frac{1}{2}=\frac{1}{\alpha}+\frac{1}{3}+\frac{1}{p}-1$. And by Young's inequality again, we have

$$\|\hat{B}\|_{L^2(Q_\rho)}\le c\|\nabla\Gamma\|_{L^{\alpha,\beta}(Q_{2\rho})}\|v\|_{L^3(Q_\rho)}\|B\|_{L^{p,q}(Q_\rho)},$$

where $\frac{1}{2}=\frac{1}{\beta}+\frac{1}{3}+\frac{1}{q}-1$. Taking $\alpha=\frac{9}{7}$, $\beta=1$, $p=\frac{18}{7}$, and $q=6$, it is clear $\frac{3}{\alpha}+\frac{2}{\beta}=\frac{13}{3}$, and $\frac{3}{p}+\frac{2}{q}=\frac{3}{2}$,

$$\|\nabla\Gamma\|_{L^{\alpha,\beta}(Q_{2\rho})}\le c\rho^{\frac{3}{\alpha}+\frac{2}{\beta}-4}=c\rho^{\frac{1}{3}}.$$

Thus

$$K(\hat{B},\rho)\le cC_1(v,\rho)^{\frac{2}{3}}\rho^{-1}\|B\|^2_{L^{p,q}(Q_\rho)}\le cC_1(v,\rho)^{\frac{2}{3}}\left[A(B,\rho)+E(B,\rho)\right].$$

Since $\tilde{B}$ meets the heat equation in Q_ρ, we have for $r\le\rho$

$$-\!\!\!\!\!\!\int_{Q_r}|\tilde{B}|^2\le c-\!\!\!\!\!\!\int_{Q_\rho}|\tilde{B}|^2,$$

i.e.

$$K(\tilde{B},r)\le c\left(\frac{r}{\rho}\right)^2K(\tilde{B},\rho),$$

so that

$$\begin{aligned}K(B,r) &\leq c\left[K(\hat{B},r)+K(\tilde{B},r)\right]\\ &\leq c\left(\frac{\rho}{r}\right)^3 C_1(v,\rho)^{\frac{2}{3}}\left[A(B,\rho)+E(B,\rho)\right]+\left(\frac{r}{\rho}\right)^2 K(B,\rho).\end{aligned}$$

The proof is complete. □

As in [14], we use following analysis Lemma 2.3 which can be found in [6] to prove local estimate Lemma 2.4.

Lemma 2.3. *Let $I(s)$ be a bounded nonnegative function in the interval $[R_1,R_2]$. Suppose that for any s, $\rho \in [R_1,R_2]$ and $s<\rho$, the following inequality holds*

$$I(s) \leq \left[a_1(\rho-s)^{-\alpha}+a_2(\rho-s)^{-\beta}+a_3(\rho-s)^{-\gamma}+a_4\right]+\theta I(\rho)$$

with $\alpha>\beta>\gamma>0$, $a_i>0$, $i=1,2,3,4$ and $\theta\in[0,1)$. Then,

$$I(R_1)\leq c(\alpha,\beta,\gamma)\left[a_1(R_2-R_1)^{-\alpha}+a_2(R_2-R_1)^{-\beta}+a_3(R_2-R_1)^{-\gamma}+a_4\right].$$

Lemma 2.4. *Let (v,B,p) be a suitable weak solution to the MHD equation* (1.1) *in $Q=\Omega\times(a,b)$. Assume that $z_0=(x_0,t_0)$ and $r>0$ with $Q_r(z_0)\subset Q$. Then the following holds:*

$$\begin{aligned}&A\left(v,B,z_0,\frac{r}{2}\right)+E\left(v,B,z_0,\frac{r}{2}\right) \qquad (2.7)\\ &\leq c\left[C(v,z_0,r)^{\frac{1}{2}}C(B,z_0,r)^{\frac{1}{2}}+C(v,B,z_0,r)^{\frac{1}{2}}+C(v,z_0,r)+D(p,z_0,r)^{\frac{7}{10}}C(v,z_0,r)^{\frac{3}{20}}\right].\end{aligned}$$

Proof. Let $\frac{r}{2}\leq s<\rho\leq r<1$, and $Q_r\subset Q_1\equiv Q$. Choosing test function $\phi(x,t)=\eta_1(x)\eta_2(t)$ with $\eta_1\in C_0^\infty(B_\rho(x_0))$, $0\leq\eta_1\leq 1$ in $\mathbb{R}^3$, $\eta_1\equiv 1$ on $B_s(x_0)$, and

$$|\nabla^\alpha\eta_1|\leq\frac{C}{(\rho-s)^{|\alpha|}}$$

for all multi-index α, with $|\alpha|\leq 3$. And $\eta_2\in C_0^\infty(t_0-\rho^2,t_0+\rho^2)$, $0\leq\eta_2\leq 1$ in $\mathbb{R}$, $\eta_2(t)\equiv 1$ for $t\in[t_0-s^2,t_0+s^2]$, with

$$|\eta_2'(t)|\leq\frac{C}{\rho^2-s^2}\leq\frac{C}{r(\rho-s)}.$$

From the local energy inequality we have

$$\begin{aligned}
&\int_\Omega \left(|v|^2+|B|^2\right)\phi dx+2\int_a^t\int_\Omega \left(|\nabla v|^2+|\nabla B|^2\right)\phi dxds \\
\leq & c\int_{t_0-\rho^2}^{t_0} \left\||v|^2+|B|^2\right\|_{W^{-1,2}(B_\rho(x_0))}\|\nabla(\phi_t+\Delta\phi)\|_{L^2(B_\rho(x_0))}dt \\
&+c\int_{t_0-\rho^2}^{t_0} \left\||v|^2\right\|_{W^{-1,2}(B_\rho(x_0))}\|\nabla(v\cdot\nabla\phi)\|_{L^2(B_\rho(x_0))}dt \\
&+c\int_{t_0-\rho^2}^{t_0} \left\||v||B|\right\|_{W^{-1,2}(B_\rho(x_0))}\left\||\nabla B||\nabla\phi|+|B||\nabla^2\phi|\right\|_{L^2(B_\rho(x_0))}dt \\
&+c\int_{t_0-\rho^2}^{t_0}\int_{B_\rho} |pv\cdot\nabla\phi|dxdt \\
=:& J_1+J_2+J_3+J_4. \qquad (2.8)
\end{aligned}$$

Here we rewrite the term

$$\int_{t_0-\rho^2}^{t_0}\int_{B_\rho}|B|^2 v\cdot\nabla\phi dxdt=\int_{t_0-\rho^2}^{t_0}\int_{B_\rho}(v\otimes B):(\nabla\phi\otimes B)dxdt.$$

Denote

$$\begin{aligned}
I(s)=&\sup_{t_0-s^2\leq t\leq t_0}\int_{B_s(x_0)}|B|^2dx+\sup_{t_0-s^2\leq t\leq t_0}\int_{B_s(x_0)}|v|^2dx \\
&+\int_{t_0-s^2}^{t_0}\int_{B_s(x_0)}|\nabla B|^2dxdt+\int_{t_0-s^2}^{t_0}\int_{B_s(x_0)}|\nabla v|^2dxdt \\
=&sA(B,s,z_0)+sA(v,s,z_0)+sE(B,s,z_0)+sE(v,s,z_0) \\
=:&I_1(B,s)+I_1(v,s)+I_2(B,s)+I_2(v,s),
\end{aligned}$$

and

$$I(v,s)=I_1(v,s)+I_2(v,s),\quad I(B,s)=I_1(B,s)+I_2(B,s).$$

Estimate J_1, J_2, J_3, and J_4 as follows. For J_1, we get

$$\begin{aligned}
J_1\leq&\frac{c\rho^{\frac{3}{2}}}{(\rho-s)^3}\int_{t_0-\rho^2}^{t_0}\left\||v|^2+|B|^2\right\|_{W^{-1,2}(B_\rho)}dt \\
\leq&\frac{c\rho^{\frac{5}{2}}}{(\rho-s)^3}\left[\int_{t_0-\rho^2}^{t_0}\left\||v|^2+|B|^2\right\|^2_{W^{-1,2}(B_\rho)}dt\right]^{\frac{1}{2}}.
\end{aligned}$$

By Young's inequality, we get

$$
\begin{aligned}
J_2 \leq & c \int_{t_0-\rho^2}^{t_0} \left[\| |v|^2 \|_{W^{-1,2}(B_\rho)} \left(\frac{\|\nabla v\|_{L^2(B_\rho)}}{\rho - s} + \frac{\|v\|_{L^2(B_\rho)}}{(\rho-s)^2} \right) \right] dt \\
\leq & \frac{c}{\rho - s} \left[\int_{t_0-\rho^2}^{t_0} \| |v|^2 \|^2_{W^{-1,2}(B_\rho)} dt \right]^{\frac{1}{2}} I_2(v,\rho)^{\frac{1}{2}} \\
& + \frac{c\rho}{(\rho-s)^2} \left[\int_{t_0-\rho^2}^{t_0} \| |v|^2 \|^2_{W^{-1,2}(B_\rho)} dt \right]^{\frac{1}{2}} I_1(v,\rho)^{\frac{1}{2}} \\
\leq & \frac{1}{4} I(v,\rho) + \left[\frac{c}{(\rho-s)^2} + \frac{c\rho^2}{(\rho-s)^4} \right] \int_{t_0-\rho^2}^{t_0} \| |v|^2 \|^2_{W^{-1,2}(B_\rho)} dt.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
J_3 \leq & c \int_{t_0-\rho^2}^{t_0} \left[\| |v||B| \|_{W^{-1,2}(B_\rho)} \left(\frac{\|\nabla B\|_{L^2(B_\rho)}}{\rho - s} + \frac{\|B\|_{L^2(B_\rho)}}{(\rho-s)^2} \right) \right] dt \\
\leq & \frac{1}{4} I(B,\rho) + \left[\frac{c}{(\rho-s)^2} + \frac{c\rho^2}{(\rho-s)^4} \right] \int_{t_0-\rho^2}^{t_0} \| |vB| \|^2_{W^{-1,2}(B_\rho)} dt.
\end{aligned}
$$

For the term J_4, using Hölder's inequality and Sobolev inequality, we have

$$
\begin{aligned}
J_4 \leq & c \int_{t_0-\rho^2}^{t_0} \|p\|_{L^{\frac{7}{5}}(B_\rho(x_0))} \|v \nabla \phi\|_{L^{\frac{7}{2}}(B_\rho(x_0))} \\
\leq & c \int_{t_0-\rho^2}^{t_0} \|p\|_{L^{\frac{7}{5}}(B_\rho)} \|\nabla (v \nabla \phi)\|^{\frac{4}{7}}_{L^2(B_\rho)} \|v \nabla \phi\|^{\frac{3}{7}}_{L^{\frac{9}{4}}(B_\rho)} \\
\leq & c \int_{t_0-\rho^2}^{t_0} \|p\|_{L^{\frac{7}{5}}(B_\rho)} \left[\|\nabla v \nabla \phi)\|_{L^2(B_\rho)} + \left\| v \nabla^2 \phi \right\|_{L^2(B_\rho)} \right]^{\frac{4}{7}} \|v \nabla \phi\|^{\frac{3}{7}}_{L^{\frac{9}{4}}(B_\rho)} \\
\leq & \frac{c}{\rho - s} \|\nabla v\|^{\frac{4}{7}}_{L^2(Q_\rho(z_0))} \left[\int_{t_0-\rho^2}^{t_0} \|p\|^{\frac{7}{5}}_{L^{\frac{7}{5}}(B_\rho)} \|v\|^{\frac{3}{5}}_{L^{\frac{9}{4}}(B_\rho)} \right]^{\frac{5}{7}} \\
& + \frac{c}{(\rho-s)^{\frac{11}{7}}} \sup_t \|v\|^{\frac{4}{7}}_{L^2(B_\rho)} \int_{t_0-\rho^2}^{t_0} \|p\|_{L^{\frac{7}{5}}(B_\rho)} \|v\|^{\frac{3}{7}}_{L^{\frac{9}{4}}(B_\rho)} \\
\leq & \frac{1}{4} I(v,\rho) + \frac{c}{(\rho-s)^{\frac{7}{5}}} \int_{t_0-\rho^2}^{t_0} \|p\|^{\frac{7}{5}}_{L^{\frac{7}{5}}(B_\rho)} \|v\|^{\frac{3}{5}}_{L^{\frac{9}{4}}(B_\rho)} \\
& + \frac{c}{(\rho-s)^{\frac{11}{5}}} \left[\int_{t_0-\rho^2}^{t_0} \|p\|_{L^{\frac{7}{5}}(B_\rho)} \|v\|^{\frac{3}{7}}_{L^{\frac{9}{4}}(B_\rho)} \right]^{\frac{7}{5}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{4}I(v,\rho)+\frac{c\rho^{\frac{16}{35}}}{(\rho-s)^{\frac{7}{5}}}\left[\int_{t_0-\rho^2}^{t_0}\|p\|^2_{L^{\frac{7}{5}}(B_\rho)}\right]^{\frac{7}{10}}\left[\int_{t_0-\rho^2}^{t_0}\|v\|^4_{L^{\frac{14}{5}}(B_\rho)}\right]^{\frac{3}{20}} \\
&\quad +\frac{c\rho^{\frac{44}{35}}}{(\rho-s)^{\frac{11}{5}}}\left[\int_{t_0-\rho^2}^{t_0}\|p\|^2_{L^{\frac{7}{5}}(B_\rho)}\right]^{\frac{7}{10}}\left[\int_{t_0-\rho^2}^{t_0}\|v\|^4_{L^{\frac{14}{5}}(B_\rho)}\right]^{\frac{3}{20}}.
\end{aligned}
$$

From (2.8), using estimates above with respect to J_1,J_2,J_3 and J_4 we get

$$
\begin{aligned}
I(s)\leq& \frac{1}{2}I(\rho)+\frac{cr^{\frac{5}{2}}}{(\rho-s)^3}\left[\int_{t_0-r^2}^{t_0}\||v|^2+|B|^2\|^2_{W^{-1,2}(B_r)}\right]^{\frac{1}{2}} \\
&+\left[\frac{c}{(\rho-s)^2}+\frac{cr^2}{(\rho-s)^4}\right]\int_{t_0-r^2}^{t_0}\left[\||vB|\|^2_{W^{-1,2}(B_r)}+\||v|^2\|^2_{W^{-1,2}(B_r)}\right] \\
&+\frac{cr^{\frac{16}{35}}}{(\rho-s)^{\frac{7}{5}}}\left[\int_{t_0-r^2}^{t_0}\|p\|^2_{L^{\frac{7}{5}}(B_r)}\right]^{\frac{7}{10}}\left[\int_{t_0-r^2}^{t_0}\|v\|^4_{L^{\frac{14}{5}}(B_r)}\right]^{\frac{3}{20}} \\
&+\frac{cr^{\frac{44}{35}}}{(\rho-s)^{\frac{11}{5}}}\left[\int_{t_0-r^2}^{t_0}\|p\|^2_{L^{\frac{7}{5}}(B_r)}\right]^{\frac{7}{10}}\left[\int_{t_0-r^2}^{t_0}\|v\|^4_{L^{\frac{14}{5}}(B_r)}\right]^{\frac{3}{20}}.
\end{aligned} \tag{2.9}
$$

By Lemma 2.3, we have

$$
\begin{aligned}
I\left(\frac{r}{2}\right)\leq& cr^{-\frac{1}{2}}\left[\int_{t_0-r^2}^{t_0}\||v|^2+|B|^2\|^2_{W^{-1,2}(B_r)}\right]^{\frac{1}{2}} \\
&+cr^{-2}\int_{t_0-r^2}^{t_0}\left(\||vB|\|^2_{W^{-1,2}(B_r)}+\||v|^2\|^2_{W^{-1,2}(B_r)}\right)dt \\
&+cr^{-\frac{33}{35}}\left[\int_{t_0-r^2}^{t_0}\|p\|^2_{L^{\frac{7}{5}}(B_r)}\right]^{\frac{7}{10}}\left[\int_{t_0-r^2}^{t_0}\|v\|^4_{L^{\frac{14}{5}}(B_r)}\right]^{\frac{3}{20}}.
\end{aligned}
$$

Finally, we get

$$
\begin{aligned}
&A\left(v,B,\frac{r}{2},z_0\right)+E\left(v,B,\frac{r}{2},z_0\right) \\
&\leq cr^{-\frac{3}{2}}\left[\int_{t_0-r^2}^{t_0}\||v|^2+|B|^2\|^2_{W^{-1,2}(B_r)}\right]^{\frac{1}{2}} \\
&\quad +cr^{-3}\int_{t_0-r^2}^{t_0}\left(\||vB|\|^2_{W^{-1,2}(B_r)}+\||v|^2\|^2_{W^{-1,2}(B_r)}\right) \\
&\quad +cr^{-\frac{68}{35}}\left[\int_{t_0-r^2}^{t_0}\|p\|^2_{L^{\frac{7}{5}}(B_r)}\right]^{\frac{7}{10}}\left[\int_{t_0-r^2}^{t_0}\|v\|^4_{L^{\frac{14}{5}}(B_r)}\right]^{\frac{3}{20}}.
\end{aligned} \tag{2.10}
$$

For $f\in L^{\frac{7}{5}}(B_r(x_0))$ and $\varphi\in C_0^\infty(B_r(x_0))$, we have

$$\begin{aligned}\left|\int_{B_r(x_0)}\varphi f(x)dx\right| &\le c\int_{B_r(x_0)}\left[\int_{B_r(x_0)}\frac{|\nabla\varphi(y)|}{|x-y|^2}dy\right]|f(x)|dx\\ &=c\int_{B_r(x_0)}|\nabla\varphi(y)|\left[\int_{B_r(x_0)}\frac{|f(x)|}{|x-y|^2}dx\right]dy\\ &\le c\|\nabla\varphi\|_{L^2(B_r(x_0))}\|\mathbf{I}_1(\chi_{B_r(x_0)}|f|)\|_{L^2(B_r(x_0))},\end{aligned}$$

where $\mathbf{I}_1$ is the first order Riesz's potential defined by

$$\mathbf{I}_1(\mu)(x)=c\int_{\mathbb{R}^3}\frac{d\mu(y)}{|x-y|^2},\quad x\in\mathbb{R}^3.$$

By using Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned}\|f\|_{W^{-1,2}(B_r(x_0))} &\le \|\mathbf{I}_1(\chi_{B_r(x_0)}|f|)\|_{L^2(B_r(x_0))}\\ &\le c\|f\|_{L^{\frac{6}{5}}(B_r(x_0))}\le cr^{\frac{5}{14}}\|f\|_{L^{\frac{7}{5}}(B_r(x_0))}.\end{aligned}\tag{2.11}$$

Applying (2.11) with $f=|v|^2+|B|^2$ and $f=|v||B|$, we obtain

$$\begin{aligned}&r^{-3}\int_{t_0-r^2}^{t_0}\big\||v|^2+|B|^2\big\|^2_{W^{-1,2}(B_r)}\\ &\le r^{\frac{-16}{7}}\int_{t_0-r^2}^{t_0}\left(\|v\|^4_{L^{\frac{14}{5}}(B_r)}+\|B\|^4_{L^{\frac{14}{5}}(B_r)}\right)=C(v,B,z_0,r),\\ &r^{-3}\int_{t_0-r^2}^{t_0}\left(\big\||vB|\big\|^2_{W^{-1,2}(B_r)}+\big\||v|^2\big\|^2_{W^{-1,2}(B_r)}\right)\\ &\le\left[C(v,z_0,r)^{\frac{1}{2}}C(B,z_0,r)^{\frac{1}{2}}+C(v,z_0,r)\right].\end{aligned}$$

The proof is complete. $\square$

We need the bounded estimates for $C(B,r)$ and $D(p,r)$ with the help of the bounded of $C(v,r)$.

Lemma 2.5. *Suppose that (v,B,p) is a suitable weak solution in $Q_1(z_0)=B_1(x_0)\times(t_0-1,t_0)$. Let*

$$C(v,z_0,r)\le M\quad \textit{for any}\quad 0<r<1$$

for $M>0$. Then for every $0<r<\frac{1}{4}$, we have the following estimates:

$$A\left(v,B,z_0,\frac{r}{2}\right)+E\left(v,B,z_0,\frac{r}{2}\right)+C\left(B,z_0,\frac{r}{2}\right)+D\left(p,z_0,\frac{r}{2}\right)$$

$$\leq c\left(M, C\left(B, z_0, \frac{1}{2}\right), D\left(p, z_0, \frac{1}{2}\right)\right),$$

$$C_1(v, B, z_0, r) + D_1(p, z_0, r) \leq c\left(M, D\left(p, z_0, \frac{1}{2}\right), C\left(B, z_0, \frac{1}{2}\right)\right).$$

Proof. Without loss of generality, we consider $z_0=(0,0)$. Suppose that η is a smooth function with

$$0 \leq \eta \leq 1, \quad |\nabla \eta| \leq \frac{c}{r}, \quad \eta \equiv 1 \quad \text{in } Q_{\frac{r}{2}} \quad \text{and} \quad \eta = 0 \quad \text{in } Q_r^c.$$

Owing to $B \in L^\infty(0,T;BMO^{-1})$, there exists $U(x,t) \in L^\infty(0,T;BMO)$ such that $B = \nabla \cdot U$. Using Hölder and Sobolev inequalities, we obtain

$$\begin{aligned}
&\frac{1}{r^{\frac{16}{7}}} \int_{-r^2}^0 \left(\int_{B_r} |B|^{\frac{14}{5}} \eta^2 dx\right)^{\frac{10}{7}} dt = \frac{1}{r^{\frac{16}{7}}} \int_{-r^2}^0 \left(\int_{B_r} \sum_{j=1}^3 \partial_j U_j \cdot B |B|^{\frac{4}{5}} \eta^2 dx\right)^{\frac{10}{7}} dt \\
\leq &\frac{1}{r^{\frac{16}{7}}} \int_{-r^2}^0 \left(\int_{B_r} |U - U_{B_r}| \left(|\nabla B||B|^{\frac{4}{5}} + |B|^{\frac{9}{5}} |\nabla \eta|\right) dx\right)^{\frac{10}{7}} dt \\
\leq &\frac{1}{r^{\frac{16}{7}}} \int_{-r^2}^0 \left(\int_{B_r} |U - U_{B_r}|^6 dx\right)^{\frac{5}{21}} \left(\int_{B_r} |\nabla B|^2 dx\right)^{\frac{5}{7}} \left(\int_{B_r} |B|^{\frac{12}{5}} dx\right)^{\frac{10}{21}} dt \\
&\quad + \frac{c}{r^{\frac{26}{7}}} \int_{-r^2}^0 \left(\int_{B_r} |U - U_{B_r}|^{\frac{14}{5}} dx\right)^{\frac{25}{49}} \left(\int_{B_r} |B|^{\frac{14}{5}} dx\right)^{\frac{45}{49}} dt \\
\leq &\frac{c}{r^{\frac{67}{49}}} \left(\iint |\nabla B|^2\right)^{\frac{5}{7}} \left(\int_{-r^2}^0 \left(\int_{B_r} |B|^{\frac{14}{5}} dx\right)^{\frac{10}{7}} dt\right)^{\frac{2}{7}} \\
&\quad + \frac{c}{r^{\frac{72}{49}}} \left(\int_{-r^2}^0 \left(\int_{B_r} |B|^{\frac{14}{5}} dx\right)^{\frac{10}{7}} dt\right)^{\frac{9}{14}}.
\end{aligned}$$

Thus, we have

$$C\left(B, \frac{r}{2}\right) \leq c\left[E(B,r)^{\frac{5}{7}} C(B,r)^{\frac{2}{7}} + C(B,r)^{\frac{9}{14}}\right]. \tag{2.12}$$

Combining with (2.7) and (2.12) we obtain

$$C\left(B, \frac{r}{2}\right) \leq c(M)\left[1 + C(B,r)^{\frac{1}{2}} + D(p,r)^{\frac{7}{10}}\right]^{\frac{5}{7}} C(B,r)^{\frac{2}{7}} + cC(B,r)^{\frac{9}{14}}. \tag{2.13}$$

Let $r=\theta\rho$ with $\theta\le\frac{1}{4}$. From (2.4) and (2.13), we have

$$\begin{aligned}C(B,r)+D(p,r)^{\frac{5}{6}}\le& c(M)\theta^{-\frac{32}{49}}\left[1+\theta^{-\frac{8}{7}}C(B,\rho)^{\frac{1}{2}}+\theta^{-\frac{8}{5}}D(p,\rho)^{\frac{7}{10}}\right]^{\frac{5}{7}}C(B,\rho)^{\frac{2}{7}}\\&+c\theta^{-\frac{72}{49}}C(B,\rho)^{\frac{9}{14}}+c\theta^{\frac{5}{3}}D(p,\rho)^{\frac{5}{6}}+c\theta^{-\frac{40}{21}}C(B,\rho)^{\frac{5}{6}}+c(M,\theta).\end{aligned}$$

Using Young's inequality, we have

$$C(B,r)+D(p,r)^{\frac{5}{6}}\le\eta C(B,\rho)+\left(\eta+c\theta^{\frac{5}{3}}\right)D(p,\rho)^{\frac{5}{6}}+c(\theta,\eta,M).$$

Set $F(r)=C(B,r)+D(p,r)^{\frac{5}{6}}$. Choose $\eta>0$ and $\theta>0$ small enough, we have

$$F(r)\le\frac{1}{2}F(\rho)+c.$$

By the standard iterating argument

$$C(B,r)+D(p,r)^{\frac{5}{6}}\le c\left(M,D\left(p,\frac{1}{2}\right),C\left(B,\frac{1}{2}\right)\right),\quad r\in(0,1/4],$$

so that for $0<r\le\frac{1}{4}$,

$$A(v,B,r)+E(v,B,r)+C(B,r)+D(p,r)\le c\left(M,C\left(B,\frac{1}{2}\right),D\left(p,\frac{1}{2}\right)\right).$$

The estimates of $C_1(v,B,r)$ and $D_1(p,r)$ are immediate results. $\square$

3 Proof of Theorem 1.1 and Theorem 1.3

We shall prove the following Proposition 3.1. Theorem 1.1 is an immediate result.

Proposition 3.1. *Let* (v,B,p) *a weak solution of* (1.1) *with*

$$\|v\|_{L^\infty(0,T;L^{(3,\infty)}(\mathbb{R}^3))}+\|B\|_{L^\infty(0,T;BMO^{-1}(\mathbb{R}^3))}\le M$$

and $B(t)\in VMO^{-1}(\mathbb{R}^3)$ *for* $t\in(0,T)$, *for* $z_0=(x_0,t_0)$ *and* $R>0$ *such that* $Q_R(z_0)\subset Q_T$, (B,p) *satisfy*

$$C(B,z_0,R)+D(p,z_0,R)\le N.$$

There exists a positive number $\varepsilon(M,N)<\frac{1}{4}$ *such that if for some* $0<r\le\frac{R}{2}$,

$$r^{-3}\left|\left\{x\in B_r(x_0):|v(x,t_0)|>\varepsilon r^{-1}\right\}\right|\le\varepsilon, \tag{3.1}$$

then there exists $\rho\in[2r\varepsilon,r]$ *such that*

$$\frac{1}{\rho^2}\int_{Q_\rho(z_0)}|v|^3+|B|^3<\varepsilon_0, \tag{3.2}$$

where ε_0 *is the same number in Proposition* 1.1.

Proof. Let (v,B,p) be a weak solution of (1.1) and assume

$$\sup_{0<r<T}\|v(t)\|_{L^{(3,\infty)}(\mathbb{R}^3)}+\|B\|_{L^\infty(0,T;BMO^{-1}(\mathbb{R}^3))}\le M. \tag{3.3}$$

Note that $B\in BMO^{-1}$, we have

$$\|B\|^4_{L^4(\mathbb{R}^3)}=\|BB\|^2_{L^2(\mathbb{R}^3)}\le c\|\nabla B\|^2_{L^2(\mathbb{R}^3)}\|B\|^2_{BMO^{-1}(\mathbb{R}^3)}.$$

Also since the real interpolation $L^4=[L^{(6,\infty)},L^{(3,\infty)}]_{\frac{1}{2},4}$ holds, then

$$\|v\|_{L^4_x}\le c\|v\|^{\frac{1}{2}}_{L_x^{(6,\infty)}}\|v\|^{\frac{1}{2}}_{L_x^{(3,\infty)}}\le c\|\nabla v\|^{\frac{1}{2}}_{L^2_x}M^{\frac{1}{2}}.$$

From energy inequality and estimates above we get

$$\|(v,B)\|_{L^4(Q_T)}\le c(M,c_0), \tag{3.4}$$

which yields that (v,B,p) is a local suitable weak solution of Eqs. (1.1) and $v\in C([0,T];L^2(\mathbb{R}^3))$.

We use a contradiction argument for $z_0=(0,0)$ and $R=1$. Fixed $N,M>0$ if the assertion of the proposition were false, then there would exist $\epsilon_k\downarrow 0$, and suitable weak solutions (v_k,B_k,p_k) of (1.1) and $r_k\le\frac{1}{2}$ such that

$$\|v_k\|_{L^\infty(-1,0;L^{3,\infty}(\mathbb{R}^3))}\le M, \tag{3.5}$$

$$C(B_k,1)+D(p_k,1)\le N, \tag{3.6}$$

$$r_k^{-3}\left|\left\{x\in B_{r_k}(0):|v_k(x,0)|>\epsilon_k r_k^{-1}\right\}\right|\le\epsilon_k, \tag{3.7}$$

and for all $\rho\in[2r_k\epsilon_k,r_k]$,

$$\frac{1}{\rho^2}\int_{Q_\rho(0)}|v_k|^3+|B_k|^3>\frac{\varepsilon_0}{2}. \tag{3.8}$$

Since for $0<r\le 1$

$$C(v_k,r)=r^{-\frac{16}{7}}\int_{-r^2}^0|B_r|^{\frac{2}{21}}\|v_k\|^4_{L^{(3,\infty)}(B_r)}$$

$$\leq c\|v_k\|^4_{L^\infty(-1,0;L^{(3,\infty)}(B_1))}\leq M^4,$$

combining the estimate and (3.6) with Lemma 2.5, we get, for $0<r\leq\frac{1}{2}$,

$$A(v_k,B_k,r)+E(v_k,B_k,r)+C(B_k,r)+D(p_k,r)\leq c(M,N).$$

Similarly, for any $z_0\in Q_{\frac{1}{2}}$ and $0<r\leq\frac{1}{2}$, we have

$$A(v_k,B_k,r,z_0)+E(v_k,B_k,r,z_0)+C(B_k,r,z_0)+D(p_k,r,z_0)\leq c(M,N).$$

Define, for $(x,t)\in Q_{r_k^{-1}}$,

$$\begin{cases} U_k(x,t)=r_kv_k\big(r_kx,r_k^2t\big), \\ D_k(x,t)=r_kB_k\big(r_kx,r_k^2t\big), \\ P_k(x,t)=r_k^2p_k\big(r_kx,r_k^2t\big). \end{cases} \tag{3.9}$$

Obviously, (U_k,D_k,P_k) are weak solutions to system (1.1). Now for $a>0$, and $ar_k\leq\frac{1}{2}$,

$$\begin{aligned}
&\|U_k\|_{L^\infty(-r_k^{-1},0;L^{(3,\infty)}(B_{r_k^{-1}}))}=\|v_k\|_{L^\infty(-1,0;L^{(3,\infty)}(B_1))}\leq M,\\
&C(D_k,a)+D(P_k,a)=C(B_k,ar_k)+D(p_k,ar_k)\leq N,\\
&C(U_k,a)=C(v_k,ar_k)\\
\leq\,&(ar_k)^{-\frac{16}{7}}\int_{-(ar_k)^2}^0|B_{ar_k}|^{\frac{2}{21}}\|v_k\|^4_{L^{3,\infty}(B_{ar_k})}\\
\leq\,&c\|v_k\|^4_{L^\infty(-1,0;L^{(3,\infty)}(B_1))}\leq M^4.
\end{aligned}$$

We have by Lemma 2.5 again

$$\begin{aligned}
&A(U_k,D_k,a)+E(U_k,D_k,a)+D(P_k,a)\\
&\quad+C_1(U_k,D_k,a)+D_1(P_k,a)\leq c(M,N),
\end{aligned} \tag{3.10}$$

so that

$$\begin{aligned}
\|U_k\|^4_{L^4(Q_a)}&\leq c\int_{-a^2}^0\|U_k\|^2_{L^{(3,\infty)}(B_a)}\|U_k\|^2_{L^6(B_a)}\\
&\leq c\int_{-a^2}^0\|U_k\|^2_{L^{(3,\infty)}(B_a)}\|\nabla U_k\|^2_{L^2(B_a)}\\
&\quad\quad+c\int_{-a^2}^0|B_a|\|U_k\|^2_{L^{(3,\infty)}(B_a)}\|U_k\|^2_{L^2(B_a)}\\
&\leq caM^2\big(A(U_k,a)+E(U_k,a)\big),\\
\|D_k\|^4_{L^4(Q_a)}&\leq ac(M,N)\big(A(D_k,a)+E(D_k,a)\big).
\end{aligned}$$

Thus the L^p estimate holds for (U_k,D_k,P_k) in Q_a, for any $a>0$,

$$\begin{aligned}\int_{Q_a}|U_k|^4+|D_k|^4+|\partial_t U_k|^{\frac{4}{3}}+|\partial_t D_k|^{\frac{4}{3}}+|\nabla^2 U_k|^{\frac{4}{3}}\\ +|\nabla^2 D_k|^{\frac{4}{3}}+|\nabla P_k|^{\frac{4}{3}}\le c_2(a,M,N).\end{aligned} \tag{3.11}$$

By Aubin-Lion's lemma, there exists a triplet (u,e,q) such that

$$\begin{cases} U_k\to u & \text{in } L^3(Q_a), \\ D_k\to e & \text{in } L^3(Q_a), \\ P_k\rightharpoonup q & \text{in } L^{\frac{3}{2}}(Q_a), \end{cases} \tag{3.12}$$

and

$$U_k\to u,\quad D_k\to e \quad \text{in } C\Big(\big[-a^2,0\big];L^{\frac{4}{3}}(B_a)\Big).$$

Using estimates above, the limit function (u,e,q) satisfy, in the sense of suitable weak solutions on $\mathbb{R}^3\times(-\infty,0)$,

$$\begin{cases} u_t-\Delta u+u\cdot\nabla u+\nabla q=e\cdot\nabla e, \\ \nabla\cdot u=0, \\ e_t-\Delta e+u\cdot\nabla e=e\cdot\nabla u. \end{cases} \tag{3.13}$$

From (3.7) and (3.8), we get

$$\big|\{x\in B(0):|U_k(x,0)|>\epsilon_k\}\big|<\epsilon_k, \tag{3.14}$$

and for $\rho\in[2\epsilon_k,1]$

$$\frac{1}{\rho^2}\int_{Q_\rho}|U_k|^3+|D_k|^3>\frac{\varepsilon_0}{2}. \tag{3.15}$$

Taking limit we get

$$u(\cdot,0)=0 \quad \text{in } B_1(0), \tag{3.16}$$

and for $\rho\in(0,1]$

$$\frac{1}{\rho^2}\int_{Q_\rho}|u|^3+|e|^3\ge\frac{\varepsilon_0}{2}. \tag{3.17}$$

The crucial point here is a reduction to backward uniqueness for the heat operator with lower order terms as [3]. Set

$$u_k=\rho_k u\big(\rho_k x,\rho_k^2 t\big),\quad e_k=\rho_k e\big(\rho_k x,\rho_k^2 t\big),\quad q_k=\rho_k^2 q\big(\rho_k x,\rho_k^2 t\big).$$

Then (u_k,e_k,q_k) satisfy (3.13), and similar to (3.10) for any $a>0$

$$A(u_k,e_k,a)+E(u_k,e_k,a)+D(q_k,a)+C_1(u_k,e_k,a)+D_1(q_k,a)\leq c(M,N). \tag{3.18}$$

As before there exists a triplet $(\widetilde{v},\widetilde{e},\widetilde{q})$ is suitable weak solution of (3.13) such that

$$\begin{cases} u_k\to\widetilde{v} & \text{in } L^3(Q_a), \\ e_k\to\widetilde{e} & \text{in } L^3(Q_a), \\ q_k\rightharpoonup\widetilde{q} & \text{in } L^{\frac{3}{2}}(Q_a), \end{cases} \tag{3.19}$$

and

$$u_k\to\widetilde{v},\quad e_k\to\widetilde{e} \quad \text{in } C\Big(\big[-a^2,0\big];L^{\frac{4}{3}}(B_a)\Big). \tag{3.20}$$

From (3.16) we get

$$\widetilde{v}(\cdot,0)=0 \quad \text{in } \mathbb{R}^3, \tag{3.21}$$

from (3.17) and take $\rho=\rho_k$

$$\int_{Q_1}|\widetilde{v}|^3+|\widetilde{e}|^3\geq\frac{\varepsilon_0}{2}. \tag{3.22}$$

On the other hand, for fixed z_0 and $0<R\leq\frac{1}{2r_k}$, as (3.18)

$$\begin{aligned} &A(u_k,e_k,z_0,R)+E(u_k,e_k,z_0,R)+D(q_k,z_0,R) \\ &\quad +C_1(u_k,e_k,z_0,R)+D_1(q_k,z_0,R)\leq c(M,N). \end{aligned} \tag{3.23}$$

By the Fubini theorem, we have, for $d_{\widetilde{v}(t)}=|\{x\in\mathbb{R}^3:|v(x,t)|>\gamma\}|$,

$$\begin{aligned} &\Big|\Big\{(x,t)\in\mathbb{R}^3\times(-T,0):|\widetilde{v}(x,t)|>\gamma\Big\}\Big| \\ &=\int_{-T}^0 d_{\widetilde{v}(t)}(\gamma)dt\leq\gamma^{-3}M^3T. \end{aligned}$$

Hence for any $\eta>0$, there exists a B_R such that

$$\Big|\Big\{(x,t)\in\big(\mathbb{R}^3\backslash B_R\big)\times(-T,0):|\widetilde{v}(x,t)|>\gamma\Big\}\Big|<\eta.$$

Let $Q_1(z_0)\subset(\mathbb{R}^3\backslash B_R)\times(-T,0]$, by (3.23), we have

$$A(\widetilde{v},\widetilde{e},z_0,\theta)+E(\widetilde{v},\widetilde{e},z_0,\theta)\leq C(M,N)$$

for any $0<\theta\leq1$. Thus, by the interpolation inequality we have

$$\theta^{-\frac{5}{3}}\int_{Q_\theta(z_0)}|\widetilde{v}|^{\frac{10}{3}}+|\widetilde{e}|^{\frac{10}{3}}\leq c(M,N).$$

Thus

$$\begin{aligned}C_1(\widetilde{v},1,z_0) &\leq \gamma^3|Q_1(z_0)| + \iint_{Q_1(z_0)\cap\{|\widetilde{v}|>\gamma\}} |\widetilde{v}|^3 dxdt\\ &\leq c\gamma^3 + \|\widetilde{v}\|_{L^{\frac{10}{3}}(Q_1(z_0))} \big|Q_1(z_0)\cap\{|\widetilde{v}|>\gamma\}\big|^{\frac{1}{10}}\\ &\leq c\left(\gamma^3+\eta^{\frac{1}{10}}\right).\end{aligned}$$

For any $\epsilon>0$ we choose γ and η such that $C_1(\widetilde{v},1,z_0)<\epsilon$.

It is easy to see that, by (2.5),

$$K(\widetilde{e},z_0,\theta) \leq c\theta^{-3}C_1(\widetilde{v},z_0,1)^{\frac{2}{3}}\big[A(\widetilde{e},z_0,1)+E(\widetilde{e},z_0,1)\big]+\theta^2 K(\widetilde{e},z_0,1). \tag{3.24}$$

Utilizing (3.24) and Hölder's inequality we have

$$\begin{aligned}C_1(\widetilde{e},z_0,\theta) &\equiv \theta^{-2}\int_{Q_\theta(z_0)}|\widetilde{e}|^3 \leq \left[\theta^{-\frac{5}{3}}\int_{Q_\theta(z_0)}|\widetilde{e}|^{\frac{10}{3}}\right]^{\frac{3}{4}}\left[\theta^{-3}\int_{Q_\theta(z_0)}|\widetilde{e}|^2\right]^{\frac{1}{4}}\\ &\leq c(M,N)K(\widetilde{e},z_0,\theta)^{\frac{1}{4}} \leq c(M,N)\left(\theta^{-\frac{3}{4}}\epsilon^{\frac{1}{8}}+\theta^{\frac{1}{2}}\right).\end{aligned} \tag{3.25}$$

Thus

$$C_1(\widetilde{v},\widetilde{e},z_0,\theta) \leq \theta^{-2}\epsilon + c(M,N)\left(\theta^{-\frac{3}{4}}\epsilon^{\frac{1}{8}}+\theta^{\frac{1}{2}}\right).$$

First, we take θ such that $c(M,N)\theta^{\frac{1}{2}} \leq \frac{\varepsilon_0}{2}$, then take ϵ such that

$$\theta^{-2}\epsilon + c(M,N)\theta^{-\frac{3}{4}}\epsilon^{\frac{1}{8}} \leq \frac{\varepsilon_0}{2},$$

i.e.

$$C_1(\widetilde{v},\widetilde{e},z_0,\theta) \leq \varepsilon_0,$$

which implies that z_0 is a regular point by Proposition 1.1. Therefore, $(\widetilde{v},\widetilde{e},\widetilde{q})$ are smooth and their derivatives are bounded in $(\mathbb{R}^3\setminus B_{2R})\times(-T/2,0)$. Next we show

$$\widetilde{e}(\cdot,0)=0. \tag{3.26}$$

For any $B(y)$ and $\phi\in C_0^\infty(B(y))$, we have

$$\left|\int_{B(y)}\widetilde{e}(x,0)\phi dx\right| \leq \int_{B(y)}|\widetilde{e}(x,0)-e_k(x,0)|dx + \left|\int_{B(y)}e_k\phi\right|.$$

Note that $e(\cdot,t)\in VMO^{-1}(\mathbb{R}^3)$ for $t\in(-T,0]$, $e(\cdot,0)=\partial_j U_j, U_j\in VMO(\mathbb{R}^3)$,

$$\begin{aligned}&\int_{B_1(y)} e_k(x,0)\varphi(x)dx\\=&r_k\int_{B_{r_k}(r_k y)}\partial_j U_j(w)\varphi\left(\frac{w}{r_k}\right)r_k^{-3}dw\\=&-r_k^{-3}\int_{B_{r_k}(r_k y)}(U_j(w)-(U_j)_{B_{r_k}})\partial_j\varphi\left(\frac{w}{r_k}\right)dw.\end{aligned}$$

From (3.20) and $U_j\in VMO(\mathbb{R}^3)$ we get

$$\widetilde{e}(x,0)=0,\quad x\in\mathbb{R}^3.$$

The backward uniqueness theorem of parabolic equations [3], we conclude

$$\widetilde{e}(x,t)=0\quad\text{in }\ \mathbb{R}^3\backslash\bar{B}_{2R}(0)\times(-T/2,0].$$

Using unique continuation theorem of parabolic equation in the bounded domain again [3], we conclude that

$$\widetilde{e}(x,t)=0\quad\text{in }\ \mathbb{R}^3\times(-T/2,0).$$

Thus, $\widetilde{v}$ satisfies Navier-Stokes equations in $\mathbb{R}^3\times(-T/2,0)$

$$\begin{cases}\partial_t\widetilde{v}-\Delta\widetilde{v}+\widetilde{v}\cdot\nabla\widetilde{v}+\nabla\widetilde{q}=0,\\ \operatorname{div}\widetilde{v}=0.\end{cases}$$

Using (3.21) and backward uniqueness of heat operator again [3], we get

$$\widetilde{v}(x,t)=0\quad\text{in }\ \mathbb{R}^3\times(-T/2,0),$$

which is a contradiction with (3.22). □

Theorem 1.3 is obvious from the process of proof before.

4 Proof of Theorem 1.2

Define for any $r>0$ such that $Q_r(z_0)\subset Q_T$,

$$C_2(v,z_0,r)=r^{-1}\int_{Q_r(z_0)}|v|^4,\quad C_2(B,z_0,r)=r^{-1}\int_{Q_r(z_0)}|B|^4.$$

By interpolation inequality we have

$$\begin{aligned}&C(B,z_0,R)\le A(B,z_0,R,)^{\frac{6}{7}}C_2(B,z_0,R)^{\frac{4}{7}}\le c(c_0,R),\\&C_2(v,z_0,R)\le cM^2\big[A(v,z_0,R)+E(v,z_0,R)\big],\end{aligned}$$

where $\|v\|_{L^\infty(-1,0;L^{3,\infty}(\mathbb{R}^3))}\le M$. On the other hand, by Calderón-Zygmund theorem,

$$\|p(t)\|^s_{L^s(\mathbb{R}^3)}\le c\|(v,B)(t)\|^{2s}_{L^{2s}(\mathbb{R}^3)}\quad\text{for}\quad 1<s<\infty,$$

and

$$(v,B)\in L^4\Big(0,T;L^3(\mathbb{R}^3)\Big),$$

which implies

$$p\in L^2\Big(0,T;L^{\frac{3}{2}}(\mathbb{R}^3)\Big),$$

we have

$$D(p,z_0,R)\le c(c_0,R).$$

Since for any $0<r\le R$,

$$C(v,z_0,r)\le cM^4,$$

by Lemma 2.5, we have for any $0<r\le\frac{R}{2}$ and $z_0\in\Omega\times(0,T)$, $\Omega\subset\subset\mathbb{R}^3$,

$$\begin{aligned}&A(v,B,z_0,r)+E(v,B,z_0,r)+C_1(v,B,z_0,r)+C_2(v,B,z_0,r)\\&\quad+C(B,z_0,r)+D(p,z_0,r)+D_1(p,z_0,r)\\&\le c(M,R,c_0)\equiv N.\end{aligned}\tag{4.1}$$

The number $\varepsilon(M,N)$ of Proposition 3.1 can be determined.

Let S be a singular points set of (v,B) at $\{(x,T):x\in\mathbb{R}^3\}$. Assume that it contains more than $M^3\varepsilon^{-4}$ elements. Letting $P=[M^3\varepsilon^{-4}]+1$, we can find P different singular points $\{(x_k,T):k=1,\ldots,P\}$ of the set S. We can choose $R_0\le R$ such that $B_{R_0}(x_k)\cap B_{R_0}(x_l)=\emptyset$, $k\neq l$, and bounded domain Ω such that $\cup_{k=1}^P B_{R_0}(x_k)\subset\Omega$. According to Proposition 3.1, for all $r\in(0,\frac{R_0}{2}]$, it holds true

$$\varepsilon\le\frac{1}{r^3}\left|\left\{x\in B_r(x_k):|v(x,T)|>\frac{\varepsilon}{r}\right\}\right|\tag{4.2}$$

for all $k=1,\ldots,P$. In particular, taking $r=r_0=\frac{R_0}{2}$, we have

$$P\varepsilon\le\sum_{k=1}^P\frac{1}{r_0^3}\left|\left\{x\in B_{r_0}(x_k):|v(x,T)|>\frac{\varepsilon}{r_0}\right\}\right|$$

$$
\begin{aligned}
&\le \frac{1}{r_0^3}\left|\left\{x\in\cup_{k=1}^{P}B_{r_0}(x_k):|v(x,T)|>\frac{\varepsilon}{r_0}\right\}\right| \\
&\le \frac{1}{r_0^3}\left|\left\{x\in\Omega:|v(x,T)|>\frac{\varepsilon}{r_0}\right\}\right| \\
&\le \varepsilon^{-3}\|v(\cdot,T)\|^3_{L^{(3,\infty)}(\Omega)}\le \varepsilon^{-3}M^3,
\end{aligned}
$$

i.e. $P\le M^3\varepsilon^{-4}<P$, which is a contradiction. □

5 Proof of Proposition 1.1

According to the L^p theorem of Stokes system in [4], if $\mathbf{f}\in W^{-1,q}(\Omega;\mathbb{R}^3)$, $1<q<\infty$, and Ω is a C^1 bounded domain, the following Stokes equations:

$$
\begin{cases}
-\Delta v+\nabla p=\mathbf{f}, \\
\operatorname{div} v=0, \quad \int_\Omega p=0, \\
v|_{\partial\Omega}=0,
\end{cases} \tag{5.1}
$$

there exists exactly one solution $(v,p)\in W^{1,q}(\Omega)\times L^q(\Omega)$, and

$$
\|\nabla v\|_{L^q(\Omega)}+\|p\|_{L^q(\Omega)}\le c(q)\|\mathbf{f}\|_{W^{-1,q}(\Omega)}. \tag{5.2}
$$

The Wolf's local pressure projection $\mathcal{P}_q$ tell us,

$$
\mathcal{P}_q: W^{-1,q}(\Omega)\to W^{-1,q}(\Omega), \quad \mathcal{P}_q(\mathbf{f})=p.
$$

As in [22], we have following lemma.

Lemma 5.1. *Let (u,B) be a weak solution of* (1.1)*, then for every C^2 bounded sub-domain Ω and any $\phi\in C_0^\infty(\Omega\times(0,T))$, there holds*

$$
\begin{aligned}
-\int_0^T\int_\Omega(u+\nabla p_h)\cdot\phi_t-\int_0^T\int_\Omega(u\otimes u-B\otimes B+p_1\mathbf{I}):\nabla\phi& \\
+\int_0^T\int_\Omega(\nabla u-p_2\mathbf{I}):\nabla\phi=0,&
\end{aligned} \tag{5.3}
$$

i.e. set $v_\Omega=v:=u+\nabla p_h$,

$$
\partial_t v+\operatorname{div}(u\otimes u)+\nabla p_1+\nabla p_2=\Delta v+B\cdot\nabla B, \tag{5.4}
$$

where $\mathbf{I}$ *is identity matrix, and*

$$\begin{cases} p_h = -\mathcal{P}_2(u), \\ p_1 = -\mathcal{P}_{\frac{3}{2}}(u\otimes u - B\otimes B), \\ p_2 = \mathcal{P}_2(\Delta u). \end{cases}$$

In addition, the following estimates hold for a.e. $t\in(0,T)$

$$\begin{cases} \|\nabla p_h(t)\|_{L^m(\Omega)} \le c\|u(t)\|_{L^m(\Omega)}, \quad 1<m\le 6, & \text{(5.5a)} \\ \|p_1(t)\|_{L^{\frac{3}{2}}(\Omega)} \le c\|u\otimes u - B\otimes B\|_{L^{\frac{3}{2}}(\Omega)}, & \text{(5.5b)} \\ \|p_2(t)\|_{L^2(\Omega)} \le c\|\nabla u(t)\|_{L^2(\Omega)}. & \text{(5.5c)} \end{cases}$$

Here $c>0$ *depends on the geometry of* Ω *and in* (5.5a) *on* m *only. In particular, if* Ω *is the ball* $B_R(x_0)$ *then* c *in* (5.5a) *depends only on* m*, while in* (5.5b) *and* (5.5c) c *is an absolute constant.*

Hence we have local energy inequality, for $\varphi\in C_0^\infty(\Omega\times(0,T))$

$$\begin{aligned} &\int_\Omega \big(|v(t)|^2+|B(t)|^2\big)\varphi + 2\int_0^t\int_\Omega \big(|\nabla v|^2+|\nabla B|^2\big)\varphi \\ \le &\int_0^t\int_\Omega \big(|v|^2+|B|^2\big)(\varphi_t+\Delta\varphi) + \int_0^t\int_\Omega \big(|v|^2u+|B|^2v\big)\cdot\nabla\varphi \\ &+\int_0^t\int_\Omega 2(p_1+p_2)v\cdot\nabla\varphi + 2\int_0^t\int_\Omega \big(u\otimes v:\nabla^2 p_h\varphi - B\otimes v: B\otimes\nabla\varphi\big). \end{aligned} \tag{5.6}$$

Note that the suitable weak solution of (1.1) satisfies the local energy inequality (5.6).

From local energy inequality we can get the Caccioppoli-type estimates

$$\|W\|^2_{L^{\frac{10}{3}}(Q_{\frac{R}{2}})} + \|\nabla W\|^2_{L^2(Q_{\frac{R}{2}})} \le cR^{-\frac{1}{3}}\|W\|^2_{L^3(Q_R)} + cR^{-1}\|W\|^3_{L^3(Q_R)}. \tag{5.7}$$

Here $W=(u,B)$, and

$$\begin{aligned} &\|W\|^2_{L^k(Q_r)} = \|u\|^2_{L^k(Q_r)} + \|B\|^2_{L^k(Q_r)}, \\ &\|\nabla W\|^2_{L^2(Q_r)} = \|\nabla u\|^2_{L^2(Q_r)} + \|\nabla B\|^2_{L^2(Q_r)}. \end{aligned}$$

Obviously,

$$C_1(W,z_0,r) = C_1(u,B,z_0,r), \quad E(W,z_0,r) = E(u,B,z_0,r).$$

Proposition 1.1 is an immediate result of following lemma.

Lemma 5.2. *Suppose that* (u,B) *is a local suitable weak solution of* (1.1). *Then there exist universal constants* $\varepsilon^*>0$ *and* $\theta\in(0,\frac{1}{4}]$ *with following property. For any* $\varepsilon\in(0,\varepsilon^*]$ *if*

$$C_1(W,z_0,1)\le\varepsilon,$$

then

$$C_1(W,z_0,\theta)\le\varepsilon.$$

Proof. We prove by contradiction. Let $\theta\in(0,\frac{1}{4}]$ be a constant to be specified later. Suppose there exist a decreasing sequence $\{\varepsilon_n\}$ converging to 0, and a sequence of pairs of local suitable weak solutions (u_n,B_n,p_n) such that

$$C_1(W_n,z_0,1)=\varepsilon_n^3, \tag{5.8}$$

$$C_1(W_n,z_0,\theta)>\varepsilon_n^3. \tag{5.9}$$

Define

$$(v_n,e_n,q_n)=\left(\frac{u_n}{\varepsilon_n},\frac{B_n}{\varepsilon_n},\frac{p_n}{\varepsilon_n}\right),$$

then they satisfy

$$\begin{aligned}&\partial_t v_n+\varepsilon_n v_n\cdot\nabla v_n+\nabla q_n=\Delta v_n-\varepsilon_n\mathrm{div}(e_n\otimes e_n),\quad \mathrm{div}v_n=0,\\&\partial_t e_n+\varepsilon_n v_n\cdot\nabla e_n-\Delta e_n=\varepsilon_n e_n\cdot\nabla v_n.\end{aligned}$$

Write $w_n=(v_n,e_n)$, then

$$C_1(w_n,z_0,1)=1, \tag{5.10}$$

$$C_1(w_n,z_0,\theta)>1. \tag{5.11}$$

Using the Caccioppoli estimate (5.7) we conclude

$$\|w_n\|_{L^{\frac{10}{3}}(Q_{\frac{1}{2}}(z_0))}+\|\nabla w_n\|_{L^2(Q_{\frac{1}{2}}(z_0))}\le c, \tag{5.12}$$

which implies

$$\|w_n\cdot\nabla w_n\|_{L^{\frac{5}{4}}(Q_{\frac{1}{2}}(z_0))}\le\|\nabla w_n\|_{L^2(Q_{\frac{1}{2}}(z_0))}\|w_n\|_{L^{\frac{10}{3}}(Q_{\frac{1}{2}}(z_0))}\le c.$$

The coercive estimate for the Stokes system (see, e.g., [13]) with a suitable cut-off function implies

$$\int_{Q_{\frac{1}{3}}(z_0)}|\partial_t w_n|^{\frac{5}{4}}+|\nabla^2 w_n|^{\frac{5}{4}}+|\nabla q_n|^{\frac{5}{4}}+|w_n|^{\frac{5}{4}}\le c,$$

where the constant c is independent of n. Thanks to the compact embedding theorem and (5.12), there exist $w\in L^3(Q_{\frac{1}{3}}(z_0))$ and $q\in L^{\frac{5}{4}}(Q_{\frac{1}{3}}(z_0))$ such that

$$\begin{aligned} &w_n\to w=(v,e) && \text{in } L^3\big(Q_{\frac{1}{3}}(z_0)\big),\\ &q_n\rightharpoonup q && \text{in } L^{\frac{5}{4}}\big(Q_{\frac{1}{3}}(z_0)\big). \end{aligned}$$

Thus (v,e,q) satisfy

$$\partial_t v-\Delta v+\nabla q=0,\quad \operatorname{div} v=0,\quad \partial_t e-\Delta e=0.$$

Moreover

$$\|w\|_{L^3(Q_{\frac{1}{3}}(z_0))}+\|q\|_{L^{\frac{5}{4}}(Q_{\frac{1}{3}}(z_0))}\le c.$$

By the classical estimate of the Stokes system [19], we get

$$\sup_{Q_{\frac{1}{3}}(z_0)}|w|\le c,$$

which implies that for $0<\theta\le\frac{1}{3}$

$$C_1(w,z_0,\theta)\le c\theta^3.$$

This contradicts (5.11), if we choose θ sufficiently small. The lemma is proved. $\square$

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References

[1] L. Caffarelli, R. Kohn and L. Nirenberg, *Partial regularity of suitable weak solution of the Navier-Stokes equations*, Comm. Pure Appl. Math. 35(6) (1982), 771–831.

[2] H. J. Choe, J. Wolf and M. Yang, *On regularity and singularity for $L^\infty(0,T;L^{3,\omega}(\mathbb{R}^3))$ solutions to the Navier-Stokes equations*, arXiv: 1611.0472v1, 2016.

[3] L. Eskauriaza, G. Seregin and V. Šverák, *$L^{3,\infty}$-solutions to the Navier-Stokes equations and backward uniqueness*, in: *Nonlinear Problems in Mathematical Physics Related Topics II*, 18(2) (2003), 353–366.

[4] G. P. Galdi, C. G. Simader and H. Sohr, *On the Stokes problem in Lipschitz domains*, Annali Di Mat. Pura Ed Appl. 167 (1994), 147–163.

[5] Y. Giga, H. Sohr, *Abstract L^p-estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains*, J. Funct. Anal. 102 (1991), 72–94.

[6] E. Giusti, *Direct Methods in the Calculus of Variations*, World Scientific Publishing, 2003.

[7] C. He and Z. P. Xin, *On the regularity of weak solutions to the magnetohydrodynamic equations*, J. Differential Equations 213(2) (2005), 235–254.

[8] E. Hopf, *Über die Anfangwertaufgaben für die hydromischen Grundgleichungen*, Math. Nach. 4 (1951), 213–321.

[9] J. Leray, *Sur le mouvement d'un liquide visqueux emlissant l'espace*, (French) Acta Math. 63(1) (1934), 193–248.

[10] F. H. Lin, *A new proof of the Caffarelli-Kohn-Nirenberg theorem*, Commun. Pure Appl. Math. 51 (1998), 241–257.

[11] X. Liu and Y. Liu, *Regularity for 3-D MHD equations in Lorentz space*, Preprint.

[12] A. Mahalov, B. Nicolaenko and T. Shilkin, *$L^{3,\infty}$-solutions to the MHD equations*, J. Math. Sci. 143(2) (2007), 2911–2923.

[13] P. Maremonti, V. A. Solonnikov, *On the estimate of solutions of evolution Stokes problem in anisotropic Sobolev spaces with a mixed norm*, Zap. Nauchn. Sem. LOMI 223 (1994), 124–150.

[14] N. C. Phuc, *The Navier-Stokes equations in nonendpoint borderline Lorentz spaces*, J. Math. Fluid Mech. 17(4) (2015), 741–760.

[15] G. Seregin, *On smoothness of $L_{3,\infty}$-solutions to the Navier-Stokes equations up to boundary*, Mathematische Annalen 332(1) (2005), 219–238.

[16] G. Seregin, *A note on weak solutions to the Navier-Stokes equations that are locally in $L^\infty(L^{3,\infty})$*, arXiv:1906.06707v1, 2019.

[17] M. Sermange, R. Teman, *Some mathematical questions related to the MHD equations*, Comm. Pure Appl. Math. 36 (1983), 635–664.

[18] J. Serrin, *On the interior regularity of weak solutions of the Navier-Stokes equations*, Archive for Rational Mechanics and Analysis 9(1) (1962), 187–195.

[19] V. A. Solonnikov, *Estimates of solutions to the linearized system of the Navier-Stokes equations*, Trudy of the Steklov Math. Inst. Vol. LXX, (1964), 213–317.

[20] W. Wang and Z. Zhang, *Regularity of weak solutions for the Navier-Stokes equations in the class $L^\infty(-1,0;BMO^{-1})$*, Commun. Contemp. Math. 14(3) (2012), 1250020, pp. 24.

[21] W. Wang and Z. Zhang, *Limiting case for the regularity criterion to the 3-D magnetohydrodynamics equations*, J. Differential Equations 252(10) (2012), 5751–5762.

[22] J. Wolf, *On the local regularity of suitable weak solutions to the generalized Navier-Stokes equations*, Ann. Univ. Ferrara 61 (2015), 149–171.

[23] J. Wu, *Regularity results for weak solutions of the 3D MHD equations*, Discrete Contin. Dyn. Syst. 10(1-2) (2012), 543–556.